On perfect, amicable, and sociable chains

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Abstract

Let $\mathbf{x} = (x_0, \ldots, x_{n-1})$ be an n-chain, i.e., an n-tuple of non-negative integers $< n$. Consider the operator $s : \mathbf{x} \mapsto \mathbf{x}' = (x'_0, \ldots, x'_{n-1})$, where x'_j represents the number of j's appearing among the components of $\mathbf{x}$. An n-chain $\mathbf{x}$ is said to be perfect if $s(\mathbf{x}) = \mathbf{x}$. For example, (2,1,2,0,0) is a perfect 5-chain. Analogously to the theory of perfect, amicable, and sociable numbers, one can define from the operator s the concepts of amicable pair and sociable group of chains. In this paper we give an exhaustive list of all the perfect, amicable, and sociable chains.

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1 Introduction

Let $n \geqslant 1$ be an integer and let $N := \{0, 1, \ldots, n-1\}$. An *$n$-chain* is an n-tuple

$$\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}),$$

with $x_i \in N$ for all $i \in N$. Since such an n-tuple can be viewed as a mapping from N into itself, the set of all n-chains will be denoted N^N, and its cardinality is $|N^N| = n^n$.

Let 2^N represent the set of all subsets of N. For any $j \in N$, define $S_j : N^N \to 2^N$ as

$$S_j(\mathbf{x}) := \{i \in N \mid x_i = j\}.$$

Clearly, for any $\mathbf{x} \in N^N$, $\{S_j(\mathbf{x}) \mid j \in N\}$ is a partition of N.

We then say that $\mathbf{x} \in N^N$ is a *perfect chain* if

$$x_j = |S_j(\mathbf{x})|, \qquad j \in N.$$

In other terms, $\mathbf{x} \in N^N$ is a perfect chain if, for any $j \in N$, x_j represents the number of j's occuring in $\{x_0, x_1, \ldots, x_{n-1}\}$. For instance

$$\mathbf{x} = (2, 1, 2, 0, 0)$$

*As this paper was accepted for publication, the author found out that the main problem (Theorem 4) was already addressed and solved by Sallows and Eijkhout [5]; see also [3, 4, 6].

is a perfect 5-chain.

We say that $\mathbf{x}, \mathbf{x}' \in N^N$ ($\mathbf{x} \neq \mathbf{x}'$) form a pair of *amicable chains* if

$$\begin{aligned} x'_j &= |S_j(\mathbf{x})|, \qquad j \in N, \\ x_j &= |S_j(\mathbf{x}')|, \qquad j \in N. \end{aligned}$$

For instance

$$\mathbf{x} = (2, 3, 0, 1, 0, 0) \quad \text{and} \quad \mathbf{x}' = (3, 1, 1, 1, 0, 0)$$

form a pair of amicable 6-chains.

Now, consider the *counting operator* $s : N^N \to \{0, 1, \ldots, n\}^N$ defined by $\mathbf{x}' = s(\mathbf{x})$ with

$$x'_j = |S_j(\mathbf{x})|, \qquad j \in N.$$

Given an integer $l \geqslant 3$, we say that the chains $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(l-1)} \in N^N$, satisfying

$$\mathbf{x}^{(k+1)} = s(\mathbf{x}^{(k)}), \qquad k \in \{0, \ldots, l-2\},$$

form a group of l *sociable chains* if they are distinct and $s(\mathbf{x}^{(l-1)}) = \mathbf{x}^{(0)}$. For instance

$$\mathbf{x}^{(0)} = (3, 3, 0, 0, 1, 0, 0), \quad \mathbf{x}^{(1)} = (4, 1, 0, 2, 0, 0, 0), \quad \mathbf{x}^{(2)} = (4, 1, 1, 0, 1, 0, 0)$$

form a group of three sociable 7-chains.

Notice that these concepts present some analogies with perfect, amicable, and sociable numbers, see e.g. [2, 7]. Consider the function $s(n) = \sigma(n) - n$, where σ denotes the divisor sum function. A positive integer n is said to be perfect if $s(n) = n$. For example, 6 is perfect. Two positive integers m and n are said to be amicable if $s(m) = n$ and $s(n) = m$. For example, 220 and 284 are amicable. An l-tuple ($l \geqslant 3$) of positive integers $(n_0, \ldots, n_{l-1})$, satisfying $n_{k+1} = s(n_k)$ for all k, is a sociable group if these integers are distinct and $s(n_{l-1}) = n_0$. For example, (12 496, 14 288, 15 472, 14 536, 14 264) is a group of 5 sociable numbers.

