THE CHISINI MEAN REVISITED

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Summary

We investigate the n-variable real functions G that are solutions of the functional equation $\mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathsf{G}(\mathbf{x}), \ldots, \mathsf{G}(\mathbf{x}))$, where F is a given function of n real variables. We provide necessary and sufficient conditions on F for the existence and uniqueness of solutions. When F is nondecreasing in each variable, we show in a constructive way that if a solution exists then a nondecreasing and idempotent solution always exists. Such solutions, called Chisini means, are then thoroughly investigated.
The full version of this article is available at `http://arxiv.org/abs/0903.1546`

Keywords: Chisini's functional equation, Chisini mean, aggregation, idempotency, quasi-idempotency.

1 INTRODUCTION

Let $\mathbb{I}$ be any nonempty real interval, bounded or not, and let $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ be any given function. We are interested in the solutions $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ of the following functional equation

$$\mathsf{F}(x_1, \ldots, x_n) = \mathsf{F}(\mathsf{G}(x_1, \ldots, x_n), \ldots, \mathsf{G}(x_1, \ldots, x_n)). \quad (1)$$

This functional equation was considered in 1929 by Chisini [4], who investigated the concept of mean as an *average* or a *numerical equalizer*. More precisely, Chisini defined a mean of n numbers $x_1, \ldots, x_n$ with respect to a function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ as a number M such that $\mathsf{F}(x_1, \ldots, x_n) = \mathsf{F}(M, \ldots, M)$. For instance, when F is the sum, the product, the sum of squares, the sum of inverses, or the sum of exponentials, the solution M of the equation above is unique and consists of the arithmetic mean, the geometric mean, the quadratic mean, the harmonic mean, and the exponential mean, respectively.

By considering the *diagonal section* of F, i.e. the one-variable function $\delta_{\mathsf{F}} : \mathbb{I} \to \mathbb{R}$ defined by $\delta_{\mathsf{F}}(x) := \mathsf{F}(x, \ldots, x)$, we can rewrite equation (1) as

$$\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}. \quad (2)$$

If, as in the examples above, we assume that F is nondecreasing (in each variable) and that δ_{F} is a bijection from $\mathbb{I}$ onto the range of F, then Chisini's equation (2) clearly has a unique solution $\mathsf{G} = \delta_{\mathsf{F}}^{-1} \circ \mathsf{F}$ which is nondecreasing and idempotent (i.e., such that $\delta_{\mathsf{G}}(x) = x$). Such a solution is then called a *Chisini mean* or a *level surface mean* (see Bullen [1, VI.4.1]).

In this paper, we consider Chisini's functional equation (2) in its full generality, i.e. without any assumption on F. We first provide necessary and sufficient conditions on F for the existence of solutions and we show how the possible solutions can be constructed (§2). We also investigate "trivial" solutions of the form $g \circ \mathsf{F}$, where g is a quasi-inverse of δ_{F} (§3). We then elaborate on the case when F is nondecreasing and we show that if a solution exists then at least one nondecreasing and idempotent solution always exists and we construct such a solution (§4). Finally, we discuss a few applications of the theory developed here to certain classes of functions (§5). In particular, we revisit the concept of Chisini mean and we extend it to the case when δ_{F} is nondecreasing but not strictly increasing.

The terminology used throughout this paper is the following. The domain and range of any function f are denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. The minimum and maximum functions are denoted by Min and Max, respectively. That is, $\mathsf{Min}(\mathbf{x}) := \min\{x_1, \ldots, x_n\}$ and $\mathsf{Max}(\mathbf{x}) := \max\{x_1, \ldots, x_n\}$ for any $\mathbf{x} \in \mathbb{R}^n$. The identity function is the function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ defined by $\mathrm{id}(x) = x$. Finally, the diagonal restriction of $\mathbb{I}^n$ is the subset $\mathrm{diag}(\mathbb{I}^n) := \{(x, \ldots, x) : x \in \mathbb{I}\}$.

2 RESOLUTION OF CHISINI'S EQUATION

In this section, we provide necessary and sufficient conditions for the existence and uniqueness of solutions of Chisini's equation and we show how the solutions can be constructed.

Let $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ be a given function and suppose that the associated Chisini equation (2) has a solution $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$. We immediately see that, for any $\mathbf{x} \in \mathbb{I}^n$, the possible values of $\mathsf{G}(\mathbf{x})$ are exactly those reals $z \in \mathbb{I}$ for which the n-tuple $(z, \ldots, z)$ belongs to the level set of F through $\mathbf{x}$. In other terms, we must have

$$\mathsf{G}(\mathbf{x}) \in \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}, \qquad \forall \mathbf{x} \in \mathbb{I}^n. \tag{3}$$

Thus a necessary condition for equation (2) to have at least one solution is

$$\operatorname{ran}(\delta_{\mathsf{F}}) = \operatorname{ran}(\mathsf{F}). \tag{4}$$

Assuming the Axiom of Choice (AC), we immediately see that condition (4) is also sufficient for equation (2) to have at least one solution. Indeed, by assuming both AC and (4), we can define a function $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ satisfying (3) and this function then solves equation (2). Note however that AC is not always required to ensure the existence of a solution. For instance, if δ_{F} is monotonic (i.e., either nondecreasing or nonincreasing), then every level set $\delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}$ is a bounded interval (except two of them at most) and for instance its midpoint could be chosen to define $\mathsf{G}(\mathbf{x})$.

