Characterization of some stable aggregation functions

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Abstract

We characterize the class of ordinaly stable, continuous, neutral (symmetric) and monotonic aggregators together with the class of associative or decomposable, continuous, neutral and monotonic aggregation operators which are stable for any positive linear transformation.

Keywords : logical connectives; aggregation functions; ordinal and interval scales; stability.

Introduction

Let us assume that a set of a finite number of elements $(x_1, \ldots, x_m) \in [0,1]^m$ is given according to some scale type as defined by Stevens [8]– see also Coombs [2] and Roberts [7]. One of the main problems for the definition of an appropriate aggregation function $M(x_1, \ldots, x_m) \in [0,1]$ in the field of multifactorial evaluation for the construction of multiple-criterion aggregation connective corresponds to the stability of the aggregators.

Let us suppose the admissible transformations related to the scale type are functions $\phi : [0,1] \to [0,1]$. Stability of M is assumed if $M[\phi(x_1), \ldots, \phi(x_m)] = \phi M(x_1, \ldots, x_m)$.

Two particular scale types are considered : the continuous ordinal scale and the interval scale where the corresponding admissible transformations ϕ are respectively the continuous strictly increasing transformations (called the family Φ) and the positive linear transformations.

The main difficulty for this kind of functions is the lack of a representation theorem for M analogous to those given by Kolmogorov [5] and Nagumo [6] for generalized means or by Aczèl [1] for associative aggregators.

The main aim in this paper is to characterize two important classes of aggregation functions : the ordinaly stable aggregators and the connectives which preserve the stability for positive linear transformations.

1 Basic definitions

We consider a vector $(x_1, \ldots, x_m) \in [0,1]^m$ and we are willing to substitute to that vector a single value $M(x_1, \ldots, x_m) \in [0,1]$ using the aggregation operator M.

The operator is called

C-operator if M is *continuous* in the arguments $x_1, \ldots, x_m$.

N-operator if M is *neutral* (commutative, symmetric), i.e. independant of the labels :

$$M(x_1, \ldots, x_m) = M(x_{i_1}, \ldots, x_{i_m})$$

where $(i_1, \ldots, i_m) = \sigma(1, \ldots, m)$ and σ represents a permutation operation.

M-operator if M is *monotonic*, which means that

$$x_i' > x_i \text{ implies } M(x_1, \ldots, x_i', \ldots, x_m) \geq M(x_1, \ldots, x_i, \ldots, x_m)$$

CNM-operator if M is a continuous, neutral and monotonic operator.

S-operator if M is *strictly monotonic* (strict), which means that

$$x_i' > x_i \text{ implies } M(x_1, \ldots, x_i', \ldots, x_m) > M(x_1, \ldots, x_i, \ldots, x_m)$$

I-operator if M is *idempotent*, i.e. if M satisfies

$$M(x, \ldots, x) = x, \text{ for all } x \in [0,1]$$

A-operator if M is *associative*. In that case, aggregation of only two arguments can be canonically extended to any finite number of arguments :

$$\begin{aligned} M(x_1, x_2, x_3) &= M(x_1, M(x_2, x_3)) = M(M(x_1, x_2), x_3) \\ M(x_1, \ldots, x_m) &= M(M(x_1, \ldots, x_{m-1}), x_m). \end{aligned}$$

D-operator if M is *decomposable* (see Kolmogorov (1930), Nagumo (1930)), i.e. each element of a subgroup of elements to be aggregated can be substituted to its partial aggregation without change :

$$M^{(m)}(x_1, \ldots, x_m) = M^{(m)}(\underbrace{\bar{x}, \ldots, \bar{x}}_{k \text{ times}}, x_{k+1}, \ldots, x_m)$$

with $\bar{x} = M^{(k)}(x_1, \ldots, x_k)$.

