Weighted Banzhaf interaction indexes and weighted least squares

J.-L. Marichal and P. Mathonet
University of Luxembourg, Mathematics Research Unit, FSTC
{jean-luc.marichal,pierre.mathonet}[at]uni.lu

In the framework of cooperative game theory, several notions of power indexes are used in order to measure the influence that a given player has on the outcome of the game, or to define a procedure to share the benefits of the game among the players. The best known power indexes are certainly due to Shapley and Shubik [9], and to Banzhaf [1], but there are many other examples of such indexes in the literature.

When one is concerned by measuring the power or analyzing the general behavior of players within a game, the information provided by a power index might not be sufficient. For instance, power indexes give no information on how players interact within the game. The notion of interaction index was introduced in order to measure such information. The first example of such an index goes back to Owen in 1972. A systematic approach was initiated by Grabisch and Roubens from 1996 and led to the definition of Shapley interaction index, Banzhaf interaction index, and many others (see [4] and the references therein).

There is no universal power or interaction index that can be used in every practical situation. The choice of such an index often depends on some desirable properties that are relevant to the problem under consideration. Therefore many axiomatic characterizations of power indexes and interaction indexes have be proposed so far [9, 3, 6].

Besides these axiomatic characterizations, the Banzhaf and Shapley power indexes were shown to be solutions of simple least squares approximation problems :

- Charnes et al. [2] considered the problem of finding the best efficient approximation of a given game by an additive one in the sense of weighted least squares. They showed that the Shapley power index appears as the solution of the approximation problem for a specified choice of weight system of the coalitions.
- Hammer and Holzman [7] considered the problem of approximating a given pseudo-Boolean function by a pseudo-Boolean function of lower degree, in the sense of standard least squares. They showed that the Banzhaf power index appears as coefficients of the linear terms in the solution of the approximation problem by functions of degree at most one. This observation was generalized in [5], where the authors showed that the Banzhaf interaction indexes appear as leading coefficients of best least squares approximations of pseudo-Boolean functions.

Recently, we used this standard least squares approximation approach to extend the notion of interaction index for classes of functions defined on the unit hypercube of $\mathbb{R}^n$. However, when we use a standard least squares problem to define an index, we implicitly consider that all coalitions of players stand on the same footing. Here we follow the Hammer and Holzman's approach, but we add a weighted probabilistic point of view : we consider the framework of pseudo-Boolean functions but we assign a weight $w(S)$ to every coalition S of players. We interpret this weight as the probability of formation of the coalition. This probabilistic point of view was considered for instance in [2, 8]. We then naturally define a weighted Banzhaf index associated to w by considering the leading coefficients of the approximation of a specified degree.

We then investigate the main properties of this class of weighted Banzhaf indexes parameterized by the possible suitable weights w, such as linearity, symmetry, and dummy player condition. We prove that they all belong to the class of so-called probabilistic interaction indexes axiomatized in [4], and that they induce linear transformations on the set of pseudo-Boolean functions.

Finally, using the results of [5], we show that the classical Banzhaf and Shapley indexes can actually be interpreted as barycenters of natural subfamilies of these weighted indexes.

Références

[1] J. F. Banzhaf. Weighted voting doesn't work : A mathematical analysis. *Rutgers Law Review*, 19 :317–343, 1965.

[2] A. Charnes, B. Golany, M. Keane, and J. Rousseau. Extremal principle solutions of games in characteristic function form : Core, Chebychev and Shapley value generalizations. In *Econometrics of planning and efficiency, Vol. dedic. Mem. G. Tintner, Adv. Stud. Theor. Appl. Econ. 11, 123–133.* Kluwer, Dordrecht, 1988.

[3] P. Dubey and L. S. Shapley. Mathematical properties of the Banzhaf power index. *Math. Oper. Res.*, 4 :99–131, 1979.

[4] K. Fujimoto, I. Kojadinovic, and J.-L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games Econom. Behav.*, 55(1) :72–99, 2006.

[5] M. Grabisch, J.-L. Marichal, and M. Roubens. Equivalent representations of set functions. *Math. Oper. Res.*, 25(2) :157–178, 2000.

[6] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. J. Game Theory*, 28(4) :547–565, 1999.

[7] P. Hammer and R. Holzman. Approximations of pseudo-Boolean functions ; applications to game theory. *Z. Oper. Res.*, 36(1) :3–21, 1992.

[8] G. Owen. Multilinear extensions of games. In *The Shapley value. Essays in honor of Lloyd S. Shapley, 139–151*, pages 139–151. Cambridge University Press, 1988.

[9] L. Shapley. A value for n-person games. In *Contributions to the Theory of Games II (Annals of Mathematics Studies 28)*, pages 307–317. Princeton University Press, 1953.