Thus we have proved the following result.

Proposition 2.1. *Let* $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ *be a function. If equation (2) has at least one solution* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *then* $\operatorname{ran}(\delta_{\mathsf{F}}) = \operatorname{ran}(\mathsf{F})$. *Under AC (not necessary if* δ_{F} *is monotonic), the converse also holds.*

Equation (3) shows how the possible solutions of the Chisini equation can be constructed. We first observe that $\mathbb{I}^n$ is a disjoint union of the level sets of F, i.e. $\mathbb{I}^n = \cup_{y\in\operatorname{ran}(\delta_{\mathsf{F}})}\mathsf{F}^{-1}\{y\}$. Thus, constructing a solution G on $\mathbb{I}^n$ reduces to constructing it on each level set $\mathsf{F}^{-1}\{y\}$, with $y \in \operatorname{ran}(\delta_{\mathsf{F}})$. That is, for every $\mathbf{x} \in \mathsf{F}^{-1}\{y\}$, we choose $\mathsf{G}(\mathbf{x}) \in \delta_{\mathsf{F}}^{-1}\{y\}$.

The next proposition yields an alternative description of the solutions of the Chisini equation through the concept of quasi-inverse function. Recall first that a function g is a *quasi-inverse* of a function f if

$$f \circ g|_{\operatorname{ran}(f)} = \operatorname{id}|_{\operatorname{ran}(f)}, \tag{5}$$

$$\operatorname{ran}(g|_{\operatorname{ran}(f)}) = \operatorname{ran}(g). \tag{6}$$

For any function f, denote by $Q(f)$ the set of its quasi-inverses. This set is nonempty whenever we assume AC, which is actually just another form of the statement "every function has a quasi-inverse." Recall also that the relation of being quasi-inverse is symmetric, i.e. if $g \in Q(f)$ then $f \in Q(g)$; moreover $\operatorname{ran}(f) \subseteq \operatorname{dom}(g)$ and $\operatorname{ran}(g) \subseteq \operatorname{dom}(f)$ (see [10, §2.1]).

By definition, if $g \in Q(f)$ then $g|_{\operatorname{ran}(f)} \in Q(f)$. Thus we can always restrict the domain of any quasi-inverse $g \in Q(f)$ to $\operatorname{ran}(f)$. These "restricted" quasi-inverses, also called *right-inverses*, are then simply characterized by condition (5), which can be rewritten as

$$g(y) \in f^{-1}\{y\}, \qquad \forall y \in \operatorname{ran}(f). \tag{7}$$

Proposition 2.2. *Let* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *be a function satisfying (4) and let* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *be any function. Then, assuming AC (not necessary if* δ_{F} *is monotonic), the following assertions are equivalent:*

(i) We have $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$.

(ii) For any $\mathbf{x} \in \mathbb{I}^n$ *we have* $\mathsf{G}(\mathbf{x}) \in \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}$.

(iii) For any $\mathbf{x} \in \mathbb{I}^n$ *there is* $g_{\mathbf{x}} \in Q(\delta_{\mathsf{F}})$ *such that* $\mathsf{G}(\mathbf{x}) = (g_{\mathbf{x}} \circ \mathsf{F})(\mathbf{x})$.

A necessary and sufficient condition for equation (2) to have a unique solution immediately follows from the assertion (ii) of Proposition 2.2.

Corollary 2.3. *Let* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *be a function satisfying (4). Then, assuming AC (not necessary if* δ_{F} *is monotonic), the associated Chisini equation (2) has a unique solution if and only if* δ_{F} *is a bijection from* $\mathbb{I}$ *onto* $\operatorname{ran}(\mathsf{F})$. *The solution is then given by* $\mathsf{G} = \delta_{\mathsf{F}}^{-1} \circ \mathsf{F}$.

3 TRIVIAL SOLUTIONS

In this section, we investigate special solutions whose construction is inspired from Proposition 2.2 (iii). These solutions are described in the following immediate result.

Proposition 3.1. *Let* $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ *be a function satisfying condition (4). Then, assuming AC (not necessary if* δ_{F} *is monotonic), for any* $g \in Q(\delta_{\mathsf{F}})$, *the function* $\mathsf{G} = g \circ \mathsf{F}$, *from* $\mathbb{I}^n$ *to* $\mathbb{I}$, *is well defined and solves Chisini's equation (2).*

Proposition 3.1 motivates the following definition. Given a function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ satisfying condition (4), we say that a function $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ is a *trivial* solution of Chisini's equation (2) if there exists $g \in Q(\delta_{\mathsf{F}})$ such that $\mathsf{G} = g \circ \mathsf{F}$.

