Pivotal decompositions of aggregation functions

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Shannon decomposition

For $f : \{0,1\}^n \to \{0,1\}$,

$$f(\mathbf{x}) \quad = \quad x_k \, f(\mathbf{x}\big|_{x_k=1}) + (1 - x_k) \, f(\mathbf{x}\big|_{x_k=0}), \qquad \forall \mathbf{x}, k.$$

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Leads to multilinear representation

$$f(\mathbf{x}) \quad = \quad \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i \prod_{i \in [n] \setminus S} (1 - x_i).$$

Median decomposition

For a Sugeno integral

$$f(\mathbf{x}) = \bigvee_{S \subseteq [n]} \mu(S) \wedge \bigwedge_{i \in S} x_i,$$

$$f(\mathbf{x}) \ = \ \mathrm{med}(x_k, f(\mathbf{x}\big|_{x_k=0}), f(\mathbf{x}\big|_{x_k=1})), \quad \forall \mathbf{x}, k$$

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Leads to median representations.

Pivotal decomposition

General assumption :

$$f : [0,1]^n \to \mathbb{R}$$

Definition

A function f is *pivotally decomposable* if there is a Φ such that

$$f(\mathbf{x}) \;=\; \Phi\Big(x_k, f(\mathbf{x}|_{x_k=0}), f(\mathbf{x}|_{x_k=1})\Big), \qquad \forall \mathbf{x}, k.$$

Φ is a *pivotal function* and f is Φ*-decomposable*.

Shannon decomposition ✓

For $f : \{0,1\}^n \to \{0,1\}$,

$$\begin{aligned} f(\mathbf{x}) &= (1 - x_k)\, f(\mathbf{x}\big|_{x_k=0}) \;+\; x_k\, f(\mathbf{x}\big|_{x_k=1}), \qquad \forall \mathbf{x}, k \\ &= \Phi\big(x_k, f(\mathbf{x}\big|_{x_k=0}), f(\mathbf{x}\big|_{x_k=1})\big) \end{aligned}$$

where

$$\Phi(x, y, z) = (1 - x)\, y + x\, z$$

Median decomposition ✓

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Multilinear polynomial functions ✓

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Conjugate weighted means

Let $\psi : [0,1] \to \mathbb{R}$ be continuous, onto, strictly monotonic,

$$f_\psi(\mathbf{x}) \;=\; \psi^{-1}\Big(\sum_{i=1}^{n} \omega_i \, \psi(x_i)\Big).$$

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$$f_\psi(\mathbf{x}) \;=\; \psi^{-1}\Big(\sum_{i=1}^{n} \omega_i\, \psi(x_i)\Big).$$

$$\begin{aligned} f(\mathbf{x}) &= \big(1-\psi(x_k)\big)\,\psi\big(f(\mathbf{x}|_{x_k=0})\big) \;+\; \psi(x_k)\,\psi\big(f(\mathbf{x}|_{x_k=1})\big), \quad \forall \mathbf{x}, k \\ &= \Phi\big(x_k, f(\mathbf{x}|_{x_k=0}), f(\mathbf{x}|_{x_k=1})\big) \end{aligned}$$

where

$$\Phi(x,y,z) = \psi\Big(\,(1-\psi(x))\psi(y) + \psi(x)\,\psi(z)\Big).$$

t-norms ✓

For a t-norm $T : [0,1]^2 \to [0,1]$,

$$\begin{aligned} T(x,y) &= T(x, T(1,y)) \\ &= \Phi(x, T(0,y), T(1,y)) \end{aligned}$$

where

$$\Phi(x,y,z) = T(x,y)$$

Discrete Choquet integrals ✗

For discrete Choquet integrals

$$f(\mathbf{x}) = \sum_{S\subset[n]} a_S \bigwedge_{i\in S} x_i, \quad a_S \in \mathbb{R},$$

no pivotal decomposition exists (in general).

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no pivotal decomposition exists (in general).

Why pivotal decomposition ?

$$f(\mathbf{x}) \;=\; \Phi\Big(x_k, f(\mathbf{x}|_{x_k=0}), f(\mathbf{x}|_{x_k=1})\Big), \quad \forall \mathbf{x}, k.$$

- Uniformly isolate the marginal contribution of a factor.
- Ease computations.
- Allow proofs by induction.
- Repeated applications of the decomposition lead to canonical forms.
- Characterizing function classes.

