Dependence between criteria and multiple criteria decision aid*

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Abstract

In this paper, we present a model allowing to determine the weights related to interactive (correlated) criteria. This is done on the basis of the knowledge of a partial ranking over a reference set of alternatives (prototypes), a partial ranking over the set of criteria, and a partial ranking over the set of interactions between pairs of criteria.

Keywords: multicriteria decision making, interactive criteria, Choquet integral.

1 Introduction

Let us consider a set of *alternatives* $A = \{a, b, c, \ldots\}$ and a set of *criteria* $N = \{1, \ldots, n\}$ in a multicriteria decision making problem. Each alternative $a \in A$ is associated with a *profile* $x(a) = (x_1(a), \ldots, x_n(a)) \in \mathbb{R}^n$ where $x_i(a)$ represents the utility of a related to the criterion i, with $x_i \in X_i$, $i = 1, \ldots, n$. We assume that all the utilities $x_i(a)$ are defined according to a same interval scale.

Suppose that the preferences over A of the decision maker are known and expressed by a binary relation $\succeq$. In the classical multiattribute utility (MAUT) model [7], the problem consists in constructing a *utility function* $U : \mathbb{R}^n \to \mathbb{R}$ representing the preference of the decision maker, that is such that

$$a \succ b \Leftrightarrow U[x(a)] > U[x(b)], \quad \forall a, b \in A.$$

The binary relation $\succeq$ on $X = \prod_i X_i$ verifies the independence in coordinates (which is proven to be equivalent to the mutually preferentially independence) iff, for all i, x, y, z_i, t_i,

$$(z_i x_{-i}) \succeq (z_i y_{-i}) \text{ implies } (t_i x_{-i}) \succeq (t_i y_{-i})$$

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if $(z_i x_{-i})$ represents the vector which has the same coordinates as $x \in X$ except for the i-th coordinative which is z_i.

For some problems this principle might be violated as it can be seen in the following example

	criterion 1 (price)	criterion 2 (consumption)	criterion 3 (comfort)
car 1	10.000 Euro	10 ℓ /100 km	very good
car 2	10.000 Euro	9 ℓ/100 km	good
car 3	30.000 Euro	10 ℓ/100 km	very good
car 4	30.000 Euro	9 ℓ/100 km	good

A decision maker might prefer car 2 to car 1 but also car 3 to car 4.

We know that independence in coordinates is a necessary condition for a utility function to be additive, i.e. it can be assumed that there exists a weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ fulfilling $\sum_i \omega_i = 1$ such that

$$U[x(a)] = \sum_{i=1}^{n} \omega_i \, x_i(a), \quad \forall a \in A. \tag{1}$$

In case of interactive criteria, the weighted arithmetic mean (1) can be extended to a Choquet integral:

$$U[x(a)] = \sum_{i=1}^{n} x_{(i)}(a) \, [\mu(A_{(i)}) - \mu(A_{(i+1)})], \tag{2}$$

where $(\cdot)$ indicates a permutation such that $x_{(1)}(a) \leq \ldots \leq x_{(n)}(a)$. Also $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \emptyset$. We thus observe that the weights ω_i related to the criteria, which were supposed independent, have been substituted by the weights $\mu(i_1, \ldots, i_k)$ related of any coalition of interactive criteria.

In this paper, we propose a model allowing to identify the weights of interactive criteria from a partial preorder over a reference set of alternatives, a partial preorder over the set of values related to each criterion, a partial preorder over interactions between pairs of criteria, and the knowlegde of the sign of some interactions between pairs of criteria. The weights can be obtained by solving a linear problem.

2 The Choquet integral as an aggregation operator

A *fuzzy measure* on the set N of criteria is a monotonic set function $\mu : 2^N \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(N) = 1$. Monotonicity means that $\mu(S) \leq \mu(T)$ whenever $S \subseteq T$.

One thinks of $\mu(S)$ as the weight of importance of the subset of criteria S. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined.

