Extensions of system signature and Barlow-Proschan importance index to dependent lifetimes

Jean-Luc Marichal*   Pierre Mathonet**

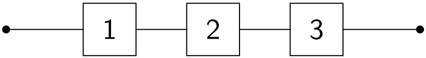
*University of Luxembourg
**University of Liège

System

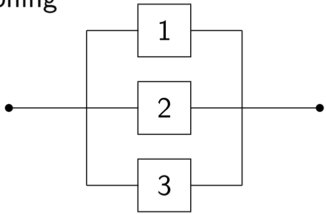
Definition. A *system* consists of several interconnected units

Assumptions:

1. The system and the units are of the crisply *on/off* kind
2. A serially connected segment of units is functioning if and only if every single unit is functioning



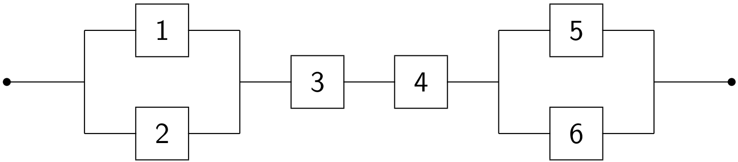
3. A system of parallel units is functioning if and only at least one unit is functioning



System

Example. Home video system

1. Blu-ray player
2. DVD player
3. LCD monitor
4. Amplifier
5. Speaker A
6. Speaker B



Structure function

Definition.
The *state of a component* $i \in [n] = \{1, \ldots, n\}$ can be represented by a Boolean variable

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is in a failed state} \end{cases}$$

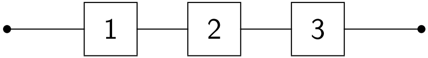
The *state of the system* is described from the component states through a Boolean function $\phi : \{0,1\}^n \to \{0,1\}$

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

This function is called the *structure function* of the system

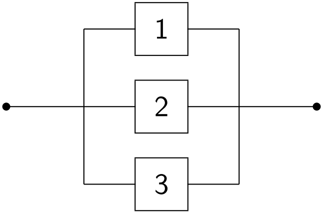
Structure function

Series structure



$$\phi(\mathbf{x}) = x_1 \; x_2 \; x_3 = \prod_{i=1}^{3} x_i$$

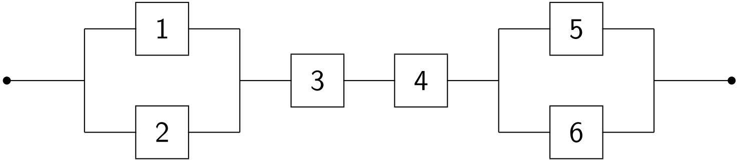
Parallel structure



$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)(1 - x_3) = \coprod_{i=1}^{3} x_i$$

Structure function

Home video system



$$\phi(\mathbf{x}) = (x_1 \amalg x_2)\ x_3\ x_4\ (x_5 \amalg x_6)$$

Lifetimes

Notation

1. X_i = random *lifetime* of component $i \in [n]$
2. T = random *lifetime* of the system
3. F = *joint c.d.f.* of the component lifetimes $X_1, \ldots, X_n$

$$F(t_1, \ldots, t_n) = \Pr(X_1 \leqslant t_1, \ldots, X_n \leqslant t_n)$$

We assume that F is absolutely continuous

System

A *system* is given by

$$\mathcal{S} = (n, \phi, F)$$

$\phi =$ structure function (design)
$F =$ distribution function (lifetimes)

System signature

Assume that the components have i.i.d. lifetimes

Let $X_{1:n}, \ldots, X_{n:n}$ are the order statistics obtained by rearranging the variables $X_1, \ldots, X_n$ in ascending order of magnitude :

$$X_{1:n} \leqslant \cdots \leqslant X_{n:n}$$

System signature (Samaniego, 1985)

$$s_k \;=\; \Pr(T = X_{k:n}) \qquad k = 1, \ldots, n$$

$$\mathbf{s} = (s_1, \ldots, s_n), \qquad \textstyle\sum_k s_k = 1$$

System signature

Explicit expression (Boland, 2001)

