k-intolerant capacities and Choquet integrals

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Abstract

We define an aggregation function to be (at most) k-intolerant if it is bounded from above by its kth lowest input value. Applying this definition to the discrete Choquet integral and its underlying capacity, we introduce the concept of k-intolerant capacities which, when varying k from 1 to n, cover all the possible capacities on n objects. Just as the concepts of k-additive capacities and p-symmetric capacities have been previously introduced essentially to overcome the problem of computational complexity of capacities, k-intolerant capacities are proposed here for the same purpose but also for dealing with intolerant or tolerant behaviors of aggregation.

Keywords: multi-criteria analysis, interacting criteria; capacities; Choquet integral.

1 Introduction

In a previous work [7] the author investigated the intolerant behavior of the discrete Choquet integral when used to aggregate interacting criteria. Roughly speaking, the Choquet integral $\mathcal{C}_v$, or equivalently its associated capacity v, has an intolerant behavior if its output (aggregated) value is often close to the lowest of its input values. More precisely, consider the domain $[0,1]^n$ of $\mathcal{C}_v$ as a probability space, with uniform distribution, and the mathematical expectation of $\mathcal{C}_v$, which expresses the typical position of $\mathcal{C}_v$ within the unit interval. A low expectation then means that the Choquet integral is rather intolerant and behaves nearly like the minimum on average. Similarly, a high expectation means that the Choquet integral is rather tolerant and behaves nearly like the maximum on average. Note that such an analysis is meaningless when criteria are independent since, in that case, the Choquet integral boils down to a weighted arithmetic mean whose expectation is always one half (neither tolerant nor intolerant.)

In this paper we pursue this idea by defining k-intolerant Choquet integrals [1]. The case $k = 1$ corresponds to the unique most intolerant Choquet integral, namely the minimum. The case $k = 2$ corresponds to the subclass of n-variable Choquet integrals that are bounded from above by their second lowest input values. Those Choquet integrals are more or less intolerant but not as much as the minimum. As an example, the following 3-variable Choquet integral

$$\mathcal{C}_v(x_1, x_2, x_3) = \frac{1}{2}\min(x_1, x_2) + \frac{1}{2}\min(x_1, x_3)$$

is clearly 2-intolerant, while being different from the minimum.

[1] Equivalently, we define k-intolerant capacities since there is a one-to-one correspondence between n-variable Choquet integrals and capacities defined on n objects.

More generally, denoting by $x_{(1)}, \ldots, x_{(n)}$ the order statistics resulting from reordering $x_1, \ldots, x_n$ in the nondecreasing order, we say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-intolerant if

$$\mathcal{C}_v(x) \leq x_{(k)} \qquad (x \in [0,1]^n) \tag{1}$$

and it is exactly k-intolerant if, in addition, there is $x^* \in [0,1]^n$ such that $\mathcal{C}_v(x^*) > x^*_{(k-1)}$, with convention that $x_{(0)} := 0$.

Interestingly, condition (1) clearly implies that the output value of $\mathcal{C}_v$ is zero whenever at least k input values are zeros. We will see in Section 3 that the converse holds true as well.

At first glance, defining k-intolerant aggregation functions may appear as a pure mathematical exercise without any real application behind. In fact, in many real-life decision problems, experts or decision-makers are or must be intolerant. This is often the case when, in a given selection problem, we search for most qualified candidates among a wide population of potential alternatives. It is then sensible to reject every candidate which fails at least k criteria.

Example 1.1. Consider a (simplified) problem of selecting candidates applying for a university permanent position and suppose that the evaluation procedure is handled by appointed expert-consultants on the basis of the following academic selection criteria:

1. Scientific value of curriculum vitae,
2. Teaching effectiveness,
3. Ability to supervise staff and work in a team environment,
4. Ability to communicate easily in English,
5. Work experience in the industry,
6. Recommendations by faculty and other individuals.

Assume also that one of the rules of the evaluation procedure states that the complete failure of any two of these criteria results in automatic rejection of the applicant. This quite reasonable rule forces the Choquet integral, when used for the aggregation procedure, to be 2-intolerant, thus restricting the class of possible Choquet integrals for such a selection problem.

