On the cardinality index of fuzzy measures and the signatures of coherent systems

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Boolean and pseudo-Boolean functions

Boolean functions:

$$\boxed{f\colon \{0,1\}^n \to \{0,1\}}$$

Pseudo-Boolean functions:

$$\boxed{f\colon \{0,1\}^n \to \mathbb{R}}$$

Set functions: $[n] = \{1, \ldots, n\}$

$$\boxed{\begin{array}{c} f\colon 2^{[n]} \to \{0,1\} \\ f\colon 2^{[n]} \to \mathbb{R} \end{array}}$$

Set functions

A *discrete fuzzy measure* on the finite set $X = \{1, \ldots, n\}$ is a nondecreasing set function $\mu\colon 2^X \to [0, 1]$ satisfying the conditions $\mu(\varnothing) = 0$ and $\mu(X) = 1$

Interpretation:
For any subset $S \subseteq X$, the number $\mu(S)$ can be interpreted as the certitude that we have that a variable will take on its value in the set $S \subseteq X$

Set functions

A *cooperative game* on a finite set of players $N = \{1, \dots, n\}$ is a set function $v: 2^N \to \mathbb{R}$ which assigns to each coalition $S \subseteq N$ of players a real number $v(S)$ which represents the *worth* of S

The game is said to be *simple* if v takes on its values in $\{0, 1\}$

The *structure of a semicoherent system* made up of n components is a set function $\phi: 2^{[n]} \to \{0, 1\}$ ...

Power indexes

Let $v\colon 2^N \to \mathbb{R}$ be a game on a set $N = \{1, \ldots, n\}$ of players
Let $j \in N$ be a player

Banzhaf power index (Banzhaf, 1965)

$$\psi_{\mathrm{B}}(v,j) \;=\; \frac{1}{2^{n-1}} \sum_{S \subseteq N \smallsetminus \{j\}} \big(v(S \cup \{j\}) - v(S)\big)$$

Shapley power index (Shapley, 1953)

$$\psi_{\mathrm{Sh}}(v,j) \;=\; \sum_{S \subseteq N \smallsetminus \{j\}} \frac{1}{n\binom{n-1}{|S|}} \big(v(S \cup \{j\}) - v(S)\big)$$

Cardinality index

Let $\mu\colon 2^X \to \mathbb{R}$ be a fuzzy measure on a set $X = \{1, \ldots, n\}$ of values
Let $k \in \{0, \ldots, n-1\}$

Cardinality index (Yager, 2002)

$$C_k = \frac{1}{(n-k)\binom{n}{k}} \sum_{\substack{S \subseteq X \\ |S|=k}} \sum_{x \in X \setminus S} \big(\mu(S \cup \{x\}) - \mu(S)\big)$$

Interpretation:
C_k is the average gain in certitude that we obtain by adding an arbitrary element to an arbitrary k-element subset

Cardinality index

Alternative formulation (game theory notation)

$$C_k = \frac{1}{(n-k)\binom{n}{k}} \sum_{|S|=k} \sum_{j \in N \setminus S} \big(v(S \cup \{j\}) - v(S)\big)$$

$$\boxed{C_k = \frac{1}{\binom{n}{k+1}} \sum_{|S|=k+1} v(S) - \frac{1}{\binom{n}{k}} \sum_{|S|=k} v(S)}$$

Interpretation:
C_k is the average gain that we obtain by adding an arbitrary player to an arbitrary k-player coalition

...when compared with the Banzhaf power index...

$$\psi_{\mathrm{B}}(v,j) = \frac{1}{2^{n-1}} \sum_{S \ni j} v(S) - \frac{1}{2^{n-1}} \sum_{S \not\ni j} v(S)$$

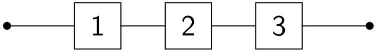
Introduction to network reliability

System

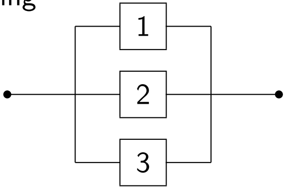
Definition. A *system* consists of several interconnected units

