On a sorting procedure in the presence of qualitative interacting points of view*

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Abstract

We present a sorting procedure for the assignment of alternatives to graded classes. The available information is given by partial evaluations of the alternatives on ordinal scales representing interacting points of view and a subset of prototypic alternatives whose assignment is imposed beforehand. The partial evaluations of each alternative are embedded in a common interval scale by means of commensurateness mappings, which in turn are aggregated by the discrete Choquet integral. The behavioral properties of this Choquet integral are then measured through importance and interaction indices.

Keywords: multi-attribute decision-making, ordinal data, interacting points of view, Choquet integral.

1   Introduction

In this paper we use the discrete Choquet integral as a discriminant function in ordinal multiattribute sorting problems in the presence of interacting (dependent) points of view. The technique we present is due to Roubens [14] and proceeds in two steps: a pre-scoring phase determines for each point of view and for each alternative a net score (the number of times a given alternative beats all the other alternatives minus the number of times that this alternative is beaten by the others) and is followed by an aggregation phase that produces a global net score associated to each alternative. These global scores are then used to assign the alternatives to graded classes.

The fuzzy measure linked to the Choquet integral can be learnt from a subset of alternatives (called prototypes) that are assigned beforehand to the classes by the decision maker.

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This leads to solving a linear constraint satisfaction problem whose unknown variables are the coefficients of the fuzzy measure.

Once a fuzzy measure (compatible with the available information on prototypes) is found, it is useful to interpret it through some behavioral parameters. We present the following two types of parameters:

1. The importance indices, which make it possible to appraise the overall importance of each point of view and each combination of points of view,

2. The interaction indices, which measure the extent to which the points of view interact (positively or negatively).

2 An ordinal sorting procedure

Let A be a set of q potential alternatives, which are to be assigned to disjoint classes, and let $N = \{1, \ldots, n\}$ be a label set of points of view to satisfy. For each point of view $i \in N$, the alternatives are evaluated according to a s_i-point ordinal performance scale; that is, a totally ordered set

$$X_i := \{g_1^i \prec_i g_2^i \prec_i \cdots \prec_i g_{s_i}^i\}.$$

We assume that each alternative $x \in A$ can be identified with its corresponding profile

$$(x_1, \ldots, x_n) \in \underset{i=1}{\overset{n}{\times}} X_i =: X,$$

where, for any $i \in N$, x_i represents the partial evaluation of x related to point of view i. In other words, each alternative is completely determined from its partial evaluations.

Through this identification, we clearly have

$$A \subseteq X \quad \text{and} \quad q \leqslant \prod_{i=1}^{n} s_i.$$

For any $x_i \in X_i$ and any $y_{-i} \in X_{-i} := \times_{j \in N \setminus \{i\}} X_j$, we set

$$x_i y_{-i} := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n) \in X.$$

Now, consider a partition of X into m nonempty classes $\{Cl_t\}_{t=1}^{m}$, which are increasingly ordered; that is, for any $r, s \in \{1, \ldots, m\}$, with $r > s$, the elements of Cl_r have a better comprehensive evaluation than the elements of Cl_s.

We also set

$$Cl_r^{\geqslant} := \bigcup_{t=r}^{m} Cl_t \qquad (r = 1, \ldots, m).$$

The *sorting* problem we actually face consists in partitioning the elements of A into the classes $\{Cl_t\}_{t=1}^{m}$. Since A is given, the problem amounts to identifying the classes themselves as a partition of X.

Greco *et al.* [5, Theorem 2.1] proved a nice representation theorem stating the equivalence between a very simple cancelation property and a general discriminant function. As we have assumed beforehand that each set X_i is endowed with a total order $\succcurlyeq_i$, we present here a slightly modified version of their result.