On the other hand, there are real-life situations where it is recommended to be tolerant, especially if the criteria are hard to meet simultaneously and if the potential alternatives are not numerous. To deal with such situations, we introduce k-tolerant aggregation functions and we will say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-tolerant if

$$\mathcal{C}_v(x) \geq x_{(n-k+1)} \qquad (x \in [0,1]^n).$$

In that case, the output value of $\mathcal{C}_v$ is one whenever at least k input values are ones.

Example 1.2. Consider a family who consults a Real Estate agent to buy a house. The parents propose the following house buying criteria:

1. Close to a school,
2. With parks for their children to play in,
3. With safe neighborhood for children to grow up in,
4. At least 100 meters from the closest major road,

5. At a fair distance from the nearest shopping mall,

6. Within reasonable distance of the airport.

Feeling that it is likely unrealistic to satisfy all six criteria simultaneously, the parents are ready to accept a house that would fully succeed any five over the six criteria. If a 6-variable Choquet integral is used in this selection problem, it must be 5-tolerant.

Considering k-intolerant and k-tolerant capacities can also be viewed as a way to make real applications easier to model from a computational viewpoint. Those "simplified" capacities indeed require less parameters than classical capacities (actually $O(n^{k-1})$ parameters instead of $O(2^n)$; see Section 3). Moreover, when varying k from 1 to n, we clearly recover all the possible capacities on n objects.

Notice however that this idea of partitioning capacities into subclasses is not new. Grabisch [3] proposed the k-additive capacities, which gradually cover all the possible capacities starting from additive capacities ($k = 1$). Later, Miranda et al. [8] introduced the p-symmetric capacities, also covering the possible capacities but starting from symmetric capacities ($p = 1$). Note also that other approaches to overcome the exponential complexity of capacities have also been previously proposed in the literature: Sugeno λ-measures [10], $\perp$-decomposable measures (see e.g. [5]), hierarchically decomposable measures [11], distorted probabilities (see e.g. [9]) to name a few.

2 Basic definitions

Let $F : [0,1]^n \to [0,1]$ be an aggregation function. By considering the cube $[0,1]^n$ as a probability space with uniform distribution, we can compute the mathematical expectation of F, that is,

$$E(F) := \int_{[0,1]^n} F(x)\,dx. \tag{2}$$

This value gives the average position of F within the interval $[0,1]$.

When F is *internal* (i.e., $\min \leq F \leq \max$) then it is convenient to rescale $E(F)$ within the interval $[E(\min), E(\max)]$. This leads to the following normalized and mutually complementary values [1, 7]:

$$\text{andness}(F) := \frac{E(\max) - E(F)}{E(\max) - E(\min)} \tag{3}$$

$$\text{orness}(F) := \frac{E(F) - E(\min)}{E(\max) - E(\min)} \tag{4}$$

Thus defined, the degree of *andness* (resp. *orness*) of F represents the degree or intensity (between 0 and 1) to which the average value of F is close to that of "min" (resp. "max"). In some sense, it also reflects the extent to which F behaves like the minimum (resp. the maximum) on average.

Define the kth order statistic function $\text{OS}_k : [0,1]^n \to [0,1]$ as

$$\text{OS}_k(x) = x_{(k)} \qquad (x \in [0,1]^n),$$

where $x_{(k)}$ is the kth lowest coordinate of x. It can be proved [7] that

$$E(\text{OS}_k) = \frac{k}{n+1} \qquad (k \in \{1,\dots,n\})$$

and hence the set $\{E(\text{OS}_k) \mid k = 1,\dots,n\}$ partitions the unit interval $[0,1]$ into $n+1$ equal-length subintervals.

Now, as mentioned in the introduction, when a function $F : [0,1]^n \to [0,1]$ is used to aggregate decision criteria, it is clear that the more $E(F)$ is low, the more F has an intolerant behavior. This suggests the following definition:

Definition 2.1. Let $k \in \{1, \ldots, n\}$. An aggregation function $F : [0,1]^n \to [0,1]$ is *at most k-intolerant* if $F \leq \mathrm{OS}_k$. It is *k-intolerant* if, in addition, $F \nleq \mathrm{OS}_{k-1}$, where $\mathrm{OS}_0 := 0$ by convention.

