ON SUGENO INTEGRAL AS AN AGGREGATION FUNCTION*

Jean-Luc Marichal
Department of Management, FEGSS, University of Liège,
Boulevard du Rectorat 7 - B31, 4000 Liège, Belgium.
Email: jl.marichal@ulg.ac.be

Abstract

The Sugeno integral, for a given fuzzy measure, is studied under the viewpoint of aggregation. In particular, we give some equivalent expressions of it. We also give an axiomatic characterization of the class of all the Sugeno integrals.

1 Introduction

Aggregation refers to the process of combining numerical values into a single one, so that the final result of aggregation takes into account all the individual values. In decision making, values to be aggregated are typically preference or satisfaction degrees and thus belong to the unit interval $[0, 1]$. For more details, see [1].

This paper aims at investigating the Sugeno integral (see [3, 4]) which can be regarded as an aggregation function. In particular, we state that any Sugeno integral is a weighted max-min function, that is, setting $X = \{1, \dots, m\}$, a function of the form

$$M^{(m)}(x_1, \dots, x_m) = \bigvee_{T \subseteq X} [a_T \wedge (\bigwedge_{i \in T} x_i)], \quad a_T \in [0, 1],$$

where a is a set function satisfying $a_\emptyset = 0$ and $\bigvee_{T \subseteq X} a_T = 1$. We then state that those functions can also be written as

$$M^{(m)}(x_1, \dots, x_m) = \bigwedge_{T \subseteq X} [b_T \vee (\bigvee_{i \in T} x_i)], \quad b_T \in [0, 1],$$

(weighted min-max functions) where b is a set function satisfying $b_\emptyset = 1$ and $\bigwedge_{T \subseteq X} b_T = 0$. The correspondance formulae $b = b(a)$ and $a = a(b)$ are given as well. For instance, we have

$$(.1 \wedge x_1) \vee (.3 \wedge x_2) \vee (x_2 \wedge x_3) = (.1 \vee x_2) \wedge (.3 \vee x_3) \wedge (x_1 \vee x_2).$$

We also propose an axiomatic characterization of this class of functions based on some aggregation properties: the increasingness and the stability for minimum and maximum with the same unit.

2 The Sugeno integral as an aggregation function

We first want to define the concept of aggregation function. Without loss of generality, we will assume that the information to be aggregated consists of numbers belonging to the interval $[0, 1]$ as required in most applications. In fact, all the definitions and results presented in this paper can be defined on any closed interval $[a, b]$ of the real line.

Let m denote any strictly positive integer.

Definition 2.1 *An* aggregation function *defined on* $[0, 1]^m$ *is a function* $M^{(m)} : [0, 1]^m \to \mathbb{R}$.

We consider a discrete set of m elements $X = \{1, \dots, m\}$, which could be players of a cooperative game, criteria, attributes or voters in a decision making problem. $\mathcal{P}(X)$ indicates the power set of X, i.e. the set of all subsets in X.

In order to avoid heavy notations, we introduce the following terminology. It will be used all along this paper.

*To be presented at EUFIT'98, Aachen, Germany, September 7–10, 1998.

- For all $T \subseteq X$, the characteristic vector of T in $\{0,1\}^m$ is defined by

$$e_T := (x_1, \ldots, x_m) \in \{0,1\}^m \text{ with } x_i = 1 \Leftrightarrow i \in T.$$

- Given a vector $(x_1, \ldots, x_m) \in [0,1]^m$, let $(\cdot)$ be the permutation on X which arranges the elements of this vector by increasing values: that is, $x_{(1)} \leq \ldots \leq x_{(m)}$.
- The notation $K \subsetneq T$ means $K \subset T$ and $K \neq T$.

In order to define the Sugeno integral, we use the concept of *fuzzy measure*.

Definition 2.2 *A* (discrete) fuzzy measure *on X is a set function $\mu : \mathcal{P}(X) \to [0,1]$ satisfying the following conditions:*
(i) $\mu(\emptyset) = 0, \mu(X) = 1,$
(ii) $R \subseteq S \subseteq X \Rightarrow \mu(R) \leq \mu(S)$.

$\mu(R)$ can be viewed as the weight of importance of the set of elements R. In the sequel we will write μ_R instead of $\mu(R)$.

We introduce now the concept of discrete Sugeno integral, viewed as an aggregation function. For this reason, we will adopt a connective-like notation instead of the usual integral form, and the integrand will be a set of m values $x_1, \ldots, x_m$ of $[0,1]$. For theorical developments, see [2, 3, 4].

