CONGRUENCE THEOREMS FOR TRAPEZOIDS

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Abstract. A congruence theorem for trapezoids provides a list of conditions that are sufficient to determine the trapezoid up to congruence. We establish such congruence theorems, the conditions being side lengths and measures of interior angles (as in the classical congruence theorems for triangles).

1. Introduction

Congruence theorems for triangles are known since the time of Euclid. For example, to determine a triangle up to congruence, it suffices to know the three side lengths. In geometry, a congruence theorem is a list of conditions that are proven to be sufficient to determine a geometrical object up to congruence. For example, to determine a square up to congruence, it suffices to know the side length. Similarly, for a rectangle it suffices to know the length of two sides that are not opposite, while for a rhombus it suffices to know the side length and the measure of one interior angle. For parallelograms, it suffices to know the length of two sides that are not opposite and the measure of one interior angle.

The aim of this work is determining all congruence theorems for trapezoids that involve side lengths (possibly specifying which side, namely whether the side is a basis or a leg) and the measure of interior angles (possibly specifying the position of the angle). Clearly, we pay attention that our list of conditions is optimal, namely that removing one of the conditions would not allow anymore to determine the trapezoid up to congruence.

Two parallel sides are called bases and the remaining two sides are called legs (unless the trapezoid is a parallelogram, there are precisely two bases and they have distinct lengths). Remark that the interior angles of a trapezoid consist of two pairs of supplementary angles. Thus, knowing two opposite angles or knowing two angles at one same basis is enough to know all four angles.

The following statement contains four congruence theorems for trapezoids involving side lengths and angle measures.

Theorem 1.1. *Any of the following list of conditions is sufficient to determine a trapezoid up to congruence (in case the measure of an angle is known, we understand that we also know the position of this angle within the trapezoid):*

(1) The length of the two bases and of the two legs, in case the trapezoid is not a parallelogram.
(2) The length of two bases, of one leg, and the measure of one interior angle at the known leg.
(3) The length of the two bases and the measure of the interior angles — in case the trapezoid is not a parallelogram.
(4) The length of one basis and of one leg — or, for a parallelogram, the length of two neighboring sides — and the measure of the interior angles.

To convey that the conditions in the above congruence theorems are optimal, we prove the following:

Theorem 1.2. *Any of the following list of conditions is not sufficient to determine a trapezoid up to congruence:*

(1a) The length of the four sides, in case of a parallelogram.

(1b) The length of three sides.
(2a) The length of three sides and the measure of one interior angle in case that the trapezoid is not a parallelogram and the unknown side is a basis.
(2b) The length of the two bases, of one leg, and the interior angles at the unknown leg.
(2c) The length of the one basis, of one leg, and the interior angles at the known leg.
(3a) The length of two parallel sides and the measure of the interior angles, in case of a parallelogram.
(3b) The length of two legs and the measure of the interior angles.
(4) The length of two bases and the list of measures of the interior angles without specifying their position.

We may count the information of being a trapezoid as one condition for convex quadrilaterals (namely, there must be a pair of opposite parallel sides). Then, recalling that the two interior angles adjacent to a basis determine the other two angles, the four congruence theorems that we presented have five conditions each, beyond possibly excluding parallelograms. This makes sense, as a quadrilateral has, up to congruence, five degrees of freedom. Indeed, up to congruence, we may fix the position of one vertex and the halfline containing an adjacent vertex; then with one parameter the adjacent vertex is determined and the four coordinates of the remaining two vertices are four additional parameters.

The outline of the paper is as follows: The four assertions in Theorem 1.1 are proven separately as four theorems in Section 3. Theorem 1.2 is proven in Section 2 (with a suitable collection of counterexamples). We conclude with a section devoted to congruence theorems for convex quadrilateral, proving in particular the following result:

Theorem 1.3. *A convex quadrilateral that is not a trapezoid is determined up to congruence if we know the lengths of two opposite sides and the interior angles (knowing their position with respect to the known sides).*

This paper can be seen as the continuation of [4], which in particular contains a list of congruence theorems for convex quadrilaterals involving sides and angles. We also point out that Theorem 1.2 (4) relates to the problem addressed in the paper [1], where the relative position of sides and angles in the quadrilateral is not fully specified. For congruence theorems for convex quadrilaterals involving heights, we refer to [2]. Finally, it is possible to formulate congruence theorems for convex polygons, see for example [3] (the convexity assumption being crucial to have a more moderate list of assumptions).

