Solving Chisini's functional equation

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Chisini mean (1929)

Let $\mathbb{I}$ be a real interval

An *average* of n numbers $x_1, \ldots, x_n \in \mathbb{I}$ with respect to a function $F\colon \mathbb{I}^n \to \mathbb{R}$ is a number $M \in \mathbb{I}$ such that

$$F(x_1, \ldots, x_n) = F(M, \ldots, M)$$

An average is also called a *Chisini mean*

Chisini mean (1929)

Example: $\mathbb{I} = [0, \infty[$

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$$

We have

$$\begin{aligned} F(x_1, \ldots, x_n) = F(M, \ldots, M) \quad &\iff \quad \sum_{i=1}^{n} x_i^2 = n\, M^2 \\ &\iff \quad M = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^2 \right)^{1/2} \end{aligned}$$

(the solution is unique)

Chisini mean (1929)

Example: $\mathbb{I} = [0,1]$

$$F(x_1, x_2) = \max(0, x_1 + x_2 - 1)$$

We have

$$F(x_1, x_2) = F(M, M)$$
$$\Updownarrow$$
$$\max(0, x_1 + x_2 - 1) = \max(0, 2M - 1)$$

- If $x_1 + x_2 - 1 \leqslant 0$ then RHS must be 0 and hence $M \leqslant \frac{1}{2}$
- If $x_1 + x_2 - 1 > 0$ then RHS must be $2M - 1 > 0$ and hence $M = \frac{x_1+x_2}{2}$

(the solution is not unique)

Chisini's functional equation

Observation:
Chisini's definition is too general. We need conditions on F to ensure the existence and uniqueness of the average M

$\longrightarrow$ Let us investigate

$$F(x_1,\ldots,x_n) = F\big(M(x_1,\ldots,x_n),\ldots,M(x_1,\ldots,x_n)\big)$$

(Chisini's functional equation)

Given: $F\colon \mathbb{I}^n \to \mathbb{R}$
Unknown: $M\colon \mathbb{I}^n \to \mathbb{I}$

Chisini's functional equation

Alternative formulation of

$$F(x_1,\ldots,x_n) = F\big(M(x_1,\ldots,x_n),\ldots,M(x_1,\ldots,x_n)\big)$$

Diagonal section of F

$$\delta_F\colon \mathbb{I} \to \mathbb{R} \qquad\qquad \delta_F(x) = F(x,\ldots,x)$$

Chisini's equation (reformulation)

$$F(x_1,\ldots,x_n) = \delta_F\big(M(x_1,\ldots,x_n)\big)$$

$$\boxed{F = \delta_F \circ M}$$

Existence of solutions

Proposition

$$F = \delta_F \circ M \text{ is solvable}$$

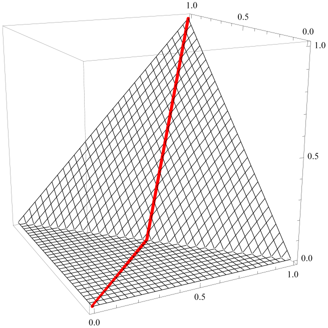
$$\Updownarrow$$

$$\mathrm{ran}(\delta_F) = \mathrm{ran}(F)$$

Existence of solutions

Example. $\mathbb{I} = [0, 1]$

$$F(x_1, x_2) = \max(0, x_1 + x_2 - 1)$$

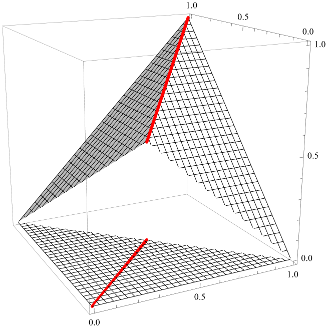


$$\mathrm{ran}(\delta_F) = \mathrm{ran}(F)$$

Existence of solutions

Example. $\mathbb{I} = [0,1]$

$$F(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 + x_2 \leqslant 1, \\ \min(x_1, x_2), & \text{otherwise.} \end{cases}$$



$$\mathrm{ran}(\delta_F) \neq \mathrm{ran}(F)$$

Resolution of Chisini's equation

Assume that Chisini's equation $F = \delta_F \circ M$ is solvable

Solutions:

$$M(\mathbf{x}) \in \delta_F^{-1}\{F(\mathbf{x})\} \qquad \forall \mathbf{x} \in \mathbb{I}^n$$

