The Chisini means revisited

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Chisini mean (1929)

Let $\mathbb{I}$ be a real interval
An *average* of n numbers $x_1, \ldots, x_n \in \mathbb{I}$ w.r.t. a function $\mathrm{F}\colon \mathbb{I}^n \to \mathbb{R}$ is a number M such that

$$\mathrm{F}(x_1, \ldots, x_n) = \mathrm{F}(M, \ldots, M)$$

An average is also called a *Chisini mean*

Example:

$$\mathrm{F}(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 \quad \Rightarrow \quad M = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

Remark: This definition is too general. We need conditions on F to ensure the existence and uniqueness of the mean M

Chisini's functional equation

$$\mathsf{F}(x_1,\ldots,x_n)=\mathsf{F}\big(\mathsf{G}(x_1,\ldots,x_n),\ldots,\mathsf{G}(x_1,\ldots,x_n)\big)$$

Given: $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$
Unknown: $\mathsf{G}\colon \mathbb{I}^n \to \mathbb{I}$

Diagonal section of F

$$\delta_{\mathsf{F}}\colon \mathbb{I} \to \mathbb{R} \qquad \delta_{\mathsf{F}}(x)=\mathsf{F}(x,\ldots,x)$$

Chisini's equation (reformulation)

$$\mathsf{F}=\delta_{\mathsf{F}}\circ\mathsf{G}$$

Existence of solutions

Proposition

Chisini's equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$ is solvable if and only if $\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F})$

Example. *Łukasiewicz t-norm*

$$\mathsf{F} \colon [0,1]^2 \longrightarrow [0,1]$$
$$\mathsf{F}(x_1, x_2) = \mathsf{Max}(0, x_1 + x_2 - 1)$$

We have

$$\delta_{\mathsf{F}}(x) = \mathsf{Max}(0, 2x - 1)$$
$$\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F}) = [0,1]$$

Existence of solutions

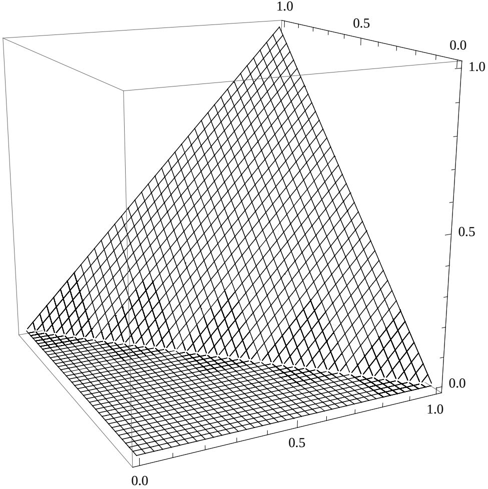


Figure: Łukasiewicz t-norm (3D plot)

Existence of solutions

Example. *Nilpotent minimum*

$$\mathsf{F}\colon [0,1]^2 \to [0,1]$$
$$\mathsf{F}(x_1,x_2) = \begin{cases} 0, & \text{if } x_1 + x_2 \leqslant 1, \\ \mathrm{Min}(x_1,x_2), & \text{otherwise.} \end{cases}$$

We have

$$\mathrm{ran}(\delta_{\mathsf{F}}) = \{0\} \cup \left]\tfrac{1}{2}, 1\right]$$
$$\mathrm{ran}(\mathsf{F}) = [0,1]$$

$\Rightarrow$ The associated Chisini equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$ is not solvable !

Existence of solutions

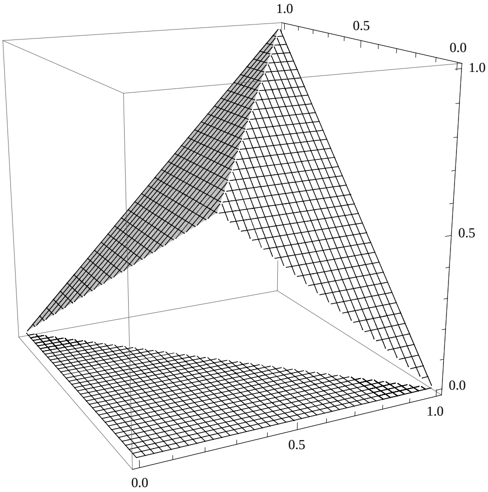


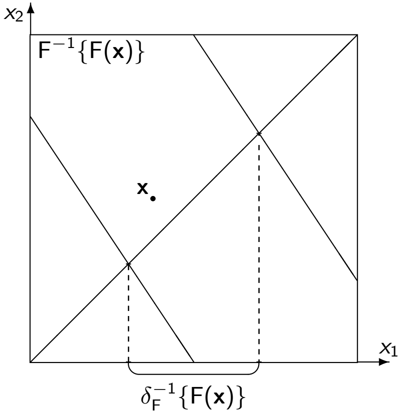
Figure: Nilpotent minimum (3D plot)

