Cumulative Distribution Functions and Moments of Weighted Lattice Polynomials

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Abstract. We give the cumulative distribution functions, the expected values, and the moments of weighted lattice polynomials when regarded as real functions. Since weighted lattice polynomial functions include Sugeno integrals, lattice polynomial functions, and order statistics, our results encompass the corresponding formulas for these particular functions.

1 Introduction

The cumulative distribution functions (c.d.f.'s) and the moments of order statistics have been discovered and studied for many years (see e.g. [4]). Their generalizations to lattice polynomial functions, which are nonsymmetric extensions of order statistics, were investigated very recently by Marichal [7] for independent variables and then by Dukhovny [5] for dependent variables.

Roughly speaking, an n-ary *lattice polynomial* is any well-formed expression involving n real variables $x_1,\ldots,x_n$ linked by the lattice operations $\wedge = \min$ and $\vee = \max$ in an arbitrary combination of parentheses. In turn, such an expression naturally defines an n-ary *lattice polynomial function*. For instance,

$$p(x_1,x_2,x_3) = (x_1 \wedge x_2) \vee x_3$$

is a 3-ary lattice polynomial function.

Lattice polynomial functions can be generalized by regarding certain variables as parameters, like in the 2-ary polynomial

$$p(x_1,x_2) = (c \wedge x_1) \vee x_2,$$

where c is a real constant. Such "parameterized" lattice polynomial functions, called *weighted lattice polynomial* functions [8, 11], are very often considered in the area of nonlinear aggregation functions as they include the whole class of discrete Sugeno integrals [12, 13].

In this paper we give a closed-form formula for the c.d.f. of any weighted lattice polynomial function in terms of the c.d.f.'s of its input variables. More precisely, considering an n-ary weighted lattice polynomial function p and n independent random variables $X_1,\ldots,X_n$, X_i $(i = 1,\ldots,n)$ having c.d.f. $F_i(x)$, we give a formula for the c.d.f. of $Y_p := p(X_1,\ldots,X_n)$. We also yield a formula for the expected value $\mathbb{E}[g(Y_p)]$, where g is any measurable function. The special cases $g(x) = x$, x^r, $[x-\mathbb{E}(Y_p)]^r$, and e^{tx} give, respectively, the expected value, the raw moments, the central moments, and the moment-generating function of Y_p.

This paper is organized as follows. In Section 2 we recall the basic material related to lattice polynomial functions and their weighted versions. In Section 3 we provide the announced results. In Section 4 we investigate the particular case where the input random variables are uniformly distributed over the unit interval. Finally, in Section 5 we provide an application of our results to the reliability analysis of coherent systems.

Weighted lattice polynomial functions play an important role in the areas of nonlinear aggregation and integration. Indeed, as we mentioned above, they include all the discrete Sugeno integrals, which are very useful aggregation functions in many areas. More details about the Sugeno integrals and their applications can be found in the remarkable edited book [6].

2 Weighted lattice polynomials

In this section we give some definitions and properties related to weighted lattice polynomial functions. More details and proofs can be found in [8].

As we are concerned with weighted lattice polynomial functions of random variables, we do not consider weighted lattice polynomial functions on a general lattice, but simply on an interval $L := [a,b]$ of the extended real number system $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Clearly, such an interval is a bounded distributive lattice, with a and b as bottom and top elements. The lattice operations $\wedge$ and $\vee$ then represent the minimum and maximum operations, respectively. To simplify the notation, we also set $[n] := \{1,\ldots,n\}$ for any integer $n \geqslant 1$.

Let us first recall the definition of a lattice polynomial (with real variables); see e.g. Birkhoff [2, §II.5].

Definition 1. *Given a finite collection of variables $x_1,\ldots,x_n \in L$, a lattice polynomial in the variables $x_1,\ldots,x_n$ is defined as follows:*

1. *the variables $x_1,\ldots,x_n$ are lattice polynomials in $x_1,\ldots,x_n$;*
2. *if p and q are lattice polynomials in $x_1,\ldots,x_n$, then $p \wedge q$ and $p \vee q$ are lattice polynomials in $x_1,\ldots,x_n$;*
3. *every lattice polynomial is formed by finitely many applications of the rules 1 and 2.*

When two different lattice polynomials p and q in the variables $x_1,\ldots,x_n$ represent the same function from L^n to L, we say that p and q are equivalent and we write $p = q$. For instance, $x_1 \vee (x_1 \wedge x_2)$ and x_1 are equivalent.

The weighted lattice polynomial functions are defined as follows.

Definition 2. *A function $p : L^n \to L$ is an n-ary weighted lattice polynomial function if there exists an integer $m \geqslant 0$, parameters $c_1,\ldots,c_m \in L$, and a lattice polynomial function $q : L^{n+m} \to L$ such that*

$$p(x_1,\ldots,x_n) = q(x_1,\ldots,x_n,c_1,\ldots,c_m) \qquad (x_1,\ldots,x_n \in L).$$

Because L is a distributive lattice, any weighted lattice polynomial function can be written in *disjunctive* and *conjunctive* forms as follows.

