# Dynamic many-valued logics for searching games with errors

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## Introduction

Providing concrete interpretations of many-valued logics has always been an intriguing problem. In [6], MUNDICI develops a model of the RÉNYI - ULAM searching games with lies in terms of ŁUKASIEWICZ logic and MV-algebras. In this game, a liar picks out a number in a given search space $M$. A detective has to guess this number by asking Yes/No questions to the liar who is allowed to lie a maximum given number of times.

In his model of the game, MUNDICI interprets the states of knowledge of the detective at a given step of the game as an element of an MV-algebra. Even though this model provides a way to interpret the effect of the liar's answers on the states of knowledge of the game, its language (the language of MV-algebras) is not rich enough to state specifications about a *whole round* of the game.

The starting point of this talk is the will to add a 'dynamic' layer to this 'static' interpretation of the game. We actually develop finitely-valued generalizations of Propositional Dynamic Logic, which is a multi-modal logic designed to reason about programs (see [2, 5]). Informally, these new logics are a mixture of many-valued modal logics (as introduced in [1, 3, 4]) and algebras of regular programs.

## $n+1$-valued KRIPKE models

We fix $n \geq 1$ for the remainder of the paper and we denote by $Ł_n$ the sub-MV-algebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$.

We denote by $\Pi$ a set of programs and by $\mathsf{Form}$ a set of formulas defined from a countable set $\mathsf{Prop}$ of propositional variables $p, q \ldots$ and a countable set $\Pi_0$ of atomic programs $a, b, \ldots$ by the following BACKUS-NAUR forms (where $\phi$ are formulas and $\alpha$ are programs) :

$$\phi ::= p \mid 0 \mid \neg\phi \mid \phi \to \phi \mid [\alpha]\phi$$
$$\alpha ::= a \mid \phi? \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^*.$$

**Definition 1.** An $n+1$-*valued* KRIPKE *model* $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ is given by a non empty set $W$, a map $R : \Pi_0 \to 2^{W\times W}$ that assigns a binary relation $R_a$ to any $a$ of $\Pi_0$ and a map $\mathrm{Val} : W \times \mathsf{Prop} \to Ł_n$ that assigns a truth value to any propositional variable $p$ of $\mathsf{Prop}$ in any world $w$ of $W$.

The maps $R$ and Val are extended by mutual induction to formulas and programs by the following rules (where $\neg^{[0,1]}$ and $\to^{[0,1]}$ denote ŁUKASIEWICZ's interpretation of $\neg$ and $\to$ on $[0, 1]$):

1. $R_{\alpha;\beta} = R_\alpha \circ R_\beta$;
2. $R_{\alpha\cup\beta} = R_\alpha \cap R_\beta$;
3. $R_{\psi?} = \{(u, u) \mid \mathrm{Val}(u, \psi) = 1\}$;
4. $R_{\alpha^*} = \bigcup_{n\in\omega}(R_\alpha)^n$;
5. $\mathrm{Val}(w, \phi \to \psi) = \mathrm{Val}(w, \phi) \to^{[0,1]} \mathrm{Val}(w, \psi)$;
6. $\mathrm{Val}(w, \neg\psi) = \neg^{[0,1]}\mathrm{Val}(w, \psi)$;
7. $\mathrm{Val}(w, [\alpha]\psi) = \bigwedge\{\mathrm{Val}(v, \psi) \mid (w, v) \in R_\alpha\}$

If $w$ is a world of a KRIPKE model $\mathcal{M}$ and if $\mathrm{Val}(w, \phi) = 1$, we write $\mathcal{M}, w \models \phi$ and say that $\phi$ *is true in* $w$. If $\phi$ is a formula that is true in each world of a model $\mathcal{M}$ then $\phi$ is *true in* $\mathcal{M}$. A formula that is true in every KRIPKE model is called a *tautology*.

Hence, we intend to interpret the operator ';' as the concatenation program operator, the operator '$\cup$' as the alternative program operator and the operator '$*$' as the KLEENE program operator.