Definition 2.3 *Let $(x_1, \ldots, x_m) \in [0,1]^m$, and μ a fuzzy measure on X. The* (discrete) Sugeno integral of $(x_1, \ldots, x_m)$ with respect to μ *is defined by*

$$\mathcal{S}_\mu^{(m)}(x_1, \ldots, x_m) := \bigvee_{i=1}^{m} [x_{(i)} \wedge \mu_{\{(i), \ldots, (m)\}}].$$

For instance, if $x_3 \leq x_1 \leq x_2$, we have

$$\mathcal{S}_\mu^{(3)}(x_1, x_2, x_3) = (x_3 \wedge \mu_{\{3,1,2\}}) \vee (x_1 \wedge \mu_{\{1,2\}}) \vee (x_2 \wedge \mu_{\{2\}}).$$

3 Weighted max-min and min-max functions

This section is devoted to weighted max-min and min-max functions. Although the coefficients involved in these functions are not really weights, but rather thresholds or aspiration degrees, we will talk in terms of weights.

The formal analogy between the weighted max-min function and the multilinear polynomial is obvious: minimum corresponds to product, maximum does to sum. Moreover, it is emphasized that weighted max-min functions can be calculated as medians, i.e., the qualitative counterparts of multilinear polynomials.

Finally, we give an axiomatic characterization of the family of weighted max-min functions.

3.1 Weighted max-min functions

Definition 3.1 *For any set function $a : \mathcal{P}(X) \to [0,1]$ such that $a_\emptyset = 0$ and $\bigvee_{T \subseteq X} a_T = 1$, the* weighted max-min *aggregation function $WMAXMIN_a^{(m)}$ associated to a is defined by*

$$WMAXMIN_a^{(m)}(x_1, \ldots, x_m) = \bigvee_{T \subseteq X} [a_T \wedge (\bigwedge_{i \in T} x_i)] \quad \forall (x_1, \ldots, x_m) \in [0,1]^m.$$

The set function a which define $\text{WMAXMIN}_a^{(m)}$ is not uniquely determined: indeed, we have, for instance, $x_1 \vee (x_1 \wedge x_2) = x_1$. The next proposition precises conditions under which two weighted max-min functions are identical.

Proposition 3.1 *Let a and a' be set functions defining $WMAXMIN_a^{(m)}$ and $WMAXMIN_{a'}^{(m)}$ respectively. Then the following four assertions are equivalent:*

$$(i) \quad WMAXMIN_{a'}^{(m)} = WMAXMIN_a^{(m)}$$

$$(ii) \quad \forall T \subseteq X : \bigvee_{K \subseteq T} a'_K = \bigvee_{K \subseteq T} a_K$$

$$(iii) \quad \forall T \subseteq X, T \neq \emptyset : \begin{cases} a'_T = a_T & \text{if } a_T > \bigvee_{K \subsetneq T} a_K \\ 0 \leq a'_T \leq \bigvee_{K \subseteq T} a_K & \text{otherwise} \end{cases}$$

Let a be any set function defining $\mathrm{WMAXMIN}_a^{(m)}$. By the third assertion of the previous proposition, each a_T is either uniquely determined or can lie in a closed interval. If a is such that

$$\forall T \subseteq X, T \neq \emptyset: \ a_T = 0 \Leftrightarrow a_T \leq \bigvee_{K \subsetneq T} a_K$$

then the a_T's are the smallest and we say that $\mathrm{WMAXMIN}_a^{(m)}$ is put under its *canonical* form. On the other hand, if a is such that

$$\forall T \subseteq X: \ a_T = \bigvee_{K \subseteq T} a_K$$

then the a_T's are the largest and we say that $\mathrm{WMAXMIN}_a^{(m)}$ is put under its *complete* form. In this case, a is a fuzzy measure since it is increasing (by inclusion).

3.2 Weighted min-max functions

By exchanging the position of the max and min operations in Definition 3.1, we can define the weighted min-max functions as follows.

Definition 3.2 *For any set function $b : \mathcal{P}(X) \to [0,1]$ such that $b_\emptyset = 1$ and $\bigwedge_{T \subseteq X} b_T = 0$, the* weighted min-max *aggregation function $WMINMAX_b^{(m)}$ associated to b is defined by*

$$WMINMAX_b^{(m)}(x_1, \ldots, x_m) = \bigwedge_{T \subseteq X} [b_T \vee (\bigvee_{i \in T} x_i)] \quad \forall (x_1, \ldots, x_m) \in [0,1]^m.$$

The set function b which define $\mathrm{WMINMAX}_b^{(m)}$ is not uniquely determined. We then have a result similar to Proposition 3.1.