SO-operator if M is *ordinaly stable*, which means that

$$M(\phi(x_1), \ldots, \phi(x_m)) = \phi M(x_1, \ldots, x_m)$$

where $\phi \in \Phi$ is a continuous strictly increasing function : $[0,1] \rightarrow [0,1]$.

SPL-operator if M is *stable for any admissible positive linear transformation*. In that case :

$$M(\alpha x_1 + t, \ldots, \alpha x_m + t) = \alpha M(x_1, \ldots, x_m) + t,$$

$\alpha > 0$, $\alpha x_k + t \in [0,1]$, for all $k \in \{1,\ldots,m\}$ and $\alpha M(x_1,\ldots,x_m) + t \in [0,1]$.

SSI-operator if M is *stable for any admissible similarity*. This property means that

$$M(\alpha x_1,\ldots,\alpha x_m) = \alpha M(x_1,\ldots,x_m),$$

$\alpha > 0$, $\alpha x_k \in [0,1]$, for all $k \in \{1,\ldots,m\}$, and $\alpha M(x_1,\ldots,x_m) \in [0,1]$.

STR-operator if M is *stable for any admissible translation*. In that case :

$$M(x_1+t,\ldots,x_m+t) = M(x_1,\ldots,x_m) + t$$

$x_k + t \in [0,1]$, for all $k \in \{1,\ldots,m\}$ and $M(x_1,\ldots,x_m) + t \in [0,1]$.

2 Characterization of ordinaly stable CNM operators

Theorem 1 *$(SO) - CNM$ operators are characterized by the family of connectives $M(x_1,\ldots,x_m)$ equal to one of its components x_r, r being independant from $(x_1,\ldots,x_m)$.*

Proof. The sufficiency part is evident. The necessary part is proved in three steps.

(1) Let us consider $(z_1,\ldots,z_m) \in [0,1]^m$, $z_1 < z_2 < \cdots < z_m$.
If $M(z_1,\ldots,z_m) = 0$, we shall prove that $z_1 = 0$. Suppose that $z_1 > 0$ and consider $\psi_i(x) = x^{1/i}$, for all $x \in [0,1]$, $i \in N_0 = \{1,2,\ldots\}$. ψ_i belongs to the family Φ.

$$M\left[\psi_i(z_1),\ldots,\psi_i(z_m)\right] = \psi_i\left\{M(z_1,\ldots,z_m)\right\} = \psi_i(0) = 0, \quad \forall i \in N_0$$

and

$$\lim_{i\to\infty} M\left[\psi_i(z_1),\ldots,\psi_i(z_m)\right] = 0.$$

Due to the continuity,

$$\begin{aligned}\lim_{i\to\infty} M\left[\psi_i(z_1),\ldots,\psi_i(z_m)\right] &= M\left[\lim_{i\to\infty}\psi_i(z_1),\ldots,\lim_{i\to\infty}\psi_i(z_m)\right]\\ &= M(1,\ldots,1), \text{ because } z_1 > 0.\end{aligned}$$

$M(1,\ldots,1) = 0$ and monotonicity imply $M(x_1,\ldots,x_m) = 0$, for all $(x_1,\ldots,x_m) \in [0,1]^m$ and $0 = M(\phi(x_1),\ldots,\phi(x_m)) = \phi\left[M(x_1,\ldots,x_m)\right] = \phi(0)$, for all $\phi \in \Phi$, which is not supposed to be the case.
If $M(z_1\ldots z_m) = 1$, we can prove with similar arguments thats $z_m = 1$.