Recall that a function $\mathsf{F}\colon \mathbb{I}^n \to \mathbb{R}$ is said to be *idempotent* if $\delta_{\mathsf{F}} = \operatorname{id}$. We say that F is *range-idempotent* if $\operatorname{ran}(\mathsf{F}) \subseteq \mathbb{I}$ and $\delta_{\mathsf{F}}|_{\operatorname{ran}(\mathsf{F})} = \operatorname{id}|_{\operatorname{ran}(\mathsf{F})}$, where the latter condition can be rewritten as $\delta_{\mathsf{F}} \circ \mathsf{F} = \mathsf{F}$. We can readily see that any trivial solution $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ of

Chisini's equation (2) is range-idempotent. Indeed, since $\mathsf{G} = g \circ \mathsf{F}$ for some $g \in Q(\delta_{\mathsf{F}})$, we simply have $\delta_{\mathsf{G}} \circ \mathsf{G} = g \circ \delta_{\mathsf{F}} \circ \mathsf{G} = g \circ \mathsf{F} = \mathsf{G}$.

An interesting feature of trivial solutions G is that, in addition of being range-idempotent, they may inherit some properties from F, such as nondecreasing monotonicity, symmetry, continuity, etc. For instance, G is nondecreasing as soon as F is either nondecreasing or nonincreasing. This property follows from the fact that if a function $f : \mathbb{I} \to \mathbb{R}$ is nondecreasing (resp. nonincreasing) then so is every $g \in Q(f)$; see [10, §4.4]. However, as the following example shows, Chisini's equation may have non-trivial solutions and the trivial solutions may be non-idempotent.

Example 3.2. The *Łukasiewicz t-norm* (see e.g. [8]) is the function $\mathsf{T}^{\mathrm{L}} : [0,1]^2 \to [0,1]$ defined as $\mathsf{T}^{\mathrm{L}}(x_1, x_2) := \mathsf{Max}(0, x_1 + x_2 - 1)$. We have $\delta_{\mathsf{T}^{\mathrm{L}}}(x) = \mathsf{Max}(0, 2x-1)$ and any $g \in Q(\delta_{\mathsf{T}^{\mathrm{L}}})$ is such that $g(x) = \frac{1}{2}(x+1)$ on $]0,1]$ and $g(0) \in [0, \frac{1}{2}]$. Thus, no function of the form $g \circ \mathsf{T}^{\mathrm{L}}$ is idempotent on $[0,1]^2$. However, the idempotent function $\mathsf{G}(x_1, x_2) = \frac{1}{2}(x_1 + x_2)$ clearly solves the Chisini equation $\mathsf{T}^{\mathrm{L}} = \delta_{\mathsf{T}^{\mathrm{L}}} \circ \mathsf{G}$.

The trivial solutions of Chisini's equation can be easily transformed into idempotent solutions. Indeed, for any $g \in Q(\delta_{\mathsf{F}})$, the function $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$, defined by

$$\mathsf{G}(\mathbf{x}) = \begin{cases} x_1, & \text{if } \mathbf{x} \in \mathrm{diag}(\mathbb{I}^n), \\ (g \circ \mathsf{F})(\mathbf{x}), & \text{otherwise,} \end{cases}$$

is an idempotent solution. However, for such solutions, some properties of the trivial solutions, such as nondecreasing monotonicity, might be lost.

This motivates the natural question whether the Chisini equation, when solvable, has nondecreasing and idempotent solutions. In the next section, we show in a constructive way that, if F is nondecreasing and satisfies condition (4), at least one such solution always exists.

4 NONDECREASING AND IDEMPOTENT SOLUTIONS

We now examine the situation when F is nondecreasing, in which case condition (4) alone ensures the solvability of Chisini's equation. Clearly δ_{F} is then nondecreasing and hence its level sets $\delta_{\mathsf{F}}^{-1}\{y\}$, $y \in \mathrm{ran}(\delta_{\mathsf{F}})$, are intervals. It follows that δ_{F} always has a nondecreasing quasi-inverse $g \in Q(\delta_{\mathsf{F}})$ (without an appeal to AC) and hence the trivial solution $\mathsf{G} = g \circ \mathsf{F}$ is also nondecreasing and even range-idempotent (see §3). However, as we observed in Example 3.2, this solution need not be idempotent.

In this section we show that, assuming condition (4), at least one nondecreasing and idempotent solution always exists and we show how to construct such a solution (see Theorem 4.2). Roughly speaking, the idea consists in constructing on each level set $\mathsf{F}^{-1}\{y\}$, $y \in \mathrm{ran}(\delta_{\mathsf{F}})$, a nondecreasing and idempotent function that assumes the value $\inf \delta_{\mathsf{F}}^{-1}\{y\}$ (resp. $\sup \delta_{\mathsf{F}}^{-1}\{y\}$) on the common edge of the level set $\mathsf{F}^{-1}\{y\}$ and the adjacent lower (resp. upper) level set.

