MEE-based Adaptive State Estimator for non-Gaussian radar Measurement

Uday Kumar Singh
Department of ECE
SRMIST, Kattankulathur
Chennai, India
udaykums@srmist.edu.in

Mohammad Alaee-Kerahroodi
SnT, University of Luxembourg
Luxembourg, Luxembourg
mohammad.alaee@uni.lu

M. R. Bhavani Shankar
SnT, University of Luxembourg
Luxembourg, Luxembourg
bhavani.shankar@uni.lu

Rama Rao Thipparaju
Department of ECE
SRMIST, Kattankulathur
Chennai, India
ramaraot@srmist.edu.in

***Abstract*—In radar, particularly in nonlinear scenarios, the estimation of targets' kinematic parameters and track updating is carried out using nonlinear extensions of the Kalman filter (KF). The two most widely employed extensions of the KF are the extended Kalman filter (EKF) and the unscented Kalman filter (UKF). Despite claims that EKF and UKF better handle nonlinearity, they rely on approximation methods involving Jacobian and sigma points, respectively. Consequently, the approximation of nonlinearity raises concerns about the suitability of EKF and UKF when dealing with highly nonlinear models. Moreover, they heavily depend on the Gaussianity assumption and employ the standard minimal mean square error (MMSE) criterion for filtering. In the literature, to address non-Gaussianity, the minimum error entropy (MEE)-based EKF has been extensively studied, purportedly offering superior performance over MMSE in handling non-Gaussianity. Nevertheless, like EKF, MEE-based EKF is also contingent on precise knowledge of the nature of nonlinearity. In addressing the aforementioned issues in EKF and MEE-based EKF, this work proposes a unified solution to handle both nonlinearity and non-Gaussianity in the standard state estimation problem. The proposed approach suggests combining the kernel recursive MEE (KRMEE)-based adaptive filter with the MEE-based EKF, resulting in a filter named EKF-KRMEE. The EKF-KRMEE filter adaptively estimates nonlinearity with each new radar measurement, and the evoked MEE criterion addresses the non-Gaussianity of the measurements. Computer simulations are employed to demonstrate the superiority of the proposed EKF-KRMEE filter over conventional EKF and MEE-based EKF.**

***Index Terms*—EKF, MMSE, MEE-based EKF, EKF-KRMEE, MEE**

I. INTRODUCTION

In radar systems, the tracker plays a crucial part in merging the discrete detections, thereby extracting meaningful tracks from noisy detections [1]. Within a tracker, post-data association, the Kalman filter is employed to update the states of the tracks and estimate the kinematic parameters of the targets, including position, velocity, and acceleration. If the radar reports detections in Euclidean space, the Kalman filter (KF) is the appropriate choice. However, when the radar reports detections in cylindrical or spherical coordinates, the nonlinear extensions of the KF, specifically the extended Kalman filter (EKF) and unscented Kalman filter (UKF), prove to be better choices over KF [2], [3]. Nevertheless, to implement the nonlinear extensions of KF (EKF and UKF), knowledge of the exact nonlinear measurement function ($\mathbf{h}(\cdot)$) in advance is essential. In most cases, this information is not known a priori. Moreover, even if $\mathbf{h}(\cdot)$ is known, EKF approximates $\mathbf{h}(\cdot)$ using Jacobian, and UKF approximates the desired distributions using sigma points. Consequently, for highly nonlinear models, the approximation leads to the suboptimal performance of EKF and UKF. The drawback of approximation is particularly severe for EKF, as linear approximations cannot effectively represent severe nonlinearities. Therefore, addressing this issue in EKF, [4] proposed the reproducing kernel Hilbert space (RKHS)-based kernel recursive least square algorithm (EKF-RKHS), which estimates/learns $\mathbf{h}(\cdot)$ using radar measurements.

The study conducted in [4] involves estimating $\mathbf{h}(\cdot)$ at each frame instant (k) and utilizes the estimated nonlinearity ($\hat{\mathbf{h}}(\cdot)$) instead of relying on the Jacobian approximation, thereby extends its applicability to any type of $\mathbf{h}(\cdot)$. Furthermore, the use of $\hat{\mathbf{h}}(\cdot)$ enhances the efficiency of EKF-RKHS compared to EKF. Beyond issues with non-linearity, EKF also encounters challenges in handling non-Gaussianity. The conventional Kalman filter KF, its nonlinear extension EKF, and its adaptive version (EKF-RKHS) inherently operate as Bayesian filters. In the realm of Bayesian filtering, the Gaussianity assumption is made because the mean and covariance alone can describe the distribution. Therefore, the mean and covariance are propagated easily through the system and measurement models to obtain a posterior probability distribution function (PDF). Also, the Gaussianity assumptions make the calculations of the posterior PDF convenient. Subsequently, the posterior PDF provides the estimates of target kinematic parameters and their corresponding covariances. Nevertheless, in practice, radar measurements result from a series of signal processing algorithms, making them not necessarily Gaussian. Apart from signal processing algorithms, various environmental factors could cast doubt on the assumption of measurement Gaussianity, such as receiver noise and surrounding clutter.

