On Choquet and Sugeno Integrals as Aggregation Functions

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Revised version

Abstract

This paper presents a synthesis on the essential mathematical properties of Choquet and Sugeno integrals viewed as aggregation functions. Some axiomatic characterizations are presented. Subfamilies of the class of those integrals are also investigated.

1 Introduction

Aggregation is a process that is used in many technologies. In the aggregation process we associate with a collection of values, called the arguments, a single value called the aggregated value. In decision making, values to be aggregated are typically preference or satisfaction degrees, see e.g. [6].

In this paper, we present some axiomatic characterizations on the well-known Choquet and Sugeno integrals. These integrals, which can be regarded as aggregation functions, have been used in many applications, see [11, 12, 15] for more details.

Let us introduce the concept of aggregation function in a formal way. We make a distinction between aggregation functions having one definite number of arguments and aggregation operators (or aggregators) defined for all number of arguments.

Let E be a non-empty real interval, finite or infinite, representing the definition set of the values to be aggregated. Let $\mathbb{N}_0$ denote the set of strictly positive integers.

Definition 1.1 *An* aggregation function *is a function* $M^{(n)} : E^n \to \mathbb{R}$*, where* $n \in \mathbb{N}_0$.

The integer n represents the number of values to be aggregated. When no confusion can arise, the aggregation functions will be written M instead of $M^{(n)}$.

Definition 1.2 *An* aggregation operator *is a sequence* $M = (M^{(n)})_{n\in\mathbb{N}_0}$ *whose the nth element is an aggregation function* $M^{(n)} : E^n \to \mathbb{R}$.

Of course, an aggregation operator can be viewed as a mapping $M : \bigcup_{n\in\mathbb{N}_0} E^n \to \mathbb{R}$. For every $n \in \mathbb{N}_0$, we then have $M(x) = M^{(n)}(x)$ for all $x \in E^n$.

$A_n(E, \mathbb{R})$ will denote the set of all aggregation functions from E^n to $\mathbb{R}$. Likewise, $A(E, \mathbb{R})$ will denote the set of all aggregation operators whose the nth element is in $A_n(E, \mathbb{R})$.

In order to avoid heavy notation, we introduce the following terminology. It will be used all along this paper.

- We will use the notation N for the set $\{1, \dots, n\}$. In a decision making problem, elements of N generally represent criteria, attributes or voters.
- Π_n denotes the set of permutations of N.
- Given a vector $(x_1, \dots, x_n) \in E^n$, let $(\cdot)$ be the permutation on N which arranges the elements of this vector by increasing values: that is, $x_{(1)} \leq \dots \leq x_{(n)}$.
- For any permutation $\pi \in \Pi_n$ and any vector $x \in E^n$, $[x_1, \dots, x_n]_\pi$ represents the vector $(x_{\pi(1)}, \dots, x_{\pi(n)})$.
- For any subset $S \subseteq N$, e_S is the *characteristic vector* of S, i.e. the vector of $\{0,1\}^n$ whose ith component is 1 if and only if $i \in S$. Geometrically, the characteristic vectors are the 2^n vertices of the hypercube $[0,1]^n$.
- $\wedge, \vee$ denote respectively the minimum and maximum operations.
- For all subsets $K, T \subseteq N$, the notation $K \subsetneq T$ means $K \subset T$ and $K \neq T$.

In order to define the Choquet and Sugeno integrals, we use the concept of *fuzzy measure*.

Definition 1.3 *A (discrete) fuzzy measure on N is a set function $\mu : 2^N \to [0,1]$ satisfying the following conditions:*
i) $\mu(\emptyset) = 0, \mu(N) = 1,$
ii) $S \subseteq T \Rightarrow \mu(S) \leq \mu(T).$

$\mu(S)$ can be viewed as the weight of importance of the set of elements S. Throughout this paper, we will often write μ_S instead of $\mu(S)$.

Definition 1.4 *A pseudo-Boolean function is a function $f : \{0,1\}^n \to \mathbb{R}$.*

Any fuzzy measure $\mu : 2^N \to \mathbb{R}$ can be assimilated unambigously with an increasing pseudo-Boolean function fulfilling the boundary conditions $f(0, \dots, 0) = 0$ and $f(1, \dots, 1) = 1$. The correspondance is straightforward: we have

$$f(x) = \sum_{T \subseteq N} \mu(T) \prod_{i \in T} x_i \prod_{i \notin T} (1 - x_i), \quad x \in \{0,1\}^n,$$

and $\mu(S) = f(e_S)$ for all $S \subseteq N$. We shall henceforth make this identification.

Hammer and Rudeanu [16] showed that any pseudo-Boolean function has a unique expression as a multilinear polynomial in n variables:

$$f(x) = \sum_{T \subseteq N} a(T) \prod_{i \in T} x_i, \quad x \in \{0,1\}^n,$$

with $a(T) \in \mathbb{R}$.

In combinatorics, a viewed as a set function on N is called the *Möbius transform* of μ (see e.g. Rota [25]), which is given by

$$a(S) = \sum_{T\subseteq S} (-1)^{|S|-|T|}\mu(T), \quad S \subseteq N.$$

The inverse transformation is then given by:

$$\mu(S) = \sum_{T\subseteq S} a(T), \quad S \subseteq N.$$

We now introduce the concept of discrete Choquet and Sugeno integrals, viewed as aggregation functions. For this reason, we will adopt a connective-like notation instead of the usual integral form, and the integrand will be a set of n values $x_1, \ldots, x_n$. For theorical developments, see [14, 22, 28, 29].

Definition 1.5 *Assume that $E \supseteq [0,1]$. Let $(x_1, \ldots, x_n) \in E^n$, and μ a fuzzy measure on N. The* (discrete) Choquet integral of $(x_1, \ldots, x_n)$ with respect to μ *is defined by*

$$\mathcal{C}_\mu(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_{(i)}[\mu_{\{(i),\ldots,(n)\}} - \mu_{\{(i+1),\ldots,(n)\}}].$$

Definition 1.6 *Let $(x_1, \ldots, x_n) \in [0,1]^n$, and μ a fuzzy measure on N. The* (discrete) Sugeno integral of $(x_1, \ldots, x_n)$ with respect to μ *is defined by*

$$\mathcal{S}_\mu(x_1, \ldots, x_n) := \bigvee_{i=1}^{n} [x_{(i)} \wedge \mu_{\{(i),\ldots,(n)\}}].$$

Of course, given a fuzzy measure μ, the Choquet and Sugeno integrals can be regarded as aggregation functions defined on E^n and $[0,1]^n$, respectively.

The paper is organized as follows. In Section 2, we list a number of mathematical properties for aggregation. In Sections 3 and 4, we investigate the Choquet and Sugeno integrals and we present some axiomatic characterizations of these families. Section 5 is devoted to the intersection of the two families, namely the class of Boolean max-min functions. As particular cases, the order statistics are also presented.

2 Aggregation properties

If we want to obtain a reasonable or satisfactory aggregation, any aggregation function should not be used. In order to eliminate the "undesirable" functions, we can adopt an axiomatic approach and impose that these functions fulfil some selected properties. Such properties can be dictated by the nature of the values to be aggregated.

The properties we consider in this paper are the following.

- $M \in A_n(E, \mathbb{R})$ is a symmetric function (Sy) if, for all $\pi \in \Pi_n$ and all $x \in E^n$, we have
$$M(x_1, \ldots, x_n) = M(x_{\pi(1)}, \ldots, x_{\pi(n)}).$$

The symmetry property essentially implies that the indexing (ordering) of the arguments does not matter. This is required when combining criteria of equal importance or anonymous expert's opinions; indeed, a symmetric function is independent of the labels.