The main aim of this paper is to determine all the perfect, amicable, and sociable chains. These are gathered in Theorem 4 below. We also investigate the counting operator and point out some of its properties.

The outline of this paper is as follows. In Section 2 we determine conditions under which the iterates of the counting operator are well defined. In Section 3 the results are presented of an exhaustive computation of all the perfect, amicable, and sociable chains. Finally, Section 4 is devoted to a description of the range of the counting operator and its iterates.

2 Preliminary results

In this section we investigate the counting operator s introduced above as well as its iterates. We first observe that this operator does not always range in N^N. For example, if $n = 4$, we have

$$s(2, 2, 2, 2) = (0, 0, 4, 0) \notin N^N.$$

We thus need to restrict the domain of s to chains $\mathbf{x}$ such that each element of the infinite sequence

$$\mathbf{x}, s(\mathbf{x}), s(s(\mathbf{x})), s(s(s(\mathbf{x}))), \ldots$$

belongs to N^N. The following results deal with this issue.

Lemma 1. *Let* $\mathbf{x} \in N^N$ *and* $\mathbf{x}' = s(\mathbf{x})$. *Then*

$$\sum_{j\in N} x'_j = n, \tag{1}$$

$$\sum_{j\in N} j\, x'_j = \sum_{j\in N} x_j. \tag{2}$$

Proof. Since $\{S_j(\mathbf{x}) \mid j \in N\}$ is a partition of N, we simply have

$$\sum_{j\in N} x'_j = \sum_{j\in N} |S_j(\mathbf{x})| = |N| = n,$$

and, by counting in two ways,

$$\sum_{j\in N} x_j = \sum_{j\in N} \sum_{i\in S_j(\mathbf{x})} x_i = \sum_{j\in N} \sum_{i\in S_j(\mathbf{x})} j = \sum_{j\in N} j\,|S_j(\mathbf{x})| = \sum_{j\in N} j\, x'_j.$$

□

Lemma 2. *Let* $\mathbf{x} \in N^N$. *The following statements hold:*

(i) $s(\mathbf{x}) \in N^N$ *if and only if* $x_0, \ldots, x_{n-1}$ *are not all equal.*

(ii) *If* $s(\mathbf{x}) \in N^N$ *then* $s(s(\mathbf{x})) \in N^N$ *if and only if* $x_0, \ldots, x_{n-1}$ *are not all distinct.*

(iii) *If* $s(\mathbf{x}), s(s(\mathbf{x})) \in N^N$ *then* $s(s(s(\mathbf{x}))) \in N^N$ *if and only if* $n \geqslant 4$.

Proof. (i) Easy.

(ii) Setting $\mathbf{x}' := s(\mathbf{x})$ and $\mathbf{x}'' := s(\mathbf{x}')$, we have

$$\begin{aligned}\mathbf{x}'' \in N^N &\Leftrightarrow x'_0, \ldots, x'_{n-1} \text{ are not all equal} \qquad (\text{by } (i))\\ &\Leftrightarrow \mathbf{x}' \neq (1, \ldots, 1) \qquad (\text{by Eq. } (1))\\ &\Leftrightarrow \{x_0, \ldots, x_{n-1}\} \neq N.\end{aligned}$$