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{\mathbf{x}\in\{0,1\}^n \\ |\mathbf{x}|=n-k+1}} \phi(\mathbf{x}) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{\mathbf{x}\in\{0,1\}^n \\ |\mathbf{x}|=n-k}} \phi(\mathbf{x})$$

where $|\mathbf{x}| = \sum_i x_i$

Set-based notation:

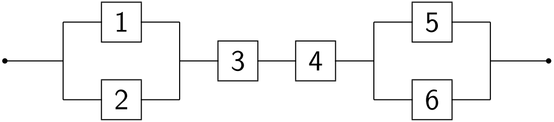
$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{A\subseteq[n] \\ |A|=n-k+1}} \phi(A) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{A\subseteq[n] \\ |A|=n-k}} \phi(A)$$

Writing convention: $\phi(\mathbf{x}) = \phi(A)$ whenever $x_i = 1 \Leftrightarrow i \in A$

$\Rightarrow$ s_k is independent of F !

System signature

Home video system



We have $\phi(A) = 1$ for the following subsets A (path sets):

$$\begin{array}{lll} \{1,3,4,5\} & \{1,2,3,4,5\} & \{1,2,3,4,5,6\} \\ \{2,3,4,5\} & \{1,3,4,5,6\} & \\ \{1,3,4,6\} & \{2,3,4,5,6\} & \\ \{2,3,4,6\} & \{1,2,3,4,6\} & \end{array}$$

$$\mathbf{s} = \left(\frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0, 0, 0\right)$$

Barlow-Proschan importance index

Assume now that the components have independent lifetimes

Importance index (Barlow-Proschan, 1975)

$$I_{\mathrm{BP}}^{(k)} = \Pr(T = X_k) \qquad k = 1, \ldots, n$$

$$\mathbf{I}_{\mathrm{BP}} = (I_{\mathrm{BP}}^{(1)}, \ldots, I_{\mathrm{BP}}^{(n)}), \qquad \sum_k I_{\mathrm{BP}}^{(k)} = 1$$

$I_{\mathrm{BP}}^{(k)}$ is an measure of importance of component k

Barlow-Proschan importance index

In the i.i.d. case:

Set $b_k = I_{\mathrm{BP}}^{(k)}$ and $\mathbf{b} = (b_1, \ldots, b_n)$

$$b_k = \sum_{A \subseteq [n] \setminus \{k\}} \frac{1}{n \binom{n-1}{|A|}} \Delta_k \phi(A)$$

where $\Delta_k \phi(A) = \phi(A \cup \{k\}) - \phi(A)$

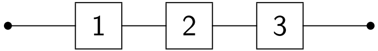
$\Rightarrow$ b_k is independent of F !

$\Rightarrow$ The n-tuple $\mathbf{b}$ defines a *structural importance index*

By analogy, the n-tuple $\mathbf{s}$ defines the *structural signature*

Barlow-Proschan importance index

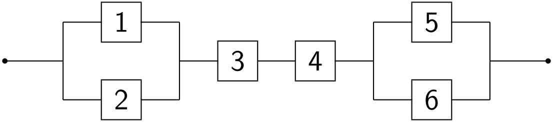
Series structure



$$\mathbf{s} = (1, 0, 0) \qquad \mathbf{b} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Barlow-Proschan importance index

Home video system



We have $\Delta_1\phi(A) = 1$ for the following subsets S (swing):

$$\{3,4,5\} \quad \{3,4,5,6\}$$
$$\{3,4,6\}$$

$$\mathbf{b} = \left(\frac{2}{30}, \frac{2}{30}, \frac{11}{30}, \frac{11}{30}, \frac{2}{30}, \frac{2}{30}\right)$$

Extension of signature to dependent lifetimes

We now only assume that F is absolutely continuous

System signature

also called "probability signature" (Navarro-Spizzichino-Balakrishnan, 2010)

$$p_k \;=\; \Pr(T = X_{k:n}) \qquad k = 1, \ldots, n$$

$$\mathbf{p} = (p_1, \ldots, p_n), \qquad \textstyle\sum_k p_k = 1$$

Can we provide an explicit expression for p_k in terms of ϕ and F ?