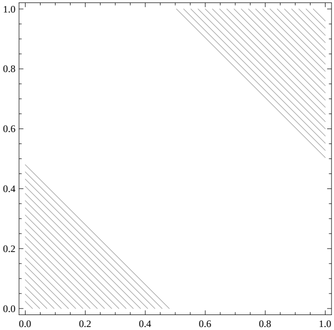
i.e.

$$M(\mathbf{x}) \in \{z \in \mathbb{I} : \delta_F(z) = F(\mathbf{x})\} \qquad \forall \mathbf{x} \in \mathbb{I}^n$$

Resolution of Chisini's equation

Example: $\mathbb{I} = [0, 1]$

$$F(x_1, x_2) = \text{median}\big(x_1 + x_2\ ,\ 1/2\ ,\ x_1 + x_2 - 1\big)$$



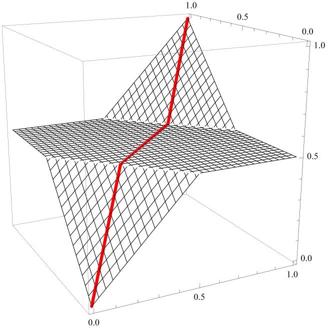
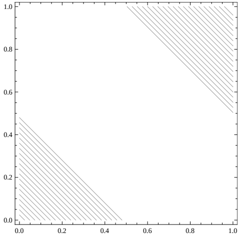
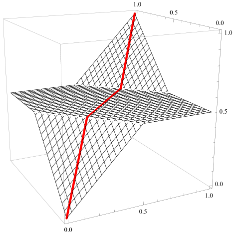


Figure: Function F (contour plot and 3D plot)

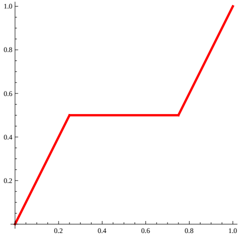
Solutions: $M(\mathbf{x}) \in \delta_F^{-1}\{F(\mathbf{x})\}$



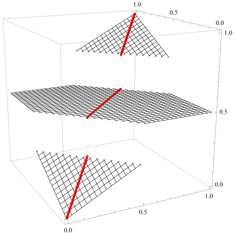
Function F (contour plot)



Function F (3D plot)



Diagonal section δ_F



One possible solution M

Uniqueness of solutions

Proposition

Assume $\mathrm{ran}(\delta_F) = \mathrm{ran}(F)$

$$F = \delta_F \circ M \text{ has a unique solution}$$
$$\Updownarrow$$
$$\delta_F \text{ is one-to-one}$$

$\Rightarrow$ Unique solution:

$$\boxed{M = \delta_F^{-1} \circ F}$$

Back to Chisini means

Chisini's equation

$$F = \delta_F \circ M$$

- Existence of M: $\operatorname{ran}(\delta_F) = \operatorname{ran}(F)$
- Uniqueness of M: δ_F is one-to-one

$$M = \delta_F^{-1} \circ F$$

- Nondecreasing monotonicity of M: F nondecreasing
  $\Rightarrow \delta_{\mathrm{F}}$ is strictly increasing

- Reflexivity of M: For free ! $\delta_M = \delta_F^{-1} \circ \delta_F = \mathrm{id}_{\mathbb{I}}$

M is a mean

A natural question

Chisini's equation

$$F = \delta_F \circ M$$

Suppose

- $\operatorname{ran}(\delta_F) = \operatorname{ran}(F)$ $\Rightarrow$ a solution M exists
- F nondecreasing $\Rightarrow$ δ_F is nondecreasing

(no further conditions on δ_F)

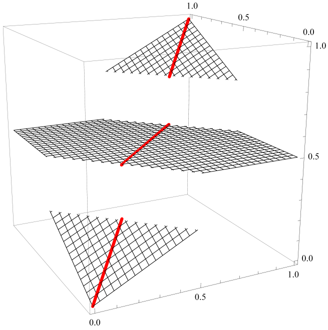
Question

Is there always a nondecreasing and reflexive solution M ?