- $M \in A_n(E, \mathbb{R})$ is a continuous function (Co) if it is continuous in the usual sense.

  A continuous aggregation function does not present any chaotic reaction to a small change of the arguments.

- $M \in A_n(E, \mathbb{R})$ is increasing (in each argument) (In) if, for all $x, x' \in E^n$, we have

$$x_i \leq x'_i \quad \forall i \in N \quad \Rightarrow \quad M(x) \leq M(x').$$

  An increasing aggregation function presents a non-negative response to any increase of the arguments. In other terms, increasing a partial value cannot decrease the result.

- $M \in A_n(E, \mathbb{R})$ is unanimously increasing (UIn) if, for all $x, x' \in E^n$, we have

  i) $x_i \leq x'_i \quad \forall i \in N \quad \Rightarrow \quad M(x) \leq M(x')$
  ii) $x_i < x'_i \quad \forall i \in N \quad \Rightarrow \quad M(x) < M(x')$.

  A unanimously increasing function is increasing and presents a positive response whenever all the arguments strictly increase.

- $M \in A_n(E, \mathbb{R})$ is idempotent (Id) if, for all $x \in E$, we have

$$M(x, \ldots, x) = x.$$

- $M \in A_n([a, b], \mathbb{R})$ is weakly idempotent (WId) if

$$M(a, \ldots, a) = a \quad \text{and} \quad M(b, \ldots, b) = b.$$

- $M \in A_n(E, \mathbb{R})$ is stable for the admissible similarity transformations (SSi) if

$$M(r\,x_1, \ldots, r\,x_n) = r\,M(x_1, \ldots, x_n)$$

  for all $x \in E^n$ and all $r > 0$ such that $r\,x_i \in E$ for all $i \in N$.

- $M \in A_n(E, \mathbb{R})$ is stable for the admissible positive linear transformations (SPL) if

$$M(r\,x_1 + s, \ldots, r\,x_n + s) = r\,M(x_1, \ldots, x_n) + s$$

  for all $x \in E^n$ and all $r > 0, s \in \mathbb{R}$ such that $r\,x_i + s \in E$ for all $i \in N$.

  The choice of the interval $[0, 1]$ is not restrictive if we consider that scores are defined up to a positive linear transformation, as it is the case for example in multiattribute utility theory.

- $M \in A_n([0, 1], \mathbb{R})$ is stable for the standard negation (SSN) if

$$M(1 - x_1, \ldots, 1 - x_n) = 1 - M(x_1, \ldots, x_n), \quad x \in [0, 1]^n.$$

  The (SSN) property means that a reversal of the scale has no effect on the evaluation. For a two-place function M, it expresses self-duality of M (compare with De Morgan laws in fuzzy sets theory, see e.g. [6]).

- $M \in A_n(E, \mathbb{R})$ is comparison meaningful (CM) if, for all $\phi \in \Phi(E)$ and all $x, x' \in E^n$, we have

  i) $M(x) = M(x') \Rightarrow M(\phi(x_1), \ldots, \phi(x_n)) = M(\phi(x'_1), \ldots, \phi(x'_n))$,

  ii) $M(x) < M(x') \Rightarrow M(\phi(x_1), \ldots, \phi(x_n)) < M(\phi(x'_1), \ldots, \phi(x'_n))$,

  where $\Phi(E)$ denotes the automorphism group of E, that is the group of all strictly increasing bijections $\phi : E \to E$.

  Comparison meaningful functions are suitable to aggregate values defined according to an ordinal scale.

- $M \in A_n(E, \mathbb{R})$ is stable for minimum with a constant vector (SMin) if

  $$M(x_1 \wedge r, \ldots, x_n \wedge r) = M(x_1, \ldots, x_n) \wedge r$$

  for all $x \in E^n$ and all $r \in E$.

- $M \in A_n(E, \mathbb{R})$ is stable for maximum with a constant vector (SMax) if

  $$M(x_1 \vee r, \ldots, x_n \vee r) = M(x_1, \ldots, x_n) \vee r$$

  for all $x \in E^n$ and all $r \in E$.

  (SMin) and (SMax) are stability properties written in a functional equation form. They were introduced by Fodor and Roubens [7] and are visibly related to an algebra which uses min and max operations instead of classical sum and product operations.

- $M \in A_n([0,1], \mathbb{R})$ is stable for minimum between Boolean and constant vectors (SMinB) if

  $$M(r\, e_T) \in \{M(e_T), r\}$$

  for all $T \subseteq N$ and all $r \in [0,1]$.

- $M \in A_n([0,1], \mathbb{R})$ is stable for maximum between Boolean and constant vectors (SMaxB) if

  $$M(e_T + r\, e_{N \setminus T}) \in \{M(e_T), r\}$$

  for all $T \subseteq N$ and all $r \in [0,1]$.

- $M \in A_n(E, \mathbb{R})$ is additive (Add) if, for all $x, x' \in E$, we have

  $$M(x_1 + x'_1, \ldots, x_n + x'_n) = M(x_1, \ldots, x_n) + M(x'_1, \ldots, x'_n).$$

- $M \in A_n(E, \mathbb{R})$ is minitive (Min) if, for all $x, x' \in E$, we have

  $$M(x_1 \wedge x'_1, \ldots, x_n \wedge x'_n) = M(x_1, \ldots, x_n) \wedge M(x'_1, \ldots, x'_n).$$

- $M \in A_n(E, \mathbb{R})$ is maxitive (Max) if, for all $x, x' \in E$, we have

  $$M(x_1 \vee x'_1, \ldots, x_n \vee x'_n) = M(x_1, \ldots, x_n) \vee M(x'_1, \ldots, x'_n).$$

- $M \in A_n(E, \mathbb{R})$ is comonotonic additive (CoAdd) if

$$M(x_1 + x'_1, \dots, x_n + x'_n) = M(x_1, \dots, x_n) + M(x'_1, \dots, x'_n)$$

for any two comonotonic vectors $x, x' \in E$, where two vectors $x, x' \in E^n$ are said to be *comonotonic* if there exists a permutation $\pi \in \Pi_n$ such that

$$x_{\pi(1)} \leq \cdots \leq x_{\pi(n)} \quad \text{and} \quad x'_{\pi(1)} \leq \cdots \leq x'_{\pi(n)}.$$

- $M \in A_n(E, \mathbb{R})$ is comonotonic minitive (CoMin) if

$$M(x_1 \wedge x'_1, \dots, x_n \wedge x'_n) = M(x_1, \dots, x_n) \wedge M(x'_1, \dots, x'_n)$$

for any two comonotonic vectors $x, x' \in E$.

- $M \in A_n(E, \mathbb{R})$ is comonotonic maxitive (CoMax) if

$$M(x_1 \vee x'_1, \dots, x_n \vee x'_n) = M(x_1, \dots, x_n) \vee M(x'_1, \dots, x'_n)$$

for any two comonotonic vectors $x, x' \in E$.

- $M \in A_n(E, E)$ is bisymmetric (B) if

$$\begin{aligned} & M(M(x_{11}, \dots, x_{1n}), \dots, M(x_{n1}, \dots, x_{nn})) \\ = \; & M(M(x_{11}, \dots, x_{n1}), \dots, M(x_{1n}, \dots, x_{nn})) \end{aligned}$$

for all square matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in E^{n \times n}.$$

- $M \in A(E, E)$ fulfils the general bisymmetry property (GB) if $M(x) = x$ for all $x \in E$, and

$$\begin{aligned} & M(M(x_{11}, \dots, x_{1n}), \dots, M(x_{p1}, \dots, x_{pn})) \\ = \; & M(M(x_{11}, \dots, x_{p1}), \dots, M(x_{1n}, \dots, x_{pn})) \end{aligned}$$

for all matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pn} \end{pmatrix} \in E^{p \times n},$$

where $n, p \in \mathbb{N}_0$.