Proposition

Let f be Φ-decomposable

- *Uniqueness of Φ,*
- *f is determined by Φ and $f\big|_{\{0,1\}^n}$.*
- *Unary sections are determined by their value on $\{0,1\}$.*

Two characterizations of classes

A function f is a multilinear polynomial function if and only if

$$f(\mathbf{x}) = x_k\, f(\mathbf{x}\big|_{x_k=1}) + (1 - x_k)\, f(\mathbf{x}\big|_{x_k=0}), \qquad \forall \mathbf{x}, lk$$

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A function f is is a lattice polynomial function if and only if

$$f(\mathbf{x}) = \mathrm{med}(x_k, f(\mathbf{x}\big|_{x_k=0}), f(\mathbf{x}\big|_{x_k=1})), \qquad \forall \mathbf{x}, \forall k$$

Hence, pivotal decompositions may lead to characterizations.

Pivotally characterized function classes

Definition

Let $D \subseteq \mathbb{R}^2$ and $\Phi : [0,1] \times D$.

Let Γ_Φ defined by

$f \in \Gamma_\Phi$ iff f is Φ-decomposable

A class C of functions is *pivotally characterized* if there is a Φ such that $C = \Gamma_\Phi$. Then, C is said to be Φ*-characterized.*

We note $f \equiv g$ if f and g are equal up to permutation of variables, addition or deletion of inessential variables.

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Let $D \subseteq \mathbb{R}^2$ and $\Phi : [0,1] \times D$.

Let Γ_Φ defined by

$$f \in \Gamma_\Phi \text{ iff } f \text{ is } \Phi\text{-decomposable (up to inessential variables).}$$

A class C of functions is *pivotally characterized* if there is a Φ such that $C = \Gamma_\Phi$. Then, C is said to be Φ*-characterized.*

Multilinear polynomial functions ✓

The class of multilinear polynomial functions is Γ_Φ for

$$\Phi(x, y, z) = (1 - x)\, u \,+\, y\, z$$

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Lattice polynomial functions ✓

The class of lattice polynomial functions is Γ_Φ for

$$\Phi(x,y,z) = \mathrm{med}(x,y,z)$$

Non-decreasing multilinear polynomial functions ✓

The class of non-decreasing multilinear polynomial functions is Γ_Φ for

$$\Phi(x, y, z) = (1 - x)\,(y \wedge z)\; +\; x\,(y \vee z)$$

t-norms

Non-decreasing multilinear polynomial functions ✓

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t-norms ✗

No pivotal characterization

Symmetric multilinear polynomials ✗

No pivotal characterization

Choquet integrals ✗

No pivotal characterization

How to recognize pivotally characterized classes ?

Problem

Given a class C of functions, can you determine if it is pivotally characterized ?

Example

Is the class of Sugeno integrals pivotally characterized ?

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Example

Is the class of Sugeno integrals pivotally characterized ?

$x \wedge y$	$x \wedge y \wedge 3$
med-decomposable	med-decomposable
Sugeno	Non Sugeno
x is unary-section	x is unary section

Function classes characterized by their unary members

Definition

A class C of functions is *UM-characterized* if for any $f : [0,1]^n \to \mathbb{R}$

> $f \in C$ if and only if so is every essential unary section of f.

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Example

Any pivotally characterized class is a UM-characterized one.

Theorem

Assume that C is a Φ-characterized and that $C' \subseteq C$. Then,

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Non-decreasing multilinear polynomial functions ✓

These are those multilinear polynomials that have non-decreasing unary sections.

It is UM-characterized inside a pivotally characterized class.

Generalizations

Definition

A function f is *componentwise pivotally decomposable* if there are some $\Phi_1, \ldots, \Phi_n$ such that

$$f(\mathbf{x}) = \Phi_k(x_k, f(\mathbf{x}\big|_{x_k=0}), f(\mathbf{x}\big|_{x_k=1})), \qquad \forall \mathbf{x}, k.$$

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Fact

A function f is pivotally decomposable
if and only if
every unary section of f is determined by its values on $\{0, 1\}$.

Binary Choquet integrals ✓

Every binary Choquet integral $f(\mathbf{x}) = a\,x_1 + b\,x_2 + c\,(x_1 \wedge x_2)$ is componentwise pivotally decomposable.

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Choquet integrals ✗

There are ternary Choquet integrals that are not componentwise pivotally decomposable.

Questions

- Find pivotal decompositions / characterizations.
- Classes characterized by componentwise pivotal decompositions.
- Two pivots decomposition

$$f(\mathbf{x}) = \Phi(x_k, x_j, f(\mathbf{x}|_{(x_k,x_j)\in\{0,1\}^2})) \qquad \forall \mathbf{x}, k.$$