Let $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ be a nondecreasing function satisfying (4). For any $y \in \mathrm{ran}(\delta_{\mathsf{F}})$, consider the corresponding lower and upper level sets of F, defined by

$$\begin{aligned} \mathsf{F}_<^{-1}(y) &:= \{\mathbf{x} \in \mathbb{I}^n : \mathsf{F}(\mathbf{x}) < y\}, \\ \mathsf{F}_>^{-1}(y) &:= \{\mathbf{x} \in \mathbb{I}^n : \mathsf{F}(\mathbf{x}) > y\}, \end{aligned}$$

respectively. Consider the Chebyshev distance between two points $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n$ and between a point $\mathbf{x} \in \mathbb{R}^n$ and a subset $S \subseteq \mathbb{R}^n$,

$$\begin{aligned} d_\infty(\mathbf{x}, \mathbf{x}') &:= \|\mathbf{x} - \mathbf{x}'\|_\infty, \\ d_\infty(\mathbf{x}, S) &:= \inf_{\mathbf{x}' \in S} \|\mathbf{x} - \mathbf{x}'\|_\infty, \end{aligned}$$

with the convention that $d_\infty(\mathbf{x}, \varnothing) = \infty$. Define also the following functions, from $\mathbb{I}^n$ to $[-\infty, \infty]$,

$$\begin{aligned} a_{\mathsf{F}}(\mathbf{x}) &:= \inf \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}, \\ b_{\mathsf{F}}(\mathbf{x}) &:= \sup \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x})\}, \end{aligned}$$

and

$$\begin{aligned} d_{\mathsf{F}}^{<}(\mathbf{x}) &:= d_\infty\big(\mathbf{x}, \mathsf{F}_<^{-1}(\mathsf{F}(\mathbf{x}))\big), \\ d_{\mathsf{F}}^{>}(\mathbf{x}) &:= d_\infty\big(\mathbf{x}, \mathsf{F}_>^{-1}(\mathsf{F}(\mathbf{x}))\big). \end{aligned}$$

The next lemma concerns the case when $d_{\mathsf{F}}^{<}(\mathbf{x}) = \infty$ (resp. $d_{\mathsf{F}}^{>}(\mathbf{x}) = \infty$), which means that $\mathsf{F}_<^{-1}(\mathsf{F}(\mathbf{x})) = \varnothing$ (resp. $\mathsf{F}_>^{-1}(\mathsf{F}(\mathbf{x})) = \varnothing$).

Lemma 4.1. *Let $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ be a nondecreasing function satisfying (4) and let $\mathbf{x} \in \mathbb{I}^n$. If $d_{\mathsf{F}}^{<}(\mathbf{x}) = \infty$ (resp. $d_{\mathsf{F}}^{>}(\mathbf{x}) = \infty$) then $a_{\mathsf{F}}(\mathbf{x}) = \inf \mathbb{I}$ (resp. $b_{\mathsf{F}}(\mathbf{x}) = \sup \mathbb{I}$). The converse holds if $\inf \mathbb{I} \notin \mathbb{I}$ (resp. $\sup \mathbb{I} \notin \mathbb{I}$).*

Now, consider the following subdomain of $\mathbb{I}^n$:

$$\Omega_{\mathsf{F}} := \{\mathbf{x} \in \mathbb{I}^n : d_{\mathsf{F}}^{>}(\mathbf{x}) + d_{\mathsf{F}}^{<}(\mathbf{x}) > 0\}$$

and define the function $\mathsf{U}_{\mathsf{F}} : \Omega_{\mathsf{F}} \to \mathbb{R}$ by

$$\mathsf{U}_{\mathsf{F}}(\mathbf{x}) := \frac{d_{\mathsf{F}}^{>}(\mathbf{x})\, a_{\mathsf{F}}(\mathbf{x}) + d_{\mathsf{F}}^{<}(\mathbf{x})\, b_{\mathsf{F}}(\mathbf{x})}{d_{\mathsf{F}}^{>}(\mathbf{x}) + d_{\mathsf{F}}^{<}(\mathbf{x})}.$$

By Lemma 4.1, we immediately observe that this function is well defined if and only if both $d_{\mathsf{F}}^{<}(\mathbf{x})$ and $d_{\mathsf{F}}^{>}(\mathbf{x})$ are bounded. By extension, when only $d_{\mathsf{F}}^{>}(\mathbf{x})$ is bounded, we naturally consider the limiting value

$$\begin{aligned} \mathsf{U}_{\mathsf{F}}(\mathbf{x}) &:= \lim_{a \to -\infty} \frac{d_{\mathsf{F}}^{>}(\mathbf{x})\, a + d_\infty(\mathbf{x}, a\mathbf{1})\, b_{\mathsf{F}}(\mathbf{x})}{d_{\mathsf{F}}^{>}(\mathbf{x}) + d_\infty(\mathbf{x}, a\mathbf{1})} \\ &= b_{\mathsf{F}}(\mathbf{x}) - d_{\mathsf{F}}^{>}(\mathbf{x}). \end{aligned}$$

