Indices de pouvoir et d'interaction en théorie des jeux coopératifs : une approche par moindres carrés

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Cooperative games and pseudo-Boolean functions

Cooperative games

Set of players : $N = \{1, \ldots, n\}$. Game : $f : 2^N \to \mathbb{R}$. For a coalition S of players $f(S)$ is the *worth* of S (usually $f(\varnothing) = 0$, not additive).

Pseudo-Boolean functions

These are functions $f : \{0,1\}^n \to \mathbb{R}$.
Identifying $S \subseteq N$ and $\mathbf{1}_S \in \{0,1\}^n$, games are pseudo-Boolean functions.
A pseudo-Boolean function can be written

$$f(\mathbf{x}) = \sum_{S \subseteq N} v(S) \prod_{i \in S} x_i \prod_{i \in N \setminus S} (1 - x_i) = \sum_{S \subseteq N} a(S) \prod_{i \in S} x_i.$$

Multilinear extensions (Owen,1972)

$$\bar{f} : [0,1]^n \to \mathbb{R} : \mathbf{x} \mapsto \sum_{S \subseteq N} a(S) \prod_{i \in S} x_i.$$

Power indexes

Two problems : to share the benefits of the game or to measure the importance/influence of a player on the outcome of the game.

The Shapley power index (L.S. Shapley 1953):

$$\phi_{\mathrm{Sh}}(f,i) = \sum_{T \not\ni i} \frac{(n-t-1)!t!}{n!} (f(T \cup i) - f(T))$$

$$= \sum_{T \not\ni i} \frac{(n-t-1)!t!}{n!} \Delta^i f(T),$$

The discrete derivative : $\Delta^i f(T) = f(T \cup i) - f(T \setminus i)$

The Banzhaf power index (J. Banzhaf 1965):

$\phi_{\mathrm{B}}(f,i) = \frac{1}{2^{n-1}} \sum_{T \not\ni i} (f(T \cup i) - f(T)) = \frac{1}{2^{n-1}} \sum_{T \not\ni i} \Delta^i f(T)$

Interaction indexes I

To measure the interaction among players i and j:

The Banzhaf interaction index (Owen (1972), Murofushi-Soneda (1993))

$$I_{\mathrm{B}}(f,\{i,j\}) = \frac{1}{2^{n-2}} \sum_{T\subset N\setminus\{i,j\}} (f(T\cup ij) - f(T\cup i) - f(T\cup j) + f(T)).$$

Note that, for $T \subseteq N \setminus \{i,j\}$,

$$\begin{aligned}
\Delta^{ij} f(T) &= f(T\cup ij) - f(T\cup i) - f(T\cup j) + f(T)\\
&= (f(T\cup ij) - f(T)) - (f(T\cup i) - f(T)) - (f(T\cup j) - f(T))\\
&= (f(T\cup ij) - f(T\cup j)) - (f(T\cup i) - f(T)).
\end{aligned}$$

For $f(\mathbf{x}) = \sum_{T\subseteq N} a(T) \prod_{i\in T} x_i$ and $S \subseteq N$,

$$\Delta^S f(\mathbf{x}) = \sum_{T\supseteq S} a(T) \prod_{i\in T\setminus S} x_i$$

Interaction indexes II

To measure the interaction among players in coalition S:

The Banzhaf interaction index of S (Roubens (1996))

$$I_{\mathrm{B}}(f,S) = \frac{1}{2^{n-s}} \sum_{T\subset N\setminus S} \Delta^S f(T).$$

The Shapley interaction index of S (Grabisch (1997))

$$I_{\mathrm{Sh}}(f,S) = \sum_{T\subset N\setminus S} \frac{(n-t-s)!\,t!}{(n-s+1)!} \Delta^S f(T).$$

Probabilistic interaction index of S (Grabisch,Roubens, see also Fujimoto, Kojadinovic, Marichal (2006))

$$I(f,S) = \sum_{T\subset N\setminus S} p_T^S \Delta^S f(T),$$

with $p_T^S \geqslant 0$ and $\sum_T p_T^S = 1$.