Constructing means

Example: $\mathbb{I} = [0,1]$

$$F(x_1, x_2) = \operatorname{median}\big(x_1 + x_2\ ,\ 1/2\ ,\ x_1 + x_2 - 1\big)$$



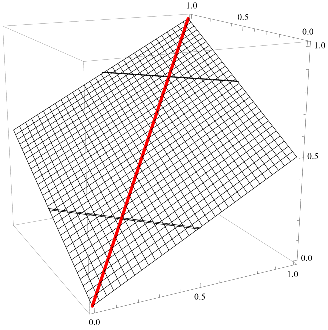


Figure: Two possible solutions M

Constructing means

Idea of the construction:

For every $\mathbf{x} \in \mathbb{I}^n$, we construct $M(\mathbf{x})$ by interpolation

Urysohn's lemma

Let A and B be disjoint closed subsets of $\mathbb{R}^n$

Let $r, s \in \mathbb{R}$, $r < s$

Then there exists a continuous function $U\colon \mathbb{R}^n \to [r, s]$ such that $U|_A = r$ and $U|_B = s$

Urysohn function:

$$U(\mathbf{x}) = r + \frac{d(\mathbf{x}, A)}{d(\mathbf{x}, A) + d(\mathbf{x}, B)}\,(s - r)$$

Constructing means

$$F(x_1, x_2) = \text{median}\big(x_1 + x_2\ ,\ 1/2\ ,\ x_1 + x_2 - 1\big)$$

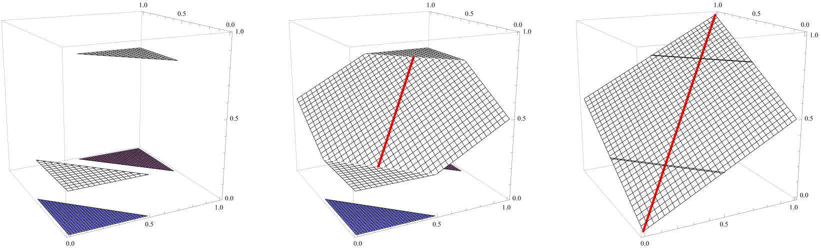


Figure: Function U (with sets A and B) and solution M (interpolation)

Constructing means

Another example: $\mathbb{I} = [0, 1]$

$$F(x_1, x_2) = \min\left(x_1 \ , \ x_2 \ , \ 1/4 + \max(x_1 + x_2 - 1 \ , \ 0)\right)$$

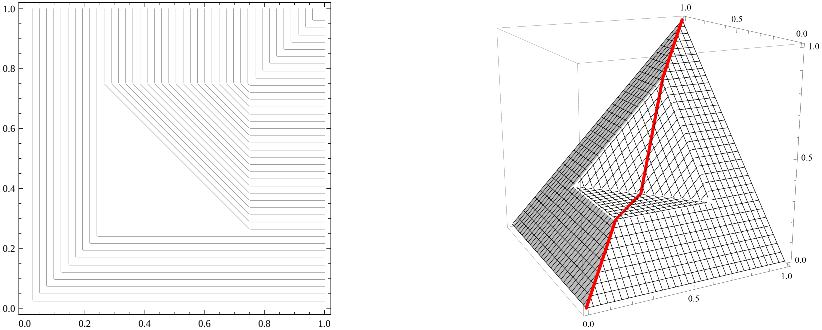


Figure: Function F (contour plot and 3D plot)