Solutions of Chisini's equation

Assume that Chisini's equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$ is solvable

Then

$$\mathsf{G}(\mathbf{x}) \in \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\} \qquad \forall \mathbf{x} \in \mathbb{I}^n$$



Uniqueness of solutions

Proposition

Assume $\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F})$

Then Chisini's equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$ has a unique solution if and only if δ_{F} is one-to-one

$\Rightarrow$ The unique solution is given by $\mathsf{G} = \delta_{\mathsf{F}}^{-1} \circ \mathsf{F}$

Back to Chisini means

Chisini's equation

$$\boxed{\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}}$$

- Existence of G: $\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F})$
- Uniqueness of G: δ_{F} is one-to one
- Nondecreasing monotonicity of G: F nondecreasing

Then it suffices to ask δ_{F} to be strictly increasing

$$\boxed{\mathsf{G} = \delta_{\mathsf{F}}^{-1} \circ \mathsf{F}}$$

- Idempotency of G: For free ! $\delta_{\mathsf{G}} = \delta_{\mathsf{F}}^{-1} \circ \delta_{\mathsf{F}} = \mathrm{id}_{\mathbb{I}}$

G is a mean

Idempotization process

A nondecreasing function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ is *idempotizable* if $\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F})$ and δ_{F} is strictly increasing

Idempotization process (Calvo et al. 2002)
Starting from an idempotizable function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$, we generate a nondecreasing and idempotent (i.e. a mean) function

Example
For the function $\mathsf{F}(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$, we have $\delta_{\mathsf{F}}(x) = nx^2$ and

$$\mathsf{G}(\mathbf{x}) = (\delta_{\mathsf{F}}^{-1} \circ \mathsf{F})(\mathbf{x}) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

Another example

On $\mathbb{I} =]-1, 1[$, consider the *Einstein sum*

$$F(x, y) = \frac{x + y}{1 + xy}$$

In fact:

$$F(x, y) = \varphi^{-1}\big(\varphi(x) + \varphi(y)\big) \qquad \varphi = \mathrm{arctanh}$$

Idempotization process:

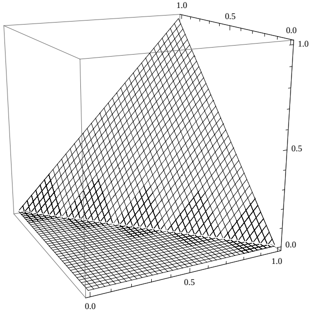
$$G(x, y) = \varphi^{-1}\Big(\frac{\varphi(x) + \varphi(y)}{2}\Big)$$

$$G(x, y) = \frac{1 + xy - \sqrt{1 - x^2}\sqrt{1 - y^2}}{x + y}$$

Solving Chisini's equation

What if δ_F is not one-to-one ?

Example: Łukasiewicz t-norm



Note: δ_F is not invertible $\rightarrow$ *But* it has at least one *right-inverse*

Solving Chisini's equation

Definition. A *right-inverse* of a function $f\colon \mathbb{I} \to \mathbb{R}$ is a function $g\colon \mathrm{ran}(f) \to \mathbb{I}$ such that

$$f \circ g = \mathrm{id}_{\mathrm{ran}(f)}$$

Proposition

Assume $\mathrm{ran}(\delta_{\mathsf{F}}) = \mathrm{ran}(\mathsf{F})$
Let $g\colon \mathrm{ran}(\delta_{\mathsf{F}}) \to \mathbb{I}$ be a right-inverse of δ_{F}

Then the function $\mathsf{G} = g \circ \mathsf{F}$ is

- well defined
- solves Chisini's equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$.

We call it a *trivial solution*

Trivial solutions

Let $\mathsf{G} = g \circ \mathsf{F}$ be a trivial solution of Chisini's equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$

- F nondecreasing $\quad \Rightarrow \quad$ G nondecreasing
- F symmetric $\quad \Rightarrow \quad$ G symmetric
- For free: G is *range-idempotent*

  i.e. $\delta_{\mathsf{G}}(x) = x \quad \forall x \in \mathrm{ran}(\mathsf{G})$

  i.e. $\delta_{\mathsf{G}} \circ \mathsf{G} = \mathsf{G}$

Proof: $\delta_{\mathsf{G}} \circ \mathsf{G} = (g \circ \delta_{\mathsf{F}}) \circ \mathsf{G} = g \circ (\delta_{\mathsf{F}} \circ \mathsf{G}) = g \circ \mathsf{F} = \mathsf{G}$

Remark.