Justification of the general bisymmetry: Consider n judges giving a score to each of p candidates. These scores, assumed to be defined on the same scale, can be put in a $p \times n$ matrix as follows:

$$\begin{array}{c} \\ C_1 \\ \vdots \\ C_p \end{array} \begin{array}{c} \begin{array}{ccc} J_1 & \cdots & J_n \end{array} \\ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pn} \end{pmatrix} \end{array}$$

Suppose now that we want to aggregate all the scores in the matrix in order to obtain a global score of the p candidates. A reasonable way to proceed could be the following: aggregate the scores of each candidate (aggregation on the rows of the matrix), and then aggregate these global values. A dual way to proceed would be: aggregate the scores given by each judge (aggregation on the columns of the matrix), and then aggregate these values.

The general bisymmetry property for an aggregation operator means that these two ways to aggregate must lead to the same global score.

- $M \in A_n(E, E)$ is bisymmetric for orderable matrices (BOM) if

$$\begin{aligned} & M([M([x_{11}, \ldots, x_{1n}]_{\pi'}), \ldots, M([x_{n1}, \ldots, x_{nn}]_{\pi'})]_{\pi}) \\ = \; & M([M([x_{11}, \ldots, x_{n1}]_{\pi}), \ldots, M([x_{1n}, \ldots, x_{nn}]_{\pi})]_{\pi'}) \end{aligned}$$

for all $\pi, \pi' \in \Pi_n$ and all ordered square matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in E^{n \times n},$$

where a matrix X is *ordered* if its elements satisfy $x_{ij} \leq x_{kl}$ for all $i \leq k$ and $j \leq l$.

A matrix is said to be *orderable* if it is possible to make it ordered by permuting some rows and/or some columns.

- $M \in A(E, E)$ fulfils the general bisymmetry for orderable matrices (GBOM) if

$$\begin{aligned} & M([M([x_{11}, \ldots, x_{1n}]_{\pi'}), \ldots, M([x_{p1}, \ldots, x_{pn}]_{\pi'})]_{\pi}) \\ = \; & M([M([x_{11}, \ldots, x_{p1}]_{\pi}), \ldots, M([x_{1n}, \ldots, x_{pn}]_{\pi})]_{\pi'}) \end{aligned}$$

for all $\pi \in \Pi_p$, all $\pi' \in \Pi_n$ and all ordered matrices

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{p1} & \cdots & x_{pn} \end{pmatrix} \in E^{p \times n}.$$

Justification of the general bisymmetry for orderable matrices: Consider the same situation as above: n judges give a score to each of p candidates. Now we start by removing some values—for example the lowest score given by each judge, or the lowest score obtained by each candidate. In general, it does not make sense anymore to aggregate as before.

However, there are situations where it still make sense: if the worst candidate is the same for each judge then, when removing this candidate, we get a score matrix for $(p-1)$ candidates and n judges, and we can aggregate as before. Likewise, if the most intolerant judge is the same for each candidate then, when removing the judge, we get a score matrix for p candidates and $(n-1)$ judges, and we can aggregate. Clearly, if we wish to take into account all the possibilities to remove judges and candidates, we have to consider orderable score matrices.

3 The Choquet integral

In this section we present some results on the Choquet integral. Geometrically, this integral corresponds to the so-called Lovász extension of the pseudo-Boolean function which represents the associated fuzzy measure.

3.1 Lovász extension

Lovász [18, §3] has observed that any $x \in (\mathbb{R}^+)^n \setminus \{0\}$ can be written uniquely in the form

$$x = \sum_{i=1}^{k} \lambda_i \, e_{S_i} \tag{1}$$

where $\lambda_1, \ldots, \lambda_k > 0$ and $\emptyset \neq S_1 \subsetneq \cdots \subsetneq S_k \subseteq N$. For example, we have

$$\begin{aligned} (1,5,3) &= 2\,(0,1,0) + 2\,(0,1,1) + 1\,(1,1,1), \\ (0,5,3) &= 2\,(0,1,0) + 3\,(0,1,1). \end{aligned}$$

Hence any function $f : \{0,1\}^n \to \mathbb{R}$ with $f(0) = 0$ can be extended to $\hat{f} : (\mathbb{R}^+)^n \to \mathbb{R}$, by $\hat{f}(0) = 0$ and

$$\hat{f}(x) = \sum_{i=1}^{k} \lambda_i \, f(e_{S_i}) \qquad (x = \sum_{i=1}^{k} \lambda_i \, e_{S_i} \in (\mathbb{R}^+)^n \setminus \{0\});$$

indeed, $\hat{f}$ is well defined (due to the uniqueness of (1)) and $\hat{f}(x) = f(x)$ for all $x \in \{0,1\}^n$. The function $\hat{f}$ is called [8] the *Lovász extension* of f.

Now, the Lovász extension of an arbitrary function $f : \{0,1\}^n \to \mathbb{R}$ is defined by

$$\hat{f}(x) = f(0) + \hat{f}_0(x), \quad x \in (\mathbb{R}^+)^n,$$

where $\hat{f}_0$ is the Lovász extension of $f_0 = f - f(0)$.

The hypercube $[0,1]^n$ can be subdivided into $n!$ simplices of the form

$$\mathcal{B}_\pi = \{x \in [0,1]^n \,|\, x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}, \quad \pi \in \Pi_n.$$

Singer [27, §2] has shown that $\hat{f}$ is defined on each cone $\mathcal{K}_\pi = \{\lambda \, \mathcal{B}_\pi \,|\, \lambda \geq 0\}$ as the unique affine function that coincides with f at the $n+1$ vertices of $\mathcal{B}_\pi$. More formally, $\hat{f}$ can be written as:

$$\hat{f}(x) = f(0) + \sum_{i=1}^{n} x_{\pi(i)} \, [f(e_{\{\pi(i),\ldots,\pi(n)\}}) - f(e_{\{\pi(i+1),\ldots,\pi(n)\}})], \quad x \in \mathcal{K}_\pi. \tag{2}$$

3.2 Properties of the Choquet integral and axiomatic characterizations

Let μ be a fuzzy measure on N. By (2), we immediately see that the Choquet integral $\mathcal{C}_\mu$, defined on $(\mathbb{R}^+)^n$, is nothing else than the Lovász extension of the pseudo-Boolean function f_μ which represents μ:

$$\mathcal{C}_\mu = \hat{f}_\mu \quad \text{on } (\mathbb{R}^+)^n.$$

Now, assume that E contains the unit interval $[0,1]$. Since the Choquet integral fulfils (SPL), it can be defined unambiguously on E^n. Thus, the Choquet integral is a piecewise affine function on E^n and we have

$$\mathcal{C}_\mu(e_S) = \mu(S), \quad S \subseteq N.$$

A practical form of $\mathcal{C}_\mu$ is the following [1].

Proposition 3.1 *Assume $E \supseteq [0,1]$. Any Choquet integral $\mathcal{C}_\mu : E^n \to \mathbb{R}$ can be written as*

$$\mathcal{C}_\mu(x) = \sum_{T \subseteq N} a_T \bigwedge_{i \in T} x_i, \quad x \in E^n,$$

where a is the Möbius representation of μ.