Alternative expressions of interactions

Expressions in terms of the Möbius transform

$$I_{\mathrm{B}}(f,S)=\sum_{T\supseteq S}\left(\frac{1}{2}\right)^{t-s}a(T),$$

$$I_{\mathrm{Sh}}(f,S)=\sum_{T\supseteq S}\frac{1}{t-s+1}a(T).$$

In terms of the derivatives of $\overline{f}$

$$I_{\mathrm{B}}(f,S)=(D^S\overline{f})(\tfrac{\mathbf{1}}{\mathbf{2}})=\int_{[0,1]^n}D^S\overline{f}(\mathbf{x})\,d\mathbf{x},$$

$$I_{\mathrm{Sh}}(f,S)=\int_{[0,1]}D^S\overline{f}(x,\ldots,x)\,dx.$$

Main properties

Alternative representations

The map $f \mapsto (I_B(f,S) : S \subseteq N)$ is a linear bijection :

$$\bar{f}(\mathbf{x}) = \sum_{S \subseteq N} I_B(f,S) \prod_{i \in S} \big(x_i - \frac{1}{2}\big).$$

Symmetry-anonymity

If $\pi \in S_n$ and $\pi(f)(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, then

$$I(\pi(f), \pi(S)) = I(f,S).$$

Dummy players

A player i is dummy in f if $f(T \cup i) = f(T) + f(i) - f(\varnothing)$ for $T \subseteq N \setminus i$.
If i is dummy, then

$$I(f,i) = f(i) \quad \text{and} \quad I(f,S) = 0 \quad \forall S \ni i.$$

Some axiomatic characterizations use these properties.

Banzhaf power index and linear model

Given a pseudo-Boolean f, consider a linear model for f :

$$f_1(\mathbf{x}) = a_\varnothing + a_1 x_1 + \cdots + a_n x_n$$

The power of i in f is given by a_i.
Randomization : all the coalitions are equally likely to form.
Least squares method : find f_1 that minimizes

$$\sum_{x \in \{0,1\}^n} (f(x) - g(x))^2$$

among all linear models g.

Theorem (Hammer-Holzman (1992))

In the solution of the least squares problem, $a_i = I_B(f, i)$.

Banzhaf interaction index and multi-linear model

V_k : space of pseudo-Boolean functions of degree k at most

$$V_k = \{g : g(\mathbf{x}) = \sum_{S \subseteq N, s \leqslant k} c(S) \prod_{i \in S} x_i\}$$

For each f, find $f_k = \sum_{s \leqslant k} a_k(S) \prod_{i \in S} x_i \in V_k$ that minimizes

$$\sum_{x \in \{0,1\}^n} (f(x) - g(x))^2$$

among all $g \in V_k$.

Theorem (Grabisch-Marichal-Roubens (2000))

$$a_k(S) = a(S) + (-1)^{k-s} \sum_{T \supseteq S, t > k} \binom{t-s-1}{k-s} \left(\frac{1}{2}\right)^{t-s} a(T).$$

$a_s(S) = I_B(f, S)$ *(!)*.

Weighted least squares

$w(S)$: the probability that coalition S forms : $w(S) = \Pr(C = S)$

Under independence

$p_i = \Pr(C \ni i) = \sum_{S \ni i} w(S) \in (0,1)$ and

$$w(S) = \prod_{i \in S} p_i \prod_{i \in N \setminus S} (1 - p_i)$$

Associated weighted least squares problem

Find the unique $f_k \in V_k$ that minimizes the (squared) distance

$$\sum_{\mathbf{x} \in \{0,1\}^n} w(\mathbf{x}) \big(f(\mathbf{x}) - g(\mathbf{x}) \big)^2 = \sum_{S \subseteq N} w(S) \big(f(S) - g(S) \big)^2$$

among all functions $g \in V_k$.

Rmk : The distance is associated to the inner product
$\langle f, g \rangle = \sum_{\mathbf{x} \in \{0,1\}^n} w(\mathbf{x}) f(\mathbf{x}) g(\mathbf{x})$, and w is defined by $\mathbf{p} = (p_1, \ldots, p_n)$.

First solution of the least squares problem

The use of independence Guoli Ding et al (2010)

$B_k = \{v_S : S \subseteq N,\ s \leqslant k\}$, where $v_S\colon \{0,1\}^n \to \mathbb{R}$ is given by

$$v_S(\mathbf{x}) = \prod_{i\in S} \frac{x_i - p_i}{\sqrt{p_i(1-p_i)}} = \sum_{T\subseteq S} \frac{\prod_{i\in S\setminus T}(-p_i)}{\prod_{i\in S}\sqrt{p_i(1-p_i)}} \prod_{i\in T} x_i$$

forms an orthonormal basis for V_k.