Constructing means

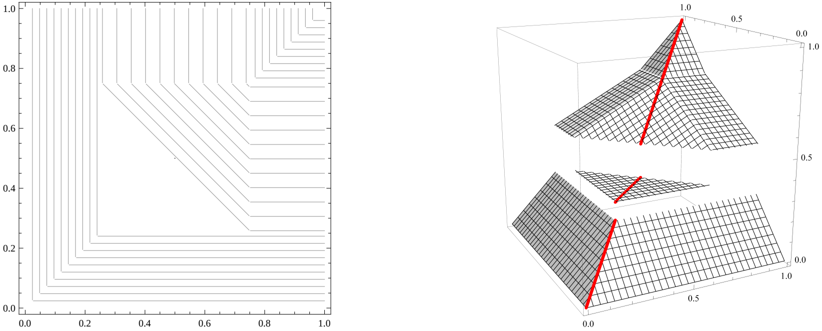


Figure: One possible solution M

Constructing means

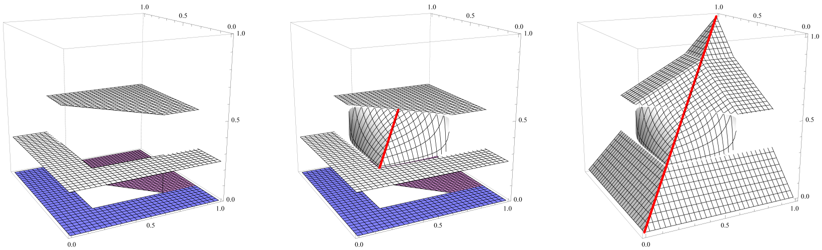


Figure: Function U (with sets A and B) and solution M (interpolation)

Constructing means

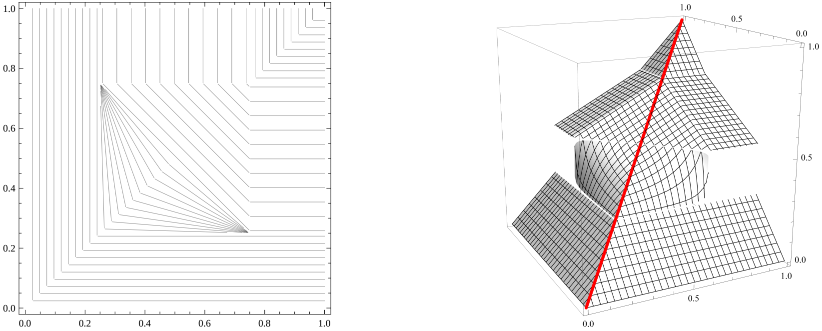


Figure: Solution M obtained by interpolation

Constructing means

Denote by M_F the solution thus constructed (by interpolation)

Theorem

Assume F is nondecreasing and $\mathrm{ran}(\delta_F) = \mathrm{ran}(F)$
Then M_F is nondecreasing, reflexive, and $F = \delta_F \circ M_F$

Further properties:

- F symmetric $\quad \Rightarrow \quad$ M_F symmetric
- $\exists\ M$ continuous solution $\quad \Rightarrow \quad$ M_F continuous

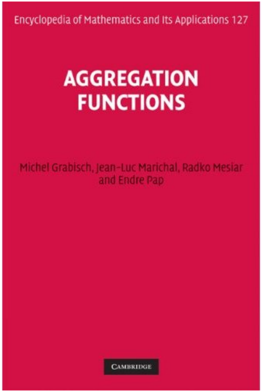
$+$ many other properties...

Book on aggregation functions

Title: *Aggregation Functions*
Authors: M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap
Publisher: Cambridge, 2009

Some chapters

- Properties for aggregation
- Conjunctive aggregation functions
- *Means and averages*
- Aggregation functions based on nonadditive integrals
- Aggregation on specific scale type
- ...



The Chisini mean revisited

Definition

A function $M\colon \mathbb{I}^n \to \mathbb{I}$ is an *average* (or *Chisini mean*) if it is a nondecreasing and reflexive solution of $F = \delta_F \circ M$ for some nondecreasing function $F\colon \mathbb{I}^n \to \mathbb{R}$

If the equation $F = \delta_F \circ M$ is solvable then M_F is a *Chisini mean*