k-intolerant capacities and Choquet integrals

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Aggregation in multicriteria decision aid

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- **Aggregation function**

$$F : [0,1]^n \to [0,1]$$

$$(x_1, \ldots, x_n) \mapsto F(x_1, \ldots, x_n)$$

Tolerant and intolerant character of F

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$$F(x) = \Big(\prod_{i=1}^{n} x_i\Big)^{1/n} \quad \longrightarrow \quad ?$$

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$$\boxed{E(F) := \int_{[0,1]^n} F(x)\, dx}$$

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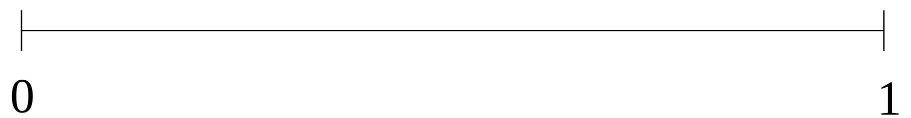

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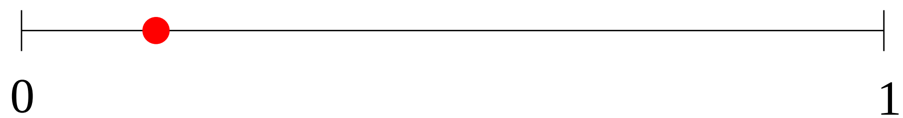

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- $E(\min) = \frac{1}{n+1}$

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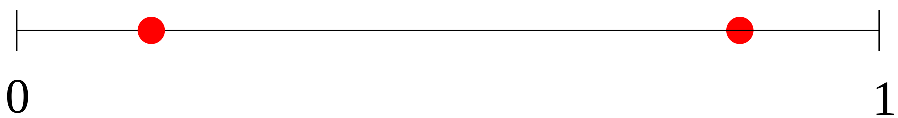

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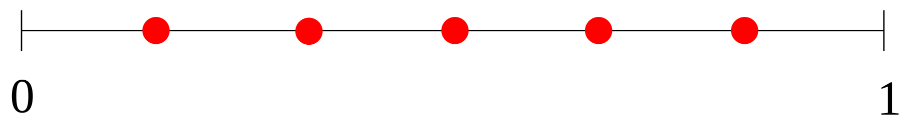

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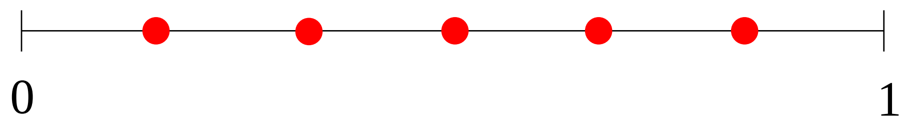

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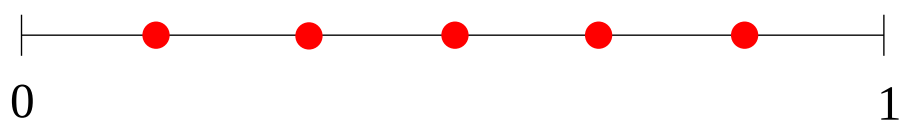

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- $E(\min) = \frac{1}{n+1}$ (most intolerant)
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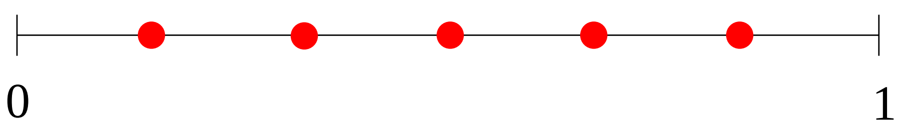