Similarly, when only $d_{\mathsf{F}}^{\leqslant}(\mathbf{x})$ is bounded, we consider the limiting value

$$\begin{aligned}\mathsf{U}_{\mathsf{F}}(\mathbf{x}) &:= \lim_{b\to+\infty} \frac{d_\infty(\mathbf{x}, b\mathbf{1})\, a_{\mathsf{F}}(\mathbf{x}) + d_{\mathsf{F}}^{\leqslant}(\mathbf{x})\, b}{d_\infty(\mathbf{x}, b\mathbf{1}) + d_{\mathsf{F}}^{\leqslant}(\mathbf{x})} \\ &= a_{\mathsf{F}}(\mathbf{x}) + d_{\mathsf{F}}^{\leqslant}(\mathbf{x}).\end{aligned}$$

Finally, when both $d_{\mathsf{F}}^{\geqslant}(\mathbf{x})$ and $d_{\mathsf{F}}^{\leqslant}(\mathbf{x})$ are unbounded (i.e. when F is a constant function), we consider

$$\begin{aligned}\mathsf{U}_{\mathsf{F}}(\mathbf{x}) &:= \lim_{b\to+\infty} \frac{d_\infty(\mathbf{x}, b\mathbf{1})(-b) + d_\infty(\mathbf{x}, -b\mathbf{1})\, b}{d_\infty(\mathbf{x}, b\mathbf{1}) + d_\infty(\mathbf{x}, -b\mathbf{1})} \\ &= \frac{\mathsf{Min}(\mathbf{x}) + \mathsf{Max}(\mathbf{x})}{2}.\end{aligned}$$

We now define the function $\mathsf{M}_{\mathsf{F}} : \mathbb{I}^n \to \mathbb{R}$ by

$$\mathsf{M}_{\mathsf{F}}(\mathbf{x}) := \begin{cases} \mathsf{U}_{\mathsf{F}}(\mathbf{x}), & \text{if } \mathbf{x} \in \Omega_{\mathsf{F}}, \\ \dfrac{a_{\mathsf{F}}(\mathbf{x}) + b_{\mathsf{F}}(\mathbf{x})}{2}, & \text{if } \mathbf{x} \in \mathbb{I}^n \setminus \Omega_{\mathsf{F}}. \end{cases} \qquad (8)$$

Even though the function M_{F} is well defined on $\mathbb{I}^n$, there are still very special situations in which this function needs to be slightly modified.

Suppose there exists $\mathbf{x}^* \in \mathbb{I}^n$ such that

$$\begin{aligned}&a_{\mathsf{F}}(\mathbf{x}^*) \notin \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x}^*)\} \text{ and} \\ &\exists \mathbf{x} \in \mathsf{F}^{-1}\{\mathsf{F}(\mathbf{x}^*)\} \cap \Omega_{\mathsf{F}} \;:\; d_{\mathsf{F}}^{\leqslant}(\mathbf{x}) = 0,\end{aligned} \qquad (9)$$

or

$$\begin{aligned}&b_{\mathsf{F}}(\mathbf{x}^*) \notin \delta_{\mathsf{F}}^{-1}\{\mathsf{F}(\mathbf{x}^*)\} \text{ and} \\ &\exists \mathbf{x} \in \mathsf{F}^{-1}\{\mathsf{F}(\mathbf{x}^*)\} \cap \Omega_{\mathsf{F}} \;:\; d_{\mathsf{F}}^{\geqslant}(\mathbf{x}) = 0.\end{aligned} \qquad (10)$$

In either case, we replace the restriction of M_{F} to the level set $\mathsf{F}^{-1}\{\mathsf{F}(\mathbf{x}^*)\}$ by

$$\tilde{\mathsf{U}}_{\mathsf{F}}(\mathbf{x}) := \frac{\tilde{d}_{\mathsf{F}}^{\geqslant}(\mathbf{x})\, a_{\mathsf{F}}(\mathbf{x}) + \tilde{d}_{\mathsf{F}}^{\leqslant}(\mathbf{x})\, b_{\mathsf{F}}(\mathbf{x})}{\tilde{d}_{\mathsf{F}}^{\geqslant}(\mathbf{x}) + \tilde{d}_{\mathsf{F}}^{\leqslant}(\mathbf{x})}$$

(or by the corresponding limiting value as defined above), where

$$\begin{aligned}\tilde{d}_{\mathsf{F}}^{\leqslant}(\mathbf{x}) &:= d_\infty(\mathbf{x}, [\inf \mathbb{I}, a_{\mathsf{F}}(\mathbf{x}^*)]^n), \\ \tilde{d}_{\mathsf{F}}^{\geqslant}(\mathbf{x}) &:= d_\infty(\mathbf{x}, [b_{\mathsf{F}}(\mathbf{x}^*), \sup \mathbb{I}]^n).\end{aligned}$$

We then note that $\tilde{d}_{\mathsf{F}}^{\geqslant}(\mathbf{x}) + \tilde{d}_{\mathsf{F}}^{\leqslant}(\mathbf{x}) > 0$ so that the new function M_{F} is well defined on $\mathbb{I}^n$.

