# Classification of associative multivariate polynomial functions

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# Semigroups

Recall that a function $f\colon \mathbb{C}^2 \to \mathbb{C}$ is *associative* if

$$f\big(f(x_1, x_2), x_3\big) = f\big(x_1, f(x_2, x_3)\big) \qquad \forall\, x_1, x_2, x_3 \in \mathbb{C}.$$

The pair $(\mathbb{C}, f)$ is called a *semigroup*

**Examples:**

$$\begin{aligned} f(x_1, x_2) &= x_1\, x_2 \\ f(x_1, x_2) &= x_1 + x_2 \end{aligned}$$

**Problem:** Classify the associative polynomial functions

# Semigroups defined by polynomials over $\mathbb{C}$

The semigroups $(\mathbb{C}, p)$, where $p\colon \mathbb{C}^2 \to \mathbb{C}$ is a polynomial function, are given by

(i) $p(x_1, x_2) = c$

(ii) $p(x_1, x_2) = x_1$

(iii) $p(x_1, x_2) = x_2$

(iv) $p(x_1, x_2) = c + x_1 + x_2$

(v) $p(x_1, x_2) = \varphi^{-1}\big(a\,\varphi(x_1)\,\varphi(x_2)\big)$, where $\varphi(x) = x + b$

# Ternary semigroups

A function $f\colon \mathbb{C}^3 \to \mathbb{C}$ is *associative* if

$$\begin{aligned} f\big(f(x_1,x_2,x_3),x_4,x_5\big) &= f\big(x_1,f(x_2,x_3,x_4),x_5\big) \\ &= f\big(x_1,x_2,f(x_3,x_4,x_5)\big) \end{aligned}$$

The pair $(\mathbb{C}, f)$ is called a *ternary semigroup* (Dörnte, 1928)

**Examples:**

$$\begin{aligned} f(x_1,x_2,x_3) &= x_1\,x_2\,x_3 \\ f(x_1,x_2,x_3) &= x_1+x_2+x_3 \\ f(x_1,x_2,x_3) &= x_1-x_2+x_3 \end{aligned}$$

**Problem:** Classify the associative ternary polynomial functions

## Ternary semigroups defined by polynomials over $\mathbb{C}$

The ternary semigroups $(\mathbb{C}, p)$, where $p\colon \mathbb{C}^3 \to \mathbb{C}$ is a polynomial function, are given by

- *(i)* $p(x_1, x_2, x_3) = c$
- *(ii)* $p(x_1, x_2, x_3) = x_1$
- *(iii)* $p(x_1, x_2, x_3) = x_3$
- *(iv)* $p(x_1, x_2, x_3) = c + x_1 + x_2 + x_3$
- *(v)* $p(x_1, x_2, x_3) = x_1 - x_2 + x_3$
- *(vi)* $p(x_1, x_2, x_3) = \varphi^{-1}\big(a\, \varphi(x_1)\, \varphi(x_2)\, \varphi(x_3)\big)$, where $\varphi(x) = x + b$

Głazek and Gleichgewicht (1985) proved this result for ternary semigroups $(R, p)$, where $R$ is an infinite commutative integral domain with identity

# $n$-ary semigroups

A function $f\colon \mathbb{C}^n \to \mathbb{C}$ is *associative* if

$$\begin{aligned} & f\big(x_1, \ldots, f(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}\big) \\ & \quad = \quad f\big(x_1, \ldots, x_i, f(x_{i+1}, \ldots, x_{i+n}), \ldots, x_{2n-1}\big), \quad i = 1, \ldots, n-1 \end{aligned}$$

The pair $(\mathbb{C}, f)$ is called an *$n$-ary semigroup* (Dörnte, 1928)

**Problem:** Classify the associative $n$-ary polynomial functions

# New results

**Theorem.** The $n$-ary semigroups $(\mathbb{C}, p)$, where $p\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial function, are given by

(*i*) $p(\mathbf{x}) = c$

(*ii*) $p(\mathbf{x}) = x_1$

(*iii*) $p(\mathbf{x}) = x_n$

(*iv*) $p(\mathbf{x}) = c + \sum_{i=1}^{n} x_i$

(*v*) $p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i$ (if $n \geqslant 3$), where $\omega^{n-1} = 1$, $\omega \neq 1$

(*vi*) $p(\mathbf{x}) = \varphi^{-1}\big(a\ \prod_{i=1}^{n} \varphi(x_i)\big)$, where $\varphi(x) = x + b$

(This classification also holds on an infinite integral domain)

# New results

**Remark on type (*v*)**

$$p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i \qquad \omega^{n-1} = 1 \qquad \omega \neq 1$$

- Case $n = 3$ reduces to

$$p(x_1, x_2, x_3) = x_1 - x_2 + x_3$$

- On $\mathbb{R}$ :

$$p(\mathbf{x}) = \begin{cases} \sum_{i=1}^{n} (-1)^{i-1} x_i & \text{if } n \text{ odd} \\ \text{nothing} & \text{if } n \text{ even} \end{cases}$$