As it can be easily verified, the Choquet integral fulfils the following aggregation properties [12]: (Co), (In), (UIn), (Id), (SPL). We shall see below that it also fulfils (CoAdd) and (BOM).

The class of Choquet integrals has been characterized by Schmeidler [26], see also [9] and [15, Theorem 8.6]. We present below a slightly different statement.

Theorem 3.1 *Assume $E \supseteq [0,1]$ and let $M \in A_n(E, \mathbb{R})$. The following statements are equivalent.*

i) M fulfils (In, SPL, CoAdd)
ii) M fulfils (In, SPL, BOM)
iii) there exists a fuzzy measure μ on N such that $M = \mathcal{C}_\mu$.

We also have the following characterization.

Theorem 3.2 *Assume $[0,1] \subseteq E \subseteq \mathbb{R}^+$. The Choquet integrals on E^n are exactly those $M \in A_n(E, \mathbb{R})$ which fulfil (In, WId) and*

$$M(\lambda\, x + (1-\lambda)\, x') = \lambda\, M(x) + (1-\lambda)\, M(x'), \quad \lambda \in [0,1],$$

for all comonotonic vectors $x, x' \in E^n$.

A Choquet integral operator $\mathcal{C}$ is an aggregation operator $M \in A(E, \mathbb{R})$ such that, for all $n \in \mathbb{N}_0$, $M^{(n)}$ is a Choquet integral. Concerning such aggregation operators, we present the following result.

Theorem 3.3 *Assume $E \supseteq [0,1]$. $M \in A(E, E)$ fulfils (In, SPL, GBOM) if and only if M is a Choquet integral operator.*

3.3 Weighted arithmetic means

The best known and most often used weighted mean in many applications is the weighted arithmetic mean function (WAM). Recall its definition.

Definition 3.1 *For any weight vector* $\omega = (\omega_1, \dots, \omega_n) \in [0,1]^n$ *such that*

$$\sum_{i=1}^{n} \omega_i = 1,$$

the weighted arithmetic mean *function* WAM_ω *associated to* ω*, is defined by*

$$\mathrm{WAM}_\omega(x) = \sum_{i=1}^{n} \omega_i \, x_i.$$

It is easy to prove that WAM functions fulfil the following properties: (Co), (In), (UIn), (Id), (SPL), (SSN), (Add), (B).

Moreover, one can readily see that any WAM_ω is a Choquet integral $\mathcal{C}_\mu$ with respect to an additive fuzzy measure (probability measure):

$$\mu_T = \sum_{i \in T} \omega_i, \quad T \subseteq N.$$

The corresponding Möbius representation is given by:

$$\begin{cases} a_i = \omega_i, & \forall i \in N, \\ a_T = 0, & \forall T \subseteq N \text{ such that } |T| \neq 1, \end{cases}$$

As a consequence, we can see that the weighted arithmetic means are the additive Choquet integrals.

Theorem 3.4 *The Choquet integral* $\mathcal{C}_\mu \in A_n(E, \mathbb{R})$ *fulfils (Add) if and only if there exists* $\omega \in [0,1]^n$ *such that* $\mathcal{C}_\mu = \mathrm{WAM}_\omega$*.*

Corollary 3.1 *Assume* $E \supseteq [0,1]$*.* $M \in A_n(E, \mathbb{R})$ *fulfils (In, SPL, Add) if and only if there exists* $\omega \in [0,1]^n$ *such that* $M = \mathrm{WAM}_\omega$*.*

The class of WAM functions includes two important special cases, namely:

- the arithmetic mean AM, when $\omega_i = 1/n$ for all i,
- the kth projection P_k, when $\omega_k = 1$.

It is clear that a WAM_ω function fulfils (Sy) if and only if $\omega_i = 1/n$ for all i (arithmetic mean).

When $E = [0,1]$, we also have the following two characterizations [21].

Theorem 3.5 $M \in A_n([0,1],[0,1])$ *fulfils (In, SSi, SSN, B) if and only if there exists* $\omega \in [0,1]^n$ *such that* $M = \mathrm{WAM}_\omega$*.*

Theorem 3.6 $M \in A([0,1],[0,1])$ *fulfils (In, SSi, SSN, GB) if and only if, for all* $n \in \mathbb{N}_0$*, there exists* $\omega \in [0,1]^n$ *such that* $M^{(n)} = \mathrm{WAM}_\omega$*.*

3.4 Ordered weighted averaging functions

The ordered weighted averaging aggregation functions (OWA) were proposed by Yager in 1988 [31]. Since their introduction, they have been applied to many fields. For a recent list of references, see [17].

These functions are defined as follows.

Definition 3.2 *For any weight vector* $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ *such that*

$$\sum_{i=1}^{n} \omega_i = 1,$$

the ordered weighted averaging *function* OWA_ω *associated to* ω*, is defined by*

$$\mathrm{OWA}_\omega(x) = \sum_{i=1}^{n} \omega_i \, x_{(i)}.$$

A fundamental aspect of such a function is the re-ordering step, in particular a score x_i is not associated with a particular weight ω_i, but rather a weight is associated with a particular ordered position of score. This ordering step introduces a non-linearity into the aggregation process.

OWA functions satisfy a number of well-known and easy-to-prove properties [2, 31], namely: (Sy), (Co), (In), (UIn), (Id), (SPL), (CoAdd), (BOM). More precisely, it is a well known fact (see e.g. [5, 23]) that OWA functions are a particular case of discrete Choquet integrals with respect to a fuzzy measure depending only on the cardinal of subsets. In fact, the class of OWA functions coincides with the class of Choquet integrals which fulfil (Sy), see [10, 11]. This result can be stated as follows.

Theorem 3.7 *Let* μ *be a fuzzy measure on* N*. Then the following assertions are equivalent.*

i) μ *depends only on the cardinality of subsets*

ii) *there exists* $\omega \in [0,1]^n$ *such that* $\mathcal{C}_\mu = \mathrm{OWA}_\omega$

iii) $\mathcal{C}_\mu$ *fulfils (Sy).*

The fuzzy measure μ associated to an OWA_ω is given by

$$\mu_T = \sum_{i=n-|T|+1}^{n} \omega_i, \quad T \subseteq N, T \neq \emptyset,$$

and its Möbius representation by [13, Theorem 1]

$$a_T = \sum_{j=0}^{|T|-1} \binom{|T|-1}{j} (-1)^{|T|-1-j} \, \omega_{n-j}, \quad T \subseteq N, T \neq \emptyset.$$

The class of OWA functions includes some important special cases:

- the min function, when $\omega_1 = 1$,
- the max function, when $\omega_n = 1$,
- the arithmetic mean $\frac{1}{n}\sum_i x_i$, when $\omega_i = 1/n$ for all i,
- the kth order statistic $x_{(k)}$, when $\omega_k = 1$,
- the median $(x_{(n/2)} + x_{(n/2+1)})/2$, when n is even and $\omega_{n/2} = \omega_{n/2+1} = 1/2$,
- the median $x_{(\frac{n+1}{2})}$, when n is odd and $\omega_{\frac{n+1}{2}} = 1$,
- the mean excluding the extremes as used by some jury of international olympic competitions, when $\omega_1 = \omega_n = 0$ and $\omega_i = \frac{1}{n-2}$ for $i \neq 1, n$.

Characterizations of OWA functions can be deduced from Theorems 3.1, 3.3 and 3.7, see also [20].

