

VARADHAN ESTIMATES FOR TIME DEPENDENT DIFFUSIONS

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Abstract

In this paper, we give small time estimates (Varadhan estimates) of the logarithm of the density of a degenerate diffusion with time dependent coefficients, by means of a semi-Riemannian distance. The main tools of the proof are the stochastic calculus of variations (Malliavin Calculus) and large derivation theory.

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1 INTRODUCTION

Consider the Markov semigroup on $\mathbb{R}^m$ associated with the second order differential operator

$$L = X_0 + \sum_{j=1}^{d} X_j^2,$$

where X_j $j = 0, ..., d$ are vector fields on $\mathbb{R}^m$ with bounded derivatives of any order.

If the vector fields X_j, $j \neq 0$ generate $\mathbb{R}^m$, one can provide $\mathbb{R}^m$ with a structure of a Riemannian manifold. In that case the second order part of L becomes the second order part of the Laplace-Beltrami operator associated with the Riemannian metric.

Moreover the semigroup admits a smooth density $p_t(x, y)$ with respect to Lebesgue measure on $\mathbb{R}^m$ and S.R.S. Varadhan [9] has proved that

$$\lim_{t \to 0} 2t \log p_t(x, y) = -d^2(x, y). \tag{1}$$

On the other hand, under the Hörmander condition i.e. if the Lie algebra generated by the vector fields X_j, $j \neq 0$ equals $\mathbb{R}^m$, the diffusion still admits a smooth density and one can define a sub-Riemannian metric associated with the subelliptic operator L (cf.[4]).

R. Léandre ([6], [7]) has proved that in this case the formula (1) remains valid.

The purpose of this paper is to generalize the results of R. Léandre to the case where the vector fields X_j, $j = 0, ..., d$ are time dependent.

It is divided in five sections organized as follows. In the second section we introduce the diffusion, the heat kernel of which we want to estimate. In the third section we recall the definitions of two metrics associated with our operator L that one can find in the literature and we show that they are in fact identical. The aim of the fourth respectively fifth section is to prove the upper respectively the lower bound of the Varadhan estimate.

2 SETTING OF THE PROBLEM

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ be a complete probability space and w a d-dimensional Brownian motion on this space. If x_t is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\circ dx_t$ (respectively dx_t) denotes its Stratonovitch (respectively Itô) differential.

Let us consider $d + 1$ time dependent vector fields $X_0, ..., X_m$ on $\mathbb{R}^m$ which we write as

$$X_j^i(t; x)\frac{\partial}{\partial x_i}, \quad x \in \mathbb{R}^m, \ j = 1, ..., d.$$

We assume that the vector fields X_j, $0 \leq j \leq d$, as well as all their derivatives in x are Hölder continuous in t uniformly in $[0,1] \times \mathcal{K}$ for any compact subset $\mathcal{K}$ in $\mathbb{R}^m$, $\mathcal{C}^\infty$ bounded in x when t is a fixed element in $[0,1]$ and that all their derivatives in x are uniformly bounded.

Denote by $x_t \in \mathbb{R}^m$ the solution of the stochastic differential equation

$$\begin{cases} dx_t = X_0(t,x_t)\,dt + X_j(t,x_t)\circ dw_t^j \\ x_0 \text{ is a fixed element in } \mathbb{R}^m \end{cases}$$

Let us suppose furthermore that a local strong Hörmander condition is satisfied i.e. that $Lie(X_1,...,X_d)(0,x_0) = \mathbb{R}^m$. Then (cf.[3]) we know that for all t in $]0,1]$ the law of the stochastic process x_t admits a smooth density $p_t(x,y)$ with respect to Lebesgue measure on $\mathbb{R}^m$.

Our aim is to study the asymptotic behaviour of $p_t(x,y)$ when t tends to 0, which amounts to studying the asymptotic behaviour when ε tends to 0 of $p_1^\varepsilon(x,y)$, where p_t^ε denotes the density of the law of the diffusion x_t^ε, solution of the stochastic differential equation

$$\begin{cases} dx_t^\varepsilon = \varepsilon^2 X_0(t,x_t^\varepsilon)\,dt + \varepsilon X_j(t,x_t^\varepsilon)\circ dw_t^j \\ x_0^\varepsilon = x \end{cases}$$

Indeed, the scaling property of the Brownian motion implies that $p_{\varepsilon^2 t}(x,y) = p_t^\varepsilon(x,y)$.