(2) Let us now consider $(z_1,\ldots,z_m) \in (0,1)^m$, $z_0 = 0 < z_1 < \cdots < z_m < z_{m+1} = 1$.
From the preceding results : $0 < M(z_1,\ldots,z_m) < 1$.
We shall prove first that there exists *one* $r \in \{0,1,\ldots,m\}$ such that $M(z_1,\ldots,z_m) = z_r$ using the fact that $M(z_1,\ldots,z_m) < 1$. We consider

$$\psi_i^*(x) = (z_{j+1} - z_j)\left(\frac{x - z_j}{z_{j+1} - z_j}\right)^i + z_j \text{ if } x \in [z_j, z_{j+1}), \; j = 0,\ldots,m.$$

$\psi_i^*(z_\ell) = z_\ell$, for all $\ell \in \{0, \dots, m\}$ and $\psi_i^* \in \Phi$.

Moreover,

$$\lim_{i \to \infty} \psi_i^*(x) = z_r \text{ if } x \in [z_r, z_{r+1}), \quad r = 0, \dots, m.$$

$M(z_1, \dots, z_m) < 1$ implies that there exists *one* $r \in \{0, \dots, m\}$ such that $M(z_1, \dots, z_m) \in [z_r, z_{r+1})$ and

$$\begin{aligned} M(z_1, \dots, z_m) &= M\left[\psi_i^*(z_1), \dots, \psi_i^*(z_m)\right] = \psi_i^*\{M(z_1, \dots, z_m)\}, \ \forall i \in N_0 \text{ and} \\ M(z_1, \dots, z_m) &= \lim_{i \to \infty} \psi_i^*\{M(z_1, \dots, z_m)\} = z_r. \end{aligned}$$

We can prove, using similar arguments, that there exists the same $r \in \{1, \dots, m+1\}$ such that $M(z_1, \dots, z_m) = z_r$ if $M(z_1, \dots, z_m) > 0$.

Finally, for any $(z_1 < \dots < z_m) \in (0,1)^m$, there exists *one* corresponding $r \in \{1, \dots, m\}$ such that $M(z_1, \dots, z_m) = z_r$.

(3) We shall now prove that $M(x_1, \dots, x_m) = x_r$, $r \in \{1, \dots, m\}$, where r corresponds to the index obtained in (2), for any vector $(x_1, \dots, x_m) \in [0,1]^m$.

M being neutral, we can reorder $(x_1, \dots, x_m) : x_1 \leq \dots \leq x_m$ without changing the connective M value.

Let us consider $\chi(x)$, a non decreasing and continuous function on $[0,1]$ such that $\chi(z_j) = x_j$, $\forall j \in \{1, \dots, m\}$ (see Fig. 1).

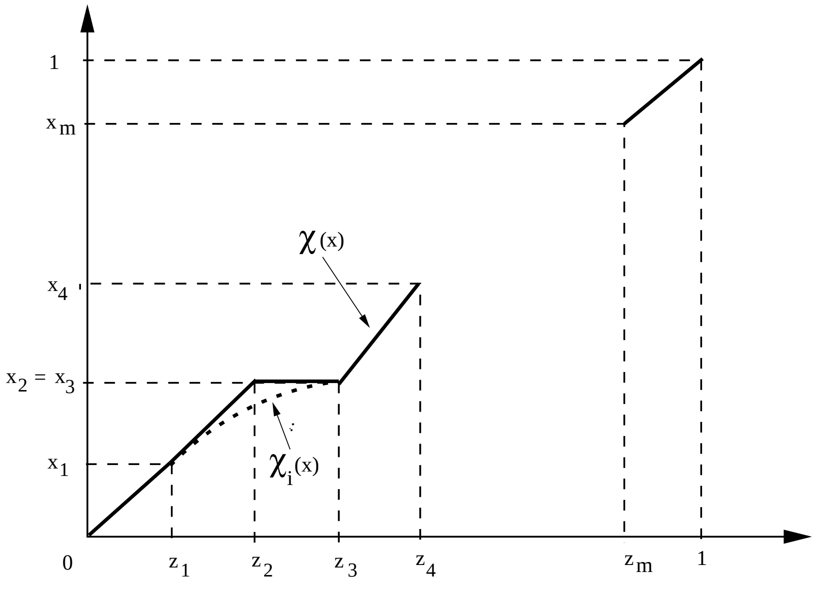

Figure 1

It is always possible to build $\chi_i \in \Phi$ such that $\lim_{i \to \infty} \chi_i(x) = \chi(x)$, for all $x \in [0,1]$.