(iii) By (i) and (ii), the numbers $x_0, \ldots, x_{n-1}$ are neither all equal nor all distinct, and hence $n \geqslant 3$. Now set $\mathbf{x}' := s(\mathbf{x})$, $\mathbf{x}'' := s(\mathbf{x}')$, and $\mathbf{x}''' := s(\mathbf{x}'')$. By ($ii$), we have

$$\mathbf{x}''' \in N^N \Leftrightarrow \{x'_0, \ldots, x'_{n-1}\} \neq N.$$

However we have

$$\begin{aligned}\{x'_0, \ldots, x'_{n-1}\} = N &\Rightarrow \sum_{j\in N} x'_j = \sum_{j\in N} j\\ &\Rightarrow n = \frac{n(n-1)}{2} \qquad (\text{by Eq. } (1))\\ &\Rightarrow n = 3\end{aligned}$$

and

$$n = 3 \Rightarrow \{x'_0, x'_1, x'_2\} = \{0, 1, 2\} = N.$$

Thus Lemma 2 is proved. □

Let $\mathcal{N}$ denote the set of all n-chains whose components are neither all equal nor all distinct. One can readily see that $|\mathcal{N}| = n^n - n! - n$. Moreover, we have the following result, which immediately follows from Lemma 2.

Proposition 3. *Let* $\mathbf{x} \in N^N$. *Then all the chains* $s(\mathbf{x}), s(s(\mathbf{x})), s(s(s(\mathbf{x}))), \ldots$ *belong to* N^N *if and only if* $\mathbf{x} \in \mathcal{N}$ *and* $n \geqslant 4$. *In that case, all these chains belong to* $\mathcal{N}$.

From now on we will assume that $n \geqslant 4$. Let $\mathbb{N}$ denote the set of non-negative integers. According to Proposition 3 we can construct from any $\mathbf{x} \in \mathcal{N}$ an infinite sequence of chains $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$ in the following way:

$$\begin{cases} \mathbf{x}^{(0)} = \mathbf{x}, \\ \mathbf{x}^{(k+1)} = s(\mathbf{x}^{(k)}), \quad k \in \mathbb{N}. \end{cases} \tag{3}$$

Since $\mathcal{N}$ is a finite set, this sequence is eventually periodic. That is, there exist $k_0, l \in \mathbb{N}$ ($l \geqslant 1$) such that

$$\mathbf{x}^{(k+l)} = \mathbf{x}^{(k)} \qquad \forall k \geqslant k_0. \tag{4}$$

If the chains $\mathbf{x}^{(k)}, \ldots, \mathbf{x}^{(k+l-1)}$ are distinct and such that $\mathbf{x}^{(k+l)} = \mathbf{x}^{(k)}$, we say that they form a *circuit* of length l. Of course, determining perfect (resp. amicable, sociable) chains amounts to identifying all the circuits of length 1 (resp. 2, $\geqslant 3$).

3 Exhaustive computation of perfect, amicable, and sociable chains

In the present section we calculate all the perfect, amicable, and sociable chains. These are given in Theorem 4 below.

Assume that $\mathbf{x}^{(k_0)} \in \mathcal{N}$ belongs to a circuit. By Proposition 3, we have $\mathbf{x}^{(k)} \in \mathcal{N}$ for all $k \geqslant k_0$. Furthermore, by Eq. (1) and (2), we have

$$\sum_{j\in N} x_j^{(k)} = n \qquad \forall k \geqslant k_0, \tag{5}$$

$$\sum_{j\in N} j\, x_j^{(k)} = n \qquad \forall k \geqslant k_0. \tag{6}$$

These identities imply trivially

$$x_0^{(k)} = \sum_{j=1}^{n-1} (j-1)\, x_j^{(k)} \qquad \forall k \geqslant k_0. \tag{7}$$

Moreover, we have

$$x_0^{(k)} \geqslant 1 \qquad \forall k \geqslant k_0. \tag{8}$$

Indeed, if $x_0^{(k)} = 0$ for some $k \geqslant k_0$ then, by Eq. (7), we have $x_2^{(k)} = \cdots = x_{n-1}^{(k)} = 0$. By Eq. (5) we then have $x_1^{(k)} = n$, a contradiction.

Theorem 4. *Let* $\|$ *denote a list, possibly empty, of zeroes.*
The perfect chains are:

$$(1, 2, 1, 0) \tag{9}$$

$$(2, 0, 2, 0) \tag{10}$$

$$(2, 1, 2, 0, 0) \tag{11}$$

$$(n-4, 2, 1 \| 1, 0, 0, 0), \qquad n \geqslant 7. \tag{12}$$

The pairs of amicable chains are:

$$(2,3,0,1,0,0), \quad (3,1,1,1,0,0) \tag{13}$$

$$(n-4,3,0,0\|0,1,0,0), \quad (n-3,1,0,1\|1,0,0,0), \qquad n \geqslant 8. \tag{14}$$

The unique group of sociable chains is:

$$(3,3,0,0,1,0,0), \quad (4,1,0,2,0,0,0), \quad (4,1,1,0,1,0,0). \tag{15}$$

There is no group of more than 3 sociable chains.