### $n + 1$-valued propositional dynamic logics

The purpose of the talk is to characterize the theory of the $n + 1$-valued KRIPKE models (Theorem 5).

**Definition 2.** An $n + 1$-*valued propositional dynamic logic* (or simply a *logic*) is a subset $\mathsf{L}$ of $\mathsf{Form}$ that is closed under the rules of *modus ponens*, uniform substitution and necessitation (generalization) and that contains the following axioms:

1. tautologies of the $n + 1$-valued ŁUKASIEWICZ logic;
2. for any program $\alpha$, axioms defining modality $[\alpha]$:
   (a) $[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$,
   (b) $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$,
   (c) $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$,
3. the axioms that define the program operators: for any programs $\alpha$ and $\beta$ of $\Pi$:

(a) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$, (d) $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$,

(b) $[\alpha; \beta]p \leftrightarrow [\alpha][\beta]p$,

(c) $[q?]p \leftrightarrow (\neg q^n \vee p)$, (e) $[\alpha^*]p \rightarrow [\alpha^*][\alpha^*]p$,

4. the induction axiom $(p \wedge [\alpha^*](p \rightarrow [\alpha]p)^n) \rightarrow [\alpha^*]p$ for any program $\alpha$.

We denote by $\mathsf{PDL}_n$ the smallest $n+1$-valued propositional dynamic logic.

As usual, a formula $\phi$ that belongs to a logic $\mathsf{L}$ is called a *theorem* of $\mathsf{L}$.

**Completeness result**

The classical construction of the canonical model can be adapted for $\mathsf{PDL}_n$. We denote by $\mathcal{F}_n$ the LINDENBAUM - TARSKI algebra of $\mathsf{PDL}_n$. The reduct of $\mathcal{F}_n$ to the language of MV-algebras is an MV-algebra. We denote by $\mathcal{MV}(\mathcal{F}_n, \text{Ł}_n)$ the set of MV-homomorphisms from the MV-reduct of $\mathcal{F}_n$ to $\text{Ł}_n$.

**Definition 3.** The *canonical model* of $\mathsf{PDL}_n$ is defined as the model $\mathcal{M}^c = \langle W^c, R^c, \mathrm{Val}^c \rangle$ where

1. $W^c = \mathcal{MV}(\mathcal{F}_n, \text{Ł}_n)$;
2. if $\alpha \in \Pi$, the relation $R^c_\alpha$ is defined by

$$R^c_\alpha = \{(u, v) \mid \forall \phi \in \mathcal{F}_n \ (u([\alpha]\phi) = 1 \Rightarrow v(\phi) = 1)\};$$

3. the map $\mathrm{Val}^c$ is defined by

$$\mathrm{Val}^c : W^c \times \mathsf{Form} : (u, \phi) \mapsto u(\phi).$$

Even though the valuation in $\mathcal{M}^c$ is defined for any formula, it turns out that it is compatible with the inductive definition of a valuation in a KRIPKE model.

**Proposition 4.** *1. If $\phi \in \mathsf{Form}$, if $\alpha \in \Pi$ and if $u$ is a world of $W^c$ then* $\mathrm{Val}^c(u, [\alpha]\phi) = \bigwedge\{\mathrm{Val}^c(v, \phi) \mid v \in R^c_\alpha u\}$.

*2. For any $\alpha \in \Pi$, the relation $R_{\alpha^*}$ is a reflexive and transitive extension of $R_\alpha$.*

According to the second item of the previous proposition, the canonical model may not be KRIPKE model. Nevertheless, it is possible to use a filtration lemma in order to use the canonical model to obtain a completeness result for $\mathsf{PDL}_n$.

**Theorem 5.** *The logic $\mathsf{PDL}_n$ is complete with respect to the $n+1$-valued* KRIPKE *models,* i.e., *a formula $\phi$ is a theorem of $\mathsf{PDL}_n$ if and only if $\phi$ is a tautology.*

## References

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