A fuzzy measure is said to be *additive* if $\mu(S \cup T) = \mu(S) + \mu(T)$ whenever $S \cap T = \emptyset$. In this case, it suffices to define the n coefficients (weights) $\mu(1), \ldots, \mu(n)$ to define the measure entirely. In general, one needs to define the 2^n coefficients corresponding to the 2^n subsets of N.

In combinatorics, a viewed as a set function on N given by

$$a(S) = \sum_{T \subseteq S} (-1)^{s-t} \mu(T), \quad \forall S \subseteq N \tag{3}$$

is called the *Möbius transform* of v (see e.g. Rota [10])

Of course, any set of 2^n coefficients $\{a(T) \mid T \subseteq N\}$ could not be the Möbius representation of a fuzzy measure: the boundary and monotonicity conditions must be ensured. In terms of the Möbius representation, those conditions can be written as follows (see [2]):

$$\begin{cases} a(\emptyset) = 0, \quad \sum\limits_{T \subseteq N} a(T) = 1, \\ \sum\limits_{T: i \in T \subseteq S} a(T) \geq 0, \quad \forall S \subseteq N, \ \forall i \in S. \end{cases} \tag{4}$$

Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and μ be a fuzzy measure on N. The (discrete) Choquet integral of $(x_1, \ldots, x_n)$ with respect to μ is defined by

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_{(i)} \, [\mu(A_{(i)}) - \mu(A_{(i+1)})],$$

where $(\cdot)$ indicates a permutation such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$, and $A_{(n+1)} = \emptyset$.

The Choquet integral has good properties for aggregation (see e.g. Grabisch [3]). For instance, it is continuous, non decreasing, comprised between *min* and *max*, stable under the same transformations of interval scales in the sense of the theory of measurement, and coincides with a weighted arithmetic mean when the fuzzy measure is additive.

In this paper, we substitute the Choquet integral to the weighted arithmetic mean whenever interactive criteria are considered.

In terms of the Möbius representation, the Choquet integral is written (see [2]):

$$C_\mu(x) = \sum_{T \subseteq N} a(T) \bigwedge_{i \in T} x_i, \quad x \in \mathbb{R}^n,$$

where $\wedge$ stands for the minimum operation.

3 The concept of interaction among criteria

The overall importance of a criterion $i \in N$ is not solely determined by the value $\mu(i)$, but also by all $\mu(S)$ such that $i \in S$. The *importance index* or *Shapley value* of criterion i with respect to μ is defined by:

$$\phi_S(i) = \sum_{T \subseteq N \setminus i} \frac{(n-t-1)!\, t!}{n!} \, [\mu(T \cup i) - \mu(T)]. \tag{5}$$

The Shapley value is a fundamental concept in game theory [12] expressing a power index. There is in fact another common way of defining a power index, due to Banzhaf [1]. The so-called Banzhaf value, defined as

$$\phi_B(i) = \frac{1}{2^{n-1}} \sum_{T \subseteq N \setminus i} [\mu(T \cup i) - \mu(T)], \tag{6}$$

can be viewed as an alternative to the Shapley value.

Now, consider a pair $\{i, j\} \subseteq N$ of criteria. The difference $a(i, j) = \mu(i, j) - \mu(i) - \mu(j)$ seems to reflect the degree of interaction between i and j. This difference is zero if i and j are independent criteria. It is positive if there is a synergy effect between i and j and negative if they are redundant. Here again, the interaction between i an j should depend on the coefficients $\mu(S)$ such that $i, j \in S$. Murofushi and Soneda [8] proposed an interaction index among a pair of criteria, based on multiattribute utility theory. More generally, Grabisch [4] introduced an interaction index among a combination S of criteria: the *Shapley interaction index* related to μ, defined by

$$I_S(S) := \sum_{T \subseteq N \setminus S} \frac{(n-t-s)!\, t!}{(n-s+1)!} \sum_{L \subseteq S} (-1)^{s-l} \mu(L \cup T), \quad \forall S \subseteq N, \tag{7}$$

that is, in terms of the Möbius representation,

$$I_S(S) = \sum_{T \supseteq S} \frac{1}{t-s+1}\, a(T), \quad \forall S \subseteq N. \tag{8}$$