The projection

$$f_k = \sum_{\substack{T\subseteq N\\ t\leqslant k}} \langle f, v_T\rangle\, v_T .$$

The index

$$I_{\mathrm{B},\mathbf{p}}(f,S) = \frac{\langle f, v_S\rangle}{\prod_{i\in S}\sqrt{p_i(1-p_i)}}$$

First properties

The index characterizes the projection :

$f_k \in V_k$ is the best kth approximation of f iff

$$I_{\mathrm{B},\mathbf{p}}(f,S) = I_{\mathrm{B},\mathbf{p}}(f_k,S) \quad \forall\, S : s \leqslant k.$$

(Hint : this is equivalent to $\langle f, v_S\rangle = \langle f_k, v_S\rangle$.)

The map $f \mapsto I_{\mathrm{B},\mathbf{p}}(f,S)$ is linear.

The map $f \mapsto (I_{\mathrm{B},\mathbf{p}}(f,S) : S \subseteq N)$ is a bijection

$$f_k(\mathbf{x}) = \sum_{T \subseteq N, t \leqslant k} I_{\mathrm{B},\mathbf{p}}(f,T) \prod_{i \in T}(x_i - p_i), \quad k = n \quad (!)$$

The index and the multilinear extension of f :

$$I_{\mathrm{B},\mathbf{p}}(f,S) = (D^S \overline{f})(\mathbf{p})$$

Explicit formulas

From $f(\mathbf{x}) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i$

Explicit expression of the index

$$I_{\mathrm{B},\mathbf{p}}(f,S) = \sum_{T \supseteq S} a(T) \prod_{i \in T \setminus S} p_i$$

Proof : Just compute the derivatives of $\bar{f}$.

Explicit expression of the approximation

$f_k(\mathbf{x}) = \sum_{S \subseteq N, s \leqslant k} a_k(S) \prod_{i \in S} x_i$

$$a_k(S) = a(S) + (-1)^{k-s} \sum_{T \supseteq S, t > k} \binom{t-s-1}{k-t} \Big(\prod_{i \in T \setminus S} p_i \Big) a(T)$$

Proof : Use expression of f_k and $I_{\mathrm{B},\mathbf{p}}(f,S)$, expand and do some algebra.

The index as an expected value

An expected value of the discrete derivative

$$I_{\mathrm{B},\mathbf{p}}(f,S) = E(\Delta^S f) = \sum_{\mathbf{x}\in\{0,1\}^n} w(\mathbf{x})\,\Delta^S f(\mathbf{x}).$$

Proof : Use $\Delta^S f(\mathbf{x}) = \sum_{T\supseteq S} a(T)\prod_{i\in T\setminus S} x_i$, independence and explicit formula for $I_{\mathrm{B},\mathbf{p}}(f,S)$.

An average (As a probabilistic interaction index)

$$I_{\mathrm{B},\mathbf{p}}(f,S) = \sum_{T\subseteq N\setminus S} p_T^S\,(\Delta^S f)(T),$$

where $p_T^S = \Pr(T \subseteq C \subseteq S \cup T) = \prod_{i\in T} p_i \prod_{i\in(N\setminus(S\cup T))}(1-p_i)$.
Proof : Use $(\Delta^S f)(T) = (\Delta^S f)(T\setminus S)$.
Interpretation : $p_T^S = \Pr(C = S\cup T \mid C \supseteq S) = \Pr(C = T \mid C \subseteq N\setminus S)$

Further properties

Null players

A player i is *null* for f if $f(T \cup i) = f(T)$ for all $T \subseteq N \setminus i$.
If S contains a null player then $I_{\mathrm{B},\mathbf{p}}(f, S) = 0$.

Dummy coalitions

$D \subseteq N$ is *dummy* for f if $f(T) = f(T \cap D) + f(T \cap (N \setminus D)) - f(\varnothing)$ for every $T \subseteq N$.
If D is dummy for f, if $K \cap D \neq \varnothing$ and $K \setminus D \neq \varnothing$: $I_{\mathrm{B},\mathbf{p}}(f, K) = 0$.