Proposition 3.2 *Let b and b' be set functions defining $WMINMAX_b^{(m)}$ and $WMINMAX_{b'}^{(m)}$ respectively. Then the following four assertions are equivalent:*

$$\begin{array}{ll}
(i) & WMINMAX_{b'}^{(m)} = WMINMAX_b^{(m)} \\
(ii) & \forall T \subseteq X: \bigwedge_{K \subseteq T} b'_K = \bigwedge_{K \subseteq T} b_K \\
(iii) & \forall T \subseteq X, T \neq \emptyset: \begin{cases} b'_T = b_T & \text{if } b_T < \bigwedge_{K \subsetneq T} b_K \\ \bigwedge_{K \subseteq T} b_K \leq b'_T \leq 1 & \text{otherwise} \end{cases}
\end{array}$$

Let b be any set function defining $\mathrm{WMINMAX}_b^{(m)}$. By the third assertion of the previous proposition, each b_T is either uniquely determined or can lie in a closed interval. If b is such that

$$\forall T \subseteq X, T \neq \emptyset: \ b_T = 1 \Leftrightarrow b_T \geq \bigwedge_{K \subsetneq T} b_K$$

then the b_T's are the largest and we say that $\mathrm{WMINMAX}_b^{(m)}$ is put under its *canonical* form. On the other hand, if b is such that

$$\forall T \subseteq X: \ b_T = \bigwedge_{K \subseteq T} b_K$$

then the b_T's are the smallest and we say that $\mathrm{WMINMAX}_b^{(m)}$ is put under its *complete* form. In this case, b is decreasing (by inclusion).

3.3 Correspondance formulae and equivalent forms

As announced at the beginning of this section, any weighted max-min function can be put under the form of a weighted min-max function and conversely. The next proposition gives the correspondance formulae.

Proposition 3.3 *For any increasing set function a defining $WMAXMIN_a^{(m)}$ and any decreasing set function b defining $WMINMAX_b^{(m)}$, we have*

$$WMINMAX_b^{(m)} = WMAXMIN_a^{(m)} \Leftrightarrow b_T = a_{X\setminus T} \quad \forall T \subseteq X.$$

The following example illustrates the use of the correspondance formulae.

Example 3.1 *Let $X = \{1, 2, 3\}$. We have*

$$(.1 \wedge x_1) \vee (.3 \wedge x_2) \vee (x_2 \wedge x_3) = (.1 \vee x_2) \wedge (.3 \vee x_3) \wedge (x_1 \vee x_2).$$

Indeed, starting from the left-hand side (a canonical form), we can compute its complete form then its dual complete form and finally its dual canonical form as follows:

$$\begin{aligned}
& (.1 \wedge x_1) \vee (.3 \wedge x_2) \vee (x_2 \wedge x_3) \\
= \; & 0 \vee (.1 \wedge x_1) \vee (.3 \wedge x_2) \vee (0 \wedge x_3) \vee (.3 \wedge x_1 \wedge x_2) \vee (.1 \wedge x_1 \wedge x_3) \vee (1 \wedge x_2 \wedge x_3) \vee (1 \wedge x_1 \wedge x_2 \wedge x_3) \\
= \; & 1 \wedge (1 \vee x_1) \wedge (.1 \vee x_2) \wedge (.3 \vee x_3) \wedge (0 \vee x_1 \vee x_2) \wedge (.3 \vee x_1 \vee x_3) \wedge (.1 \vee x_2 \vee x_3) \wedge (0 \vee x_1 \vee x_2 \vee x_3) \\
= \; & (.1 \vee x_2) \wedge (.3 \vee x_3) \wedge (x_1 \vee x_2).
\end{aligned}$$

Now, we state that any $\text{WMAXMIN}_a^{(m)}$ function can be written under equivalent forms involving at most m variable coefficients. These coefficients only depend on the order of the x_i's.

Theorem 3.1 *(i) For any increasing set function a defining $WMAXMIN_a^{(m)}$, we have, for all $(x_1, \ldots, x_m) \in [0,1]^m$,*

$$\begin{aligned}
WMAXMIN_a^{(m)}(x_1, \ldots, x_m) &= \bigvee_{i=1}^{m} [x_{(i)} \wedge a_{\{(i),\ldots,(m)\}}] \\
&= median(x_1, \ldots, x_m, a_{\{(2),\ldots,(m)\}}, a_{\{(3),\ldots,(m)\}}, \ldots, a_{\{(m)\}}).
\end{aligned}$$

(ii) For any decreasing set function b defining $WMINMAX_b^{(m)}$, we have, for all $(x_1, \ldots, x_m) \in [0,1]^m$,

$$\begin{aligned}
WMINMAX_b^{(m)}(x_1, \ldots, x_m) &= \bigwedge_{i=1}^{m} [x_{(i)} \vee b_{\{(1),\ldots,(i)\}}] \\
&= median(x_1, \ldots, x_m, b_{\{(1)\}}, b_{\{(1),(2)\}}, \ldots, b_{\{(1),\ldots,(m-1)\}}).
\end{aligned}$$

3.4 Axiomatic characterization of the family of weighted max-min functions

According to Proposition 3.3, the set of weighted max-min functions and the set of weighted min-max functions represent the same family of functions. This family can be characterized with the help of some selected properties. These are presented in the next definition.