Theorem 3.8 *Assume* $E \supseteq [0,1]$ *and let* $M \in A_n(E, \mathbb{R})$. *The following statements are equivalent.*

i) *M fulfils (Sy, In, SPL, CoAdd)*

ii) *M fulfils (Sy, In, SPL, BOM)*

iii) *there exists* $\omega \in [0,1]^n$ *such that* $M = \mathrm{OWA}_\omega$.

Theorem 3.9 *Assume* $E \supseteq [0,1]$. $M \in A(E, E)$ *fulfils (Sy, In, SPL, GBOM) if and only if, for all* $n \in \mathbb{N}_0$, *there exists* $\omega \in [0,1]^n$ *such that* $M^{(n)} = \mathrm{OWA}_\omega$.

4 The Sugeno integral

We now investigate the Sugeno integral under the viewpoint of aggregation. In particular, it will be shown that this integral can be written in the form of a weighted max-min function, which will be introduced and studied hereafter.

The formal analogy between the weighted max-min function and the multilinear polynomial is obvious: minimum corresponds to product, maximum does to sum. Moreover, it is emphasized that weighted max-min functions can be calculated as medians, i.e., the qualitative counterparts of multilinear polynomials.

Most of the results presented here can be found in Marichal [19].

4.1 Weighted max-min functions

If f_μ is the pseudo-Boolean function which represents a given fuzzy measure μ, then we can write

$$f_\mu(x) = \bigvee_{T \subseteq N} \Big[\mu_T \wedge \big(\bigwedge_{i \in T} x_i\big)\Big], \quad x \in \{0,1\}^n.$$

However, such an expression can sometimes be simplified as the following example shows: assuming that $N = \{1,2\}$ and $\mu_1 = 1$, $\mu_2 = 0$, we have

$$f_\mu(x) = x_1 \vee (x_1 \wedge x_2) = x_1. \tag{3}$$

Thus, in a more general way, we see that there exist several set functions $c : 2^N \to [0,1]$ fulfilling $c_\emptyset = 0$ and

$$\bigvee_{T \subseteq N} c_T = 1$$

such that

$$f_\mu(x) = \bigvee_{T \subseteq N} \Big[c_T \wedge \big(\bigwedge_{i \in T} x_i\big)\Big], \quad x \in \{0,1\}^n.$$

We now investigate a natural extension of such a pseudo-Boolean function: the weighted max-min function.

Definition 4.1 *For any set function* $c : 2^N \to [0,1]$ *such that* $c_\emptyset = 0$ *and*

$$\bigvee_{T \subseteq N} c_T = 1,$$

the weighted max-min function $\mathrm{W}_c^{\vee\wedge}:[0,1]^n\to[0,1]$ *associated to* c *is defined by*

$$\mathrm{W}_c^{\vee\wedge}(x)=\bigvee_{T\subseteq N}\Big[c_T\wedge(\bigwedge_{i\in T}x_i)\Big],\quad x\in[0,1]^n.$$

Observe that we have

$$\mathrm{W}_c^{\vee\wedge}(e_S)=\bigvee_{T\subseteq S}c_T,\quad S\subseteq N.$$

As already observed in (3), the set function c which defines $\mathrm{W}_c^{\vee\wedge}$ is not uniquely determined. The next proposition precises conditions under which two weighted max-min functions are identical.

Proposition 4.1 *Let* c *and* c' *be set functions defining* $\mathrm{W}_c^{\vee\wedge}$ *and* $\mathrm{W}_{c'}^{\vee\wedge}$ *respectively. Then* $\mathrm{W}_{c'}^{\vee\wedge}=\mathrm{W}_c^{\vee\wedge}$ *if and only if for all* $T\subseteq N$, $T\neq\emptyset$, *we have*

$$\begin{cases}c'_T=c_T, & \text{if } c_T>\bigvee_{K\subsetneq T}c_K,\\ 0\leq c'_T\leq\bigvee_{K\subseteq T}c_K, & \text{if } c_T\leq\bigvee_{K\subsetneq T}c_K.\end{cases}$$

Let c be any set function defining $\mathrm{W}_c^{\vee\wedge}$ and let $T\subseteq N$, $T\neq\emptyset$. If $c_T>\bigvee_{K\subsetneq T}c_K$ then c_T cannot be modified without altering $\mathrm{W}_c^{\vee\wedge}$. In the other case, it can be replaced by any value lying between 0 and $\bigvee_{K\subseteq T}c_K$.

If c is such that

$$\forall\, T\subseteq N, T\neq\emptyset:\; c_T=0\;\Leftrightarrow\; c_T\leq\bigvee_{K\subsetneq T}c_K$$

then all the c_T's are taken as small as possible and we say that $\mathrm{W}_c^{\vee\wedge}$ is put in its *canonical* form. By contrast, if c is such that

$$\forall\, T\subseteq N:\; c_T=\bigvee_{K\subseteq T}c_K$$

then the c_T's are taken as large as possible and we say that $\mathrm{W}_c^{\vee\wedge}$ is put in its *complete* form. In this case, c is a fuzzy measure since it is increasing. In fact, $\mathrm{W}_c^{\vee\wedge}$ is put in its complete form if and only if c is increasing.

For instance, all the possible expressions of $x_1\vee(x_1\wedge x_2)$ as a two-place weighted max-min function are given by

$$x_1\vee(\lambda\wedge x_1\wedge x_2),\quad \lambda\in[0,1].$$

The case $\lambda=0$ corresponds to the canonical form and the case $\lambda=1$ corresponds to the complete form.

4.2 Weighted min-max functions

By exchanging the position of the max and min operations in Definition 4.1, we can define the weighted min-max functions as follows.

Definition 4.2 *For any set function $d : 2^N \to [0,1]$ such that $d_\emptyset = 1$ and*

$$\bigwedge_{T\subseteq N} d_T = 0,$$

the weighted min-max function $\mathrm{W}_d^{\wedge\vee} : [0,1]^n \to [0,1]$ associated to d is defined by

$$\mathrm{W}_d^{\wedge\vee}(x) = \bigwedge_{T\subseteq N} \Big[d_T \vee (\bigvee_{i\in T} x_i)\Big], \quad x \in [0,1]^n.$$

Observe that we have

$$\mathrm{W}_d^{\wedge\vee}(e_S) = \bigwedge_{T\subseteq N\setminus S} d_T, \quad S \subseteq N.$$

Moreover, the set function d which defines $\mathrm{W}_d^{\wedge\vee}$ is not uniquely determined; indeed, we have, for instance, $x_1 \wedge (x_1 \vee x_2) = x_1$. We then have a result similar to Proposition 4.1.

Proposition 4.2 *Let d and d' be set functions defining $\mathrm{W}_d^{\wedge\vee}$ and $\mathrm{W}_{d'}^{\wedge\vee}$ respectively. Then $\mathrm{W}_{d'}^{\wedge\vee} = \mathrm{W}_d^{\wedge\vee}$ if and only if for all $T \subseteq N$, $T \neq \emptyset$, we have*

$$\begin{cases} d'_T = d_T, & \text{if } d_T < \bigwedge_{K\subsetneq T} d_K, \\ \bigwedge_{K\subseteq T} d_K \leq d'_T \leq 1, & \text{if } d_T \geq \bigwedge_{K\subsetneq T} d_K. \end{cases}$$

Let d be any set function defining $\mathrm{W}_d^{\wedge\vee}$ and let $T \subseteq N$, $T \neq \emptyset$. If $d_T < \bigwedge_{K\subsetneq T} d_K$ then d_T cannot be modified without altering $\mathrm{W}_d^{\wedge\vee}$. In the other case, it can be replaced by any value lying between $\bigwedge_{K\subseteq T} d_K$ and 1.