Proof. Let $\mathbf{x}^{(k_0)} \in \mathcal{N}$ belong to a circuit. Choose $k \geqslant k_0$ such that $x_0^{(k+1)} \leqslant x_0^{(k)}$. Such a k exists for otherwise $\mathbf{x}^{(k_0)}$ would not belong to a circuit.

Set $p := x_0^{(k)}$. By Eq. (8), we have $1 \leqslant p \leqslant n-1$. Moreover, since $0 \in S_p(\mathbf{x}^{(k)})$, we have

$$x_p^{(k+1)} = |S_p(\mathbf{x}^{(k)})| \geqslant 1.$$

Using Eq. (7), we have

$$x_0^{(k+1)} = \sum_{j=1}^{n-1} (j-1)\, x_j^{(k+1)} \geqslant (p-1) + \sum_{\substack{j=1 \\ j \neq p}}^{n-1} (j-1)\, x_j^{(k+1)},$$

and hence,

$$1 \geqslant 1 + x_0^{(k+1)} - p \geqslant \sum_{\substack{j=1 \\ j \neq p}}^{n-1} (j-1)\, x_j^{(k+1)} \geqslant 0, \tag{16}$$

implying $x_0^{(k+1)} = p$ or $x_0^{(k+1)} = p-1$. We now investigate these two cases separately.

1. *Case* $x_0^{(k+1)} = p$.

   By Eq. (16), we have

   $$x_j^{(k+1)} = 0 \qquad \forall j \in N \setminus \{0,1,2,p\}.$$

   (a) *Case* $p = 1$.
   Using Eq. (7) and (5), we obtain $x_2^{(k+1)} = 1$ and $x_1^{(k+1)} = n-2$, so that

   $$\mathbf{x}^{(k+1)} = (1, n-2, 1\|)$$

   and $\{x_1^{(k)}, \ldots, x_{n-1}^{(k)}\} = \{2,1,\ldots,1,0\}$. By Eq. (6), we have

   $$n = \sum_{j \in N} j\, x_j^{(k)} \geqslant 2 + \sum_{j=2}^{n-2} j = \frac{1}{2}(n-2)(n-1) + 1,$$

   that is $n = 4$. This leads to the circuit (9).

   (b) *Case* $p = 2$.
   Using Eq. (7) and (5), we obtain $x_2^{(k+1)} = 2$ and $x_1^{(k+1)} = n-4$, so that

   $$\mathbf{x}^{(k+1)} = (2, n-4, 2\|)$$

   and $\{x_1^{(k)}, \ldots, x_{n-1}^{(k)}\} = \{2,1,\ldots,1,0,0\}$. By Eq. (6), we have

   $$n = \sum_{j \in N} j\, x_j^{(k)} \geqslant 2 + \sum_{j=2}^{n-3} j = \frac{1}{2}(n-3)(n-2) + 1,$$

   that is $n \in \{4,5\}$. This leads to the circuits (10) and (11).

(c) *Case* $p \geqslant 3$.

By Eq. (7), we have

$$p = x_2^{(k+1)} + (p-1)\, x_p^{(k+1)},$$

which implies $x_2^{(k+1)} = x_p^{(k+1)} = 1$. By Eq. (5), we then have $x_1^{(k+1)} = n - p - 2$, and hence

$$\mathbf{x}^{(k+1)} = (\underbrace{p, n-p-2, 1\|1}_{p+1}\|),$$

with $n \geqslant p + 2$.

i. *Case* $n = p + 2$ $(\geqslant 5)$.

We have

$$\mathbf{x}^{(k+1)} = (n-2, 0, 1\|1, 0), \quad \mathbf{x}^{(k+2)} = (n-3, 2, 0\|1, 0).$$

For $n = 5$, we get the circuit (11). For $n \geqslant 6$, we have

$$\mathbf{x}^{(k+3)} = (n-3, 1, 1\|1, 0, 0), \quad \mathbf{x}^{(k+4)} = (n-4, 3, 0\|1, 0, 0).$$

For $n = 6$, $n = 7$ and $n \geqslant 8$, we get the circuits (13), (15) and (14) respectively.

ii. *Case* $n = p + 3$ $(\geqslant 6)$.