$$\mathcal{S} = (n, \phi, F)$$

Extension of signature to dependent lifetimes

Define the *relative quality function* $q\colon 2^{[n]} \to [0,1]$ by

$$\begin{aligned} q(A) &= \Pr\left(X_i < X_j : i \notin A,\ j \in A\right) \\ &= \Pr\left(\max_{i\notin A} X_i < \min_{j\in A} X_j\right) \end{aligned}$$

The number $q(A)$ is the probability that the best $|A|$ components (those having the longest lifetimes) are exactly A

$\rightarrow$ $q(A)$ measures the overall *quality* of the components A *when compared with* the components $[n] \setminus A$

Remark: q is independent of ϕ (q depends only on n and F)

Extension of signature to dependent lifetimes

Properties of q :

1. We have
$$\sum_{A\subseteq[n]\,:\,|A|=k} q(A) = 1 \qquad k = 1, \ldots, n$$

2. For $k \in \{1, \ldots, n\}$ and $A \subseteq [n]$ such that $|A| \leqslant k$,
$$\sum_{\substack{B\supseteq A\\|B|=k}} q(B)$$
is the probability that A be among the best k components

3. $q(\{i\})$ is the probability that i is the best component

4. $q([n] \setminus \{i\})$ is the probability that i is the worst component

Extension of signature to dependent lifetimes

Theorem

$$p_k \;=\; \sum_{\substack{\mathbf{x}\in\{0,1\}^n\\ |\mathbf{x}|=n-k+1}} q(\mathbf{x})\,\phi(\mathbf{x}) \;-\; \sum_{\substack{\mathbf{x}\in\{0,1\}^n\\ |\mathbf{x}|=n-k}} q(\mathbf{x})\,\phi(\mathbf{x})$$

Set-based notation:

$$p_k \;=\; \sum_{\substack{A\subseteq[n]\\ |A|=n-k+1}} q(A)\,\phi(A) \;-\; \sum_{\substack{A\subseteq[n]\\ |A|=n-k}} q(A)\,\phi(A)$$

This explicit expression for p_k extends Boland's formula (obtained in the i.i.d. case)

Extension of signature to dependent lifetimes

Proposition

If $X_1, \ldots, X_n$ are exchangeable, then q is symmetric

$$q(A) = \frac{1}{\binom{n}{|A|}}$$

The converse does not hold

→ Back to Boland's formula

$$p_k = s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{\mathbf{x}\in\{0,1\}^n \\ |\mathbf{x}|=n-k+1}} \phi(\mathbf{x}) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{\mathbf{x}\in\{0,1\}^n \\ |\mathbf{x}|=n-k}} \phi(\mathbf{x})$$

$$\mathbf{p} = \mathbf{s}$$

Extension of signature to dependent lifetimes

A system $\mathcal{S} = (n, \phi, F)$ is said to be *semicoherent* if

- $\phi(0, \ldots, 0) = 0$
- $\phi(1, \ldots, 1) = 1$
- ϕ is nondecreasing :

$$x_i \leqslant x_i' \quad \Rightarrow \quad \phi(x_1, \ldots, x_n) \leqslant \phi(x_1', \ldots, x_n')$$

Theorem

The identity $\mathbf{p} = \mathbf{s}$ holds for every n-component semicoherent system *if and only if* q is symmetric

Extension of BP index to dependent lifetimes

For every $k \in [n]$, define the function $q_k \colon 2^{[n]\setminus\{k\}} \to [0,1]$ by

$$q_k(A) \;=\; \Pr\Big(\max_{i \notin A\cup\{k\}} X_i < X_k < \min_{j \in A} X_j\Big)$$

The number $q_k(A)$ is the probability that the components that are better than component k are precisely A.