If d is such that

$$\forall\, T \subseteq N, T \neq \emptyset :\ d_T = 1 \ \Leftrightarrow\ d_T \geq \bigwedge_{K\subsetneq T} d_K$$

then all the d_T's are taken as large as possible and we say that $\mathrm{W}_d^{\wedge\vee}$ is put in its *canonical* form. By contrast, if d is such that

$$\forall\, T \subseteq N :\ d_T = \bigwedge_{K\subseteq T} d_K$$

then the d_T's are taken as small as possible and we say that $\mathrm{W}_d^{\wedge\vee}$ is put in its *complete* form. In this case, d is decreasing. In fact, $\mathrm{W}_d^{\wedge\vee}$ is put in its complete form if and only if d is decreasing.

4.3 Correspondance formulas and equivalent forms

Any weighted max-min function can be put under the form of a weighted min-max function and conversely. The next proposition gives the correspondance formulas.

Proposition 4.3 *Let c and d be set functions defining $\mathrm{W}_c^{\vee\wedge}$ and $\mathrm{W}_d^{\wedge\vee}$ respectively. Then we have*

$$\mathrm{W}_c^{\vee\wedge} = \mathrm{W}_d^{\wedge\vee} \quad\Leftrightarrow\quad \bigvee_{K\subseteq T} c_K = \bigwedge_{K\subseteq N\setminus T} d_K \quad \forall T \subseteq N.$$

When $\mathrm{W}_c^{\vee\wedge}$ and $\mathrm{W}_d^{\wedge\vee}$ are put in their complete forms, the correspondance formulas become simpler.

Corollary 4.1 *For any increasing set function c defining $\mathrm{W}_c^{\vee\wedge}$ and any decreasing set function d defining $\mathrm{W}_d^{\wedge\vee}$, we have*

$$\mathrm{W}_c^{\vee\wedge} = \mathrm{W}_d^{\wedge\vee} \quad \Leftrightarrow \quad c_T = d_{N\setminus T} \quad \forall T \subseteq N.$$

$\mathrm{W}_c^{\vee\wedge}$ and $\mathrm{W}_d^{\wedge\vee}$ can be written under equivalent forms involving at most n variable coefficients. These coefficients only depend on the order of the x_i's.

This result can be formulated as follows.

Theorem 4.1 *i) For any increasing set function c defining $\mathrm{W}_c^{\vee\wedge}$, we have, for all $x \in [0,1]^n$,*

$$\begin{aligned}\mathrm{W}_c^{\vee\wedge}(x) &= \bigvee_{i=1}^{n} [x_{(i)} \wedge c_{\{(i),\ldots,(n)\}}] \\ &= \mathrm{median}(x_1,\ldots,x_n, c_{\{(2),\ldots,(n)\}}, c_{\{(3),\ldots,(n)\}},\ldots, c_{\{(n)\}}).\end{aligned}$$

ii) For any decreasing set function d defining $\mathrm{W}_d^{\wedge\vee}$, we have, for all $x \in [0,1]^n$,

$$\begin{aligned}\mathrm{W}_d^{\wedge\vee}(x) &= \bigwedge_{i=1}^{n} [x_{(i)} \vee d_{\{(1),\ldots,(i)\}}] \\ &= \mathrm{median}(x_1,\ldots,x_n, d_{\{(1)\}}, d_{\{(1),(2)\}},\ldots, d_{\{(1),\ldots,(n-1)\}}).\end{aligned}$$

4.4 Alternative expressions of the Sugeno integral and axiomatic characterizations

Theorem 4.1 shows that the class of the Sugeno integrals coincides with the family of weighted max-min functions which, in turn, coincides with the family of weighted min-max functions. This allows to derive alternative expressions of the Sugeno integral.

Theorem 4.2 *Let $x \in [0,1]^n$ and μ be a fuzzy measure on N. Then we have*

$$\begin{aligned}\mathcal{S}_\mu(x) &= \bigvee_{i=1}^{n} [x_{(i)} \wedge \mu_{\{(i),\ldots,(n)\}}] \\ &= \bigwedge_{i=1}^{n} [x_{(i)} \vee \mu_{\{(i+1),\ldots,(n)\}}] \\ &= \bigvee_{T\subseteq N} \Big[\mu_T \wedge \big(\bigwedge_{i\in T} x_i\big)\Big] \\ &= \bigwedge_{T\subseteq N} \Big[\mu_{N\setminus T} \vee \big(\bigvee_{i\in T} x_i\big)\Big] \\ &= \mathrm{median}(x_1,\ldots,x_n, \mu_{\{(2),\ldots,(n)\}}, \mu_{\{(3),\ldots,(n)\}},\ldots, \mu_{\{(n)\}}).\end{aligned}$$

Let us consider an example. Assume that $N = \{1,2,3\}$ and $x \in [0,1]^3$ with $x_3 \le x_1 \le x_2$. Then

$$\begin{aligned}\mathcal{S}_\mu(x_1,x_2,x_3) &= x_3 \vee (x_1 \wedge \mu_{\{1,2\}}) \vee (x_2 \wedge \mu_{\{2\}}) \\ &= (x_3 \vee \mu_{\{1,2\}}) \wedge (x_1 \vee \mu_{\{2\}}) \wedge x_2 \\ &= \mathrm{median}(x_1,x_2,x_3,\mu_{\{1,2\}},\mu_{\{2\}}).\end{aligned}$$

We can observe that, as an aggregation function, the Sugeno integral with respect to a measure μ is an extension on the entire hypercube $[0,1]^n$ of the pseudo-Boolean function f_μ which defines μ.

As the following theorem shows, the class of Sugeno integrals can be characterized by means of some selected properties.

Theorem 4.3 *Let $M : [0,1]^n \to \mathbb{R}$. Then the following assertions are equivalent:*

i) *M fulfils (In, Id, CoMin, CoMax)*
ii) *M fulfils (In, SMin, SMax)*
iii) *M fulfils (In, Id, SMinB, SMaxB)*
iv) *There exists a set function $c : 2^N \to [0,1]$ such that $M = \mathrm{W}_c^{\vee\wedge}$*
v) *There exists a set function $d : 2^N \to [0,1]$ such that $M = \mathrm{W}_d^{\wedge\vee}$*
vi) *There exists a fuzzy measure μ on N such that $M = \mathcal{S}_\mu$*

4.5 Weighted maximum and minimum functions

Min and max functions have been extended by Dubois and Prade [3], in a way which is consistent with possibility theory: the weighted minimum (wmin) and maximum (wmax).