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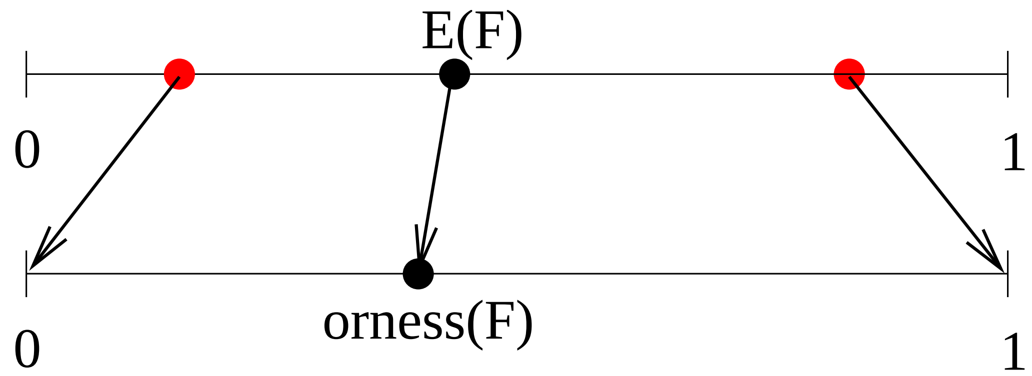

$$\text{orness}(F) \quad := \quad \frac{E(F) - E(\min)}{E(\max) - E(\min)}$$

(Dujmović, 1974)

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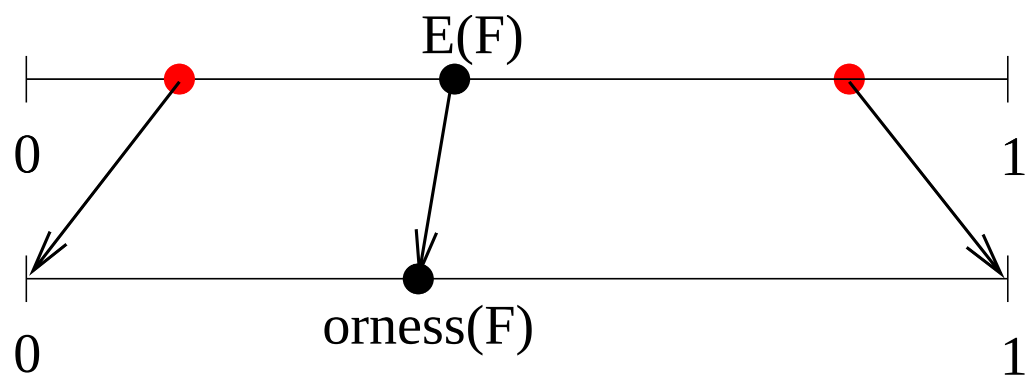

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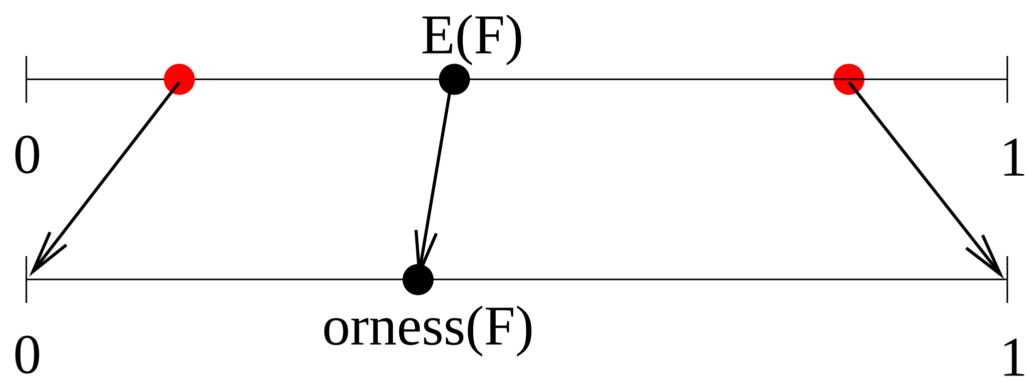

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$$\text{andness}(F) + \text{orness}(F) = 1$$

Intolerant behavior : application

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Selection of candidates for a university permanent position

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Academic selection criteria

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Example of procedure rules

The complete failure of any two of these criteria results in automatic rejection of the applicant

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$$\boxed{F(x) \leqslant x_{(k)} \qquad (x \in [0,1]^n)}$$

Choquet integral

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Capacity on N

$$v : 2^N \to [0,1], \text{ monotone, } v(\varnothing) = 0, \text{ and } v(N) = 1$$

$$\mathcal{F}_n := \{\text{capacities on } N\}$$

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Choquet integral of $x \in [0,1]^n$ w.r.t. v

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)} \Big[v[(i),\ldots,(n)] - v[(i+1),\ldots,(n)] \Big]$$

with the convention that $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$.

