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SMOOTHNESS OF THE DENSITY FOR THE FILTER UNDER INFINITE DIMENSIONAL NOISE AND UNBOUNDED OBSERVATION COEFFICIENTS

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Abstract

The purpose of this paper is to prove that the unnormalized filter associated with a nonlinear filtering problem with unbounded observation coefficients and infinite dimensional dependent noise admits a smooth density with respect to the Lebesgue measure.

1 Introduction

The purpose of this paper is to prove that the unnormalized filter associated with a nonlinear filtering problem with infinite dimensional dependent noises and a scalar observation process the coefficients of which are unbounded admits a smooth density with respect to the Lebesgue measure.

This problem has already been investigated when the noise appearing in the system process is finite dimensional by many authors. Michel [7] and Bismut and Michel [1] have solved this problem under a local Hörmander condition in the case of systems with dependent noises and bounded coefficients. The case of independent noises and unbounded observation coefficients has been handled by Ferreyra [2] whereas Florchinger [4] has treated the case of dependent noises.

In [5], by means of the Malliavin calculus, it is proved, under a local Hörmander condition, that the filter associated with nonlinear filtering systems with independent noises, bounded coefficients and a state process driven by infinite dimensional noises admits a smooth density with respect to the Lebesgue measure.

This paper is divided in three sections organized as follows. In section one, we introduce the nonlinear filtering problem studied in this paper and we recall some notations that we need in the sequel. In section two, we define an unnormalized filter linked with the filter defined in the previous section by means of a Kallianpur–Striebel formula. In section three, we state and prove the main result of the paper.

2 Setting of the problem

Let $(\Omega, \mathcal{F}, P)$ be an usual probability space, $\{w_t^k, t \in [0,T], k \in \mathbb{N}^*\}$ a sequence of independent standard Wiener processes and v a standard Wiener process independent of the processes w^k, $k \in \mathbb{N}^*$. Denote by $||.||$ the norm on the space of matrices in $\mathbb{R}^n \times \mathbb{R}^{\mathbb{N}}$ given by

$$||M|| = \left(\sum_{i=1}^{n} \sum_{j=0}^{+\infty} (m_{ij})^2 \right)^{1/2}$$

Assume that we are given a family $\{X_k, k \geq 0\}$ of vector fields on $\mathbb{R}^n$ which we write as

$$X_k(x) = \sum_{i=1}^{n} X_k^i(x) \frac{\partial}{\partial x_i} \quad \forall x \in \mathbb{R}^n \ , \ \forall k \in \mathbb{N},$$

such that the map $X : \mathbb{R}^n \to (\mathbb{R}^n)^{\mathbb{N}}$, $X = \{X_k, k \geq 0\}$, is a C^∞ function with bounded derivatives of any order.

Consider the nonlinear filtering problem associated with the system process–observation pair $(x_t, y_t) \in \mathbb{R}^n \times \mathbb{R}$ solution of the stochastic differential system,

$$\left\{\begin{array}{l} x_t = x_0 + \int_0^t X_0(x_s)ds \\ \qquad + \sum_{i=1}^{+\infty} \int_0^t X_i(x_s) \circ dw_s^i \\ \qquad + \int_0^t \bar{X}(x_s) \circ (h(x_s)ds + dv_s) \\ \\ y_t = \int_0^t h(x_s)ds + v_t \end{array}\right. \tag{1}$$

where,

1. x_0 has distribution m_0.
2. $\bar{X}$ is a C_b^∞ vector fields on $\mathbb{R}^n$ which we write as
$$\bar{X}(x) = \sum_{j=1}^{n} \bar{X}^j(x)\frac{\partial}{\partial x_j}.$$
3. h is a function in $C^\infty(\mathbb{R}^n, \mathbb{R})$ which is as well as its derivatives of any order of less than exponential growth.
4. $X_0 + h\bar{X}$ is of less than linear growth (in order to avoid explosives solutions of the system (1)).

As usually in nonlinear filtering problems define the filter associated with the system (1) as follows.

Definition 2.1 *For every t in $[0, T]$, denote by π_t the filter associated with the system (1) defined for any function ψ in $C_b(\mathbb{R}^n, \mathbb{R})$ by*

$$\pi_t\psi = E\left[\psi(x_t)/\mathcal{Y}_t\right]$$

where $\mathcal{Y}_t = \sigma(y_s \ / \ 0 \leq s \leq t)$.

3 The unnormalized filter

The assumptions on function h do not allow to define, as usually in nonlinear filtering problems, a reference probability measure; indeed the Girsanov exponential associated with the system (1) is not necessarily a martingale and Girsanov theorem cannot be applied. To get round this difficulty, we define a formal unnormalized filter and prove a Kallianpur–Striebel formula.

In order to get rid of the dependence of the noises, introduce the following definitions and notations.

Definition 3.1 *Denote by Φ_t the deterministic flow associated with the vector field $\bar{X}$ (i.e. Φ_t is the unique solution of the deterministic equation $\Phi_t = x + \int_0^t \bar{X}(\Phi_s(x))ds$).*

Convention In the rest of this paper, the summation sign is omitted for repeated indices appearing once at the top and once at the bottom.