Due to ordinal stability and results from (2),

$$M\left[\chi_i(z_1), \dots, \chi_i(z_m)\right] = \chi_i\left[M(z_1, \dots, z_m)\right] = \chi_i(z_r).$$

$$\lim_{i \to \infty} M\left[\chi_i(z_1), \dots, \chi_i(z_m)\right] = \lim_{i \to \infty} \chi_i(z_r) = x_r.$$

Finally, continuity gives

$$\begin{aligned} M(x_1,\ldots,x_m) &= M\left[\chi(z_1),\ldots,\chi(z_m)\right] = M\left[\lim_{i\to\infty}\chi_i(z_1),\ldots,\lim_{i\to\infty}\chi_i(z_m)\right] \\ &= \lim_{i\to\infty} M\left[\chi_i(z_1),\ldots,\chi_i(z_m)\right] = x_r. \end{aligned}$$

■

Corollary 1 *The class of* $(A\&SO)-CNM$ *operators or* $(D\&SO)-CNM$ *operators are reduced to*

$$M(x_1,\ldots,x_m) = \left[\min_i x_i\right] \ \mathit{or}\ \left[\max_i x_i\right].$$

Proof. Evident because $x_{(r)}$ is not associative nor decomposable for $r \neq 1, m$, if $x_{(1)} \leq \cdots \leq x_{(m)}$.

3 Characterization of associative or decomposable $(SPL)-CNM$ operators

It is known from results of Fung and Fu [4] and Dubois and Prade [3] that $(A\&I)-CNM$ operators are characterized by

$$M(x_1,\ldots,x_m) = \mathit{median}(\min_i x_i, \max_i x_i, \alpha), \quad \alpha \in [0,1].$$

The class of $(D\&I)-CNM$ operators has not been identified up to now but from results due to Kolmogorov [5] and Nagumo [6], we already know that the class of $(D\&I\&S)-CNM$ operators correspond to the generalized means

$$M(x_1,\ldots,x_m) = f^{-1}\left[\frac{1}{m}\sum_i f(x_i)\right]$$

where f is any continuous strictly monotonic function on $[0,1]$.

If STR property is added, the class of generalized means can be reduced to (see Nagumo [6])

$$M(x_1,\ldots,x_m) = \left[\frac{1}{m}\sum_i x_i\right] \text{ or } \left[\frac{1}{\lambda}\log\left(\frac{1}{m}\sum_i e^{\lambda x_i}\right)\right], \quad \lambda \neq 0$$

when SSI property is introduced, the restriction is focused on (see Nagumo [6])

$$M(x_1,\ldots,x_m) = \left[\left(\prod_i x_i\right)^{1/m}\right] \text{ or } \left[\left(\frac{1}{m}\sum_i x_i^{\lambda}\right)^{1/\lambda}\right], \quad \lambda \neq 0.$$

We shall characterize two classes of connectives : the $(A\&SPL)-CNM$ and $(D\&SPL)-CNM$ operators.

Let us first prove some lemmas.

Lemma 1 *The* $(SPL)-CNM$ *operators are idempotent.*

Proof. We consider $\alpha_i > 0$ such that $\lim_{i\to\infty} \alpha_i = 0$, $i \in N_0$.

$$M(\alpha_i x_1, \ldots, \alpha_i x_m) = \alpha_i M(x_1, \ldots, x_m) \quad \text{(stability)}.$$