Assumptions:

1. The system and the units are of the crisply *on/off* kind
2. A serially connected segment of units is functioning if and only if every single unit is functioning



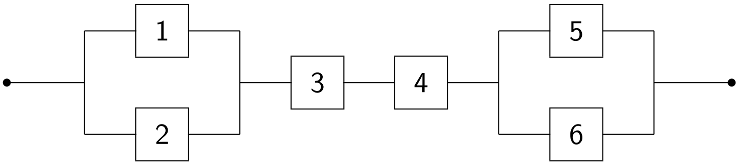
3. A system of parallel units is functioning if and only at least one unit is functioning



System

Example. Home video system

1. Blu-ray player
2. DVD player
3. LCD monitor
4. Amplifier
5. Speaker A
6. Speaker B



Structure function

Definition.
The *state of a component* $j \in [n] = \{1, \ldots, n\}$ can be represented by a Boolean variable

$$x_j = \begin{cases} 1 & \text{if component } j \text{ is functioning} \\ 0 & \text{if component } j \text{ is in a failed state} \end{cases}$$

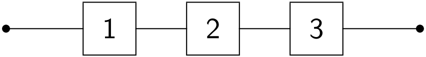
The *state of the system* is described from the component states through a Boolean function $\phi : \{0,1\}^n \to \{0,1\}$

$$\phi(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is in a failed state} \end{cases}$$

This function is called the *structure function* of the system

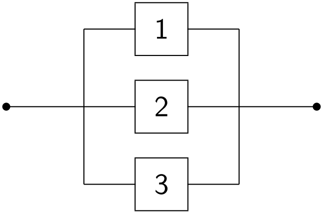
Structure function

Series structure



$$\phi(x_1, x_2, x_3) \;=\; x_1\; x_2\; x_3 \;=\; \prod_{j=1}^{3} x_j$$

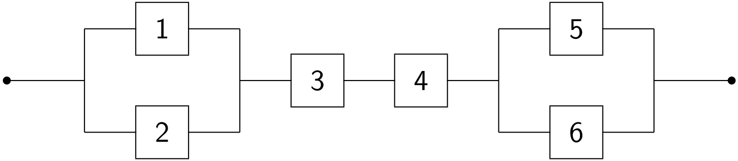
Parallel structure



$$\phi(x_1, x_2, x_3) \;=\; 1 - (1 - x_1)(1 - x_2)(1 - x_3) \;=\; \coprod_{j=1}^{3} x_j$$

Structure function

Home video system



$$\phi(x_1, \dots, x_6) \;=\; (x_1 \amalg x_2)\; x_3\; x_4\; (x_5 \amalg x_6)$$

Lifetimes

Notation

1. T_j = *lifetime* of component $j \in [n]$
2. T = *lifetime* of the system

We assume that the component lifetimes $T_1, \ldots, T_n$ are continuous and i.i.d.

Barlow-Proschan importance index

Importance index (Barlow-Proschan, 1975)

$$I_{\mathrm{BP}}^{(j)} = \Pr(T = T_j) \qquad j \in [n]$$

$I_{\mathrm{BP}}^{(j)}$ is a measure of importance of component j

In the i.i.d. case:

$$I_{\mathrm{BP}}^{(j)} = \sum_{S \subseteq [n] \setminus \{j\}} \frac{1}{n\binom{n-1}{|S|}} \big(\phi(S \cup \{j\}) - \phi(S)\big) = \psi_{\mathrm{Sh}}(\phi, j)$$

System signature

Let $T_{(1)} \leqslant \cdots \leqslant T_{(n)}$ be the order statistics obtained from the variables $T_1, \ldots, T_n$

System signature (Samaniego, 1985)