# *n*-ary groups

The pair $(\mathbb{C}, f)$ is an *n-ary quasigroup* if, for every $a_1, \ldots, a_n, b \in \mathbb{C}$ and every $i \in \{1, \ldots, n\}$, the equation

$$f(a_1, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_n) = b$$

has a unique solution $z \in \mathbb{C}$

The pair $(\mathbb{C}, f)$ is an *n-ary group* if it is an $n$-ary semigroup and an $n$-ary quasigroup

**Remark:** Any 2-ary group is a group

# $n$-ary groups

**Corollary.** The $n$-ary groups $(\mathbb{C}, p)$, where $p\colon \mathbb{C}^n \to \mathbb{C}$ is a polynomial function, are given by

(*iv*) $p(\mathbf{x}) = c + \sum_{i=1}^{n} x_i$

(*v*) $p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i$ (if $n \geqslant 3$), where $\omega^{n-1} = 1$, $\omega \neq 1$

(*vi*) $p(\mathbf{x}) = \varphi^{-1}\big(a \prod_{i=1}^{n} \varphi(x_i)\big)$, where $\varphi(x) = x + b$

# Reducibility

From a semigroup $(\mathbb{C}, g)$ we can define an $n$-ary semigroup $(\mathbb{C}, f)$ by

$$f(x_1, \ldots, x_n) = g(\cdots g(g(g(x_1, x_2), x_3), x_4), \ldots, x_n)$$

We then say that the $n$-ary semigroup $(\mathbb{C}, f)$ is *reducible to* or *derived from* $(\mathbb{C}, g)$

**Examples:**

$f(x_1, x_2, x_3) = x_1\, x_2\, x_3$ is reducible to $g(x_1, x_2) = x_1\, x_2$

$f(x_1, x_2, x_3) = x_1 + x_2 + x_3$ is reducible to $g(x_1, x_2) = x_1 + x_2$

Is $f(x_1, x_2, x_3) = x_1 - x_2 + x_3$ reducible ?

# Reducibility for polynomial functions over $\mathbb{C}$

$(i)$ $p(\mathbf{x}) = c$ is reducible to $g(x_1, x_2) = c$

$(ii)$ $p(\mathbf{x}) = x_1$ is reducible to $g(x_1, x_2) = x_1$

$(iii)$ $p(\mathbf{x}) = x_n$ is reducible to $g(x_1, x_2) = x_2$

$(iv)$ $p(\mathbf{x}) = c + \sum_{i=1}^{n} x_i$ is reducible to

$$g(x_1, x_2) = \tfrac{c}{n-1} + x_1 + x_2$$

$(v)$ $p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i$ (if $n \geqslant 3$) is not reducible !!

$(vi)$ $p(\mathbf{x}) = \varphi^{-1}\big(a \prod_{i=1}^{n} \varphi(x_i)\big)$ is reducible to

$$g(x_1, x_2) = \varphi^{-1}\big(\alpha\, \varphi(x_1)\varphi(x_2)\big)$$

where $\alpha \in \mathbb{C}$ is such that $\alpha^{n-1} = a$

We have extended these results to the case of an infinite integral domain

## Irreducibility of $p(\mathbf{x}) = \sum_{i=1}^{n} \omega^{i-1} x_i$

**Proof.** Suppose $p$ is reducible to $g$. Then $y = p(y, 0, \ldots, 0)$. Therefore

$$\begin{aligned} g(x,y) &= g(x, p(y,0,\ldots,0)) = g(x, g(\cdots g(g(y,0),0),\ldots,0) \\ &= p(x, g(y,0), 0, \ldots, 0) \end{aligned}$$

Then we have

$$g(x,y) = x + \omega\, g(y,0) \qquad x,y \in \mathbb{C} \tag{1}$$

and hence

$$g(0,0) = \omega\, g(0,0) \qquad (\text{implying} \quad g(0,0) = 0) \tag{2}$$

By (1) and (2), we obtain

$$g(x,0) = x + \omega\, g(0,0) = x \tag{3}$$

Combining (1) with (3) produces

$$g(x,y) = x + \omega\, y \qquad (\omega \neq 1)$$

and this polynomial function is not a semigroup ! $\rightarrow$ Contradiction $\square$

# Medial $n$-ary semigroup structures

An $n$-ary semigroup $(\mathbb{C}, f)$ is *medial* if $f$ satisfies the bisymmetry functional equation, i.e., the expression

$$f\big(f(x_{11},\ldots,x_{1n}),\ldots,f(x_{n1},\ldots,x_{nn})\big)$$

remains invariant when replacing $x_{ij}$ by $x_{ji}$ for all $i,j=1,\ldots,n$

**Proposition.** (straightforward)
Every $n$-ary semigroup defined by a polynomial function over $\mathbb{C}$ is medial

**A natural question.** Describe the class of all $n$-ary polynomial functions over $\mathbb{C}$ (or an integral domain) satisfying the bisymmetry equation