The next theorem essentially states that, thus defined, the function $\mathsf{M}_{\mathsf{F}} : \mathbb{I}^n \to \mathbb{R}$ is a nondecreasing and idempotent solution to Chisini's equation (2).

Theorem 4.2. *For any nondecreasing function* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *satisfying (4), we have* $\operatorname{ran}(\mathsf{M}_{\mathsf{F}}) \subseteq \mathbb{I}$ *and* $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}_{\mathsf{F}}$. *Moreover,* M_{F} *is nondecreasing and idempotent.*

Example 4.3. Consider the continuous function $\mathsf{F} : [0,1]^2 \to [0,1]$ defined by

$$\mathsf{F}(x_1, x_2) := \mathsf{Min}\big(x_1, x_2, \tfrac{1}{4} + \mathsf{Max}(0, x_1 + x_2 - 1)\big).$$

Thus defined, F is an *ordinal sum* constructed from the Łukasiewicz t-norm; see e.g. [8]. Figure 1 shows the 3D plot of F. Figure 2 shows that of M_{F}. Note that the restriction of M_{F} to the open triangle of vertices $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{4}, \frac{1}{4})$, $(\frac{3}{4}, \frac{1}{4})$ is the function U_{F}, with $a_{\mathsf{F}}(x_1, x_2) = \frac{1}{4}$, $b_{\mathsf{F}}(x_1, x_2) = \frac{1}{2}$, $d_{\mathsf{F}}^{\leqslant}(x_1, x_2) = \mathsf{Min}(x_1, x_2) - \frac{1}{4}$, and $d_{\mathsf{F}}^{\geqslant}(x_1, x_2) = \frac{1}{2} - \frac{1}{2}(x_1 + x_2)$.

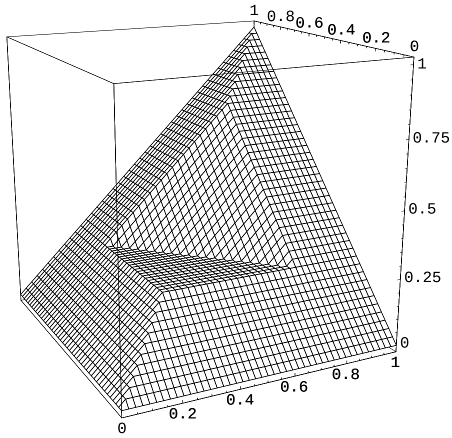

Figure 1: Function F of Example 4.3 (3D plot)

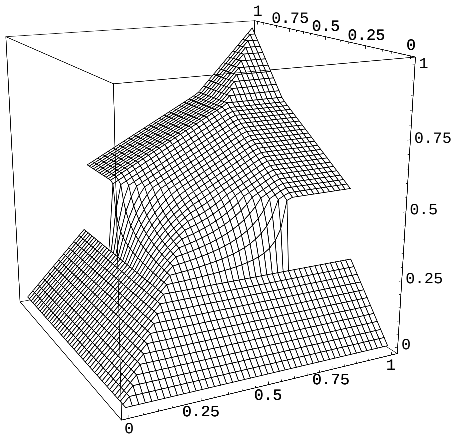

Figure 2: Function M_{F} of Example 4.3 (3D plot)

The following straightforward result shows that M_{F} can also be obtained from any increasing transformation of F.

Proposition 4.4. *Let* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *be a nondecreasing function satisfying condition (4). For any strictly increasing function* $g : \operatorname{ran}(\delta_{\mathsf{F}}) \to \mathbb{R}$*, we have* $\mathsf{M}_{\mathsf{F}} = \mathsf{M}_{g\circ\mathsf{F}}$.

Remark 4.5. Proposition 4.4 has an important application. Any $g \in Q(\delta_{\mathsf{F}})$ such that $\text{dom}(g) = \text{ran}(\delta_{\mathsf{F}})$ is strictly increasing and $g \circ \mathsf{F}$ is range-idempotent (see §3). The calculation of M_{F} can then be greatly simplified if we consider $\mathsf{M}_{g\circ\mathsf{F}}$ instead. Observe for instance that, for every $\mathbf{x} \in \mathbb{I}^n$ such that $a_{\mathsf{F}}(\mathbf{x}) = b_{\mathsf{F}}(\mathbf{x})$, we have $\mathsf{M}_{g\circ\mathsf{F}}(\mathbf{x}) = \mathsf{M}_{\mathsf{F}}(\mathbf{x}) = (g \circ \mathsf{F})(\mathbf{x})$.

We now investigate the effect of *dualization* of F over M_{F} when F is defined on a compact domain $[a, b]^n$. Recall first the concepts of *dual* and *self-dual* functions (see [6] for a recent background). The *dual* of a function $\mathsf{F} : [a, b]^n \to [a, b]$ is the function $\mathsf{F}^d : [a, b]^n \to [a, b]$, defined by $\mathsf{F}^d = \psi \circ \mathsf{F} \circ (\psi, \ldots, \psi)$, where $\psi : [a, b] \to [a, b]$ is the order-reversing involutive transformation $\psi(x) = a + b - x$ ($\psi^{-1} = \psi$). A function $\mathsf{F} : [a, b]^n \to [a, b]$ is said to be *self-dual* if $\mathsf{F}^d = \mathsf{F}$.