We have

$$\mathbf{x}^{(k+1)} = (n-3, 1, 1\|1, 0, 0),$$

which leads to a previous case.

iii. *Case* $n = p + 4$ $(\geqslant 7)$.

We have

$$\mathbf{x}^{(k+1)} = (n-4, 2, 1\|1, 0, 0, 0),$$

which leads to the circuit (12).

iv. *Case* $n = p + 5$ $(\geqslant 8)$.

We have

$$\mathbf{x}^{(k+1)} = (n-5, 3, 1\|1, 0, 0, 0, 0).$$

For $n = 8$, we get the circuit (14). For $n \geqslant 9$, we have

$$\mathbf{x}^{(k+2)} = (n-4, 2, 0, 1\|1, 0, 0, 0, 0),$$

retrieving the circuit (12).

v. *Case* $n = p + r$ $(\geqslant 3 + r)$*, with* $r \geqslant 6$.

We have

$$\mathbf{x}^{(k+1)} = (\underbrace{n-r, r-2, 1\|1}_{n-r+1}\|).$$

If $n - r < r - 2$ then

$$\mathbf{x}^{(k+2)} = (\underbrace{\overbrace{n-4, 2, 0\|1}^{n-r+1}\|1}_{r-1}\|),$$

which leads to a previous case.
If $n - r = r - 2$ then

$$\mathbf{x}^{(k+2)} = (\underbrace{n-4, 2, 0\|2}_{n-r+1}\|), \quad \mathbf{x}^{(k+3)} = (n-3, 0, 2\|1, 0, 0, 0),$$

which leads to a previous case.
If $n - r > r - 2$ then

$$\mathbf{x}^{(k+2)} = (\underbrace{\overbrace{n-4, 2, 0\|1}^{r-1}\|1}_{n-r+1}\|),$$

which leads to a previous case.

2. *Case* $x_0^{(k+1)} = p - 1$.

   By Eq. (16), we have

   $$x_j^{(k+1)} = 0 \qquad \forall j \in N \setminus \{0, 1, p\},$$

   with $p = x_0^{(k+1)} + 1 \geqslant 2$. Using Eq. (5) and (7), we obtain $x_p^{(k+1)} = 1$ and $x_1^{(k+1)} = n - p$, so that

   $$\mathbf{x}^{(k+1)} = (\underbrace{p-1, n-p\|1}_{p+1}\|).$$

   (a) *Case* $p = 2$.
   We have

   $$\mathbf{x}^{(k+1)} = (1, n-2, 1\|),$$

   that is a case previously encountered.

   (b) *Case* $p \geqslant 3$.

   i. *Case* $n = p + 1$ $(\geqslant 4)$.
   We have

   $$\mathbf{x}^{(k+1)} = (n-2, 1\|1),$$

   which leads to a previous case.

   ii. *Case* $n = p + r$ $(\geqslant 3 + r)$, *with* $r \geqslant 2$.
   We have

   $$\mathbf{x}^{(k+1)} = (\underbrace{n-r-1, r\,\|1}_{n-r+1}\|).$$

   If $n - r - 1 < r$ then

   $$\mathbf{x}^{(k+2)} = (\underbrace{\overbrace{n-3, 1\|1}^{n-r}\|1}_{r+1}\|),$$

   which leads to a previous case.
   If $n - r - 1 = r$ then

   $$\mathbf{x}^{(k+2)} = (\underbrace{n-3, 1\|2}_{n-r}\|),$$

which leads to a previous case.
If $n-r-1>r$ then

$$\mathbf{x}^{(k+2)} = (\underbrace{\overbrace{n-3,1\|1}^{r+1}\|1}_{n-r+1}\|),$$

which leads to a previous case.

Theorem 4 is now proved. □

Corollary 5. *Any circuit of length* $\geqslant 2$ *contains the chain* $(n-4,3\|1,0,0)$.