Theorem 2.1 *The following two assertions are equivalent:*

1. For all $i \in N$, $t \in \{1, \ldots, m\}$, $x_i, x_i' \in X_i$, $y_{-i} \in X_{-i}$, we have

$$x_i' \succcurlyeq_i x_i \quad \text{and} \quad x_i y_{-i} \in Cl_t \quad \Rightarrow \quad x_i' y_{-i} \in Cl_t^{\geqslant}.$$

2. There exist

- *functions $g_i : X_i \to \mathbb{R}$ $(i \in N)$, increasing, called criteria,*
- *a function $f : \mathbb{R}^n \to \mathbb{R}$, increasing in each argument, called discriminant function,*
- *$m - 1$ ordered thresholds $\{z_t\}_{t=2}^m$ satisfying*

$$z_2 \leqslant z_3 \leqslant \cdots \leqslant z_m$$

such that, for any $x \in X$ and any $t \in \{2, \ldots, m\}$, we have

$$f[g_1(x_1), g_2(x_2), \ldots, g_n(x_n)] \geqslant z_t \quad \Leftrightarrow \quad x \in Cl_t^{\geqslant}.$$

Theorem 2.1 states that, under a simple condition of monotonicity, it is possible to find a discriminant function that strictly separates the classes $Cl_1, \ldots, Cl_m$ by thresholds. This result is very general and imposes no particular forms to criteria and discriminant functions.

For a practical use of this result and in order to produce a meaningful result, Roubens [14] restricted the family of possible discriminant functions to the class of n-place Choquet integrals and the criteria functions to normalized scores. We now present the sorting procedure in this particular case.

2.1 Normalized scores as criteria

In order to locate x_i in the scale X_i we define a mapping $\mathrm{ord}_i : A \to \{1, \ldots, s_i\}$ as

$$\mathrm{ord}_i(x) = r \quad \Leftrightarrow \quad x_i = g_r^i.$$

For each point of view $i \in N$, the order $\preccurlyeq_i$ defined on X_i can be characterized by a valuation $R_i : A \times A \to \{0, 1\}$ defined as

$$R_i(x, y) := \begin{cases} 1, & \text{if } x_i \succcurlyeq_i y_i, \\ 0, & \text{otherwise.} \end{cases}$$

From each of these valuations we determine a partial *net score* $S_i : A \to \mathbb{R}$ as follows:

$$S_i(x) := \sum_{y \in A} [R_i(x, y) - R_i(y, x)] \qquad (x \in A).$$

In the particular case where

$$A = \underset{i=1}{\overset{n}{\times}} X_i,$$

then it is easy to see that

$$S_i(x) = q \left(\frac{2 \, \mathrm{ord}_i(x) - 1}{s_i} - 1 \right) \qquad (i \in N).$$

Indeed, there are $(\mathrm{ord}_i(x) - 1)\, q/s_i$ alternatives $y \in A$ such that $x_i \succ_i y_i$ and $(s_i - \mathrm{ord}_i(x))\, q/s_i$ alternatives $y \in A$ such that $y_i \succ_i x_i$.

The integer $S_i(x)$ represents the number of times that x is prefered to any other alternative minus the number of times that any other alternative is prefered to x for point of view i.

On can easily show that the partial net scores identify the corresponding partial evaluations. That is,

$$x_i \succeq_i y_i \quad \Leftrightarrow \quad S_i(x) \geqslant S_i(y). \tag{1}$$

Thus aggregating the partial evaluations of a given alternative amounts to aggregating the corresponding partial scores. This latter aggregation makes sense since, contrary to the partial evaluations, the partial scores are *commensurable*, that is, each partial score can be compared with any other partial score, even along a different point of view.

Clearly, the partial scores are defined according to the same interval scale. As positive linear transformations are meaningful with respect to such a scale, we can normalize these scores so that they range in the unit interval. We thus define normalized partial scores $S_1^N, \ldots, S_n^N$ as

$$S_i^N(x) := \frac{S_i(x) + (q-1)}{2(q-1)} \in [0,1] \qquad (i \in N).$$

Throughout the paper, we will use the notation $S^N(x) := (S_1^N(x), \ldots, S_n^N(x))$.