3 SOME SUBRIEMANNIAN GEOMETRY

Definition 3.1 *An $\mathbb{R}^m$-valued Lipschitzian path γ, parametrized by $[0,\rho]$ is said to be subunit if for a.e. t and all ξ in $\mathbb{R}^m$*

$$\langle\dot\gamma(t),\xi\rangle^2 \le \sum_{j=1}^d \langle X_j(t,\gamma(t)),\xi\rangle^2.$$

We set

$$d_1(x,y) = \inf\{\rho\,/\,\exists \text{ a subunit path } \gamma \text{ s.t. } \gamma(0)=x \text{ and } \gamma(\rho)=y\}$$

Then, (cf.[2]) the function $(x,y)\to d_1(x,y)$ is well defined and continuous on $\mathbb{R}^m\times\mathbb{R}^m$.

On the other hand, consider for any $h\in H := L_2([0,1],\mathbb{R}^d)$ the deterministic process $\gamma_t^h(x)$, solution of the ordinary differential system

$$\begin{cases} d\gamma_t^h = \sum_{j=1}^d X_j(t,\gamma_t^h)\,h_t^j\,dt \\ \gamma_0^h = x \end{cases}$$

$\gamma_t^h(x)$ is said to be a horizontal curve issued from x and we denote by $I(\gamma^h)$ the energy of γ^h defined by

$$I(h) = \int_0^1 |h_s|^2\,ds = \sum_{j=1}^d (h_s^j)^2\,ds.$$

Set $\dfrac{\partial\gamma_t^h}{\partial x} = \Phi_t^h$. Then, Φ_t^h is the solution of the ordinary differential system

$$\begin{cases} d\Phi_t^h = \sum_{j=1}^d \dfrac{\partial X_j}{\partial x}(t,\gamma_t^h)\,\Phi_t^h\,h_t^j\,dt \\ \Phi_0^h = Id_{\mathbb{R}^m} \end{cases}$$

The function $(x,h)\to[t\to(\gamma_t^h(x)\,\Phi_t^h,(\Phi_t^h)^{-1})]$ is of class C^∞ from $\mathbb{R}^m\times H$ into the space of functions from $[0,1]$ into $\mathbb{R}^m\times Gl(\mathbb{R}^m)\times Gl(\mathbb{R}^m)$ equipped with the uniform topology.

Notation 3.2 *If $\tilde\gamma$ is a flow with differential $\tilde\Phi_t$ and Y a vector field on $[0,1]\times\mathbb{R}^m$, then we set*

$$(\tilde\Phi_t^*)^{-1}Y(t,x) = \left(\frac{\partial\tilde\gamma_t(x)}{\partial x}\right)^{-1} Y(t,\tilde\gamma_t(x)).$$

Theorem 3.3 *([2]) The function $h\to\gamma_1^h$ from H to $\mathbb{R}$ is a submersion if and only if the quadratic form*

$$M(x,h) = \sum_{j=1}^d \int_0^1 \langle(\Phi_s^{h*})^{-1}X_j(t,x),\cdot\rangle^2$$

is invertible. $M(x,h)$ is said to be the Malliavin quadratic form associated with the horizontal curve $\gamma(x,h)$.

Moreover, for every integer k, denote by $C_k(t,x)$ the vector space generated by the Lie brackets of the vector fields X_j, $j=1,...,d$ of length smaller then k, taken in (t,x).

Suppose furthermore that the following hypothesis is satisfied:

(H) For any x in $\mathbb{R}^m$, there exists an integer $r(x)$ such that for any t in $[0,1]$, $C_{r(x)}(t,x) = R^m$.

Under this assumption (cf.[2]) it is possible to connect any couple of points by a horizontal curve.