Symmetry of the index

An index I is symmetric if $I(\pi(f), \pi(S)) = I(f, S)$ for all permutations π.
$I_{\mathrm{B},\mathbf{p}}$ is symmetric if and only w or is symmetric i.e. $p_1 = \cdots = p_n$.

Back to Banzhaf and Shapley

A link with the Banzhaf index

$$I_{\mathrm{B},\mathbf{p}'}(f,S) = \sum_{T \supseteq S} I_{\mathrm{B},\mathbf{p}}(f,T) \prod_{i \in T \setminus S} (p'_i - p_i), \quad \text{set } p_i \text{ or } p'_i \text{ to } \tfrac{1}{2} \quad (!)$$

Another link with the Banzhaf index

$$I_{\mathrm{B}}(f,S) = \int_{[0,1]^n} I_{\mathrm{B},\mathbf{p}}(f,S)\, d\mathbf{p}$$

Proof : Just use explicit expressions and integrate.
Interpretation : take an average over **p** if it is not known.

A link with the Shapley index

$$I_{\mathrm{Sh}}(f,S) = \int_0^1 I_{\mathrm{B},(p,\ldots,p)}(f,S)\, dp.$$

Interpretation : average if the players behave in the same (unknown) way

Part II : Interactions among variables of functions over $[0, 1]^n$.

Problem I: Influence of variables

The problem

Consider $f \colon [0,1]^n \to \mathbb{R}$.
How to measure the influence of variable x_k over $f(x_1, \ldots, x_k, \ldots, x_n)$?

Applications

x_i : partial score of an alternative with respect to criterion i.
$f(x_1, \ldots, x_n)$: global score.

A simple example

$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} c_i x_i + c_0$.

A simple answer

$I(f, k) = c_k$.

More examples

- The geometric mean $f(x_1, \ldots, x_n) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}$;
- Weighted geometric means $f(x_1, \ldots, x_n) = \prod_{i=1}^n x_i^{c_i}$;
- Weighted maxima $f(x_1, \ldots, x_n) = \max_{i \in N} \min(c_i, x_i)$;
- Lovasz extensions $f(x_1, \ldots, x_n) = \sum_{T \subseteq N} a_T \min_{i \in T} x_i$
- ...

A solution to problem I

Step I

Given a function f, approximate f by a linear model of the form $f_1 = c_0 + c_1 x_1 + \cdots + c_n x_n$. Restriction: $f \in L^2([0,1]^n)$.

Step II

Define $I(f,k) = c_k$.

Explicit expression

$$I(f,k) = 12 \int_{[0,1]^n} f(\mathbf{x})\big(x_k - \frac{1}{2}\big)\, d\,\mathbf{x}.$$

Alternative definition

Consider a linear model of the form $f_{\{k\}} = c_0 + c_k x_k$.

The "worth to improve" index

The concept of influence of subsets of variables was defined by M. Grabisch and C. Labreuche. Axioms :

- $I(\cdot, k)$ is linear on $L^2([0,1]^n)$;
- $I(\cdot, k)$ is continuous on $L^2([0,1]^n)$;
- Step evaluation : value of the index on particular threshold functions;
- Normalization: value of the index for weighted means.

The result

The influence index (worth to improve) criterion k:

$$W_k = 6 \int_{[0,1]^n} f(\mathbf{x})(2x_k - 1)d\,\mathbf{x}.$$

In general for $S \subseteq \{1, \ldots, n\}$,

$$W_S = 3 \times 2^{|S|} \int_{[0,1]^n} f(\mathbf{x})\Big(\prod_{i \in S} x_i - \prod_{i \in S}(1 - x_i)\Big)d\,\mathbf{x}.$$

Problem II : Interactions among variables

The problem

Consider $f\colon [0,1]^n \to \mathbb{R}$.
How to measure the interactions among $\{x_i : i \in S\}$ in $f(x_1, \ldots, x_k, \ldots, x_n)$?

A simple example

Consider a multilinear function

$$f\colon [0,1]^2 \to \mathbb{R}\colon (x_1, x_2) \mapsto c_0 + c_1x_1 + c_2x_2 + c_{12}x_1x_2.$$

A simple answer

The interaction between x_1 and x_2 within f is defined by c_{12}.