Viewed as a set function, the Shapley interaction index coincides on singletons with the Shapley value (5). Roubens [11] developed a parallel notion of interaction index, based on the Banzhaf value (6): the Banzhaf interaction index, defined by

$$I_B(S) := \frac{1}{2^{n-s}} \sum_{T \subseteq N \setminus S} \sum_{L \subseteq S} (-1)^{s-l} v(L \cup T), \quad \forall S \subseteq N, \tag{9}$$

that is, in terms of the Möbius representation,

$$I_B(S) = \sum_{T \supseteq S} (\frac{1}{2})^{t-s}\, a(T), \quad \forall S \subseteq N. \tag{10}$$

It should be noted that the interaction indices I_B and I_S have been axiomatically characterized by Grabisch and Roubens [5].

4 The 2-order model

We know that a problem involving n criteria requires 2^n coefficients in $[0, 1]$ in order to define the fuzzy measure μ on every coalition. Of course, a decision maker is not able to give such an amount of information. Moreover, the meaning of the numbers $\mu(S)$ and $a(S)$ for $|S| > 2$ is not so clear for the decision maker.

To overcome this problem, Grabisch [4] proposed to use the concept of *k-order fuzzy measure*. We may think of a fuzzy measure having a polynomial representation of degree 2, or 3, or any fixed integer k. Such a fuzzy measure is naturally called k-order fuzzy measure since it represents a k-order approximation of its polynomial expression.

We now confine to the 2-order case, which seems to be the most interesting in practical applications, since it permits to model interaction between criteria while remaining very simple. Indeed, only $n + \binom{n}{2} = \frac{n(n+1)}{2}$ coefficients are required to define the fuzzy measure:

$$\mu(S) = \sum_{i \in S} a(i) + \sum_{\{i,j\} \subseteq S} a(i, j), \quad \forall S \subseteq N.$$

Note that the 2-order case is equivalent to suppose that the Shapley and Banzhaf interaction indices are zero for subsets of at least 3 elements. In this case, the Choquet integral becomes

$$C_\mu(x) = \sum_{i\in N} a(i)\, x_i + \sum_{\{i,j\}\subseteq N} a(i,j)\,(x_i \wedge x_j), \quad x \in \mathbb{R}^n. \tag{11}$$

Moreover, the interaction indices coincide ($I_S = I_B = I$) and we have immediately:

$$\begin{aligned} I(i) &= a(i) + \frac{1}{2}\sum_{j\in N\setminus i} a(i,j), \quad i \in N, & (12)\\ I(i,j) &= a(i,j), \quad i,j \in N, & (13)\\ I(S) &= 0, \quad \forall S \subseteq N,\ |S| > 2. & (14) \end{aligned}$$

5 Identification of weights

We address now the problem of identification of weights of interactive criteria. More precisely, we are interested in finding a 2-order fuzzy measure on the basis of a partial ranking over a set alternatives (prototypes).

In this section, we suppose that we have at our disposal an expert or decision maker who is able to tell the relative importance of criteria, and the kind of interaction between them, if any. Formally, the input data of the problem can be summarized as follows:

- The set A of alternatives and the set N of criteria,
- A table of scores (utilities) $\{x_i(a)\,|\, i \in N, a \in A\}$,
- A partial preorder $\succeq_A$ on A (ranking of alternatives),
- A partial preorder $\succeq_N$ on N (ranking of criteria),
- A partial preorder $\succeq_P$ on the set of pairs of criteria (ranking of interaction indices),
- The sign of some interactions $a(i,j)$: positive, nul, negative (translating synergy, independence or redundancy).