Definition 4.3 *For any weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that*

$$\bigvee_{i=1}^{n} \omega_i = 1,$$

the weighted maximum *function* wmax_ω *associated to ω is defined by*

$$\mathrm{wmax}_\omega(x) = \bigvee_{i=1}^{n} (\omega_i \wedge x_i), \quad x \in [0,1]^n.$$

For any weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that

$$\bigwedge_{i=1}^{n} \omega_i = 0,$$

the weighted minimum *function* wmin_ω *associated to ω is defined by*

$$\mathrm{wmin}_\omega(x) = \bigwedge_{i=1}^{n} (\omega_i \vee x_i), \quad x \in [0,1]^n.$$

Any wmax_ω function is a $\mathrm{W}_c^{\vee\wedge}$ function whose canonical form is defined by:

$$\begin{cases} c_i = \omega_i, & \forall i \in N, \\ c_T = 0, & \forall T \subseteq N \text{ such that } |T| \neq 1, \end{cases}$$

and complete form by:

$$c_T = \bigvee_{i \in T} \omega_i, \quad \forall T \subseteq N.$$

In this case, if c is increasing then it represents a *possibility measure* π which is characterized by the following property:

$$\pi(S \cup T) = \pi(S) \vee \pi(T), \quad \forall S, T \subseteq N.$$

Likewise, any wmin_ω function is a $\mathrm{W}_d^{\wedge\vee}$ function whose canonical form is defined by:

$$\begin{cases} d_i = \omega_i, & \forall i \in N, \\ d_T = 1, & \forall T \subseteq N \text{ such that } |T| \neq 1, \end{cases}$$

and complete form by:

$$d_T = \bigwedge_{i \in T} \omega_i, \quad \forall T \subseteq N.$$

In this case, if d is decreasing then the set function c', defined by $c'_T = d_{N \setminus T}$ for all $T \subseteq N$, represents a *necessity measure* $\mathcal{N}$ which is characterized by the following property:

$$\mathcal{N}(S \cap T) = \mathcal{N}(S) \wedge \mathcal{N}(T), \quad \forall S, T \subseteq N.$$

The functions wmax_ω and wmin_ω have been characterized by Fodor and Roubens [7]. We present below a slightly more general statement.

Theorem 4.4 *i) $M \in A_n([0,1], \mathbb{R})$ fulfils (WId, SMinB, Max) if and only if there exists $\omega \in [0,1]^n$ such that $M = \mathrm{wmax}_\omega$.*

ii) $M \in A_n([0,1], \mathbb{R})$ fulfils (WId, SMaxB, Min) if and only if there exists $\omega \in [0,1]^n$ such that $M = \mathrm{wmin}_\omega$.

We know that the weighted minimum and maximum functions are particular Sugeno integrals. More precisely, we have the following.

Theorem 4.5 *Let μ be a fuzzy measure on N. Then the following assertions are equivalent.*

i) μ is a possibility measure

ii) there exists $\omega \in [0,1]^n$ such that $\mathcal{S}_\mu = \mathrm{wmax}_\omega$

iii) $\mathcal{S}_\mu$ fulfils (Max).

The following assertions are equivalent.

iv) μ is a necessity measure

v) there exists $\omega \in [0,1]^n$ such that $\mathcal{S}_\mu = \mathrm{wmin}_\omega$

vi) $\mathcal{S}_\mu$ fulfils (Min).

4.6 Ordered weighted maximum and minimum functions

Dubois *et al.* [4] used the ordered weighted maximum (owmax) and minimum (owmin) for modelling soft partial matching. The basic idea of owmax (and owmin) is the same as in the OWA function introduced by Yager [31]. That is, in both papers weights are associated with a particular rank rather than a particular element.

Definition 4.4 *For any weight vector $\omega = (\omega_1, \ldots, \omega_n) \in [0,1]^n$ such that*

$$1 = \omega_1 \geq \ldots \geq \omega_n,$$

the ordered weighted maximum *function* owmax_ω *associated to ω is defined by*

$$\mathrm{owmax}_\omega(x) = \bigvee_{i=1}^{n} (\omega_i \wedge x_{(i)}), \quad x \in [0,1]^n.$$

For any weight vector $\omega' = (\omega'_1, \ldots, \omega'_n) \in [0,1]^n$ such that

$$\omega'_1 \geq \ldots \geq \omega'_n = 0,$$

the ordered weighted minimum *function* owmin_ω *associated to* ω *is defined by*

$$\mathrm{owmin}_\omega(x) = \bigwedge_{i=1}^{n} (\omega'_i \vee x_{(i)}), \quad x \in [0,1]^n.$$

In Definition 4.4, the inequalities $\omega_1 \geq \ldots \geq \omega_n$ and $\omega'_1 \geq \ldots \geq \omega'_n$ are not restrictive. Indeed, if there exists $i \in \{1, \ldots, n-1\}$ such that $\omega_i \leq \omega_{i+1}$ and $\omega'_i \leq \omega'_{i+1}$ then we have

$$\begin{aligned} (\omega_i \wedge x_{(i)}) \vee (\omega_{i+1} \wedge x_{(i+1)}) &= \omega_{i+1} \wedge x_{(i+1)}, \\ (\omega'_i \vee x_{(i)}) \wedge (\omega'_{i+1} \vee x_{(i+1)}) &= \omega'_i \vee x_{(i)}. \end{aligned}$$

This means that ω_i can be replaced by ω_{i+1} in owmax_ω and ω'_{i+1} by ω'_i in owmin_ω.

Any owmax_ω function is a $\mathrm{W}_c^{\vee\wedge}$ function whose canonical form is defined by:

$$\forall T \subseteq N, T \neq \emptyset : c_T = \begin{cases} 0, & \text{if } \omega_{n-|T|+1} = \omega_{n-|T|+2}, \\ \omega_{n-|T|+1}, & \text{else,} \end{cases}$$

and complete form by:

$$\forall T \subseteq N, T \neq \emptyset : c_T = \omega_{n-|T|+1}.$$

Likewise, any owmin_ω function is a $\mathrm{W}_d^{\wedge\vee}$ function whose canonical form is defined by:

$$\forall T \subseteq N, T \neq \emptyset : d_T = \begin{cases} 1, & \text{if } \omega'_{|T|} = \omega'_{|T|-1}, \\ \omega'_{|T|}, & \text{else,} \end{cases}$$

and complete form by:

$$\forall T \subseteq N, T \neq \emptyset : d_T = \omega'_{|T|}.$$

The next proposition shows that any ordered weighted maximum function can be put in the form of an ordered weighted minimum function and conversely.

Proposition 4.4 *Let* ω *and* ω' *be weight vectors defining* owmax_ω *and* $\mathrm{owmin}_{\omega'}$ *respectively. Then we have*

$$\mathrm{owmin}_{\omega'} = \mathrm{owmax}_\omega \quad \Leftrightarrow \quad \omega'_i = \omega_{i+1} \quad \forall i \in \{1, \ldots, n-1\}.$$

We also have, for all $x \in [0,1]^n$,

$$\begin{aligned} \mathrm{owmax}_\omega(x) &= \mathrm{median}(x_1, \ldots, x_n, \omega_2, \ldots, \omega_n), \\ \mathrm{owmin}_{\omega'}(x) &= \mathrm{median}(x_1, \ldots, x_n, \omega'_1, \ldots, \omega'_{n-1}). \end{aligned}$$

The owmax_ω and $\mathrm{owmin}_{\omega'}$ functions are exactly those weighted max-min functions (or Sugeno integrals) which fulfil (Sy):

Theorem 4.6 *Let* μ *be a fuzzy measure on* N*. Then the following assertions are equivalent.*

i) μ *depends only on the cardinality of subsets*
ii) *there exists* $\omega \in [0,1]^n$ *such that* $\mathcal{S}_\mu = \mathrm{owmax}_\omega$
iii) *there exists* $\omega' \in [0,1]^n$ *such that* $\mathcal{S}_\mu = \mathrm{owmin}_{\omega'}$
iv) $\mathcal{S}_\mu$ *fulfils (Sy).*

Note that other characterizations of these families have been obtained in [7] by means of ordered versions of (SMin), (SMax), (Min) and (Max), which seem to be unappealing properties.

5 Common area between the two classes of integrals

5.1 Boolean max-min and min-max functions

The Boolean max-min and min-max functions are defined as follows.