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Example

If $x_3 \leqslant x_1 \leqslant x_2$, we have

$$\mathcal{C}_v(x_1, x_2, x_3) = x_3[v(3,1,2) - v(1,2)] + x_1[v(1,2) - v(2)] + x_2 v(2)$$

k-intolerant capacities and Choquet integrals

Definition

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Then the following assertions are equivalent:

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Recruiting problem
The global evaluation is bounded above by $x_{(2)}$

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Definition
$v \in \mathcal{F}_n$ is k-intolerant if $iv)$ holds for k and not for $k-1$.

Tolerant behavior : application

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Parents want to buy a house

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House buying criteria

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3. With safe neighborhood for children to grow up in

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The parents are ready to consider a house that would fully succeed any five over the six criteria

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When $F \equiv \mathcal{C}_v$

we have a similar proposition as for intolerance...

Intolerance and tolerance indices

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Given a Choquet integral $\mathcal{C}_v$

$\mathrm{andness}(\mathcal{C}_v)$ measures the degree to which $\mathcal{C}_v$ is intolerant

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How can we measure the degree to which the inequality $\mathcal{C}_v \leqslant \mathrm{OS}_k$ holds ?

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Idea : defined from the conditional expectation $E(\mathcal{C}_v \mid x_{(k)} = 0)$

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In terms of v, this index reads

$$\mathrm{intol}_k(\mathcal{C}_v) = 1 - \frac{1}{n-k} \sum_{t=0}^{n-k} \frac{1}{\binom{n}{t}} \sum_{\substack{T \subseteq N \\ |T|=t}} v(T)$$

Intolerance and tolerance indices

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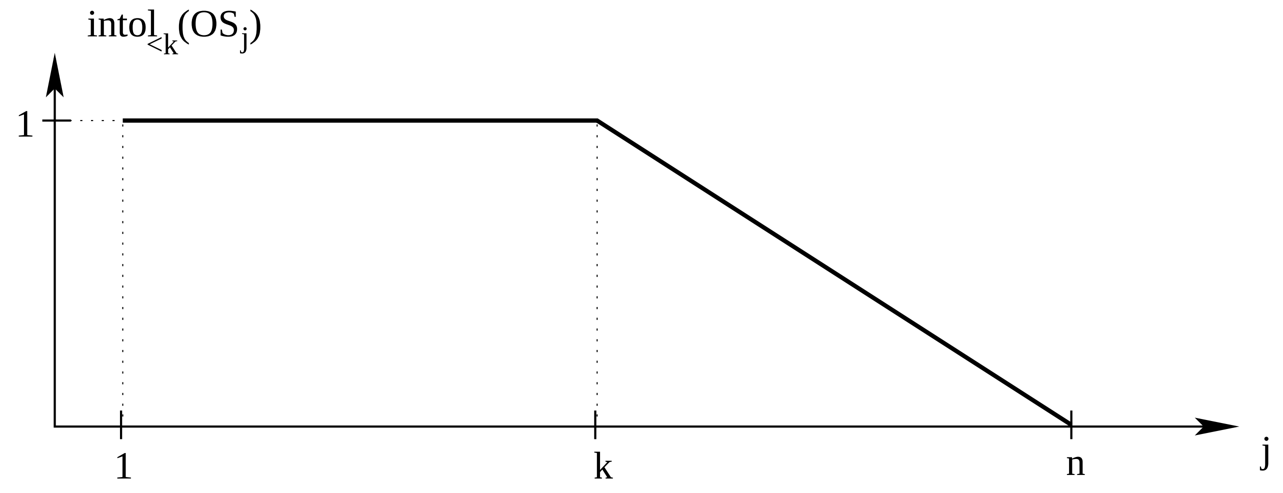

