Pivotal decompositions of aggregation functions

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1 Preliminaries

A remarkable (though immediate) property of Boolean functions is the so-called *Shannon decomposition* [9], also called *pivotal decomposition* [1]. This property states that, for every n-ary Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ and every $k \in [n] = \{1,\ldots,n\}$, the following decomposition formula holds

$$f(\mathbf{x}) \;=\; \overline{x}_k\, f(\mathbf{x}_k^0) + x_k\, f(\mathbf{x}_k^1)\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{1}$$

where $\overline{x}_k = 1 - x_k$ and $\mathbf{x}_k^0$ (resp. $\mathbf{x}_k^1$) is the n-tuple whose i-th coordinate is 0 (resp. 1), if $i = k$, and x_i, otherwise. Here the '$+$' sign represents the classical addition for real numbers.

As it is well known, repeated applications of (1) show that any n-ary Boolean function can always be expressed as the multilinear polynomial function

$$f(\mathbf{x}) \;=\; \sum_{S\subseteq[n]} f(\mathbf{1}_S) \prod_{i\in S} x_i \prod_{i\in[n]\setminus S} \overline{x}_i\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{2}$$

where $\mathbf{1}_S$ is the characteristic vector of S in $\{0,1\}^n$, that is, the n-tuple whose i-th coordinate is 1, if $i \in S$, and 0, otherwise.

One can easily show that, if f is nondecreasing (in each variable), decomposition formula (1) reduces to

$$f(\mathbf{x}) \;=\; \mathrm{med}(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1))\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{3}$$

or, equivalently,

$$f(\mathbf{x}) \;=\; \overline{x}_k\,(f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1)) + x_k\,(f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1))\,, \qquad \mathbf{x} \in \{0,1\}^n. \tag{4}$$

where $\wedge$ (resp. $\vee$) is the minimum (resp. maximum) operation and med is the ternary median operation.

Actually, any of the decomposition formulas (3)–(4) exactly expresses the fact that f should be nondecreasing and hence characterizes the subclass of nondecreasing n-ary Boolean functions.

Decomposition property (1) also holds for functions $f\colon \{0,1\}^n \to \mathbb{R}$, called *n-ary pseudo-Boolean functions*. As a consequence, these functions also have the representation given in (2). Moreover, formula (4) clearly characterizes the subclass of nondecreasing n-ary pseudo-Boolean functions.

The *multilinear extension* of an n-ary pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$ is the function $\hat{f}\colon [0,1]^n \to \mathbb{R}$ defined by (see Owen [7, 8])

$$\hat{f}(\mathbf{x}) \;=\; \sum_{S\subseteq[n]} f(\mathbf{1}_S)\prod_{i\in S} x_i \prod_{i\in[n]\setminus S}(1-x_i)\,, \qquad \mathbf{x}\in[0,1]^n.$$

Thus defined, one can easily see that the class of multilinear extensions and that of nondecreasing multilinear extensions can be characterized as follows.

Proposition 1. *A function $f\colon [0,1]^n \to \mathbb{R}$ is a multilinear extension if and only if it satisfies*

$$f(\mathbf{x}) \;=\; (1-x_k)\,f(\mathbf{x}_k^0) + x_k\,f(\mathbf{x}_k^1)\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

Proposition 2. *A function $f\colon [0,1]^n \to \mathbb{R}$ is a nondecreasing multilinear extension if and only if it satisfies*

$$f(\mathbf{x}) \;=\; \overline{x}_k\,(f(\mathbf{x}_k^0)\wedge f(\mathbf{x}_k^1)) + x_k\,(f(\mathbf{x}_k^0)\vee f(\mathbf{x}_k^1))\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

The decomposition formulas considered in this introduction share an interesting common feature, namely the fact that any variable, here denoted x_k and called *pivot*, can be isolated from the others in the evaluation of functions. This feature may be useful when for instance the values $f(\mathbf{x}_k^0)$ and $f(\mathbf{x}_k^1)$ are much easier to compute than that of $f(\mathbf{x})$. In addition to this, such (pivotal) decompositions may facilitate inductive proofs and may lead to canonical forms such as (2).

In this note we define a more general concept of pivotal decomposition for various functions $f\colon [0,1]^n \to \mathbb{R}$, including certain aggregation functions. We also introduce pivotal characterizations of classes of such functions.

2 Pivotal decompositions of functions

The examples presented in the previous section motivate the following definition.

Definition 1. *We say that a function $f\colon [0,1]^n \to \mathbb{R}$ is* pivotally decomposable *if there exists a subset D of $\mathbb{R}^3$ and a function $\Phi\colon D\to\mathbb{R}$, called* pivotal function*, such that*

$$D \supseteq \{(f(\mathbf{x}_k^0), z, f(\mathbf{x}_k^1)) : z\in[0,1],\, \mathbf{x}\in[0,1]^n\}\,, \qquad k\in[n]$$

and

$$f(\mathbf{x}) \;=\; \Phi(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1))\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

In this case, we say that f is Φ-decomposable.

Example 1 (Lattice polynomial functions). Recall that a *lattice polynomial function* is simply a composition of projections, constant functions, and the fundamental lattice operations $\wedge$ and $\vee$; see, e.g., [3, 4]. An n-ary function $f\colon [0,1]^n \to [0,1]$ is a lattice polynomial function if and only if it can be written in the (disjunctive normal) form

$$f(\mathbf{x}) \;=\; \bigvee_{S\subseteq[n]} f(\mathbf{1}_S)\wedge\bigwedge_{i\in S} x_i\,, \qquad \mathbf{x}\in[0,1]^n.$$

The so-called *discrete Sugeno integrals* are exactly those lattice polynomial functions which are idempotent (i.e., $f(x,\ldots,x)=x$ for all $x\in[0,1]$).