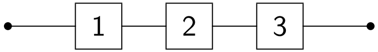
$$s_k = \Pr(T = T_{(k)}) \qquad k = 1, \ldots, n$$

In the i.i.d. case (Boland, 2001)

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{\substack{S \subseteq [n] \\ |S| = n-k+1}} \phi(S) - \frac{1}{\binom{n}{n-k}} \sum_{\substack{S \subseteq [n] \\ |S| = n-k}} \phi(S) = C_{n-k}$$

B-P importance index and system signature

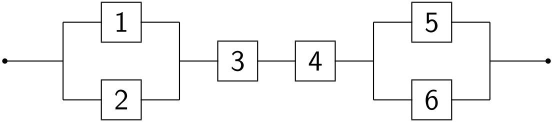
Series structure



$$I_{\mathrm{BP}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad s = (1, 0, 0)$$

B-P importance index and system signature

Home video system



$$I_{\mathrm{BP}} = \left(\frac{1}{15}, \frac{1}{15}, \frac{11}{30}, \frac{11}{30}, \frac{1}{15}, \frac{1}{15}\right)$$

$$s = \left(\frac{1}{3}, \frac{2}{5}, \frac{4}{15}, 0, 0, 0\right)$$

Manual computation of the cardinality index

Manual computation

Any set function $f\colon 2^{[n]} \to \mathbb{R}$ can be represented as a multilinear polynomial

$$f(x_1, \ldots, x_n) \;=\; \sum_{S \subseteq [n]} f(S) \prod_{j \in S} x_j \prod_{j \in [n] \setminus S} (1 - x_j)$$

$$f(x_1, \ldots, x_n) \;=\; \sum_{S \subseteq [n]} c(S) \prod_{j \in S} x_j$$

Manual computation

The *multilinear extension* of f (Owen, 1972) is the function $\overline{f}\colon [0,1]^n \to \mathbb{R}$ defined by

$$\overline{f}(x_1,\ldots,x_n) = \sum_{S\subseteq[n]} f(S) \prod_{j\in S} x_j \prod_{j\in[n]\setminus S} (1-x_j)$$

$$\overline{f}(x_1,\ldots,x_n) = \sum_{S\subseteq[n]} c(S) \prod_{j\in S} x_j$$

Example:
$\max(x_1, x_2) = x_1 \amalg x_2 = 1-(1-x_1)(1-x_2) = x_1 + x_2 - x_1 x_2$

Manual computation: Banzhaf and Shapley power indexes

$$\psi_{\mathrm{B}}(f,j) \;=\; \Big(\frac{\partial}{\partial x_j}\,\overline{f}\Big)\big(\tfrac{1}{2},\ldots,\tfrac{1}{2}\big)$$

$$\psi_{\mathrm{Sh}}(f,j) \;=\; \int_0^1 \Big(\frac{\partial}{\partial x_j}\,\overline{f}\Big)\big(x,\ldots,x\big)\,dx$$

(Owen, 1972)

Manual computation: Cardinality index

With any n-degree polynomial $p\colon \mathbb{R} \to \mathbb{R}$ we associate the *reflected* polynomial $R^n p\colon \mathbb{R} \to \mathbb{R}$ defined by

$$(R^n p)(x) = x^n \, p\left(\tfrac{1}{x}\right)$$

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad \Rightarrow \quad (R^n p)(x) = a_n + a_{n-1} x + \cdots + a_0 x^n$$

(M. and Mathonet, 2013)