2.2 The Choquet integral as a discriminant function

As mentioned in the beginning of this section, the partial scores of a given alternative x can be aggregated by means of a Choquet integral [1], namely

$$\mathcal{C}_v(S^N(x)) := \sum_{i=1}^n S_{(i)}^N(x)\, [v(A_{(i)}) - v(A_{(i+1)})]$$

where v represents a fuzzy measure on N; that is, a monotone set function $v : 2^N \to [0,1]$ fulfilling $v(\emptyset) = 0$ and $v(N) = 1$. This fuzzy measure merely expresses the importance of each subset of points of view. The parentheses used for indices represent a permutation on N such that

$$S_{(1)}^N(x) \leqslant \cdots \leqslant S_{(n)}^N(x),$$

and $A_{(i)}$ represents the subset $\{(i), \ldots, (n)\}$.

We note that for additive measures ($v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$) the Choquet integral coincides with the usual discrete Lebesgue integral and the set function v is simply determined by the importance of each point of view: $v(1), \ldots, v(n)$. In this particular case

$$\mathcal{C}_v(S^N(x)) = \sum_{i=1}^n v(i)\, S_i^N(x) \qquad (x \in A),$$

which is the natural extension of the *Borda score* as defined in voting theory if alternatives play the role of candidates and points of view represent voters.

If points of view cannot be considered as being independent, importance of combinations $S \subseteq N$, $v(S)$, has to be taken into account.

Some combinations of points of view might present a positive interaction or *synergy*. Although the importance of some points of view, members of a combination S, might be low, the importance of a pair, a triple,..., might be substantially larger and $v(S) > \sum_{i \in S} v(i)$.

In other situations, points of view might exhibit negative interaction or *redundancy*. The union of some points of view do not have much impact on the decision and for such combinations S, $v(S) < \sum_{i \in S} v(i)$. In this perspective, the use of the Choquet integral is recommended.

The Choquet integral presents standard properties for aggregation (see [3, 8]): it is continuous, non decreasing, located between min and max.

We will now indicate an axiomatic characterization of the class of all Choquet integrals with n arguments. This result is due to Marichal [8]. Let e_S denote the characteristic vector of S in $\{0,1\}^n$, i.e., the vector of $\{0,1\}^n$ whose ith component is one if and only if $i \in S$.

Theorem 2.2 *The operators $M_v : \mathbb{R}^n \to \mathbb{R}$ (v being a fuzzy measure on N) are*

- *linear w.r.t. the fuzzy measures, that is, there exist 2^n functions $f_T : \mathbb{R}^n \to \mathbb{R}$ ($T \subseteq N$), such that*

$$M_v = \sum_{T \subseteq N} v(T) \, f_T \,,$$

- *non decreasing in each argument,*

- *stable for the admissible positive linear transformations, that is,*

$$M_v(r \, x_1 + s, \ldots, r \, x_n + s) = r \, M_v(x_1, \ldots, x_n) + s$$

for all $x \in \mathbb{R}^n$, $r > 0$, $s \in \mathbb{R}$,

- *properly weighted by v, that is,*

$$M_v(e_S) = v(S),$$

if and only if $M_v = \mathcal{C}_v$ for all fuzzy measure v on N.

This important characterization clearly justifies the way the partial scores have been aggregated.

The first axiom is proposed to keep the aggregation model as simple as possible. The second axiom says that increasing a partial score cannot decrease the global score. The third axiom only demands that the aggregated value is stable with respect to any change of scale. Finally, assuming that the partial score scale is embedded in $[0, 1]$, the fourth axiom suggests that the weight of importance of any subset S of criteria is defined as the global evaluation of the alternative that completely satisfies points of view S and totally fails to satisfy the others.

The fourth axiom is fundamental. It gives an appropriate definition of the weights of subsets of points of view, interpreting them as global evaluation of particular alternatives.

The major advantage linked to the use of the Choquet integral derives from the large number of parameters ($2^n - 2$) associated with a fuzzy measure but this flexibility can be also considered as a serious drawback when assessing real values to the importance of all possible combinations. We will come back to the important question the next section.