It follows immediately from this definition that, for any k-intolerant function F, we have $E(F) \leq E(\mathrm{OS}_k)$ and, if F is internal, we have $\mathrm{andness}(F) \geq \mathrm{andness}(\mathrm{OS}_k)$ and $\mathrm{orness}(F) \leq \mathrm{orness}(\mathrm{OS}_k)$.

Example 2.1. The product $F(x) = \prod_i x_i$, defined on $[0,1]^n$, is 1-intolerant and we have $E(F) = 1/2^n$.

By duality, we can also introduce k-tolerant functions as follows:

Definition 2.2. Let $k \in \{1, \ldots, n\}$. An aggregation function $F : [0,1]^n \to [0,1]$ is *at most k-tolerant* if $F \geq \mathrm{OS}_{n-k+1}$. It is *k-tolerant* if, in addition, $F \ngeq \mathrm{OS}_{n-k+2}$, where $\mathrm{OS}_{n+1} := 1$ by convention.

It is immediate to see that when a function $F : [0,1]^n \to [0,1]$ is k-intolerant, its *dual* $F^* : [0,1]^n \to [0,1]$, defined by

$$F^*(x_1, \ldots, x_n) := 1 - F(1 - x_1, \ldots, 1 - x_n) \qquad (x \in [0,1]^n) \tag{5}$$

is k-tolerant and vice versa.

In the next section we investigate the particular case where F is the Choquet integral and we define the concepts of k-intolerant and k-tolerant capacities.

3 Case of Choquet integrals and capacities

The use of the Choquet integral has been proposed by many authors as an adequate substitute to the weighted arithmetic mean to aggregate interacting criteria; see e.g. [2, 6]. In the weighted arithmetic mean model, each criterion is given a weight representing the importance of this criterion in the decision. In the Choquet integral model, where criteria can be dependent, a capacity is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria.

Let us first recall the formal definitions of these concepts. Throughout, we will use the notation $N := \{1, \ldots, n\}$ for the set of criteria.

Definition 3.1. A *capacity* on N is a set function $v : 2^N \to [0,1]$, that is nondecreasing with respect to set inclusion and such that $v(\varnothing) = 0$ and $v(N) = 1$.

Definition 3.2. Let v be a capacity on N. The *Choquet integral* of $x : N \to \mathbb{R}$ with respect to v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)} \, [v(A_{(i)}) - v(A_{(i+1)})], \tag{6}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Furthermore $A_{(i)} := \{(i), \ldots, (n)\}$ and $A_{(n+1)} := \varnothing$.

In this section we apply the ideas of k-intolerance and k-tolerance (cf. Definitions 2.1 and 2.2) to the Choquet integral. Since this integral is internal, it can be seen as a function from $[0,1]^n$ to $[0,1]$.

Let us denote by $\mathcal{F}_N$ the set of all capacities on N. The following proposition, inspired from [7, §4], gives equivalent conditions for a Choquet integral to be at most k-intolerant.

Proposition 3.1. *Let $k \in \{1, \ldots, n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \le x_{(k)}\ \forall x \in [0,1]^n$,

ii) $v(T) = 0\ \forall T \subseteq N$ *such that* $|T| \le n-k$,

iii) $\mathcal{C}_v(x) = 0\ \forall x \in [0,1]^n$ *such that* $x_{(k)} = 0$,

iv) $\mathcal{C}_v(x)$ *is independent of* $x_{(k+1)}, \ldots, x_{(n)}$,

v) $\exists \lambda \in [0,1)$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(k)} \le \lambda \Rightarrow \mathcal{C}_v(x) \le \lambda$,

As we can see, some assertions of Proposition 3.1 are natural and can be interpreted easily. Some others are more surprising and show that the Choquet integral may have an unexpected behavior.

First, assertion (*ii*) enables us to define k-intolerant capacities as follows:

Definition 3.3. Let $k \in \{1, \ldots, n\}$. A capacity $v \in \mathcal{F}_N$ is *k-intolerant* if $v(T) = 0$ for all $T \subseteq N$ such that $|T| \le n-k$ and there is $T^* \subseteq N$, with $|T^*| = n-k+1$, such that $v(T^*) \neq 0$.