Using continuity,

$$M(0, \ldots, 0) = 0$$

and

$$M(t, \ldots, t) = t \ \ \text{(stability)}.$$

■

Lemma 2 *For any* (SPL) *operator and* $m = 2$*, we have*

$$M(x_1, x_2) = \theta x_1 + (1-\theta)x_2 \ \ \textit{if} \ \ 0 \le x_1 \le x_2 \le 1 \ \ \textit{and} \ \ \theta \in [0,1]\,.$$

Proof. Consider $x_1 \le x_2$,

$$M(x_1, x_2) - x_1 = M(0, x_2 - x_1) = (x_2 - x_1)M(0,1) \quad \text{(stability)}.$$

Finally, $M(x_1, x_2) = \theta x_1 + (1-\theta)x_2$, with $\theta = 1 - M(0,1)$.

■

Lemma 3 *For any* $(D\&I)-CNM$ *operator,*

$$M(x_1, \ldots, x_m) = M(p.x_1, \ldots, p.x_m), \quad \textit{with}\, p \in N_0.$$

Proof.

$$\begin{aligned}
M(p.x_1, \ldots, p.x_m) &= M(\underbrace{x_1, \ldots, x_1}_{p \text{ times}}, \ldots, \underbrace{x_m, \ldots, x_m}_{p \text{ times}}) \ \ \text{(notation)} \\
&= M(x_1, \ldots, x_m, \ldots, x_1, \ldots, x_m) \ \ \text{(commutativity)} \\
&= M(m.M(x_1, \ldots, x_m), \ldots, m.M(x_1, \ldots, x_m)) \ \ \text{(decomposability)} \\
&= M(x_1, \ldots, x_m) \ \ \text{(idempotency)}.
\end{aligned}$$

■

Theorem 2 (i) $(A\&SPL)-CNM$ *operators correspond to the class of*

$$M(x_1, \ldots, x_m) = \left[\min_i x_i\right] \ \ \textit{or} \ \ \left[\max_i x_i\right].$$

(ii) $(D\&SPL)-CNM$ *operators correspond to the class of*

$$M(x_1, \ldots, x_m) = \left[\min_i x_i\right] \ \ \textit{or} \ \ \left[\max_i x_i\right] \ \ \textit{or} \ \ \left[\frac{1}{m}\sum_i x_i\right].$$

Proof. Sufficient part of the theorem is evident. Let us turn to the necessary part.

Let us consider first (i).
Associativity implies :

$$M(z_1, M(z_2, z_3)) = M(M(z_1, z_2), z_3).$$

If $z_1 \leq z_2 \leq z_3$, lemma 2 gives

$$\theta z_1 + \theta(1-\theta)z_2 + (1-\theta)^2 z_3 = \theta^2 z_1 + \theta(1-\theta)z_2 + (1-\theta)z_3$$

or

$$\theta(1-\theta)(z_3 - z_1) = 0 \ , \quad \text{for all } z_3 \geq z_1.$$

As a consequence, $\theta = 0$ or 1 and $M(x_1, x_2) = \min(x_1, x_2) \vee \max(x_1, x_2)$.
The same values for θ are still obtained in a recurrent way for $m > 2$.

We turn now to (ii)
(ii-1) : Let us first prove that for $m = 3$,

$$M(x_1, x_2, x_3) = \frac{\theta^2 x_1 + \theta(1-\theta)x_2 + (1-\theta)^2 x_3}{\theta^2 + \theta(1-\theta) + (1-\theta)^2}, \quad (x_1 \leq x_2 \leq x_3) \in [0,1]^3 \ , \ \theta \in [0,1]$$

$$\begin{aligned} M(x_1, x_2, x_3) &= M(2.x_1, 2.x_2, 2.x_3) \quad \text{(lemma 3)} \\ &= M(x_1, x_2, x_1, x_3, x_2, x_3) \quad \text{(commutativity)} \\ &= M(2.M(x_1, x_2), 2.M(x_1, x_3), 2.M(x_2, x_3)) \quad \text{(decomposability)} \\ &= M(\theta x_1 + (1-\theta)x_2, \theta x_1 + (1-\theta)x_3, \theta x_2 + (1-\theta)x_3) \quad \text{(lemmas 2 \& 3)} \end{aligned}$$

$$\begin{aligned} M(x) &= M(x_1, x_2, x_3) = M(xA) \\ &= M\left[(x_1, x_2, x_3)\begin{pmatrix} \theta & \theta & 0 \\ 1-\theta & 0 & \theta \\ 0 & 1-\theta & 1-\theta \end{pmatrix}\right] = M(x^{(1)}). \end{aligned}$$