Let v be a fuzzy measure on N. The Möbius transform of v is a set function $m : 2^N \to \mathbb{R}$ defined by

$$m(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \, v(T) \qquad (S \subseteq N).$$

This transformation is invertible and thus constitutes an equivalent form of a fuzzy measure and v can be recovered from m using

$$v(S) = \sum_{T \subseteq S} m(T) \qquad (S \subseteq N).$$

This transformation can be used to redefine the Choquet integral without reordering the partial scores:

$$\mathcal{C}_v(S^N(x)) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} S_i^N(x).$$

A fuzzy measure v is k-additive [3] if its Möbius transform m satisfies $m(S) = 0$ for S such that $|S| > k$ and there exists at least one subset S such that $|S| = k$ and $m(S) \neq 0$.

Thus, k-additive fuzzy measures can be represented by at most $\sum_{i=1}^{k} \binom{n}{i}$ coefficients.

For a k-additive fuzzy measure,

$$\mathcal{C}_v(S^N(x)) = \sum_{\substack{T \subseteq N \\ |T| \leqslant k}} m(T) \bigwedge_{i \in T} S_i^N(x).$$

In order to assure boundary and monotonicity conditions imposed on v, the Möbius transform of a k-additive fuzzy measure must satisfy:

$$\begin{cases} m(\emptyset) = 0, \quad \displaystyle\sum_{\substack{T \subseteq N \\ |T| \leqslant k}} m(T) = 1 \\ \displaystyle\sum_{\substack{T | i \in T \subseteq S \\ |T| \leqslant k}} m(T) \geqslant 0, \quad \forall S \subseteq N, \ \forall i \in S \end{cases}$$

2.3   Assessment of fuzzy measures

Assume that all the alternatives of $A \subseteq X$ are already sorted into classes $Cl_1, \ldots, Cl_m$. In some particular cases there exist a fuzzy measure v on N and $m-1$ ordered thresholds $\{z_t\}_{t=2}^m$ satisfying

$$z_2 \leqslant z_3 \leqslant \cdots \leqslant z_m$$

such that for any $x \in A$, and any $t \in \{2, \ldots, m\}$, we have

$$\mathcal{C}_v(S^N(x)) \geqslant z_t \quad \Leftrightarrow \quad x \in Cl_t^{\geqslant}.$$

Of course, if such a fuzzy measure does exist then the thresholds may be defined by

$$z_t := \min_{x \in Cl_t^{\geqslant}} \mathcal{C}_v(S^N(x)) \qquad (t = 2, \ldots, m).$$

Conversely, the knowledge of the fuzzy measure v associated to the sorting problem completely determines the assignment.

In real situations, the assignment of all alternatives is not known but has to be determined. However, this assignment, or equivalently the fuzzy measure v, can be learnt from a reference set of prototypes, which have been sorted beforehand by the decision maker.

Practically, the decision maker is asked to provide a set of prototypes $P \subseteq A$ and the assignment of each of these prototypes to a given class; that is, a partition of P into

prototypic classes $\{P_t\}_{t=1}^m$, where $P_t := P \cap Cl_t$ for all $t \in \{1, \ldots, m\}$. Here some prototypic classes may be empty.

As the Choquet integral is supposed to strictly separate the classes Cl_t, we must impose the following necessary condition

$$\mathcal{C}_v(S^N(x)) - \mathcal{C}_v(S^N(x')) \geqslant \varepsilon, \tag{2}$$

for each ordered pair $(x, x') \in P_t \times P_{t-1}$ and each $t \in \{2, \ldots, m\}$, where ε is a given strictly positive threshold.

These separation conditions, put together with the boundary and monotonicity constraints on the fuzzy measure, form a linear constraint satisfaction problem whose unknowns are the coefficients of the fuzzy measure. Thus the sorting problem consists in finding a feasible solution satisfying all these constraints. If ε has been chosen too big, the problem might have no solution. To avoid this, we can consider ε as a non-negative variable to be maximized. In this case its optimal value must be strictly positive for the problem to have a solution.

In the resolution of this problem, we use the principle of parsimony: If no solution is found for $k = 1$, we turn to $k = 2$. If no solution is still found, we turn to $k = 3$, and so on, until $k = n$. Notice however that an empty solution set for $k = n$ is necessarily due to an incompatibility between the assignment of the given prototypes and the assumption that the discriminant function is a Choquet integral.