Definition 3.4 *For any x and y in R^m denote by $d_2(x,y)$ the distance defined by*

$$d_2^2(x,y) = \inf_{\substack{\gamma^h\ horizontal \\ \gamma_0^h=x,\ \gamma_1^h=y}} I(\gamma^h).$$

Then (cf.[2]) the function $(x,y)\to d_2(x,y)$ is continuous and we have the following easy extension of proposition 3.1. from [4]

Theorem 3.5 *The two distances previously introduced are identical i.e. for any x and y in $\mathbb{R}^m$,*

$$d_1(x,y) = d_2(x,y).$$

From now on, we denote this distance by d and we call it the sub-Riemannian distance associated with the vector fields X_j, $j = 1, ..., d$.

4 THE UPPER BOUND

In this section we give the upper bound of the density. We have the following result

Theorem 4.1 *Under hypothesis (H), we have for any multi-index of integers α*

$$\overline{\lim_{t\to 0}}\, 2t \log\left(\left|\frac{\partial^\alpha}{\partial y^\alpha} p_t(x,y)\right|\right) \le -d^2(x,y).$$

In particular,

$$\overline{\lim_{t\to 0}}\, 2t \log p_t(x,y) \le -d^2(x,y).$$

To prove the theorem, we need the following two lemmas:

Lemma 4.2 *([1]) If ε denotes a fixed strictly positive real number,*

$$\overline{\lim_{t\to 0}} 2t \log\left(\int_{|z-y|\le\varepsilon} p_t(x,z)\,dz\right) \le -\inf_{|z-y|\le\varepsilon} d^2(x,z) \quad (2)$$

uniformly in (x,y) on any compact set K in $\mathbb{R}^m \times \mathbb{R}^m$. if ε is a fixed strictly positive real number.

Lemma 4.3 *([5]) For any multi-index of integers α, there exists an integer $N(\alpha)$ and a real number $C(\alpha)$ such that*

$$\sup_{\substack{x\in\mathbb{R}^m\\ y\in\mathbb{R}^m}} \left|\frac{\partial^\alpha}{\partial y^\alpha} p_t(x,y)\right| \le C(\alpha)\, t^{-N(\alpha)}. \quad (3)$$

For any $\eta > 0$, there exists a strictly positive real number $C(\eta)$, a real number $C(\alpha)$ and an integer $N(\alpha)$ such that

$$\sup_{\substack{x\in\mathbb{R}^m\\ |x-y|\ge\eta}} \left|\frac{\partial^\alpha}{\partial y^\alpha} p_t(x,y)\right| \le C(\alpha)\, t^{-N(\alpha)} \exp\left[-\frac{C(\nu)}{2t}\right]. \quad (4)$$

proof of the theorem:

The proof is essentially the same as the one of proposition 1 of [7]. We recall it for the sake of completeness.

Fix a compact K in $\mathbb{R}^m \times \mathbb{R}^m$. Since the function $(x,y) \to d^2(x,y)$ is continuous, we have that for any $\varepsilon > 0$, there exists an $\eta > 0$ such that for all (x,y) in K, $d^2(x,y) \le d^2(x,z) + \varepsilon$, as soon as $|x-y| \le \eta$.

By means of relation (2), we can then find a real number t_0 such that for any (x,y) in K and all $0 < t_1 < t_0$,

$$J_1(x,y) := \int_{|z-y|\le\eta} p_{t_1}(x,z)\,dz \le \exp\left[-\frac{d^2(x,y)-\varepsilon}{2t_1}\right] \quad (5)$$

Let $K_1 \times K_2$ be a compact in $\mathbb{R}^m \times R^m$ which contains K. Set

$$J_2(x,y) = \int_{\mathbb{R}^m\setminus K_2} p_{t_1}(x,z)\,dz = P(x_{t_1} \in K_2^c).$$

Consequently by means of the exponential bound of Stroock-Varadhan (cf.[8]), we have

$$J_2(x,y) \le C \exp\left[-\frac{C\,\Lambda(x,K_2^c)}{2t_1}\right],$$

where $\Lambda(x,K_2^c) := \inf_{y\in K_2^c} \|x-y\|^2$.