Proposition 1. *Let $p : L^n \to L$ be any weighted lattice polynomial function. Then there are set functions $\alpha : 2^{[n]} \to L$ and $\beta : 2^{[n]} \to L$ such that*

$$p(x) = \bigvee_{S\subseteq[n]} \Big[\alpha(S) \wedge \bigwedge_{i\in S} x_i\Big] = \bigwedge_{S\subseteq[n]} \Big[\beta(S) \vee \bigvee_{i\in S} x_i\Big].$$

Proposition 1 naturally includes the classical lattice polynomial functions. To see it, it suffices to consider nonconstant set functions $\alpha : 2^{[n]} \to \{a,b\}$ and $\beta : 2^{[n]} \to \{a,b\}$, with $\alpha(\varnothing) = a$ and $\beta(\varnothing) = b$.

The set functions α and β that disjunctively and conjunctively generate the polynomial function p in Proposition 1 are not unique. For example, as we have already observed above, we have

$$x_1 \vee (x_1 \wedge x_2) = x_1 = x_1 \wedge (x_1 \vee x_2).$$

However, it can be shown that, from among all the possible set functions that disjunctively generate a given weighted lattice polynomial function, only one is nondecreasing. Similarly, from among all the possible set functions that conjunctively generate a given weighted lattice polynomial function, only one is nonincreasing. These particular set functions are given by

$$\alpha(S) = p(\mathbf{e}_S) \quad \text{and} \quad \beta(S) = p(\mathbf{e}_{[n]\setminus S}),$$

where, for any $S \subseteq [n]$, $\mathbf{e}_S$ denotes the characteristic vector of S in $\{a,b\}^n$, i.e., the n-dimensional vector whose ith component is a, if $i \in S$, and b, otherwise. Thus, an n-ary weighted lattice polynomial function can always be written as

$$p(x) = \bigvee_{S\subseteq[n]} \Big[p(\mathbf{e}_S) \wedge \bigwedge_{i\in S} x_i\Big] = \bigwedge_{S\subseteq[n]} \Big[p(\mathbf{e}_{[n]\setminus S}) \vee \bigvee_{i\in S} x_i\Big].$$

The best known instances of weighted lattice polynomial functions are given by the discrete *Sugeno integrals*, which consist of a nonlinear discrete integration with respect to a *fuzzy measure*.

Definition 3. *An L-valued fuzzy measure on $[n]$ is a nondecreasing set function $\mu : 2^{[n]} \to L$ such that $\mu(\varnothing) = a$ and $\mu([n]) = b$.*

The Sugeno integrals can be presented in various equivalent forms. The next definition introduce them in one of their simplest forms (see [12]).

Definition 4. *Let μ be an L-valued fuzzy measure on $[n]$. The Sugeno integral of a function $x : [n] \to L$ with respect to μ is defined by*

$$\mathcal{S}_\mu(x) := \bigvee_{S\subseteq[n]} \Big[\mu(S) \wedge \bigwedge_{i\in S} x_i\Big].$$

Thus, any function $f : L^n \to L$ is an n-ary Sugeno integral if and only if it is a weighted lattice polynomial function fulfilling $f(\mathbf{e}_\varnothing) = a$ and $f(\mathbf{e}_{[n]}) = b$.

3 Cumulative distribution functions and moments

Consider n independent random variables $X_1,\ldots,X_n$, X_i $(i \in [n])$ having c.d.f. $F_i(x)$, and set $Y_p := p(X_1,\ldots,X_n)$, where $p : L^n \to L$ is any weighted lattice polynomial function. Let $H : \overline{\mathbb{R}} \to \{0,1\}$ be the Heaviside step function defined by $H(x) = 1$, if $x \geqslant 0$, and 0, otherwise. For any $c \in \overline{\mathbb{R}}$, we also define the function $H_c(x) = H(x-c)$.

The c.d.f. of Y_p is given in the next theorem.

Theorem 1. *Let $p : L^n \to L$ be a weighted lattice polynomial function. Then, the c.d.f. of Y_p is given by*

$$F_p(y) = 1 - \sum_{S\subseteq[n]} \big[1 - H_{p(\mathbf{e}_S)}(y)\big] \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)].$$

As a corollary, we retrieve the c.d.f. of any lattice polynomial function; see [7].

Corollary 1. *Let $p : L^n \to L$ be a lattice polynomial function. Then, the c.d.f. of Y_p is given by*

$$F_p(y) = 1 - \sum_{\substack{S\subseteq[n] \\ p(\mathbf{e}_S)=b}} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1 - F_i(y)].$$

Let us now consider the expected value $\mathbb{E}[g(Y_p)]$, where $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is any measurable function. From its expression we can compute the expected value and the moments of Y_p.

By definition, we simply have

$$\mathbb{E}[g(Y_p)] = \int_{-\infty}^{\infty} g(y)\,\mathrm{d}F_p(y).$$

Using integration by parts, we can derive an alternative expression of $\mathbb{E}[g(Y_p)]$. We then have the following result.