Setting $p(x) = \frac{d}{dx}\big(\overline{f}(x,\ldots,x)\big)$, we have

$$(R^{n-1} p)(x+1) = \sum_{k=1}^{n} s_k \binom{n}{k} k \, x^{k-1}$$

Manual computation: Cardinality index

Example. Home video system

$$\phi(x_1, \ldots, x_6) = (x_1 \amalg x_2)\ x_3\ x_4\ (x_5 \amalg x_6)$$

$$\begin{aligned}\overline{\phi}(x_1, \ldots, x_6) = {} & x_1x_3x_4x_5 + x_2x_3x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_4x_6 \\ & -x_1x_2x_3x_4x_5 - x_1x_2x_3x_4x_6 - x_1x_3x_4x_5x_6 - x_2x_3x_4x_5x_6 \\ & +x_1x_2x_3x_4x_5x_6\end{aligned}$$

$$\overline{\phi}(x, \ldots, x) = 4x^4 - 4x^5 + x^6$$

Manual computation: Cardinality index

$$\overline{\phi}(x,\ldots,x) \;=\; 4x^4 - 4x^5 + x^6$$

$$p(x) \;=\; \tfrac{d}{dx}\big(\overline{\phi}(x,\ldots,x)\big) \;=\; 16x^3 - 20x^4 + 6x^5$$

$$(R^5 p)(x) \;=\; 6 - 20x + 16x^2$$

$$(R^5 p)(x+1) \;=\; 2+12x+16x^2 \;=\; s_1\binom{6}{1} + s_2\binom{6}{2}\,2\,x + s_2\binom{6}{3}\,3\,x^2 + \cdots$$

$$\Rightarrow \qquad s \;=\; \left(\tfrac{1}{3}, \tfrac{2}{5}, \tfrac{4}{15}, 0, 0, 0\right)$$

$$\Rightarrow \qquad C \;=\; \left(0, 0, 0, \tfrac{4}{15}, \tfrac{2}{5}, \tfrac{1}{3}\right)$$

Least squares approximation problems

Least squares approximation problems

Denote by V the set of games g on N of the form

$$g(\mathbf{x}) \;=\; c_0 + \sum_{j\in N} c_j\, x_j\,, \qquad c_0, c_1, \ldots, c_n \in \mathbb{R}$$

Approximation problem (Hammer and Holzman, 1992)
Given a game f on N, the *best first-degree approximation* of f is the game f^* on N that minimizes the square distance

$$\|f-g\|^2 \;=\; \sum_{\mathbf{x}\in\{0,1\}^n} \big(f(\mathbf{x})-g(\mathbf{x})\big)^2$$

from among all games $g \in V$.

We have

$$\boxed{c_j^* \;=\; \psi_{\mathrm{B}}(f,j) \qquad j\in N}$$

Least squares approximation problems

Denote by V_c the set of games g on N of the form

$$g(\mathbf{x}) = \sum_{j \in N} c_j x_j \quad \text{such that} \quad g(1,\ldots,1) = f(1,\ldots,1)$$

Approximation problem (Charnes et al., 1988)
For a given game f on N, the *best c-approximation* of f is the unique game f^* on N that minimizes the square distance

$$\|f - g\|_c^2 = \sum_{T \subseteq N} \frac{1}{\binom{n-2}{|T|-1}} \big(f(T) - g(T)\big)^2$$

from among all games $g \in V_c$.

We have

$$\boxed{c_j^* = \psi_{\mathrm{Sh}}(f,j) \qquad j \in N}$$

Least squares approximation problems

Denote by V_s the set of *symmetric* games g on N, that is, of the form

$$g(\mathbf{x}) \;=\; c_0 + \sum_{k=1}^{n} c_k\, x_{(k)}\,, \qquad c_0 = 0$$

Approximation problem (M. and Mathonet, 2012)
For a given game f on N, the *best symmetric approximation* of f is the unique game f^* on N that minimizes the square distance

$$\|f - g\|_s^2 \;=\; \sum_{T\subseteq N} \frac{1}{\binom{n}{|T|}}\,\big(f(T) - g(T)\big)^2$$

from among all games $g \in V_s$.

We have

$$c_k^* \;=\; s_k \;=\; C_{n-k} \qquad k = 1, \ldots, n$$

Thank you for your attention!

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