Notation 3.2 *If X denotes a vector field on $\mathbb{R}^n$ such that for any x in $\mathbb{R}^n$, $X(x) = X^i(x)\frac{\partial}{\partial x_i}$ and F is a function in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ then, for every x in $\mathbb{R}^n$, set*

$$XF(x) = X^i(x)\frac{\partial F}{\partial x_i}(x)$$

and,

$$(\Phi_t^{\star -1}X)F(x) = (D\Phi_t(x))^{-1}X(\Phi_t(x))DF(x).$$

Then, since the observation process y_t is one dimensional, one can substitute the time parameter in the expression of the deterministic flow Φ_t by y_t, and introduce, as in [1], a stochastic process $\bar{x}_t \in \mathbb{R}^n$ solution of the stochastic differential equation

$$\begin{aligned} \bar{x}_t &= x_0 + \int_0^t (\Phi_{y_s}^{\star -1}X_0)(\bar{x}_s)ds \\ &+ \sum_{i=1}^{+\infty} \int_0^t (\Phi_{y_s}^{\star -1}X_i)(\bar{x}_s) \circ dw_t^i. \end{aligned} \tag{2}$$

Then, the following result holds.

Proposition 3.3 *(cf. [1]) For all t in $[0, T]$, $x_t = \Phi_{y_t}(\bar{x}_t)$.*

Furthermore, as usually in nonlinear filtering problems, define for any t in $[0, T]$, the Girsanov exponential associated with the system (1) by :

$$Z_t = \exp\left(\int_0^t h(x_s)dv_s + \frac{1}{2}\int_0^t h^2(x_s)ds\right). \tag{3}$$

Therefore, according with proposition 3.3 and the definition of the stochastic process y_t one can prove that for any t in $[0,T]$, $Z_t = \exp V_t$ where

$$V_t = \int_0^t h(\Phi_{y_s}(\bar{x}_s))dy_s - \frac{1}{2}\int_0^t h^2(\Phi_{y_s}(\bar{x}_s))ds. \quad (4)$$

Furthermore, since h is unbounded and the noises appearing in the system process and the observation process are dependent, the stochastic process Z_t^{-1} is not necessarily a $\mathcal{F}_t$ martingale and consequently, we cannot apply Girsanov theorem and define as usually an unnormalized filter. Nevertheless, one can define a formal unnormalized filter associated with the system (1) by means of the expression of the stochastic process Z_t obtained after an integration by parts in the stochastic integral appearing in the expression of the stochastic process V_t. Moreover, in order to prove an integrability result for the stochastic process Z_t, assume (as in the rest of this paper) that for all $r > 0$ and $\epsilon > 0$, there exists $K_\epsilon > 0$ such that

$$|\tilde{X}h| + \sup_{|s|\leq r} |G(h \circ \Phi_s)| + \sum_{i=1}^{+\infty} \sup_{|s|\leq r} |X_i(h \circ \Phi_s)|^2 \leq \epsilon h^2 + K_\epsilon$$

where G is the second order differential operator defined by

$$G = X_0 + \frac{1}{2}\sum_{i=1}^{+\infty} X_i^2.$$

Then,

Theorem 3.4 *(cf. [3]) For any $p > 0$, the stochastic process Z_t is in $L^p(W \otimes m_0)$ almost surely in y (here, W denotes the Wiener measure defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$).*

Hence, one can define an unnormalized filter as follows.

Definition 3.5 *For any t in $[0,T]$, define for any function Ψ in $C_b(\mathbb{R}^n, \mathbb{R})$, the unnormalized filter associated with the system (1) by*

$$\rho_t\Psi = E^w(\Psi(x_t)Z_t) \quad (5)$$

where E^w denotes the integration with respect to $W \otimes m_0$.

Furthermore, the unnormalized filter ρ_t is linked with the filter π_t by means of the following Kallianpur–Striebel like formula.

Theorem 3.6 *(cf. [3]) For any t in $[0,T]$ and any function Ψ in $C_b(\mathbb{R}^n, \mathbb{R})$ one has,*

$$\pi_t\Psi = \frac{\rho_t\Psi}{\rho_t 1}. \quad (6)$$

4 Existence of a regular density for the filter

Since the filter π_t is linked with the unnormalized filter ρ_t by the Kallianpur–Striebel formula it is equivalent to prove the existence of a smooth density for the filter π_t or the unnormalized filter ρ_t.

In that aim it is enough to prove that all the derivatives of the process ρ_t in the sense of distributions are bounded measures. Thus, since the Malliavin calculus allows to proceed to an integration by parts on the Wiener space the result will follow from the following lemma.

Lemma 4.1 *(cf. [6]) Let ν be a finite Radon measure on $\mathbb{R}^n$. Assume that for all multi-indices α there exists a finite constant C_α such that for all function $\Psi \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ one has*

$$|\int_{\mathbb{R}^n} \nabla^\alpha \Psi(x)\nu(dx)| \leq C_\alpha ||\Psi||_\infty,$$

then the measure ν admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Then, the main theorem of the paper can be stated as follows.