Every lattice polynomial function is Φ-decomposable with $\Phi\colon [0,1]^3\to\mathbb{R}$ defined by $\Phi(r,z,s)=\mathrm{med}(r,z,s)$; see, e.g., [6].

Example 2 (Lovász extensions). Recall that the *Lovász extension* of a pseudo-Boolean function $f\colon \{0,1\}^n\to\mathbb{R}$ is the function

$$L_f(\mathbf{x}) \;=\; \sum_{S\subseteq[n]} a(S)\bigwedge_{i\in S} x_i\,,$$

where the set function $a\colon 2^{[n]}\to\mathbb{R}$ is defined by $a(S)=\sum_{T\subseteq S}(-1)^{|S|-|T|}f(\mathbf{1}_T)$; see, e.g., [5]. The so-called *discrete Choquet integrals* are exactly those Lovász extensions which are nondecreasing and idempotent.

There are ternary Lovász extensions $L_f\colon [0,1]^3\to\mathbb{R}$ that are not pivotally decomposable, e.g., $L_f(x_1,x_2,x_3)=x_1\wedge x_2+x_2\wedge x_3$.

Example 3 (T-norms). A *t-norm* is a binary function $T\colon [0,1]^2\to[0,1]$ that is symmetric, nondecreasing, associative, and such that $T(1,x)=x$. Every t-norm $T\colon [0,1]^2\to[0,1]$ is Φ-decomposable with $\Phi\colon [0,1]^3\to\mathbb{R}$ defined by $\Phi(r,z,s)=T(z,s)$.

Example 4 (Conjugate functions). Given a function $f\colon [0,1]^n\to[0,1]$ and a strictly increasing bijection $\phi\colon [0,1]\to[0,1]$, the ϕ-conjugate of f is the function $f_\phi=\phi^{-1}\circ f\circ(\phi,\ldots,\phi)$. One can easily show that f is Φ-decomposable for some pivotal function Φ if and only if f_ϕ is Φ_ϕ-decomposable, where $\Phi_\phi=\phi^{-1}\circ\Phi\circ(\phi,\phi,\phi)$. Combining this for instance with Proposition 2 shows that every *quasi-linear mean function* (i.e., ϕ-conjugate of a weighted arithmetic mean) is pivotally decomposable.

For every $k\in[n]$, and every $\mathbf{a}\in[0,1]^n$, we define the *unary section* $f_k^{\mathbf{a}}\colon [0,1]\to\mathbb{R}$ of f by setting $f_k^{\mathbf{a}}(x)=f(\mathbf{a}_k^x)$. The kth argument of f is said to be *inessential* if $f_k^{\mathbf{a}}$ is constant for every $\mathbf{a}\in[0,1]^n$. Otherwise, it is said to be *essential*. We say that a unary section $f_k^{\mathbf{a}}$ of f is *essential* if the kth argument of f is essential.

For every function $f\colon X^n\to Y$ and every map $\sigma\colon [n]\to[m]$, we define the function $f_\sigma\colon X^m\to Y$ by $f_\sigma(\mathbf{a})=f(\mathbf{a}\sigma)$, where $\mathbf{a}\sigma$ denotes the n-tuple $(a_{\sigma(1)},\ldots,a_{\sigma(n)})$.

Define on the set $U=\cup_{n\geqslant 1}\mathbb{R}^{[0,1]^n}$ the equivalence relation $\equiv$ as follows: For functions $f\colon [0,1]^n\to\mathbb{R}$ and $g\colon [0,1]^m\to\mathbb{R}$, we write $f\equiv g$ if there exist maps $\sigma\colon [m]\to[n]$ and $\mu\colon [n]\to[m]$ such that $f=g_\sigma$ and $g=f_\mu$. Equivalently, $f\equiv g$ means that f can be obtained from g by permuting arguments or by adding or deleting inessential arguments.

Definition 2. *Let $\Phi\colon D\to\mathbb{R}$ be a pivotal function. We denote by C_Φ the class of all the functions $f\colon [0,1]^n\to\mathbb{R}$ (where $n\geqslant 0$) that are $\equiv$-equivalent to a Φ-decomposable function with no essential argument or no inessential argument. We say that a class $C\subseteq U$ is* pivotally characterizable *if there exists a pivotal function Φ such that $C=C_\Phi$. In that case, we say that C is* Φ-characterized.

Proposition 3. *Let Φ be a pivotal function.*

(i) A nonconstant function $f\colon [0,1]^n\to\mathbb{R}$ is in C_Φ if and only if so are its essential unary sections.

(ii) A constant function $f\colon [0,1]^n \to \{c\}$ is in C_Φ if and only if $\Phi(c,z,c) = c$ for every $z \in [0,1]$.

Example 5 (Lattice polynomial functions). The class of lattice polynomial functions is Φ-characterized for the pivotal function $\Phi\colon [0,1]^3 \to \mathbb{R}$ defined by $\Phi(r,z,s) = \mathrm{med}(r,z,s)$.

3 Classes characterized by their unary members

Proposition 3 shows that a class C_Φ is characterized by the essential unary sections of its members. This observation motivates the following definition, which is inspired from [2].

Definition 3. *A class $C \subseteq U$ is* characterized by its unary members *if it satisfies the following conditions:*

(i) A nonconstant function f is in C if and only if so are its essential unary sections.
(ii) If f is a constant function in C and $g \equiv f$, then g is in C.

We denote by CU *the family of classes characterized by their unary members.*

Theorem 1. *Let Φ be a pivotal function. A nonempty subclass of C_Φ is characterized by its unary members if and only if it is pivotally characterizable.*

Theorem 2. *The family* CU *can be endowed with a complete and atomic Boolean algebra structure.*

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