Definition 3.3 *The aggregation function $M^{(m)}$ defined on $[0,1]^m$ is*

- increasing (In) *if $M^{(m)}$ is increasing in each argument, i.e. if, for all $(x_1, \ldots, x_m)$, $(x'_1, \ldots, x'_m) \in [0,1]^m$, we have*
$$x_i \leq x'_i \ \forall i \in X \Rightarrow M^{(m)}(x_1, \ldots, x_m) \leq M^{(m)}(x'_1, \ldots, x'_m).$$
- stable for minimum with the same unit (SMINU) *if, for all $(x_1, \ldots, x_m) \in [0,1]^m$ and all $r \in [0,1]$, we have*
$$M^{(m)}(x_1 \wedge r, \ldots, x_m \wedge r) = M^{(m)}(x_1, \ldots, x_m) \wedge r.$$
- stable for maximum with the same unit (SMAXU) *if, for all $(x_1, \ldots, x_m) \in [0,1]^m$ and all $r \in [0,1]$, we have*
$$M^{(m)}(x_1 \vee r, \ldots, x_m \vee r) = M^{(m)}(x_1, \ldots, x_m) \vee r.$$

Now, we state that the family of $\text{WMAXMIN}_a^{(m)}$ functions can be characterized by means of these three propreties.

Theorem 3.2 *Let $M^{(m)}$ be any aggregation function defined on $[0,1]^m$. Then the following three assertions are equivalent:*

(i) $M^{(m)}$ *fulfils (In, SMINU, SMAXU)*
(ii) *There exists a set function a such that* $M^{(m)} = WMAXMIN_a^{(m)}$
(iii) *There exists a set function b such that* $M^{(m)} = WMINMAX_b^{(m)}$

4 Back to the Sugeno integral

According to some results from the previous section, we can see that the class of the Sugeno integrals coincides with the family of weighted max-min functions. By using Theorem 3.1, we are then allow to derive equivalent forms of the Sugeno integral. The next theorem deals with this issue.

Theorem 4.1 *Let* $(x_1, \ldots, x_m) \in [0,1]^m$ *and* μ *a fuzzy measure on* X. *Then we have*

$$\begin{aligned} \mathcal{S}_\mu^{(m)}(x_1, \ldots, x_m) &= \bigvee_{i=1}^{m} [x_{(i)} \wedge \mu_{\{(i),\ldots,(m)\}}] = \bigwedge_{i=1}^{m} [x_{(i)} \vee \mu_{\{(i+1),\ldots,(m)\}}] \\ &= \bigvee_{T \subseteq X} [\mu_T \wedge (\bigwedge_{i \in T} x_i)] = \bigwedge_{T \subseteq X} [\mu_{X \setminus T} \vee (\bigvee_{i \in T} x_i)] \\ &= \mathit{median}(x_1, \ldots, x_m, \mu_{\{(2),\ldots,(m)\}}, \mu_{\{(3),\ldots,(m)\}}, \ldots, \mu_{\{(m)\}}). \end{aligned}$$

In addition to the previous result, Theorem 3.2 leads us to an axiomatic characterization of the class of Sugeno integrals. We state it as follows:

Theorem 4.2 *The aggregation function* $M^{(m)}$ *defined on* $[0,1]^m$ *fulfils (In, SMINU, SMAXU) if and only if there exists a fuzzy measure* μ *on* X *such that* $M^{(m)} = \mathcal{S}_\mu^{(m)}$.

References

[1] J. Fodor and M. Roubens (1994), *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer, Dordrecht.

[2] M. Grabisch, T. Murofushi and M. Sugeno (1992), *Fuzzy measure of fuzzy events defined by fuzzy integrals*, Fuzzy Sets and Systems, **50**: 293-313.

[3] M. Sugeno (1974), *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, Tokyo.

[4] M. Sugeno (1977), *Fuzzy measures and fuzzy integrals: a survey*, in: M.M. Gupta, G.N. Saridis and B.R. Gaines (Eds.), Fuzzy Automata and Decision Processes, (North-Holland, Amsterdam): 89-102.