The approximation problems

Denote by M_s the space of mutlilinear functions $g\colon [0,1]^n \to \mathbb{R}$ of degree less than or equal to s, i.e. of the form

$$g(\mathbf{x}) = \sum_{T \subseteq N, |T| \leqslant s} c(T) \prod_{i \in T} x_i.$$

The standard L^2 distance of functions $f, h \in L^2([0,1]^n)$ is

$$d(f,h)^2 = \int_{[0,1]^n} \big(f(\mathbf{x}) - h(\mathbf{x})\big)^2 d\,\mathbf{x}.$$

For $f \in L^2([0,1]^n)$, find $f_s \in M_s$ that minimizes $d(f,g)$ for g in M_s.
Alternative problem : Define M_S as the space of multilinear functions

$$g(\mathbf{x}) = \sum_{T \subseteq S} c(T) \prod_{i \in T} x_i.$$

For every $f \in L^2([0,1]^n)$, compute f_S.

Interaction indexes

First index

For $f \in L^2([0,1]^n)$ and $s \leqslant n$ write

$$f_s(\mathbf{x}) = \sum_{T \subseteq N, |T| \leqslant s} a_s(T) \prod_{i \in T} x_i.$$

Define

$$\mathcal{I}_1(f,S) = a_s(S) \quad \text{for} \quad S \subseteq N, \quad s = |S|.$$

Second index

For $f \in L^2([0,1]^n)$ and $S \subseteq N$ write

$$f_S(\mathbf{x}) = \sum_{T \subseteq S} a_S(T) \prod_{i \in T} x_i.$$

Define

$$\mathcal{I}_2(f,S) = a_S(S) \quad \text{for} \quad S \subseteq N.$$

Computations

The result

For every $f \in L^2([0,1]^n)$ and $S \subseteq N$ we have

$$\mathcal{I}_1(f,S) = \mathcal{I}_2(f,S) := \mathcal{I}(f,S) = 12^{|S|} \int_{[0,1]^n} f(\mathbf{x}) \prod_{i \in S} \left(x_i - \frac{1}{2}\right) d\,\mathbf{x}.$$

Proof

The space $L^2([0,1]^n)$ is an inner product space w.r.t.

$$\langle f, g \rangle = \int_{[0,1]^n} f(\mathbf{x}) g(\mathbf{x}) d\,\mathbf{x}.$$

$$\text{Define} \quad v_T(\mathbf{x}) = 12^{|T|/2} \prod_{i \in T} \left(x_i - \frac{1}{2}\right), \quad T \subseteq N.$$

$$\text{Then} \quad f_s = \sum_{|T| \leqslant s} \langle f, v_T \rangle v_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} \langle f, v_T \rangle v_T.$$

Examples

$$\boxed{f(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i.}$$

- $\mathcal{I}(f,k) = \frac{1}{n}$;
- $\mathcal{I}(f,S) = 0 \quad \text{if} \quad |S| \geqslant 2.$

$$\boxed{f(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}}$$

- $\mathcal{I}(f,k) = \left(\frac{1}{n+1}\right)^n \frac{6}{2n+1}$;
- $\mathcal{I}(f,S) = \left(\frac{1}{n+1}\right)^n \left(\frac{6}{2n+1}\right)^s$.

$$\boxed{f(\mathbf{x}) = \prod_{i=1}^{n} x_i^{c_i}}$$

$$\mathcal{I}(f,S) = \prod_{i \in N} \frac{1}{c_i+1} \prod_{i \in S} \frac{6c_i}{c_i+2}.$$

Interpretations as average of derivatives

For $\mathbf{x} \in [0,1]^n$,

- $D_k f(\mathbf{x})$ = local influence of x_k to f at $\mathbf{x}$;
- $D_k D_j f(\mathbf{x})$ = local interaction of x_j and x_k within f at $\mathbf{x}$;

Theorem

If $f \in L^2([0,1]^n)$ is smooth enough, then

$$\mathcal{I}(f,S) = \int_{[0,1]^n} q_S(\mathbf{x})\, D^S f(\mathbf{x})\, d\mathbf{x},$$

where $q_S(\mathbf{x}) = 6^{|S|} \prod_{i \in S} x_i(1-x_i)$ is a p.d.f. on $[0,1]^n$.

Proof : Integration by parts.