Theorem 4.2 *Assume that the vector space spaned by the vector fields*

$$X_k,\ k \geq 1;\ [X_{k_1}, X_{k_2}],\ k_1, k_2 \geq 0; ...$$

$$[...[X_{k_{j-1}}, X_{k_j}]...],\ k_1, ..., k_j \geq 0; ...$$

evaluated at x_0 has dimension n. Then, for any $t \in]0,T]$, the unnormalized filter ρ_t admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Proof Since for any $t \in [0,T]$, one has

$$\rho_t\Psi = E^w(\Psi \circ \Phi_{y_t}(\bar{x}_t)Z_t) \quad (7)$$

and, according with [3], ρ_t is continuous with respect to the paths of the observation process y_t one can fix the path of the process y in (7) and then work out a Malliavin calculus only on functionals of w and apply the results of proved in [8].

By means of arguments similar to those used in [4], on can prove that for all function Ψ in $C_b^\infty(\mathbb{R}^n, \mathbb{R})$ and all t in $[0,T]$, $\Psi \circ \Phi_{y_t}(\bar{x}_t)$ and Z_t are in $D_\infty(W)$ for all y in $C_0([0,T],\mathbb{R})$.

Furthermore, denoting by M_t the Malliavin covariance matrix associated with the stochastic process $\bar{x}_t$, one can prove the following integration by parts formula.

Lemma 4.3 *For all t in $[0,T]$, H in D_∞ and all i, $1 \leq i \leq n$, one has*

$$E^w\left(\nabla^i \Psi \circ \Phi_{y_t}(\bar{x}_t) Z_t M_t H\right) = E^w\left(\Psi \circ \Phi_{y_t}(\bar{x}_t) Z_t \left(-\int_0^t \sum_{k=1}^{+\infty} D_s^k H D_s^k \bar{x}_t ds - H\int_0^t \sum_{k=1}^{+\infty} D_s^k(\log Z_t) D_s^k \bar{x}_t ds + H\int_0^t \sum_{k=1}^{+\infty} D_s^k(\bar{x}_t) dw_s^k\right)\right). \quad (8)$$

Moreover since the hypothesis of the theorem imply that the vector space spaned by the vector fields

$$\Phi_{y_t}^{\star-1}X_k,\ k \geq 1;\ \left[\Phi_{y_t}^{\star-1}X_{k_1}, \Phi_{y_t}^{\star-1}X_{k_2}\right], k_1, k_2 \geq 0;$$

$$\ldots, \left[\ldots\left[\Phi_{y_t}^{\star-1}X_{k_{j-1}}, \Phi_{y_t}^{\star-1}X_{k_j}\right]\ldots\right], k_1, \ldots, k_j \geq 0; \ldots$$

evaluated at $(0, x_0)$ has dimension n, one can deduce from the results proved in [8] that for all $t \in]0,T]$, $(\det M_t)^{-1}$ is in $L^p(W \otimes m_0)$ for all p in $\mathbb{N}^\star$.

Then, according with Lemma 4.1 and equality (8) one can deduce easily that the unnormalized filter ρ_t admits a density with respect to the Lebesgue measure.

Furthermore, iterating like in [1] the integration by parts formula one can deduce that the density of the unnormalized filter is smooth.

This concludes the proof of Theorem 4.2.

References

[1] J.M.Bismut,D.Michel, Diffusions conditionnelles. *Journal of Functional Analysis.* Part I **44** (1981) 174–211, Part II **4** (1982) 274–292.

[2] G.S.Ferreyra, Smoothness of the unnormalized conditional measures of stochastic nonlinear filtering. *Proceedings of 23rd IEEE Conference on Decision and Control* Las Vegas (NV) (1984).

[3] P.Florchinger, Continuité par rapport à la trajectoire de l'observation du filtre associé à des systèmes corélés à coefficients de l'observation non bornés. *Stochastics and Stochastics Reports* **28** (1989) 21–64.

[4] P.Florchinger, Malliavin calculus with time dependent coefficients and application to nonlinear filtering. *Probability Theory and Related Fields* **86** (1990) 203–223.

[5] P.Florchinger, Existence of a smooth density for the filter in nonlinear filtering with infinite dimensional noise. *Systems and Control Letters* **16** 2 (1991) 131–137.

[6] P.Malliavin, Stochastic calculus of variations and hypoelliptic operators. *Proceedings International Conference on Stochastic Differential Equations* Kyoto 1976, 195–263. Tokyo : Kinokuniya; New York : Wiley (1978).

[7] D.Michel, Régularité des lois conditionnelles en théorie du filtrage non linéaire et calcul des variations stochastiques. *Journal of Functional Analysis* **41** (1981) 3–36.

[8] J.Schiltz, Calcul de Malliavin appliqué à une classe d'EDS avec coefficients dépendant du temps. Submitted.