$x_1^{(1)} \leq x_2^{(1)} \leq x_3^{(1)}$ since $x_1 \leq x_2 \leq x_3$ and lemma 2.
By iteration, $M(x) = M(xA^i) = M(x^{(i)})$ with $x_1^{(i)} \leq x_2^{(i)} \leq x_3^{(i)}, \ \ \forall i \in N_0$.
The diagonalization of A gives

$$\lim_{i \to \infty} A^i = \frac{1}{D}\begin{pmatrix} \theta^2 & \theta^2 & \theta^2 \\ \theta(1-\theta) & \theta(1-\theta) & \theta(1-\theta) \\ (1-\theta)^2 & (1-\theta)^2 & (1-\theta)^2 \end{pmatrix} \ , \quad D = \theta^2 + \theta(1-\theta) + (1-\theta)^2.$$

Finally,

$$\begin{aligned} M(x) &= \lim_{i \to \infty} M(x^{(i)}) = M\left(x \lim_{i \to \infty} A^i\right) = M\left(3.\frac{\theta^2 x_1 + \theta(1-\theta)x_2 + (1-\theta)^2 x_3}{D}\right) \\ &= \frac{\theta^2 x_1 + \theta(1-\theta)x_2 + (1-\theta)^2 x_3}{D} \quad \text{(idempotency, see lemma 1).} \qquad \blacksquare \end{aligned}$$

(ii-2) : We now prove that $\theta \in \{1, 0, 1/2\}$, i.e.

$$M(x_1, x_2) = \min(x_1, x_2) \text{ or } \max(x_1, x_2) \text{ or } \left(\frac{x_1 + x_2}{2}\right).$$

Let us consider $0 \leq z_1 \leq z_2 \leq z_3 \leq 1$.
Decomposability implies

$$M(z_1, z_2, z_3) = M\left[M(z_1, z_3), M(z_1, z_3), z_2\right].$$

If $M(z_1, z_3) \leq z_2$,

$$\theta^2 z_1 + \theta(1-\theta)z_2 + (1-\theta)^2 z_3 = \theta^2 M(z_1, z_3) + \theta(1-\theta)M(z_1, z_3) + (1-\theta)^2 z_2$$

or $(1-\theta)(1-2\theta)(z_3 - z_2) = 0$.
As a consequence, $\theta = 1$ or $1/2$.
If $M(z_1, z_3) \geq z_2$,

$$\theta^2 z_1 + \theta(1-\theta)z_2 + (1-\theta)^2 z_3 = \theta^2 z_2 + \theta(1-\theta)M(z_1, z_3) + (1-\theta)^2 M(z_1, z_3),$$

or $\theta(1-2\theta)(z_2 - z_1) = 0$.
We can conclude that $\theta \in \{0, 1, 1/2\}$.