Define the discrete derivative $\Delta_{\mathbf{h}}^{S} f(\mathbf{x})$ of f at $\mathbf{x} \in [0,1]^n$ by

$$\Delta_{\mathbf{h}}^{\{i\}} f(\mathbf{x}) = f(\mathbf{x} + h_i e_i) - f(\mathbf{x}).$$

$+$ induction. The difference quotient is then

$$Q_{\mathbf{h}}^{S} f(\mathbf{x}) = \frac{\Delta_{\mathbf{h}}^{S} f(\mathbf{x})}{\prod_{i \in S} h_i}$$

Theorem

For every $S \subseteq N$, we have

$$\mathcal{I}(f,S) = \int_{\mathbf{x}\in[0,1]^n} \int_{\mathbf{y}_S \in [\mathbf{x}_S, \mathbf{1}]} p_S(\mathbf{x}, \mathbf{y}_S)\, Q_{\mathbf{y}-\mathbf{x}}^{S} f(\mathbf{x})\, d\mathbf{y}_S\, d\mathbf{x},$$

where $p_S(\mathbf{x}, \mathbf{y}_S) = 6^{|S|} \prod_{i \in S}(y_i - x_i)$ is a p.d.f on the integration domain.

Properties I

The index is linear and continuous.

Link with the Banzhaf index

If $g : \{0,1\}^n \to \mathbb{R}$ is pseudo-Boolean and $f = \bar{g}$ is the multilinear extension, then

$$\mathcal{I}(f, S) = I_B(g, S).$$

Symmetry

The index $\mathcal{I}$ is symmetric: we have $\mathcal{I}(\pi(f), \pi(S)) = \mathcal{I}(f, S)$.

Properties II

Ineffective variables

If x_i is ineffective for f, then $\mathcal{I}(f, S) = 0$ whenever $i \in S$.

Dummy subsets of variables

$D \subseteq N$ is *dummy* for $f : [0,1]^n \to \mathbb{R}$ if

$$f(\mathbf{x}) = f(\mathbf{0}_D \mathbf{x}_{N\setminus D}) + f(\mathbf{0}_{N\setminus D}\mathbf{x}_D) - f(\mathbf{0}).$$

Then if $S \cap D \neq \varnothing$ and $S \setminus D \neq \varnothing$, $\mathcal{I}(f, S) = 0$.

S-Increasing functions

If f is S-increasing for some $S \subseteq N$ ($\Delta^S_{\mathbf{y}-\mathbf{x}} f(\mathbf{x}) \geqslant 0$ for $\mathbf{y} \geqslant \mathbf{x}$), then $\mathcal{I}(f, S) \geqslant 0$.

More examples

Multiplicative functions

If f is multiplicative : $f(\mathbf{x}) = \prod_{i\in N} f_i(x_i)$ then

$$\frac{\mathcal{I}(f,S)}{\mathcal{I}(f,\varnothing)} = \prod_{i\in S} \frac{\mathcal{I}(f_i,i)}{\mathcal{I}(f_i,\varnothing)}.$$

Discrete Choquet integrals

If $f(\mathbf{x}) = \sum_{T\subseteq N} a(T)\,\min_{i\in T} x_i$, then

$$\mathcal{I}(f,S) = 6^{|S|} \sum_{T\supseteq S} a(T)\, B(|S|+1,|T|+1),$$

where

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt = \frac{(p-1)!(q-1)!}{(p+q-1)!}\,.$$

Normalized index

Goal : to compare indexes for different functions.
Problem : we need to normalize the index.
Observation : the index is defined (for $S \neq \varnothing$) as a covariance :

$$\mathcal{I}(f,S) = \langle f - E(f), 12^{|S|/2}(v_T - E(v_T))\rangle$$

We define the normalized index as the Pearson coefficient:

$$r(f,S) = \frac{\mathcal{I}(f,S)}{12^{|S|/2}\,\sigma(f)}.$$

- $r(AM, i) = 1/\sqrt{n}$;
- $r(\min_N x_k, i) = \frac{\sqrt{3}}{\sqrt{n(n+2)}}$;
- $r(GM, \{i\}) = \frac{\sqrt{3}}{2n+1}\left(\left(\frac{(n+1)^2}{n(n+2)}\right)^n - 1\right)^{-1/2}$.

Thanks for your attention

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