Immediate property:

$$\sum_{A \subseteq [n]\setminus\{k\}} q_k(A) = 1 \qquad k \in [n]$$

Extension of BP index to dependent lifetimes

The functions q_k are related to the function q :

$$\begin{aligned} q(A) &= \sum_{k \notin A} q_k(A) \qquad A \neq [n] \\ q(A) &= \sum_{k \in A} q_k(A \setminus \{k\}) \qquad A \neq \varnothing \end{aligned}$$

Extension of BP index to dependent lifetimes

Theorem

$$I_{\mathrm{BP}}^{(k)} = \sum_{A\subseteq[n]\setminus\{k\}} q_k(A)\,\Delta_k\phi(A)$$

$\rightarrow$ weighted mean (since $\sum_{A\subseteq[n]\setminus\{k\}} q_k(A) = 1$)

In the i.i.d. case:

$$I_{\mathrm{BP}}^{(k)} = b_k = \sum_{A\subseteq[n]\setminus\{k\}} \frac{1}{n\binom{n-1}{|A|}}\,\Delta_k\phi(A)$$

Extension of BP index to dependent lifetimes

Proposition

If $X_1, \ldots, X_n$ are exchangeable, then

$$q_k(A) = \frac{1}{n\binom{n-1}{|A|}}$$

$$I_{\mathrm{BP}}^{(k)} \;=\; b_k \;=\; \sum_{A \subseteq [n] \setminus \{k\}} \frac{1}{n\binom{n-1}{|A|}}\, \Delta_k \phi(A)$$

$$\boxed{\mathbf{I}_{\mathrm{BP}} = \mathbf{b}}$$

Extension of BP index to dependent lifetimes

Theorem

The identity $\mathbf{I}_{\mathrm{BP}} = \mathbf{b}$ holds for every n-component semicoherent system *if and only if*

$$q_k(A) = \frac{1}{n\binom{n-1}{|A|}}$$

Case of independent lifetimes

We now assume that $X_1, \ldots, X_n$ are *independent* lifetimes

Every X_i has a
- a p.d.f. f_i
- a c.d.f. F_i with $F_i(0) = 0$

Theorem

$$q(A) = \sum_{j \in A} \int_0^\infty f_j(t) \prod_{i \notin A} F_i(t) \prod_{i \in A \setminus \{j\}} \overline{F}_i(t)\, dt \qquad (A \neq \varnothing)$$

where $\overline{F}_i(t) = 1 - F_i(t)$

$\rightarrow$ provides an explicit expression for the signature in the independent case

Case of independent lifetimes

Example: *independent Weibull* lifetimes

$$F_i(t) = 1 - e^{-(\lambda_i t)^\alpha} \qquad \alpha > 0,\ \lambda_i > 0$$

(exponential model if $\alpha = 1$)

Corollary

$$q(A) \ = \ \sum_{B \subseteq [n] \setminus A} (-1)^{|B|} \frac{\lambda_\alpha(A)}{\lambda_\alpha(A \cup B)} \qquad (A \neq \varnothing)$$

where $\lambda_\alpha(A) = \sum_{i \in A} \lambda_i^\alpha$

Case of independent lifetimes

The ratio

$$\frac{\lambda_\alpha(\{i\})}{\lambda_\alpha([n])} \;=\; q([n] \setminus \{i\})$$

is the probability that i is the worst component

More generally,

$$\frac{\lambda_\alpha(A)}{\lambda_\alpha([n])} \;=\; \sum_{i \in A} q([n] \setminus \{i\})$$

is the probability that the worst component is in A

Case of independent lifetimes

Theorem

$$q_k(A) \;=\; \int_0^\infty f_k(t) \prod_{i\notin A\cup\{k\}} F_i(t) \prod_{i\in A} \overline{F}_i(t)\, dt$$

→ provides an explicit expression for Barlow–Proschan index in the independent case

Corollary

For independent Weibull lifetimes

$$q_k(A) \;=\; \sum_{B\subseteq[n]\setminus(A\cup\{k\})} (-1)^{|B|} \frac{\lambda_\alpha(\{k\})}{\lambda_\alpha(A\cup B\cup\{k\})}$$

Thank you for your attention !