Definition 5.1 *i) For any set function $c : 2^N \to \{0,1\}$ such that $c_\emptyset = 0$ and*

$$\bigvee_{T \subseteq N} c_T = 1,$$

the Boolean max-min function $\mathrm{B}_c^{\vee\wedge}$ *associated to c is defined by $\mathrm{B}_c^{\vee\wedge} = \mathrm{W}_c^{\vee\wedge}$.*

ii) For any set function $d : 2^N \to \{0,1\}$ such that $d_\emptyset = 1$ and

$$\bigwedge_{T \subseteq N} d_T = 0,$$

the Boolean min-max function $\mathrm{B}_d^{\wedge\vee}$ *associated to d is defined by $\mathrm{B}_d^{\wedge\vee} = \mathrm{W}_d^{\wedge\vee}$.*

Thus defined, a Boolean max-min function (resp. Boolean min-max function) is nothing else than a weighted max-min function (resp. weighted min-max function) whose canonical and complete forms are defined by set functions taking their values in $\{0,1\}$. Moreover, we can write, for any $x \in [0,1]^n$,

$$\begin{aligned} \mathrm{B}_c^{\vee\wedge}(x) &= \bigvee_{\substack{T \subseteq N \\ c_T = 1}} \bigwedge_{i \in T} x_i \in \{x_1, \ldots, x_n\}, \qquad &\text{(disjunctive normal form)} \\ \mathrm{B}_d^{\wedge\vee}(x) &= \bigwedge_{\substack{T \subseteq N \\ d_T = 0}} \bigvee_{i \in T} x_i \in \{x_1, \ldots, x_n\}, \qquad &\text{(conjunctive normal form).} \end{aligned}$$

In terms of fuzzy measures, if the set function c is increasing, it represents a 0-1 fuzzy measure. More precisely, Murofushi and Sugeno [23, §2] showed the following result.

Proposition 5.1 *If μ is a 0-1 fuzzy measure on N then the Choquet and the Sugeno integral take the following form:*

$$\mathcal{C}_\mu = \mathcal{S}_\mu = \mathrm{B}_\mu^{\vee\wedge}.$$

We now present a stronger result: the intersection of the class of Choquet integrals and the class of Sugeno integrals coincides with the class of Boolean max-min functions.

Theorem 5.1 *Let $M \in A_n([0,1], \mathbb{R})$. Then the following assertions are equivalent.*

i) There exists a 0-1 fuzzy measure μ on N such that $M = \mathcal{S}_\mu$.

ii) There exist fuzzy measures μ and ν on N such that $M = \mathcal{C}_\mu = \mathcal{S}_\nu$.

iii) M fulfils (UIn) and there exists a fuzzy measure μ on N such that $M = \mathcal{S}_\mu$.

iv) There exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$.

v) There exists a set function $d : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_d^{\wedge\vee}$.

Since any $\mathrm{B}_c^{\vee\wedge}$ function is a Choquet integral (see Proposition 5.1), it fulfils (SPL) and thus it can be defined on any E^n, where $E \supseteq [0,1]$. We then have the following result.

Theorem 5.2 *Let $M \in A_n(E, \mathbb{R})$, with $E \supseteq [0,1]$. Then the following assertions are equivalent.*

i) *There exists a 0-1 fuzzy measure μ on N such that $M = \mathcal{C}_\mu$.*

ii) *M fulfils (In, SPL) and $M(e_T) \in \{0,1\}$ for all $T \subseteq N$.*

iii) *There exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$.*

iv) *There exists a set function $d : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_d^{\wedge\vee}$.*

We also have the following result.

Theorem 5.3 *Let $M \in A_n(E, \mathbb{R})$. Then the following assertions are equivalent.*

i) *M fulfils (Co, Id, CM).*

ii) *There exists a set function $c : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_c^{\vee\wedge}$.*

iii) *There exists a set function $d : 2^N \to \{0,1\}$ such that $M = \mathrm{B}_d^{\wedge\vee}$.*

5.2 Order statistics and medians

The *order statistics* are defined as follows (cf. van der Waerden [30, §17]).

Definition 5.2 *For any $k \in N$, the* order statistic *function* OS_k *associated to the kth argument is defined by*

$$\mathrm{OS}_k(x) = x_{(k)}.$$

Any order statistic OS_k is a $\mathrm{B}_c^{\vee\wedge}$ whose canonical form is defined by

$$\forall T \subseteq N, T \neq \emptyset : c_T = \begin{cases} 1, & \text{if } |T| = n-k+1, \\ 0, & \text{otherwise,} \end{cases}$$

and complete form by:

$$\forall T \subseteq N, T \neq \emptyset : c_T = \begin{cases} 1, & \text{if } |T| \geq n-k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, it is also a $\mathrm{B}_d^{\wedge\vee}$ function whose canonical form is defined by:

$$\forall T \subseteq N, T \neq \emptyset : d_T = \begin{cases} 0, & \text{if } |T| = k, \\ 1, & \text{otherwise,} \end{cases}$$

and complete form by:

$$\forall T \subseteq N, T \neq \emptyset : d_T = \begin{cases} 0, & \text{if } |T| \geq k, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, we have:

$$x_{(k)} = \bigvee_{\substack{T \subseteq N \\ |T| = n-k+1}} \bigwedge_{i \in T} x_i = \bigwedge_{\substack{T \subseteq N \\ |T| = k}} \bigvee_{i \in T} x_i, \quad k \in N.$$

By Theorem 4.2, we also have

$$x_{(k)} = \mathrm{median}(x_1, \ldots, x_n, \underbrace{1, \ldots, 1}_{k-1}, \underbrace{0, \ldots, 0}_{n-k}), \quad k \in N.$$

It is known [24, Theorem 4.3] that the order statistics on any E^n form the class of functions satisfying (Sy, Co, Id, CM). They are also the Boolean max-min functions fulfilling (Sy). As a consequence, we have the following result.

Theorem 5.4 *Let $M \in A_n([0,1], \mathbb{R})$. Then the following assertions are equivalent.*

i) *M fulfils (Sy) and there exists a 0-1 fuzzy measure μ such that $M = \mathcal{C}_\mu$.*

ii) *M fulfils (Sy) and there exists a 0-1 fuzzy measure μ such that $M = \mathcal{S}_\mu$.*

iii) *M fulfils (UIn) and there exists $\omega \in [0,1]^n$ such that $M = \mathrm{owmax}_\omega$.*

iv) *M fulfils (Sy) and there exists a set function c such that $M = \mathrm{B}_c^{\vee\wedge}$.*

v) *There exists $k \in N$ such that $M = \mathrm{OS}_k$.*

A particular case of order statistic is the so-called median of an odd number of scores. If $x_1, \ldots, x_{2k-1} \in E$, we have

$$\begin{aligned}\mathrm{median}(x_1, \ldots, x_{2k-1}) = x_{(k)} &= \bigvee_{1 \leq i_1 < \cdots < i_k \leq 2k-1} (x_{i_1} \wedge \cdots \wedge x_{i_k}) \\ &= \bigwedge_{1 \leq i_1 < \cdots < i_k \leq 2k-1} (x_{i_1} \vee \cdots \vee x_{i_k}).\end{aligned}$$

For instance, we have

$$\begin{aligned}\mathrm{median}(x_1, x_2, x_3) &= (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) \\ &= (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3).\end{aligned}$$

Regarding medians, we have the following immediate characterization.

Theorem 5.5 *Let $k \in \mathbb{N}_0$ and $M \in A_{2k-1}([0,1], \mathbb{R})$. Then M is an order statistic fulfilling (SSN) if and only if $M = \mathrm{median}$.*

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