We choose K_2 large enough in order to ensure that $C\,\Lambda(x,K_2^c) \ge \sup_{(x,y)\in K} d^2(x,y) + \varepsilon$. Hence

$$J_2(x,y) \le C \exp\left[-\frac{d^2(x,y)-\varepsilon}{2\,t_1}\right] \quad (6)$$

Finally, we may suppose that $\{z \,/\, |z-y| \le \eta\} \subset K_2$. Set

$$J_3(x,y) = \int_{K_2\setminus\{z\,/\,|z-y|\le\eta\}} \left|\frac{\partial^\alpha}{\partial y^\alpha} p_{t-t_1}(z,y)\right| dz$$

Relation (4) then implies that

$$J_3(x,y) \le C(\alpha,t,t_1,K_2) \exp\left[-\frac{C(\eta)}{2(t-t_1)}\right]. \quad (7)$$

Moreover by Kolmogorov's formula, we have for any $t_1 < t$

$$\frac{\partial^\alpha}{\partial y^\alpha} p_t(x,y) = \int_{\mathbb{R}^m} p_{t_1}(x,z) \frac{\partial^\alpha}{\partial y^\alpha} p_{t-t_1}(z,y)\,dz. \quad (8)$$

But,

$$\mathbb{R}^m = \{z\,/\,|z-y|\le\eta\} \cup K_2\setminus\{z\,/\,|z-y|\le\eta\} \cup \mathbb{R}^m\setminus K_2.$$

Choose N such that $N\,C(\eta) > \sup_{(x,y)\in K} d^2(x,y) + \varepsilon$ and $N > 1$ and take $t_1 = t(1-\frac{1}{N}) < t$. The formulae (8), (5), (6) and (7) then imply that

$$\overline{\lim_{t\to 0}}\, 2t \log\left(\left|\frac{\partial^\alpha}{\partial y^\alpha} p_t(x,y)\right|\right) \le -\frac{d^2(x,y)+\varepsilon}{1-\frac{1}{N}} \le -d^2(x,y)+\varepsilon$$

Hence the result follows by letting ε tend to 0.

□

5 THE LOWER BOUND

In this section we give the lower bound of the density. For that, we will need the following proposition.

Proposition 5.1 *For every strictly positive ε one can find a horizontal curve issued from x with invertible Malliavin quadratic form and an energy smaller than ε.*

proof:

Let $g \in C^\infty(\mathbb{R}, [0,1])$ be a function such that $g'(x) > 0$ and $g(x) \leq 1$, for all x in $\mathbb{R}$ (for instance $g(x)$=Arctan x). Consider d independent Brownian motions v_t^j, $j = 1, ..., d$. Let $\gamma_t^g(x)$ be the horizontal curve defined as the solution of the ordinary differential equation with random coefficients

$$\begin{cases} d\gamma_t^g(x) = \sqrt{\varepsilon}\, X_j\left(t, \gamma_t^g(x)\right) g(v_t^j)\, dt \\ \gamma_0^g(x) = x. \end{cases}$$

Thus $I\left(\gamma^g(x)\right) = \varepsilon \int_0^1 g^2(v_t^j)\, dt \leq \varepsilon$ and Φ_t^g is the solution of the differential equation with random coefficients

$$\begin{cases} d\Phi_t^g = \sqrt{\varepsilon}\, \dfrac{\partial X_j}{\partial x}\left(t, \gamma_t^g(x)\right) \Phi_t^g\, g(v_t^j)\, dt \\ \Phi_0^g(x) = Id_{\mathbb{R}^m}. \end{cases}$$

Hence

$$M(x,g) = \varepsilon \sum_{j=1}^d \int_0^1 \langle (\Phi_t^{g*})^{-1} X_j(t,x), \cdot \rangle^2\, dt.$$

Moreover, the quadratic form M is invertible, since if we take $u \in S^{m-1}$, there cannot exist an a.s. strictly positive stopping time T_u such that $\langle (\Phi_t^{g*})^{-1} X_j(t,x), u \rangle = 0$ on $[\![0, T_u]\!]$, for $j = 1, ..., d$. Indeed, otherwhise this would imply that $\sum_{j=1}^d \langle (\Phi_t^{g*})^{-1} [X_i, X_j](t,x), u \rangle g(v_t^j) = 0$ on $[[0, T_u]]$. Consequently the martingale part of the last process would be zero. But the stochastic variation of this martingale part is equal to $\sum_{j=1}^d \int_0^t \langle (\phi_t^{g*})^{-1} [X_i, X_j](t,x), u \rangle^2 \left(g'(v_t^j)\right)^2 dt$. Since $g'(x) > 0$, $\langle (\Phi_t^{g*})^{-1} X_j(t,x), u \rangle$ would be zero on $[\![0, T_u]\!]$. By induction, one would get that if Y denotes a Lie bracket of order k, $\langle (\Phi_t^{g*})^{-1} u \rangle$ would be zero on $[\![0, T_u]\!]$, which leads to a contradiction.