■

(ii-3) : We finally prove in a recurrent way that

$$M(x_1, \ldots, x_m) = \left[\min_i x_i\right] \text{ or } \left[\max_i x_i\right] \text{ or } \left[\frac{1}{m}\sum_i x_i\right].$$

Suppose $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq 1$, $m \geq 3$.

$$\begin{aligned} M(x_1, \ldots, x_m) &= M\left[(m-1).x_1, \ldots, (m-1).x_m\right] \\ &= M\left[x_1, x_2, \ldots, x_{m-1}, x_1, x_2, \ldots, x_{m-2}, x_m, \ldots, x_2, \ldots, x_m\right] \\ &= M\left[(m-1).M(x_1, \ldots, x_{m-1}), \ldots, (m-1).M(x_2, \ldots, x_m)\right] \\ &= M\left[M(x_1, \ldots, x_{m-1}), \ldots, M(x_2, \ldots, x_m)\right]. \end{aligned}$$

Using recurrency,

$$M(x_1, \ldots, x_m) = M\left[(x_1, \ldots, x_m)A_{m\times m}\right] = M(x_1^{(1)}, \ldots, x_m^{(1)})$$

where

$$A_{m\times m} = A_{\min,\, m\times m} \text{ or } A_{\max,\, m\times m} \text{ or } A_{mean,\, m\times m},$$

and

$$A_{\min,\, m\times m} = \left(\begin{array}{ccccc|c} 1 & \ldots & \ldots & \ldots & 1 & 0 \\ \hline & & & & & 1 \\ \cline{6-6} & & & & & \\ & & 0 & & & \\ & & & & & \end{array}\right)$$

$$A_{\max,\, m\times m} = \left(\begin{array}{cccccc} & & \multicolumn{2}{c}{0} & & \\ & & & & & \\ \cline{1-1} 1 & \multicolumn{1}{|c}{} & & & & \\ \cline{2-6} 0 & 1 & \ldots & \ldots & \ldots & 1 \end{array}\right)$$

$$A_{mean,\, m\times m} = \frac{1}{m-1}\left(\begin{array}{cccccc} & & & & & 0 \\ & 1 & & & & \\ & & & 0 & & \\ & & & & 1 & \\ 0 & & & & & \end{array}\right)$$

Due to monotonicity,

$$x_1^{(1)} = M(x_1, \ldots, x_{m-2}, x_{m-1}) \leq x_2^{(1)} = M(x_1, \ldots, x_{m-2}, x_m)$$

and $x_1^{(1)} \leq \cdots \leq x_m^{(1)}$.

By iteration,

$$\begin{aligned} M(x_1, \ldots, x_m) &= M\left[(x_1, \ldots, x_m)A_{m\times m}\right] = M\left[(x_1, \ldots, x_m)A^i_{m\times m}\right] \\ &= M(x_1^{(i)}, \ldots, x_m^{(i)}), \quad \text{for all } i \in N_0 \end{aligned}$$

with $x_1^{(i)} \leq \cdots \leq x_m^{(i)}$.

It is easily shown that

$$\lim_{i\to\infty} A^i_{\min,\, m\times m} = \left(\begin{array}{ccc} 1 & \ldots & 1 \\ \hline & & \\ & 0 & \\ & & \end{array}\right)$$

$$\lim_{i\to\infty} A^i_{\max,\, m\times m} = \left(\begin{array}{ccc} & 0 & \\ & & \\ & & \\ \hline 1 & \ldots & 1 \end{array}\right)$$

$$\lim_{i\to\infty} A^i_{mean,\, m\times m} = \frac{1}{m}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

Consequently,

$$M(x_1, \ldots, x_m) = \lim_{i\to\infty} M\left[(x_1, \ldots, x_m)A^i_{m\times m}\right] = M\left[(x_1, \ldots, x_m)A^\infty_{m\times m}\right]$$

where

$$A^\infty_{m\times m} = \left[\lim_{i\to\infty} A^i_{\min,\, m\times m}\right] \text{ or } \left[\lim_{i\to\infty} A^i_{\max,\, m\times m}\right] \text{ or } \left[\lim_{i\to\infty} A^i_{mean,\, m\times m}\right]$$

■

Corollary 2 *The class of $(D\&S\&SPL) - CNM$ operators is reduced to*

$$M(x_1, \ldots, x_m) = \frac{1}{m}\sum x_i.$$

Proof. Evident.

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