2. Counterexamples to congruence theorems for trapezoids

In this section we collect counterexamples to potential congruence theorems for trapezoids. Combining them proves Theorem 1.2.

Remark 2.1 (1a, 3a)**.** *A parallelogram is not determined up to congruence if we know its side lengths. Indeed, it is possible — fixing one basis — to rotate the two neighboring sides by the same angle (as to keep them parallel), thus obtain again a parallelogram with the same side lengths as before and with arbitrary interior angles. Alternatively, if we know the angles of the parallelogram and the length of two opposite sides, then it would be possible to increase or decrease the length of the other pair of opposite sides, resulting in a non-congruent parallelogram.*

Remark 2.2 (1b)**.** *Knowing the length of three sides is not enough to determine a trapezoid up to congruence. By Remark 2.1 we may suppose that the trapezoid is not a parallelogram. If the unknown side is a basis, fix the known basis and the length of the two legs. Any sufficiently small rotation of the two legs (making two compatible rotations so that the two bases are still parallel) changes the trapezoid height, the interior angles, and the length of the unknown basis. See Figure 1,* *where the points A, B and the lengths a, c, d are fixed. If the unknown side is a leg, fix one basis. A sufficiently small rotation of the known leg (rotating also the other basis so that it is still parallel to the opposite side) changes the trapezoid height, the interior angles, and the length of the unknown leg. See Figure 2.*

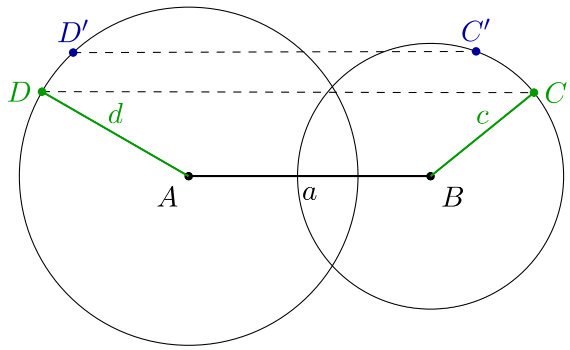

FIGURE 1. Knowing the length of two legs and one basis is not sufficient.

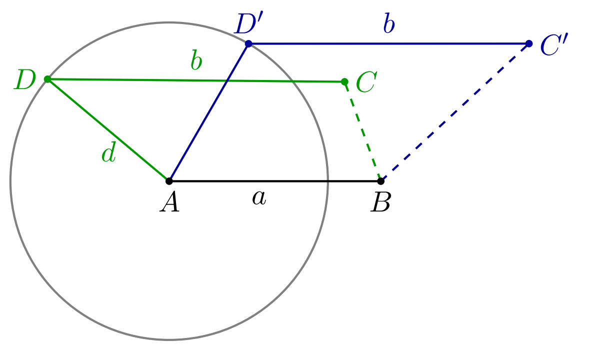

FIGURE 2. Knowing the length of two bases and one leg is not sufficient.

Remark 2.3 (3b)**.** *Knowing the length of one basis and the measure of the interior angles is not enough to determine a trapezoid up to congruence, as we could increase/decrease the trapezoid height with a parallel shift of the other basis (while fixing the halflines containing the legs). See Figure 3, where the quantities a, α, β are fixed. In a completely analogous way, knowing the length of two legs and the interior angles is not sufficient, as we could make a parallel shift of one leg without changing neither the length of the legs nor the interior angles.*

Remark 2.4 (2c)**.** *Knowing the length of one basis, one leg and the interior angle in between is not enough to determine a trapezoid up to congruence. Indeed, it would be possible to shorten or to extend the other basis without affecting the given quantities. Similarly, if we would know instead the other angle at the known basis, we could rotate the known leg. See Figure 4, where the quantities a, b, α are fixed.*
Moreover, knowing the lengths of the two legs and the interior angles is not enough to determine the trapezoid, as we could make a parallel shift of one leg preserving the interior angles, the trapezoid height, and the length of the legs.