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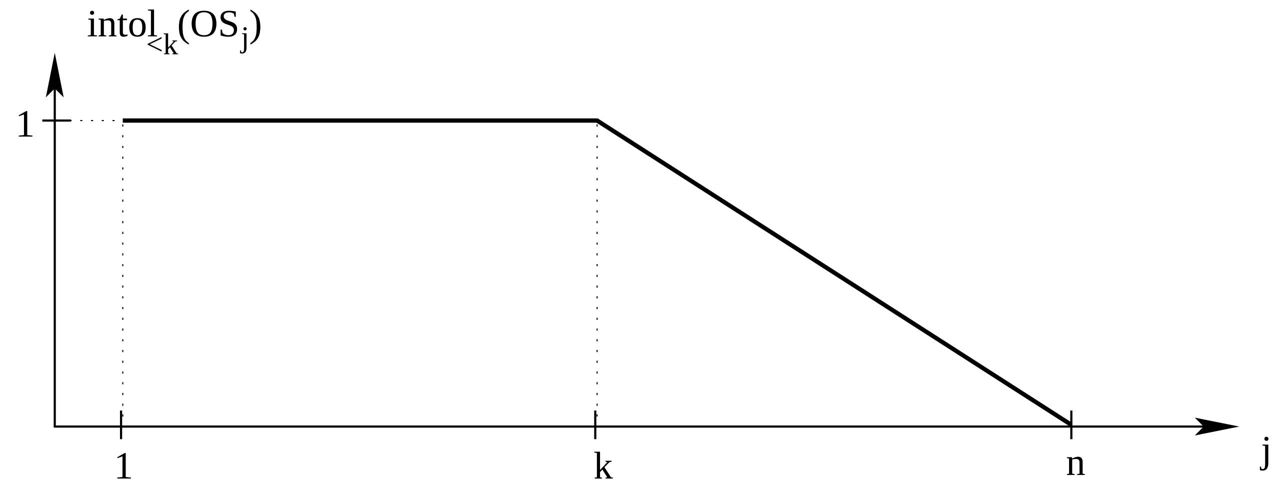

$$\mathrm{OS}_j \leqslant \mathrm{OS}_k \quad \Longleftrightarrow \quad j \leqslant k$$

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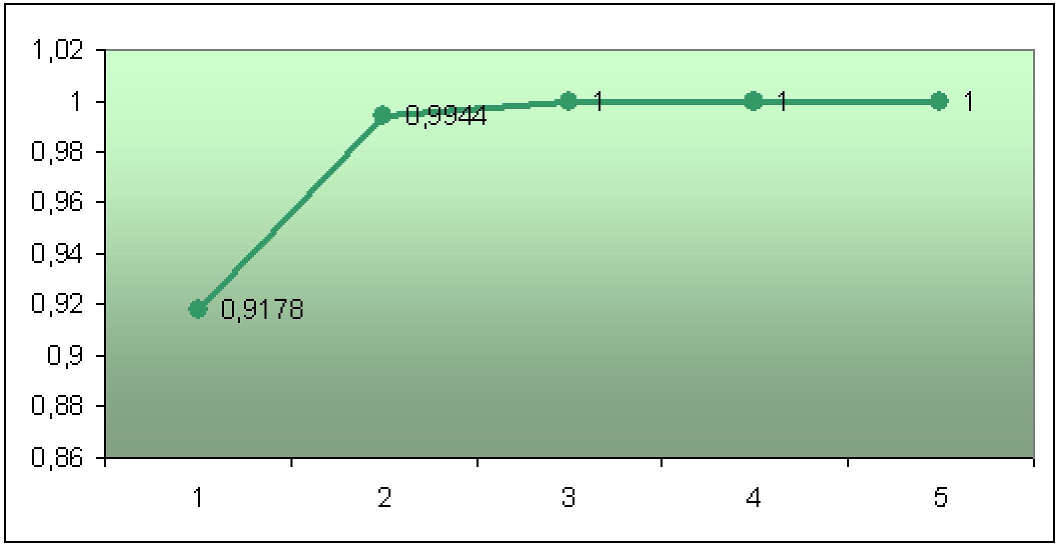
Sequence $\mathrm{intol}_k(\mathcal{C}_v)$ for $k = 1, \ldots, 5$



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...with similar motivation, characterization, properties.

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- Identification of capacities :
  - by optimization
  - learning data
  - constraints on behavioral parameters...