Due to the increasing monotonicity of the Choquet integral, the number of separation constraints (2) can be reduced significantly. For example, if $x'' \in P_{t-1}$ is such that $\mathcal{C}_v(S^N(x')) \geqslant \mathcal{C}_v(S^N(x''))$ then, by transitivity, the constraint

$$\mathcal{C}_v(S^N(x)) - \mathcal{C}_v(S^N(x'')) \geqslant \varepsilon$$

is redundant.

Now, on the basis of orders $\succcurlyeq_i$ $(i \in N)$, we can define a dominance relation D on X as follows: For each $x, y \in X$,

$$xDy \quad \Leftrightarrow \quad x_i \succcurlyeq_i y_i \ \forall i \in N.$$

By (1), this is equivalent to

$$xDy \quad \Leftrightarrow \quad S_i^N(x) \geqslant S_i^N(y) \ \forall i \in N.$$

Being an intersection of complete orders, the binary relation D is a partial order, i.e., it is reflexive, antisymmetric, and transitive. Furthermore we clearly have

$$xDy \quad \Rightarrow \quad \mathcal{C}_v(S^N(x)) \geqslant \mathcal{C}_v(S^N(y)).$$

It is then useful to define, for each $t \in \{1, \ldots, m\}$, the set of non-dominating alternatives of P_t,

$$Nd_t := \{x \in P_t \mid \nexists\, x' \in P_t \setminus \{x\} : xDx'\},$$

and the set of non-dominated alternatives of P_t,

$$ND_t := \{x \in P_t \mid \nexists\, x' \in P_t \setminus \{x\} : x'Dx\},$$

and to consider only the constraint (2) for each ordered pair $(x, x') \in Nd_t \times ND_{t-1}$ and each $t \in \{2, \ldots, m\}$. The total number of separation constraints boils down to

$$\sum_{t=2}^m |Nd_t| \, |ND_{t-1}|.$$

Now, suppose that there exists a k-additive fuzzy measure v^* that solves the above problem. Then any alternative $x \in A$ will be assigned to

- the class Cl_t if

$$\min_{y \in Nd_t} \mathcal{C}_{v^*}(S^N(y)) \leqslant \mathcal{C}_{v^*}(S^N(x)) \leqslant \max_{y \in ND_t} \mathcal{C}_{v^*}(S^N(y)),$$

- one of the classes Cl_t or Cl_{t-1} if

$$\max_{y \in ND_{t-1}} \mathcal{C}_{v^*}(S^N(y)) < \mathcal{C}_{v^*}(S^N(x)) < \min_{y \in Nd_t} \mathcal{C}_{v^*}(S^N(y)).$$

3 Behavioral analysis of aggregation

Now that we have a sorting model for assigning alternatives to classes, an important question arises: How can we interpret the behavior of the Choquet integral or that of its associated fuzzy measure? Of course the meaning of the values $v(T)$ is not always clear for the decision maker. These values do not give immediately the global importance of the points of view, nor the degree of interaction among them.

In fact, from a given fuzzy measure, it is possible to derive some indices or parameters that will enable us to interprete the behavior of the fuzzy measure. These indices constitute a kind of *identity card* of the fuzzy measure. In this section, we present two types of indices: importance and interaction. Other indices, such as tolerance and dispersion, were proposed and studied by Marichal [6, 7].

3.1 Importance indices

The overall importance of a point of view $i \in N$ into a decision problem is not solely determined by the number $v(\{i\})$, but also by all $v(T)$ such that $i \in T$. Indeed, we may have $v(\{i\}) = 0$, suggesting that element i is unimportant, but it may happen that for many subsets $T \subseteq N$, $v(T \cup \{i\})$ is much greater than $v(T)$, suggesting that i is actually an important element in the decision.