Remark 2.5 (2a)**.** *Knowing the length of two legs, of one basis, and one interior angle is not sufficient to determine the trapezoid. As shown in Figure 5, if the leg opposite to the known angle is longer than the trapezoid height (and the known basis is sufficiently long) one may flip this leg*

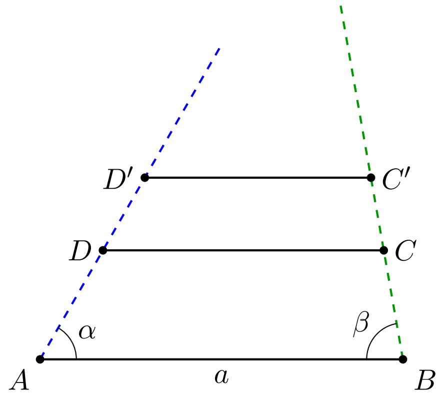

FIGURE 3. Knowing the length of one basis and the interior angles is not sufficient.

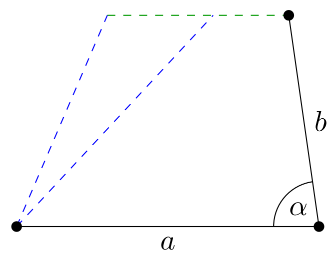

FIGURE 4. Knowing the length of one basis and of one leg and the measure of the interior angle in between is not sufficient.

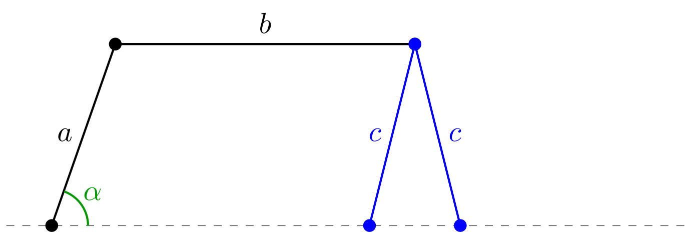

FIGURE 5. Knowing three sides and one angle is not sufficient.

(at the vertex on the known basis): this preserves the known quantities but it gives a trapezoid with a different length of the unknown basis.

Remark 2.6 (2b)**.** *A trapezoid is not determined up to congruence if we know the lengths of the two bases, one leg, and the interior angles at the unknown leg. Indeed, we can construct a counterexample as follows. Start with an acute isosceles triangle ADD' with apex A. Consider a halfline through A, obtained by rotating the halfline containing AD' while enlarging the angle at A. Then choose a point B on this halfline so that the segment AB crosses the line containing DD'. Then consider the line ℓ through B that is parallel to DD'. The line ℓ then contains points C and C' such that CD and $C'D'$ are parallel to AB: given the parallelism, the segments CD and $C'D'$ have the same length. In this way, we have constructed two trapezoids $ABCD$ and $AB'CD'$ with the base AB in common, such that the opposite bases and the legs AD and AD' pairwise have the same lengths. Moreover, the angle at B is common, and its supplementary angles at C and C' are the same. Nevertheless, the two trapezoids are not congruent because they don't have the same interior angles (their respective angles at A are different). See Figure 6.*

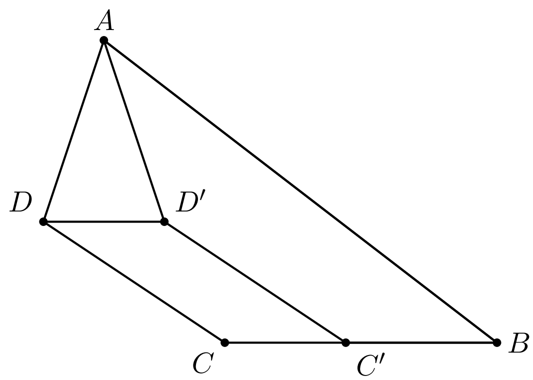

FIGURE 6. Knowing three sides and two angles is not sufficient.

Remark 2.7 (4)**.** *Knowing the lengths of the two bases and the list of values of the interior angles is not sufficient to determine a trapezoid up to congruence. Indeed, let* $\alpha, \beta, \pi-\alpha, \pi-\beta$ *be the angle values, with* $\alpha, \beta \leq \frac{\pi}{2}$*. We may suppose that* $\alpha \neq \beta$ *so that the trapezoid is not a parallelogram. Suppose that* $\beta \neq \frac{\pi}{2}$ *and that* α *is so small that both* $(\pi-\alpha)+\beta$ *and* $(\pi-\alpha)+(\pi-\beta)$ *are larger than* π*. Then at the smaller base* AB*, we could have the pair of angles* $\pi-\alpha$ *and* $\pi-\beta$ *or the pair* $\pi-\alpha$ *and* β*. In the former case, we obtain an acute trapezoid, and in the latter case, an obtuse trapezoid. These two trapezoids have the smaller base in common and the same list of values for the interior angles. By adjusting the heights of the two trapezoids, we can achieve that the length of the larger base* CD *and* CD' *is also the same. See Figure 7.*

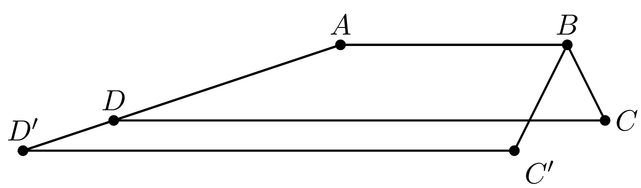

FIGURE 7. Knowing two bases and the list of the angle measures is not sufficient.