Assertion (*iii*) says that the output value of the Choquet integral is zero whenever at least k input values are zeros. This is actually a straightforward consequence of k-intolerance.

Assertion (*iv*) is more surprising. It says that the output value of the Choquet integral does not take into account the values of $x_{(k+1)}, \ldots, x_{(n)}$. Back to Example 1.1, only the two lowest scores are taken into account to provide a global evaluation, regardless of the other scores.

Assertion (*v*) is also of interest. By imposing that $\mathcal{C}_v(x) \le \lambda$ whenever $x_{(k)} \le \lambda$ for a given threshold $\lambda \in [0,1)$, we necessarily force $\mathcal{C}_v$ to be at most k-intolerant. For instance, consider the problem of evaluating students with respect to different courses and suppose that it is decided that if the lowest k marks obtained by a student are less than 18/20 then his/her global mark must be less than 18/20. In this case, the Choquet integral utilized is at most k-intolerant.

Proposition 3.1 can be easily rewritten for k-tolerance by considering the dual $\mathcal{C}_v^*$ of the Choquet integral $\mathcal{C}_v$ as defined in Eq. (5). On this issue, Grabisch et al. [4, §4] showed that the dual $\mathcal{C}_v^*$ of $\mathcal{C}_v$ is the Choquet integral $\mathcal{C}_{v^*}$ defined from the *dual capacity* v^*, which is constructed from v by

$$v(T) = 1 - v(N \setminus T) \qquad (T \subseteq N).$$

We then have

$$\mathcal{C}_v \ge \mathrm{OS}_{n-k+1} \quad \Leftrightarrow \quad \mathcal{C}_{v^*} \le \mathrm{OS}_k.$$

Proposition 3.2. *Let $k \in \{1, \ldots, n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \ge x_{(n-k+1)}\ \forall x \in [0,1]^n$,

ii) $v(T) = 1\ \forall T \subseteq N$ *such that* $|T| \ge k$,

iii) $\mathcal{C}_v(x) = 1\ \forall x \in [0,1]^n$ *such that* $x_{(n-k+1)} = 1$,

iv) $\mathcal{C}_v(x)$ *is independent of* $x_{(1)}, \ldots, x_{(n-k)}$,

v) $\exists \lambda \in (0,1]$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(n-k+1)} \ge \lambda \Rightarrow \mathcal{C}_v(x) \ge \lambda$,

Here again, some assertions are of interest. First, assertion (*ii*) enables us to define k-tolerant capacities as follows:

Definition 3.4. Let $k \in \{1, \ldots, n\}$. A capacity $v \in \mathcal{F}_N$ is *k-tolerant* if $v(T) = 1$ for all $T \subseteq N$ such that $|T| \ge k$ and there is $T^* \subseteq N$, with $|T^*| = k-1$, such that $v(T^*) \neq 1$.

Assertion (*iii*) says that the output value of the Choquet integral is one whenever at least k input values are ones.

Assertion (*iv*) says that the output value of the Choquet integral does not take into account the values of $x_{(1)}, \ldots, x_{(n-k)}$. As an application, consider students who are evaluated according to n homework assignments and assume that the evaluation procedure states that the two lowest homework scores of each student are dropped, which implies that each student can miss two homework assignments without affecting his/her final grade. If a n-variable Choquet integral is used to aggregate the homework scores, it should not take $x_{(1)}$ and $x_{(2)}$ into consideration and hence it is at most $(n-2)$-tolerant.

4 Conclusion

In this paper, which can be considered as the sequel of [7], we have proposed the concepts of k-intolerant and k-tolerant Choquet integrals and capacities. Besides the obvious computational advantage of these concepts (comparable to that of k-additive and p-symmetric capacities), they can be easily interpreted in practical decision problems where the decision makers must be intolerant or tolerant. In an extended version of this paper, we also introduce axiomatically intolerance and tolerance indices which measure the degree to which the Choquet integral is k-intolerant and k-tolerant. These indices, when varying k from 1 to $n-1$, make it possible to identify and measure the intolerant or tolerant character of the decision maker.

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