Shapley [15] proposed in 1953 a definition of a coefficient of importance, based on a set of reasonable axioms. The *importance index* or *Shapley value* of point of view i with respect to v is defined by:

$$\phi(v, \{i\}) := \sum_{T \subseteq N \setminus \{i\}} \frac{(n - |T| - 1)! \, |T|!}{n!} [v(T \cup \{i\}) - v(T)]. \tag{3}$$

The Shapley value is a fundamental concept in game theory expressing a power index. It can be interpreted as a weighted average value of the marginal contribution $v(T \cup \{i\}) - v(T)$ of element i alone in all combinations. To make this clearer, it is informative to rewrite the index as follows:

$$\phi(v, \{i\}) = \frac{1}{n} \sum_{t=0}^{n-1} \frac{1}{\binom{n-1}{t}} \sum_{\substack{T \subseteq N \setminus \{i\} \\ |T| = t}} [v(T \cup \{i\}) - v(T)].$$

Thus, the average value of $v(T \cup \{i\}) - v(T)$ is computed first over the subsets of same size t and then over all the possible sizes. Consequently, the subsets containing about $n/2$

points of view are the less important in the average, since they are numerous and a same point of view j is very often involved into them.

The use of the Shapley value in multicriteria decision making was proposed in 1992 by Murofushi [10]. It is worth noting that a basic property of the Shapley value is

$$\sum_{i=1}^{n} \phi(v, \{i\}) = 1.$$

Note also that, when v is additive, we clearly have $v(T \cup \{i\}) - v(T) = v(\{i\})$ for all $i \in N$ and all $T \subseteq N \setminus \{i\}$, and hence

$$\phi(v, \{i\}) = v(\{i\}), \quad i \in N. \tag{4}$$

If v is non-additive then some points of view are dependent and (4) generally does not hold anymore. This shows that it is sensible to search for a coefficient of overall importance for each point of view.

In terms of the Möbius representation, the Shapley value takes a very simple form [15]:

$$\phi(v, \{i\}) = \sum_{T \ni \{i\}} \frac{1}{|T|} m(T). \tag{5}$$

Now, the concept of importance index can be easily generalized to subsets of points of view. The *importance index* of subset $S \subseteq N$ with respect to v is defined by

$$\phi(v, S) := \sum_{T \subseteq N \setminus S} \frac{(n - |T| - |S|)! \, |T|!}{(n - |S| + 1)!} \, [v(T \cup S) - v(T)].$$

This index, introduced by Marichal [9] as the *influence index* of points of view S, measures the overall importance of subset S of points of view.

In terms of the Möbius representation, it is given by

$$\phi(v, S) = \sum_{\substack{T \subseteq N \\ T \cap S \neq \emptyset}} \frac{1}{|T \setminus S| + 1} m(T).$$

It was shown [9] that this expression is also the average amplitude of the range of $\mathcal{C}_v$ that points of view S may control when assigning partial scores in $[0, 1]$ to the points of view in $N \setminus S$ at random. That is,

$$
\begin{aligned}
\phi(v, S) &= \int_0^1 \cdots \int_0^1 \Big[\lim_{\substack{x_j \to 1 \\ j \in S}} \mathcal{C}_v(x) - \lim_{\substack{x_j \to 0 \\ j \in S}} \mathcal{C}_v(x) \Big] \, \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_{n-s}} \\
&= \int_{[0,1]^n} \Big[\lim_{\substack{x_j \to 1 \\ j \in S}} \mathcal{C}_v(x) - \lim_{\substack{x_j \to 0 \\ j \in S}} \mathcal{C}_v(x) \Big] \, \mathrm{d}x \,,
\end{aligned}
$$

where $N \setminus S = \{i_1, \ldots, i_{n-s}\}$.

3.2 Interaction indices

Another interesting concept is that of *interaction* among points of view. We have seen that when the fuzzy measure is not additive then some points of view interact. Of course, it would be interesting to appraise the degree of interaction among any subset of points of view.

Consider first a pair $\{i, j\} \subseteq N$ of points of view. It may happen that $v(\{i\})$ and $v(\{j\})$ are small and at the same time $v(\{i, j\})$ is large. Clearly, the number $\phi(v, \{i\})$ merely measures the average contribution that point of view i brings to all possible combinations, but it gives no information on the phenomena of interaction existing among points of view.