3. Congruence Theorems

We let A, B, C, D be the vertices of a trapezoid in cyclic order, with AB and CD the bases. Up to congruence, we may suppose that (in a Cartesian coordinate system) A is the origin and B is on the positive halfline of the x-axis.

Theorem 3.1 (BBLL non-parallelogram)**.** *A non-parallelogram trapezoid is determined up to congruence if we know the lengths of the two bases and the two legs.*

Proof. Let AB and CD have length a, and b, respectively and call α the angle $\widehat{DAB}$. We call L the length of the leg DA and (as the trapezoid is not a parallelogram) we have $a \neq b$. With a suitable choice of coordinates, we have $A = (0,0)$ and $B = (a, 0)$ and $D = (L\cos\alpha, L\sin\alpha)$ and $C = (b + L\cos\alpha, L\sin\alpha)$.

The length of the segment BC is for any $\alpha \in (0,\pi)$

$$\begin{aligned} f(\alpha) &= ||(b + L\cos\alpha, L\sin\alpha) - (a,0)|| \\ &= \sqrt{(b + L\cos\alpha - a)^2 + (L\sin\alpha)^2} \\ &= \sqrt{a^2 + b^2 + L^2 - 2ab + 2(b-a)L\cos\alpha}\,. \end{aligned}$$

The first derivative of the function f is

$$f'(\alpha) = \frac{2(a-b)L\sin\alpha}{2\sqrt{a^2 + b^2 + L^2 - 2ab + 2(b-a)L\cos\alpha}} \neq 0\,.$$

Since $f(\alpha)$ is strictly monotone, the value of α is determined by the length of BC. Thus the positions of A, B, C, D are determined. □

Theorem 3.2 (BBLA known-leg)**.** *A trapezoid is determined up to congruence if we know the length of the two bases, the length of one leg, and the measure of one interior angle at the known leg (knowing its position with respect to the bases).*

Proof. Call α and $\pi - \alpha$ the interior angles at the known leg. Up to congruence, we may fix the position of the known leg and the half-plane (with respect to such leg) in which the trapezoid lies. Then we can also fix the position of the two bases because we know their length and the interior angle they form with the known leg. Thus the position of all vertices of the trapezoid is determined, see Figure 8.

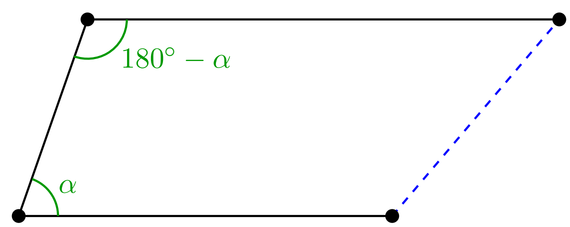

FIGURE 8. A trapezoid is determined by the length of the bases and one leg and the measure of one interior angle at this leg.

□

Theorem 3.3 (BBAA non-parallelogram)**.** *A non-parallelogram trapezoid is determined up to congruence by the lengths of its bases and the values of the interior angles, knowing the positions of these angles with respect to the two bases.*

Proof. Up to congruence, we may fix the larger base and the halflines containing the two legs. By varying the trapezoid's height, the length of the smaller base changes. Thus, the height is determined as we know the lengths of this base. We deduce that the trapezoid is determined, as we cannot change the positions of its four vertices. □

Remark 3.1. *The following congruence theorem is immediate: a parallelogram is determined up to congruence by its side lengths and the values of its interior angles. We observe that knowing the relative position of the angles of the parallelogram does not matter (up to congruence) because at each side there are two supplementary angles.*

Theorem 3.4 (BLAA non-parallelogram)**.** *A non-parallelogram trapezoid is determined up to congruence by the length of one base, the length of one leg, and the value of the interior angles (specifying the position of the angles with respect to the bases).*