Theorem 2. *Let $p : L^n \to L$ by any weighted lattice polynomial function. For any measurable function $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ such that*

$$\lim_{y\to\infty} g(y)[1-F_i(y)] = 0 \quad (i \in [n]),$$

then

$$\mathbb{E}[g(Y_p)] = \lim_{y\to-\infty} g(y) + \sum_{S\subseteq[n]} \int_{-\infty}^{p(\mathbf{e}_S)} \prod_{i\in[n]\setminus S} F_i(y) \prod_{i\in S}[1-F_i(y)]\,\mathrm{d}g(y).$$

4 The case of uniformly distributed variables on the unit interval

We now examine the case where the random variables $X_1,\ldots,X_n$ are uniformly distributed on $[0,1]$. We also assume $L = [0,1]$.

Recall that the *incomplete Beta function* is defined, for any $u, v > 0$, by

$$B_z(u,v) := \int_0^z t^{u-1}(1-t)^{v-1}\,\mathrm{d}t \qquad (z \in \mathbb{R}),$$

and the *Beta function* is defined, for any $u, v > 0$, by $B(u,v) := B_1(u,v)$.

According to Theorem 2, for any weighted lattice polynomial function $p : [0,1]^n \to [0,1]$ and any measurable function $g : [0,1] \to \overline{\mathbb{R}}$, we have

$$\mathbb{E}[g(Y_p)] = g(0) + \sum_{S\subseteq[n]} \int_0^{p(\mathbf{e}_S)} y^{n-|S|}(1-y)^{|S|}\,\mathrm{d}g(y).$$

Let us now examine the case of the Sugeno integrals. As these integrals are usually considered over the domain $[0,1]^n$, we naturally calculate their expected values when their input variables are uniformly distributed over $[0,1]^n$. Since any Sugeno integral is a particular weighted lattice polynomial, its expected value then writes

$$\int_{[0,1]^n} \mathcal{S}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} B_{\mu(S)}(n-|S|+1, |S|+1).$$

Surprisingly, this expression is very close to that of the expected value of the Choquet integral with respect to the same fuzzy measure.

Let us recall the definition of the Choquet integrals (see [3]). Just as for the Sugeno integrals, the Choquet integrals can be expressed in various equivalent forms. We present them in one of their simplest forms; see e.g. [9].

Definition 5. *Let μ be an $[0,1]$-valued fuzzy measure on $[n]$. The* Choquet integral *of a function $x:[n]\to[0,1]$ with respect to μ is defined by*

$$\mathcal{C}_\mu(x) := \sum_{S\subseteq[n]} \mu(S)\Big[\sum_{T\supseteq S}(-1)^{|T|-|S|}\bigwedge_{i\in T} x_i\Big].$$

For comparison purposes, the expected value of $\mathcal{C}_\mu$ is given by (see e.g. [10])

$$\int_{[0,1]^n} \mathcal{C}_\mu(x)\,\mathrm{d}x = \sum_{S\subseteq[n]} \mu(S)\,B(n-|S|+1,|S|+1).$$

5 Application to reliability theory

In this final section we show how the results derived here can be applied to the reliability analysis of certain coherent systems. For a reference on reliability theory, see e.g. [1].

Consider a system made up of n independent components, each component C_i ($i\in[n]$) having a lifetime X_i and a reliability $r_i(t) := \Pr[X_i > t]$ at time $t>0$. Additional components, with constant lifetimes, may also be considered.

We assume that, when components are connected in series, the lifetime of the subsystem they form is simply given by the minimum of the component lifetimes. Likewise, for a parallel connection, the subsystem lifetime is the maximum of the component lifetimes.

It follows immediately that, for a system mixing series and parallel connections, the system lifetime is given by a weighted lattice polynomial function

$$Y_p = p(X_1,\ldots,X_n)$$

of the component lifetimes. We then have explicit formulas for the c.d.f., the expected value, and the moments of the system lifetime.

For example, the system reliability at time $t>0$ is given by

$$R_p(t) := \Pr[Y_p > t] = \sum_{S\subseteq[n]} \big[1-H_{p(\mathbf{e}_S)}(y)\big]\prod_{i\in S} r_i(t)\prod_{i\in[n]\setminus S}[1-r_i(t)].$$

Moreover, for any measurable function $g:[0,\infty]\to\overline{\mathbb{R}}$ such that

$$\lim_{t\to\infty} g(t)r_i(t) = 0 \quad (i\in[n]),$$

we have, by Theorem 2,

$$\mathbb{E}[g(Y_p)] = g(0) + \int_0^\infty R_p(t)\,\mathrm{d}g(t).$$

Example 1. If $r_i(t) = e^{-\lambda_i t}$ ($i\in[n]$), we can show that

$$\mathbb{E}[Y_p] = p(\mathbf{e}_\varnothing) + \sum_{\substack{S\subseteq[n]\\ S\neq\varnothing}} \sum_{T\subseteq S} (-1)^{|S|-|T|}\,\frac{1-e^{-\lambda(S)\,p(\mathbf{e}_T)}}{\lambda(S)}\,,$$

where $\lambda(S) := \sum_{i\in S}\lambda_i$.

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