All these data can be formulated with the help of linear equalities or inequalities. Strict inequalities can be converted into vague inequalities by introducing a positive slack quantity as the following immediate proposition shows.

$x \in \mathbb{R}^n$ is a solution of the linear system

$$\begin{cases} \sum_{j=1}^n a_{ij}\, x_j \le b_i, & i = 1,\dots,p,\\ \sum_{j=1}^n c_{ij}\, x_j < d_i, & i = 1,\dots,q, \end{cases}$$

if and only if there exists $\varepsilon > 0$ such that

$$\begin{cases} \sum_{j=1}^n a_{ij}\, x_j \le b_i, & i = 1,\dots,p,\\ \sum_{j=1}^n c_{ij}\, x_j \le d_i - \varepsilon, & i = 1,\dots,q. \end{cases}$$

In particular, a solution exists if and only if the following linear program

$$\max z = \varepsilon$$

subject to

$$\begin{cases} \sum_{j=1}^n a_{ij}\, x_j \le b_i, & i = 1,\dots,p,\\ \sum_{j=1}^n c_{ij}\, x_j \le d_i - \varepsilon, & i = 1,\dots,q, \end{cases}$$

has an optimal solution $x^* \in \mathbb{R}^n$ with an optimal value $\varepsilon^* > 0$. In this case, x^* is a solution of the first system.

Thus, the problem of finding a 2-order fuzzy measure can be formalized with the help of a linear program. It is obvious that the more the input information is poor, the more the solution set is big. Hence, it is desirable that the information is as complete as possible. However, if this information contains incoherences then the solution set could be empty.

Now, a model for identifying weights could be as follows:

$$\max z = \varepsilon$$

subject to

$$\left.\begin{array}{ll} C(a) - C(b) \geq \delta + \varepsilon & \text{if } a \succ_A b \\ -\delta \leq C(a) - C(b) \leq \delta & \text{if } a \sim_A b \end{array}\right\} \text{partial semiorder with threshold } \delta$$

$$\left.\begin{array}{ll} I(i) - I(j) \geq \varepsilon & \text{if } i \succ_N j \\ I(i) = I(j) & \text{if } i \sim_N j \end{array}\right\} \text{ranking of criteria}$$

$$\left.\begin{array}{ll} a(i,j) - a(k,l) \geq \varepsilon & \text{if } \{i,j\} \succ_P \{k,l\} \\ a(i,j) = a(k,l) & \text{if } \{i,j\} \sim_P \{k,l\} \end{array}\right\} \text{ranking of pairs of criteria}$$

$$\left.\begin{array}{ll} a(i,j) \geq \varepsilon \text{ (resp. } \leq -\varepsilon) & \text{if } a(i,j) > 0 \text{ (resp. } < 0) \\ a(i,j) = 0 & \text{if } a(i,j) = 0 \end{array}\right\} \text{sign of interactions}$$

$$\left.\begin{array}{ll} \sum_i I(i) = 1 & \\ a(i) \geq 0 & \forall i \in N \\ a(i) + \sum_{j \in T} a(i,j) \geq 0 & \forall i \in N,\ \forall T \subseteq N \setminus i \end{array}\right\} \begin{array}{l}\text{boundary and monotonicity}\\ \text{conditions}\end{array}$$

$$\left.\begin{array}{ll} I(i) = a(i) + \frac{1}{2} \sum_{j \in N \setminus i} a(i,j) & \forall i \in N \\ C(a) = \sum_{i \in N} a(i)\, x_i(a) + \sum_{\{i,j\} \subseteq N} a(i,j)\, [x_i(a) \wedge x_j(a)] & \forall a \in A \end{array}\right\} \text{definitions}$$

It seems natural to assume that the ranking over A is translated into a partial semiorder over the set of the global evaluations given by the Choquet integral. This partial semiorder has a fixed threshold δ, which can be tuned as wished.

In order to illustrate the model, two small examples will be presented. They are constructed in such a way that no generalized weighted arithmetic mean can be used as utility function.

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