Clearly, if the marginal contribution of j to every combination of points of view that contains i is greater (resp. less) than the marginal contribution of j to the same combination when i is excluded, the expression

$$[v(T \cup \{i, j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)]$$

is positive (resp. negative) for any $T \subseteq N \setminus \{i, j\}$. We then say that i and j positively (resp. negatively) interact.

This latter expression is called the *marginal interaction* between i and j, conditioned to the presence of elements of the combination $T \subseteq N \setminus \{i, j\}$. Now, an interaction index for $\{i, j\}$ is given by an average value of this marginal interaction. Murofushi and Soneda [11] proposed in 1993 to calculate this average value as for the Shapley value. Setting

$$(\Delta_{ij} v)(T) := v(T \cup \{i, j\}) - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T),$$

the *interaction index* of points of view i and j related to v is then defined by

$$I(v, \{i, j\}) := \sum_{T \subseteq N \setminus \{i, j\}} \frac{(n - |T| - 2)! \, |T|!}{(n - 1)!} \, (\Delta_{ij} v)(T). \tag{6}$$

It should be mentioned that, historically, the interaction index (6) was first introduced in 1972 by Owen (see Eq. (28) in [13]) in game theory to express a degree of complementarity or competitiveness between elements i and j.

The interaction index among a combination S of points of view was introduced by Grabisch [3] as a natural extension of the case $|S| = 2$. The *interaction index* of S ($|S| \geq 2$) related to v, is defined by

$$I(v, S) := \sum_{T \subseteq N \setminus S} \frac{(n - |T| - |S|)! \, |T|!}{(n - |S| + 1)!} \, (\Delta_S v)(T),$$

where we have set

$$(\Delta_S v)(T) := \sum_{L \subseteq S} (-1)^{|S| - |L|} v(L \cup T).$$

In terms of the Möbius representation, this index is written [3]

$$I(v, S) = \sum_{T \supseteq S} \frac{1}{|T| - |S| + 1} \, m(T), \qquad S \subseteq N. \tag{7}$$

Viewed as a set function, it coincides on singletons with the Shapley value (3).

In terms of the Choquet integral, we have [4, Proposition 4.1]

$$
\begin{aligned}
I(v, S) &= \int_0^1 \cdots \int_0^1 (\Delta_S \mathcal{C}_v)(x) \, dx_{i_1} \cdots dx_{i_{n-s}} \\
&= \int_{[0,1]^n} (\Delta_S \mathcal{C}_v)(x) \, dx \,,
\end{aligned}
$$

where $N \setminus S = \{i_1, \ldots, i_{n-s}\}$ and

$$
(\Delta_S \mathcal{C}_v)(x) := \sum_{L \subseteq S} (-1)^{|S|-|L|} \lim_{\substack{x_i \to 1 \\ i \in L}} \lim_{\substack{x_j \to 0 \\ j \in S \setminus L}} \mathcal{C}_v(x).
$$

It was proved in [4, Proposition 5.1] that the transformation (7) is invertible and its inverse is written as

$$
m(S) = \sum_{T \supseteq S} B_{|T|-|S|} \, I(v, T), \qquad S \subseteq N, \tag{8}
$$

where B_n is the nth Bernoulli number, that is the nth element of the numerical sequence $\{B_n\}_{n \in \mathbb{N}}$ defined recursively by

$$
\begin{cases}
B_0 = 1, \\
\sum_{k=0}^n \binom{n+1}{k} B_k = 0, \qquad n \in \mathbb{N} \setminus \{0\}.
\end{cases}
$$

4 Concluding remarks

We have described a sorting procedure which aggregates interacting points of view measured on qualitative scales. The aggregation function that is used is the discrete Choquet integral whose parameters are learnt form a reference set of alternatives.

The motivation of this approach is based mainly on a very general representation theorem pointing out the use of a discriminant function, but also on an axiomatic characterization of the class of Choquet integrals having a fixed number of arguments.

The use of some indices is proposed to appraise the overall importance of points of view as well as the interaction existing among them.

The next step will be to measure the quality of the sorting procedure with respect to the choice of the prototypic alternatives and their assignment. A research is now in progress along this line.

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