Proposition 4.6. *Let* $\mathsf{F} : [a, b]^n \to [a, b]$ *be a nondecreasing function satisfying condition (4). Then* $\mathsf{M}_{\mathsf{F}^d} = \mathsf{M}_{\mathsf{F}}^d$. *In particular, if* F *is self-dual then so is* M_{F}.

5 APPLICATIONS

We briefly describe three applications for these special solutions of Chisini's equation: revisiting the concept of Chisini mean, proposing and investigating generalizations of idempotency, and extending the idempotization process to nondecreasing functions whose diagonal section is not strictly increasing.

5.1 THE CONCEPTS OF MEAN AND AVERAGE REVISITED

The study of Chisini's functional equation enables us to better understand the concepts of mean and average. Already discovered and studied by the ancient Greeks, the concept of mean has given rise today to a very wide field of investigation with a huge variety of applications. For general background, see [1, 7].

The first modern definition of mean was probably due to Cauchy [3] who considered in 1821 a mean as an *internal* function, i.e. a function $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ satisfying $\mathsf{Min} \leqslant \mathsf{M} \leqslant \mathsf{Max}$. As it is natural to ask a mean to be nondecreasing, we say that a function $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ is a *mean* in $\mathbb{I}^n$ if it is nondecreasing and internal. As a consequence, every mean is idempotent. Conversely, any nondecreasing and idempotent function is internal and hence is a mean. This well-known fact follows from the immediate inequalities $\delta_{\mathsf{M}} \circ \mathsf{Min} \leqslant \mathsf{M} \leqslant \delta_{\mathsf{M}} \circ \mathsf{Max}$. Moreover, if $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ is a mean in $\mathbb{I}^n$ then, for any subinterval $\mathbb{J} \subseteq \mathbb{I}$, M is also a mean in $\mathbb{J}^n$.

The concept of mean as an average is usually ascribed to Chisini [4, p. 108], who defined in 1929 a mean associated with a function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ as a solution $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ of the equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}$. Unfortunately, as noted by de Finetti [5] in 1931, Chisini's definition is so general that it does not even imply that the "mean" (provided there exists a unique solution to Chisini's equation) satisfies the internality property. To ensure existence, uniqueness, nondecreasing monotonicity, and internality of the solution of Chisini's equation it is enough to assume that F is nondecreasing and that δ_{F} is a bijection from $\mathbb{I}$ onto $\text{ran}(\mathsf{F})$ (see Corollary 2.3). Thus, we say that a function $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ is an *average* in $\mathbb{I}^n$ if there exists a nondecreasing function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$, whose diagonal section δ_{F} is a bijection from $\mathbb{I}$ onto $\text{ran}(\mathsf{F})$, such that $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}$. In this case, we say that $\mathsf{M} = \delta_{\mathsf{F}}^{-1} \circ \mathsf{F}$ is the *average associated with* F (or the F-*level mean* [1, VI.4.1]) in $\mathbb{I}^n$.

Thus defined, the concepts of mean and average coincide. Indeed, any average is nondecreasing and idempotent and hence is a mean. Conversely, any mean is the average associated with itself.

Now, by relaxing the strict increasing monotonicity of δ_{F} into condition (4), the existence (but not the uniqueness) of solutions of the Chisini equation is still ensured (see Proposition 2.1) and we have even seen that, if F is nondecreasing, there are always means among the solutions (see Theorem 4.2). This motivates the following general definition.

Definition 5.1. A function $\mathsf{M} : \mathbb{I}^n \to \mathbb{I}$ is an *average* (or a *Chisini mean* or a *level surface mean*) in $\mathbb{I}^n$ if it is a nondecreasing and idempotent solution of the equation $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{M}$ for some nondecreasing function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$. In this case, we say that M is an *average associated with* F (or an F-*level mean*) in $\mathbb{I}^n$.

5.2 QUASI-IDEMPOTENCY AND RANGE-IDEMPOTENCY

Let $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ be a function satisfying (4). We have seen in §3 that, assuming AC (not necessary if δ_{F} is monotonic), there exists an idempotent function $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ such that $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$. This result motivates the following definition. We say that a function $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ satisfying condition (4) is *quasi-idempotent* if δ_{F} is monotonic. We say that it is *idempotizable* if δ_{F} is strictly monotonic.

Proposition 5.2. *Let* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *be a function. Then the following assertions are equivalent:*

(i) F *is quasi-idempotent.*

(ii) δ_{F} *is monotonic and there is a function* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *such that* $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$.

(iii) δ_{F} *is monotonic and there is an idempotent function* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *such that* $\mathsf{F} = \delta_{\mathsf{F}} \circ \mathsf{G}$.