Before closing this section, we present the following open problem. For any $\mathbf{x}\in\mathcal{N}$, we denote by $\mathcal{C}(\mathbf{x})$ the circuit obtained from the infinite sequence $(\mathbf{x}^{(k)})_{k\in\mathbb{N}}$. The question then arises of determining the length of the non-periodic part of this sequence; that is, the number of elements that do not belong to $\mathcal{C}(\mathbf{x})$:

$$\Psi(\mathbf{x}) := \min\{k\in\mathbb{N} \mid \mathbf{x}^{(k)}\in\mathcal{C}(\mathbf{x})\}.$$

Interestingly enough, the following sequence:

$$\psi(n) := \max_{\mathbf{x}\in\mathcal{N}} \Psi(\mathbf{x}), \qquad n\geqslant 4,$$

has a rather strange behavior. Its first values (for $4\leqslant n\leqslant 44$) are: 3, 4, 7, 4, 7, 7, 7, 6, 7, 6, 7, 7, 7, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 8.

We conjecture that the elements of this sequence can be arbitrary large; that is, for any $M\geqslant 3$ there exists $n\geqslant 4$ such that $\psi(n)\geqslant M$.

4 Range of the counting operator and its iterates

For any $k\in\mathbb{N}$, let $s^{(k)}$ denote the kth iterate of the operator s. It is clear that we have

$$s^{(k+1)}(\mathcal{N}) \subseteq s^{(k)}(\mathcal{N}), \qquad k\in\mathbb{N}.$$

In this final section we intend to describe the subset $s^{(k)}(\mathcal{N})$ for each $k\in\mathbb{N}$. The case $k=1$ is dealt with in the next proposition.

Proposition 6. *We have*

$$s(\mathcal{N}) = \Big\{\mathbf{x}\in\mathcal{N} \,\Big|\, \sum_{j\in N} x_j = n\Big\}.$$

Proof. ($\subseteq$) Follows from Eq. (1).
($\supseteq$) Let $\mathbf{x}\in\mathcal{N}$ such that $\sum_{j\in N} x_j = n$. Setting

$$\mathbf{z} := (\underbrace{0,\ldots,0}_{x_0},\underbrace{1,\ldots,1}_{x_1},\ldots,\underbrace{n-1,\ldots,n-1}_{x_{n-1}}),$$

we have $\mathbf{z}\in\mathcal{N}$ and $s(\mathbf{z})=\mathbf{x}$, and hence $\mathbf{x}\in s(\mathcal{N})$. □

Let the operator $r : s(\mathcal{N}) \to \mathcal{N}$ be defined by

$$r(\mathbf{x}) = (\underbrace{0,\ldots,0}_{x_0},\underbrace{1,\ldots,1}_{x_1},\ldots,\underbrace{n-1,\ldots,n-1}_{x_{n-1}}).$$

Let Π_N be the set of all the permutations on N and define the operator $q : \mathcal{N} \to \mathcal{N}$ by

$$q(\mathbf{x}) = (x_{\nu(0)},\ldots,x_{\nu(n-1)}),$$

where $\nu \in \Pi_N$ is such that $x_{\nu(0)} \leqslant \cdots \leqslant x_{\nu(n-1)}$. One can easily see that $s \circ r = \mathrm{id}$ and $r \circ s = q$, thus showing that s is not invertible.

For any $\pi \in \Pi_N$, we define $r_\pi : s(\mathcal{N}) \to \mathcal{N}$ by

$$r_\pi(\mathbf{x}) = (r(\mathbf{x})_{\pi(0)},\ldots,r(\mathbf{x})_{\pi(n-1)}).$$

For any $\mathbf{x} \in s(\mathcal{N})$, we clearly have $s^{(-1)}(\mathbf{x}) = \{r_\pi(\mathbf{x}) \mid \pi \in \Pi_N\}$. Moreover, we have the following result.

Proposition 7. *For any $k \in \mathbb{N}$, we have*

$$s^{(k+1)}(\mathcal{N}) = \Big\{\mathbf{x} \in s^{(k)}(\mathcal{N}) \,\Big|\, \exists\, \pi_1,\ldots,\pi_k \in \Pi_N : \sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_k})(\mathbf{x})_j = n\Big\}.$$

Proof. We proceed by induction over $k \in \mathbb{N}$. By Proposition 6, the result holds for $k = 0$. Assume that it also holds for $k = 0,\ldots,K-1$, with a given $K \geqslant 1$. We now show that it still holds for $k = K$.