– The trivial solutions are not always idempotent

– Some solutions may be not trivial

Trivial solutions

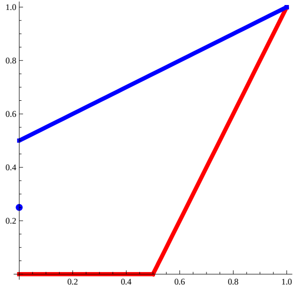
Example: Łukasiewicz t-norm $\quad \mathsf{F}(x, y) = \mathsf{Max}(0, x + y - 1)$

$$\delta_{\mathsf{F}}(x) = \mathsf{Max}(0, 2x - 1)$$

Every right-inverse g of δ_{F} satisfies

$g(x) = \frac{1}{2}(x + 1)$ on $]0, 1]$

$g(0) \in [0, \frac{1}{2}]$



Trivial solutions

Trivial solutions:

$$G(x,y) = \begin{cases} g(0), & \text{if } x + y \leqslant 1, \\ \frac{x+y}{2}, & \text{otherwise.} \end{cases}$$

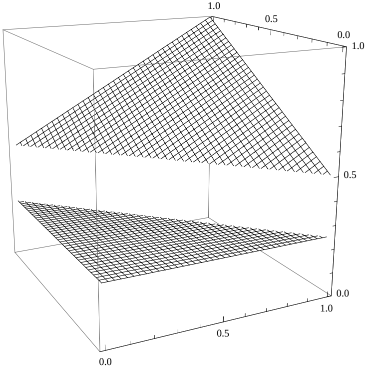


Figure: Trivial solution with $g(0) = \frac{1}{4}$

Trivial solutions

A non-trivial and idempotent solution:

$$\boxed{\mathsf{G}(x,y) = \frac{x+y}{2}}$$

Indeed,

$$\begin{aligned}(\delta_{\mathsf{F}} \circ \mathsf{G})(x,y) &= \mathsf{Max}(0, 2\,\mathsf{G}(x,y) - 1) \\ &= \mathsf{Max}(0, 2\,\frac{x+y}{2} - 1) \\ &= \mathsf{Max}(0, x+y-1) \\ &= \text{Łukasiewicz t-norm}\end{aligned}$$

Constructing means

Question

Let $F\colon \mathbb{I}^n \longrightarrow \mathbb{R}$ be a nondecreasing function whose corresponding Chisini's equation is solvable

Is there always a nondecreasing and idempotent solution ?

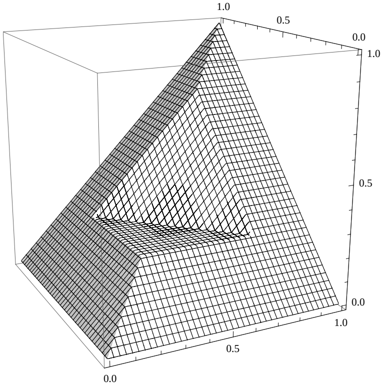
Answer: YES !

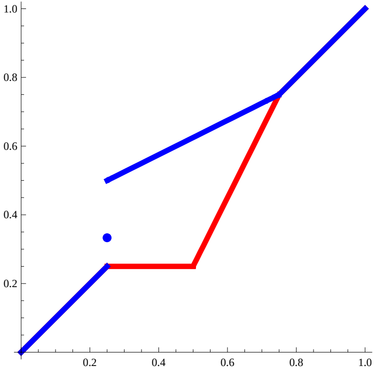
We can even construct a nondecreasing and idempotent solution having many other nice properties

Constructing means

Yet another example

$$F(x, y) = \mathrm{Min}\big(x, y, 1/4 + \mathrm{Max}(x + y - 1, 0)\big)$$





Constructing means

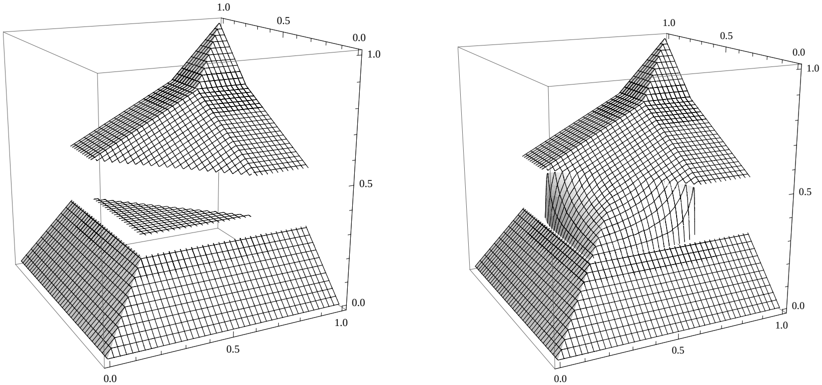


Figure: Trivial and idempotent solutions

Constructing means

Idea of the construction:
For every $\mathbf{x} \in \mathbb{I}^n$, we construct $\mathrm{G}(\mathbf{x})$ by interpolation (based on Urysohn's lemma)