(iv) δ_F *is monotonic and there are functions* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *and* $f : \mathrm{ran}(\mathsf{G}) \to \mathbb{R}$ *such that* $\mathrm{ran}(\delta_\mathsf{G}) = \mathrm{ran}(\mathsf{G})$ *and* $\mathsf{F} = f \circ \mathsf{G}$.

(v) δ_F *is monotonic and there are functions* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ *and* $f : \mathrm{ran}(\mathsf{G}) \to \mathbb{R}$ *such that* G *is idempotent and* $\mathsf{F} = f \circ \mathsf{G}$. *In this case,* $f = \delta_\mathsf{F}$.

Corollary 5.3. *A function* $\mathsf{F} : \mathbb{I}^n \to \mathbb{R}$ *is idempotizable if and only if* δ_F *is a strictly monotonic bijection from* $\mathbb{I}$ *onto* $\mathrm{ran}(\mathsf{F})$ *and there is a unique idempotent function* $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$, *namely* $\mathsf{G} = \delta_\mathsf{F}^{-1} \circ \mathsf{F}$, *such that* $\mathsf{F} = \delta_\mathsf{F} \circ \mathsf{G}$.

Recall that a function $\mathsf{F} : \mathbb{I}^n \to \mathbb{I}$ is *range-idempotent* if $\delta_\mathsf{F} \circ \mathsf{F} = \mathsf{F}$ (see §3). In this case, $f := \delta_\mathsf{F}$ necessarily satisfies the functional equation $f \circ f = f$, called the *idempotency equation* [9, §11.9E]. We can easily see that a function $f : \mathbb{I} \to \mathbb{R}$ solves this equation if and only if $f|_{\mathrm{ran}(f)} = \mathrm{id}|_{\mathrm{ran}(f)}$; see also [10, §2.1]. The next two results characterize the family of nondecreasing solutions and the subfamily of nondecreasing and continuous solutions of the idempotency equation.

Proposition 5.4. *A nondecreasing function* $f : \mathbb{I} \to \mathbb{R}$ *satisfies* $f \circ f = f$ *if and only if the following conditions hold:*

(i) *If* f *is strictly increasing on* $\mathbb{J} \subseteq \mathbb{I}$ *(*$\mathbb{J}$ *not a singleton) then* $f|_\mathbb{J} = \mathrm{id}|_\mathbb{J}$.

(ii) *If* $f = c_\mathbb{J}$ *is constant on* $\mathbb{J} \subseteq \mathbb{I}$ *then* $c_\mathbb{J} \in f^{-1}\{c_\mathbb{J}\}$.

Corollary 5.5. *A nondecreasing and continuous function* $f : \mathbb{I} \to \mathbb{R}$ *satisfies* $f \circ f = f$ *if and only if there are* $a, b \in \mathbb{I} \cup \{-\infty, \infty\}$, $a \leqslant b$, *with* $a < b$ *if* $a \notin \mathbb{I}$ *or* $b \notin \mathbb{I}$, *such that* $f(x) = \mathsf{Max}(a, \mathsf{Min}(x, b))$.

It is an immediate fact that a range-idempotent function $\mathsf{F} : \mathbb{I}^n \to \mathbb{I}$ with monotonic δ_F is quasi-idempotent. Therefore, by combining Proposition 5.2 and Corollary 5.5, we see that a function $\mathsf{F} : \mathbb{I}^n \to \mathbb{I}$ is range-idempotent with nondecreasing and continuous δ_F if and only if there are $a, b \in \mathbb{I} \cup \{-\infty, \infty\}$, $a \leqslant b$, with $a < b$ if $a \notin \mathbb{I}$ or $b \notin \mathbb{I}$, and an idempotent function $\mathsf{G} : \mathbb{I}^n \to \mathbb{I}$ such that $\mathsf{F}(\mathbf{x}) = \mathsf{Max}(a, \mathsf{Min}(\mathsf{G}(\mathbf{x}), b))$.

5.3 IDEMPOTIZATION PROCESS

Corollary 5.3 makes it possible to define an idempotent function G from any idempotizable function F (see §5.2), simply by writing $\mathsf{G} = \delta_\mathsf{F}^{-1} \circ \mathsf{F}$, hence the name "idempotizable". This generation process is known as the *idempotization process*; see [2, §3.1]. Of course, if F is nondecreasing then so is G and hence G is a mean, namely the F-level mean M_F (see §5.1). For instance, from the *Einstein sum*, defined on $]-1, 1[^2$ by $\mathsf{F}(x_1, x_2) = \varphi^{-1}(\varphi(x_1) + \varphi(x_2))$, where $\varphi = \mathrm{arctanh}$, that is $\mathsf{F}(x_1, x_2) = \frac{x_1 + x_2}{1 + x_1 x_2}$, we generate the quasi-arithmetic mean $\mathsf{M}_\mathsf{F}(x_1, x_2) = \varphi^{-1}(\frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2))$. Clearly, we can extend this process to any nondecreasing and quasi-idempotent function F simply by considering any F-level mean (e.g., M_F). We call this process the *generalized idempotization process*.

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