($\subseteq$) Let $\mathbf{x} \in s^{(K+1)}(\mathcal{N})$. Take $\pi_K \in \Pi_N$ and set $\mathbf{z} := r_{\pi_K}(\mathbf{x})$. We have $\mathbf{x} = s(\mathbf{z})$ and hence $\mathbf{z} \in s^{(K)}(\mathcal{N})$. By induction hypothesis, there exist $\pi_1,\ldots,\pi_{K-1} \in \Pi_N$ such that

$$\sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_K})(\mathbf{x})_j = \sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_{K-1}})(\mathbf{z})_j = n.$$

($\supseteq$) Let $\mathbf{x} \in s^{(K)}(\mathcal{N})$ and assume that there exist $\pi_1,\ldots,\pi_K \in \Pi_N$ such that

$$\sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_K})(\mathbf{x})_j = n.$$

We only have to prove that $\mathbf{x} \in \{s(\mathbf{z}) \mid \mathbf{z} \in s^{(K)}(\mathcal{N})\}$. Set $\mathbf{z} := r_{\pi_K}(\mathbf{x})$. We have $\mathbf{x} = s(\mathbf{z})$ and hence $\mathbf{z} \in s^{(K-1)}(\mathcal{N})$. Moreover, we have

$$\sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_{K-1}})(\mathbf{z})_j = \sum_{j\in N} (r_{\pi_1} \circ \cdots \circ r_{\pi_K})(\mathbf{x})_j = n,$$

and hence $\mathbf{z} \in s^{(K)}(\mathcal{N})$ by induction hypothesis. □

The case $k = 2$ is particularly interesting. One can easily see that, for any $\mathbf{x} \in \mathcal{N}$ and any $j \in N$, $s^{(2)}(\mathbf{x})_j$ represents the number of distinct values occuring j times in $\{x_0,\ldots,x_{n-1}\}$. Moreover, we have the following proposition.

Proposition 8. *We have*

$$s^{(2)}(\mathcal{N}) = \Big\{\mathbf{x} \in s(\mathcal{N}) \,\Big|\, \sum_{j\in N} j\, x_j = n\Big\}.$$

Proof. For any $\pi \in \Pi_N$, we have

$$\sum_{j\in N} r_\pi(\mathbf{x})_j = \sum_{j\in N} j\, x_j.$$

We then conclude by Proposition 7. $\square$

Now, from the identity

$$\Big|\Big\{\mathbf{x}\in\mathbb{N}^n \,\Big|\, \sum_{j=1}^{n} j\, x_j = n\Big\}\Big| = P(n),$$

where $P(n)$ is the number of unrestricted partitions of the integer n (see e.g. [1]), we can easily show that $|s^{(2)}(\mathcal{N})| = P(n) - 2$. Similarly, from the well-known identity

$$\Big|\Big\{\mathbf{x}\in\mathbb{N}^n \,\Big|\, \sum_{j=1}^{n} x_j = n\Big\}\Big| = \tbinom{2n-1}{n},$$

we can readily see that $|s(\mathcal{N})| = \binom{2n-1}{n} - n - 1$.

Finally, from the identities $r \circ s = q$ and $r \circ s^{(2)} = q \circ s$, we clearly have

$$\begin{aligned} r(s(\mathcal{N})) &= \{\mathbf{x}\in\mathcal{N} \mid x_0 \leqslant \cdots \leqslant x_{n-1}\},\\ r(s^{(2)}(\mathcal{N})) &= \Big\{\mathbf{x}\in\mathcal{N} \,\Big|\, \sum_{j\in N} x_j = n \text{ and } x_0 \leqslant \cdots \leqslant x_{n-1}\Big\}, \end{aligned}$$

and, since r is an injection, we have

$$|r(s(\mathcal{N}))| = |s(\mathcal{N})| \quad\text{and}\quad |r(s^{(2)}(\mathcal{N}))| = |s^{(2)}(\mathcal{N})|.$$

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