> *Let A and B be disjoint closed subsets of $\mathbb{R}^n$*
> *Let $a, b \in \mathbb{R}$, $a < b$*
> *Then there exists a continuous function $\mathrm{U}\colon \mathbb{R}^n \to [a, b]$ such that $\mathrm{U}|_A = a$ and $\mathrm{U}|_B = b$*

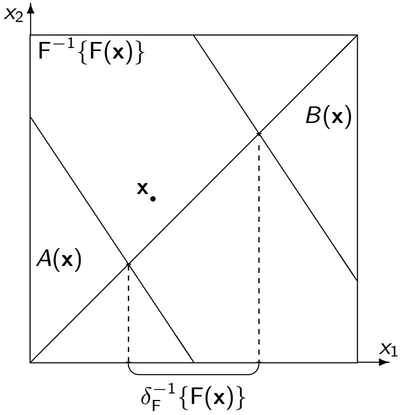
Urysohn function:

$$\mathrm{U}(\mathbf{x}) = a + \frac{d(\mathbf{x}, A)}{d(\mathbf{x}, A) + d(\mathbf{x}, B)}\,(b - a)$$

Constructing means

We choose:

- $A(\mathbf{x}) = \{\mathbf{z} \in \mathbb{I}^n : \mathsf{F}(\mathbf{z}) < \mathsf{F}(\mathbf{x})\}$ (lower level set)
- $a(\mathbf{x}) = \inf \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}$



Constructing means

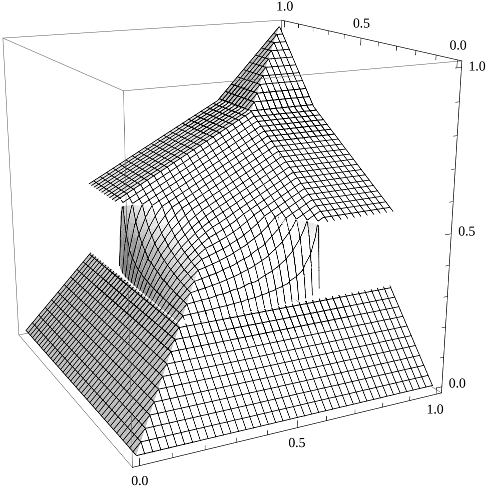


Figure: Idempotent solution (interpolation)

Constructing means

Denote by M_F the solution thus constructed

Theorem

Let $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ be nondecreasing and such that $\mathrm{ran}(\delta_\mathsf{F}) = \mathrm{ran}(\mathsf{F})$
Then M_F is nondecreasing, idempotent, and $\mathsf{F} = \delta_\mathsf{F} \circ \mathsf{M}_\mathsf{F}$

Some properties:

- $\mathsf{M}_\mathsf{F} = \mathsf{M}_{g\circ\mathsf{F}}$ for every right-inverse g of δ_F
- F symmetric $\quad\Rightarrow\quad$ M_F symmetric
- F^d dual of F $\quad\Rightarrow\quad$ $\mathsf{M}_{\mathsf{F}^d} = \mathsf{M}_\mathsf{F}^d$
  (interpolation and dualization commute)

The Chisini mean revisited

Definition

A function $\mathsf{M}\colon \mathbb{I}^n \to \mathbb{I}$ is an *average* (or *Chisini mean*) if it is a nondecreasing and idempotent solution of $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}$ for some nondecreasing function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$

If the equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}$ is solvable then M_{F} is a *Chisini mean*

$\Rightarrow$ Generalized idempotization process
(constructing means from nondecreasing functions)

Continuous solutions

Theorem

Let $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ be nondecreasing and such that $\mathrm{ran}(\delta_\mathsf{F}) = \mathrm{ran}(\mathsf{F})$. Then the equation $\mathsf{F} = \delta_\mathsf{F} \circ \mathsf{G}$ has a continuous solution G if and only if M_F is continuous

Theorem

Let $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ be nondecreasing and continuous.
Then the following assertions are equivalent:

1. The equation $\mathsf{F} = \delta_\mathsf{F} \circ \mathsf{G}$ has a continuous solution
2. M_F is continuous
3. $d\big(\mathbf{x}, A(\mathbf{x})\big) = d\big(\mathbf{x}, B(\mathbf{x})\big) = 0 \;\Rightarrow\; \delta_\mathsf{F}^{-1}\{\mathsf{F}(\mathbf{x})\}$ is a singleton

Continuous solutions

Transformed continuous functions

Find necessary and sufficient conditions on a nondecreasing function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ for its factorization as

$$\mathsf{F} = f \circ \mathsf{G}$$

where f is one-place and nondecreasing and G is nondecreasing and continuous

- If G satisfies condition 3 above: SOLVED !
- General case: OPEN !

Thank you for your attention

The full paper is available at:

http://arxiv.org/abs/0903.1546