Proof. Up to congruence, we may fix the known base AB and the halflines containing the two legs. Assuming that we know the length of BC, we have also determined the position of C. Finally, the remaining trapezoid vertex D is the intersection of the halfline containing the leg AD with the line from C that is parallel to AB. □

4. Congruence theorems for convex quadrilaterals

Now we consider some further congruence theorems for convex quadrilaterals involving side lenghts and measures of interior angles. We let $ABCD$ be a convex quadrilateral, where the vertices are in cyclic order and we call α (respectively, β, γ, δ) the interior angle at A (respectively, at B,C,D). We observe that knowing three angles is equivalent to knowing four angles because we have $\alpha + \beta + \gamma + \delta = 2\pi$. Moreover, the interior angles are strictly between 0 and π so knowing their value is equivalent to knowing the value of their cosine.

Theorem 4.1 (4S1A)**.** *A convex quadrilateral is determined up to congruence if we know the four side lengths and one angle (knowing the angle position with respect to the sides).*

Proof. Suppose that the angle at A is known. As the sides AB and AD are known, by the SAS congruence theorem for triangles, we know the triangle ABD up to congruence, and in particular the length of BD (which is a diagonal of the convex quadrilateral). Then for the triangle BCD, we know all side lengths, and hence, by the SSS congruence theorem, this triangle is also determined up to congruence. We deduce that the union of the triangles ABD and BCD, namely the convex quadrilateral, is determined up to congruence. □

Theorem 4.2 (3S2A with specific angles)**.** *A convex quadrilateral is determined up to congruence if we know three side lengths and two angles (knowing the angle's position with respect to the sides), provided that none or both of the known angles are at the unknown side.*

Proof. Let AD be the unknown side. If we know the angles β and γ, then, up to congruence, we may fix the side BC and then also the vertices A and D (because we know the halflines containing AB and CD and the lengths of these two segments).

Now suppose that we know the angles α and δ. Up to congruence, we may fix the side AB and the halfline containing AD. Extending the segment AD, while keeping the angle δ, has the effect that DC undergoes a parallel shift, moving away from AB. In particular, the distance from B to C increases. Considering that we know the length of BC, we cannot change the position of D, meaning that we know all side lengths of $ABCD$, and we may conclude by the previous congruence theorem $4S1A$. □

Remark 4.1. *A congruence quadrilateral $ABCD$ is not determined up to congruence if we know the lengths of AB, BC, and CD and the angles γ and δ. Indeed, consider the foot F of the perpendicular from B to the line containing AD. By flipping A at F while preserving the line containing AD we have not changed the length of AB (nor of BC and CD, nor the angles γ and δ). So it is possible to construct two non-congruent convex quadrilaterals satisfying the same set of conditions. See Figure 9.*

Theorem 4.3 (2S4A neighboring)**.** *A convex quadrilateral is determined up to congruence if we know the lengths of two neighboring sides and the interior angles (knowing their position with respect to the known sides).*

Proof. Suppose that we know AB, AD, and α. Then, up to congruence, we may fix the position of the vertices A, B, D. As we know the angles, we can also determine the halflines containing BC and CD, and hence we obtain the remaining vertex C as the intersection of these two halflines. □

Theorem 4.4 (2S4A opposite, non-trapezoid)**.** *A convex quadrilateral that is not a trapezoid is determined up to congruence if we know the lengths of two opposite sides and the interior angles (knowing their position with respect to the known sides).*

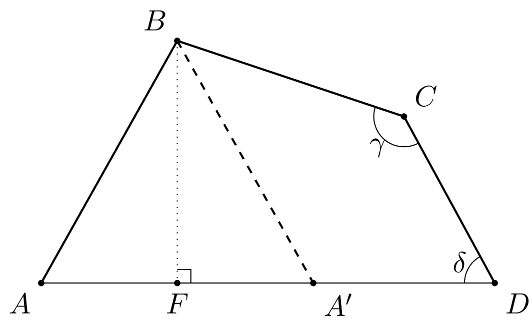

FIGURE 9. Knowing three sides and two angles is not sufficient to determine a convex quadrilateral.

Proof. Up to congruence, we may fix a known side AB and the halflines containing AD and BC. Since all interior angles are known, we can only make a parallel shift of the side CD. Considering that the sides AD and BC are not parallel, we have $\alpha + \beta \neq \pi$. Then, a parallel shift of CD that changes the position of C on the halflines AC would change the length of CD, so it is not possible. We deduce that the positions of C and D are also determined. □

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