□

That allows us to prove the main theorem of this section.

Theorem 5.2 *Uniformly on any compact subset of $\mathbb{R}^m \times \mathbb{R}^m$, we have*

$$\underline{\lim}_{t \to 0} 2t \log p_t(x,y) \geq -d^2(x,y).$$

proof:

Actually it is equivalent to prove that on any compact subset of $\mathbb{R}^m \times \mathbb{R}^m$,

$$\underline{\lim}_{t \to 0} 2\varepsilon^2 \log p_1^\varepsilon(x,y) \geq -d^2(x,y).$$

It is sufficient to prove that for every choice of an x_0 and an y_0 in $\mathbb{R}^m$ and a strictly positive real η, we can find some neighbourhoods $V(x_0)$ and $V(y_0)$ such that there exists a strictly positive ε_0 such that for any $\varepsilon \leq \varepsilon_0$,

$$2\varepsilon^2 \log p_1^\varepsilon(x,y) \geq -d^2(x,y) - 4\eta,$$

for any x in $V(x_0)$ and y in $V(y_0)$.

Lemma 5.3 *([7]) Consider an (x_0, y_0) in $\mathbb{R}^m \times \mathbb{R}^m$ and a strictly positive real number η. There exists a neighbourhood $V(x_0)$ of x_0 and a neighbourhood $V(y_0)$ of y_0 and an open subset V_1 of $\mathbb{R}^m$, such that for any x in $V(x_0)$ and any y in $V(y_0)$*

$$\inf_{z \in V_1} d^2(x,z) \leq d^2(x,y) + \eta, \tag{9}$$

and

$$\overline{\lim_{\varepsilon \to 0}} 2\varepsilon^2 \log\{\inf_{z \in V_1, y \in V(y_0)} p_1^\varepsilon(z,y)\} \geq -\eta. \tag{10}$$

Consider the neighbourhoods $V(x_0)$ and $V(y_0)$ from lemma 5.3 and introduce the associated neighbourhood V_1. Consider two positive real numbers ε_1 and ε_2 such that $\varepsilon_1^2 + \varepsilon_2^2 = \varepsilon^2$. The Markov property implies that

$$\begin{aligned} p_1^\varepsilon(x,y) &= \int_{\mathbb{R}^m} p_1^\varepsilon(x,z)\, p_1^\varepsilon(z,y)\, dz \\ &\geq \int_{V_1} \int_{\mathbb{R}^m} p_1^\varepsilon(x,z)\, p_1^\varepsilon(z,y)\, dz.\ cr \end{aligned}$$

Lemma 5.3 then implies that there exists a real number ε_0 such that for any $\varepsilon \leq \varepsilon_0$, for any y in $V(y_0)$ and any z in V_1,

$$p_1^\varepsilon(x,y) \geq \int_{V_1} p_1^\varepsilon(x,z) \exp\left[-\frac{\eta}{\varepsilon_2^2}\right] dz. \tag{11}$$

On the other hand, [1] and formula (10) imply that there exists an ε_0' such that for any $\varepsilon \leq \varepsilon_0'$ and all x in $V(x_0)$,

$$\int_{V_1} p_1^\varepsilon(x,z)\, dz \geq \exp\left[-\frac{d^2(x,y)+\eta}{2\varepsilon_1^2}\right]. \tag{12}$$

Hence the relations (11) and (12) allow us to conclude, if we choose

$$\varepsilon_1^2 = \varepsilon^2 \frac{d^2(x,y)+\eta}{d^2(x,y)+3\eta} \quad \text{and} \quad \varepsilon_2^2 = \varepsilon^2 \frac{2\eta}{d^2(x,y)+3\eta}.$$

□

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