When EKF and its adaptive version EKF-RKHS [4] are exposed to non-Gaussianity, their performances deteriorate as only the mean and covariance are not sufficient to describe the non-Gaussian distribution. Although particle filters are capable of dealing with nonlinearity and non-Gaussainity, they suffer from a huge computational compexity burden. Addressing this issue, in [5], an information-theoretic non-MMSE criterion called the maximum correntropy criterion (MCC)-based EKF is proposed. In [5], it is shown that MCC is a suitable choice over the conventional minimum mean square error (MMSE) criterion when dealing with non-Gaussian measurements. The literature claims this because MCC considers the higher-order statistics of estimation error in filtering. However, like EKF, the MCC-based EKF is also dependent on the approximation of $\mathbf{h}(\cdot)$ and is prone to producing inaccurate estimates under complex $\mathbf{h}(\cdot)$. Therefore, in the same line of research as EKF-RKHS, the adaptive version of MCC-based EKF (EKF-MCC-RKHS) is proposed in [6]. With the help of MCC, the EKF-MCC-RKHS provides a better estimate of $\mathbf{h}(\cdot)$, which in turn produces a better estimation of target kinematic parameters.

Apart from MCC, another non-MMSE criterion is the minimum error entropy (MEE) criterion. The MEE criterion is a significant learning criterion that has found successful applications in robust regression, classification, system identification, and adaptive filtering. Numerous experimental results demonstrate that MEE can outperform MCC in many situations, although its computational complexity is slightly higher. The MEE-based EKF is proposed in [7], where the superior performance of MEE-based EKF over EKF and MCC-based EKF is established for non-Gaussian measurements. To date, the attempt to make MEE-based EKF adaptive so that it would not rely on knowing the exact form of $\mathbf{h}(\cdot)$ is not explored. Therefore, this work aims to make the MEE-based EKF adaptive by employing the kernel recursive minimum error entropy (KRMEE) algorithm. Consequently, similar to EKF-RKHS and EKF-MCC-RKHS, the proposed adaptive version of the MEE-based EKF, named EKF-KRMEE, utilizes the estimate of $\mathbf{h}(\cdot)$ for filtering. Unlike EKF-RKHS and EKF-MCC-RKHS, EKF-KRMEE exploits MEE to address the effects of non-Gaussianity. In summary, this work aims to generalize the solution to the state estimation problem in terms of measurement non-linearity and non-Gaussianity. The following summarizes the main contributions of this work:

1) The superiority of MEE-based EKF over EKF and MCC-based EKF is established for non-Gaussian radar measurements.
2) To estimate $\mathbf{h}(\cdot)$, the MEE-based EKF is made adaptive by combining MEE-based EKF with RKHS-based KRMEE; hence, the resulting adaptive filter is named EKF-KRMEE.

The organization of the paper is as follows. In Section II, the problem statement is described in detail. Next, Section III, covers the description of the proposed filtering scheme. Further, in section IV, simulations are performed over the practical target motion model. Lastly, conclusions are drawn in Section V.

Notations: Scalar variables (constants) are denoted by lower (upper) case letters. Vectors (matrices) are denoted by boldface lower (upper) case letters. Superscripts $(.)^T$, $(.)^H$ and $(.)^*$ denote matrix/vector transpose, complex conjugate transpose, and scalar complex conjugation operation respectively. $\mathbb{E}[.]$ denotes statistical expectation and $\mathbb{R}$ denote the set of real numbers. $\mathbf{I}_n$ denotes the identity matrix of cardinality n and $\otimes$ denotes the Kronecker product.

II. PROBLEM FORMULATION

Let at k^{th} frame instant the radar sensor is reporting the noisy measurement $\mathbf{y}_k \in \mathbb{R}^{m\times 1} = [\hat{r}_k, \hat{\theta}_k, \hat{\phi}_k]^T$, where $\hat{r}_k \in \mathbb{R}$, $\hat{\theta}_k \in \mathbb{R}$, and $\hat{\phi}_k \in \mathbb{R}$ are the estimates of targets radial range, azimuth angle, and elevation angle, respectively. After observing $\mathbf{y}_k$, the challenge is to extract the positions ($[x_k, y_k, z_k]^T$), velocities ($[v_{xk}, v_{y_k}, v_{zk}]^T$), and accelerations ($[a_{xk}, a_{y_k}, a_{zk}]^T$) of the target in Cartesian coordinate. Therefore, along with reducing the effect of the perturbations, the estimator has to do the coordinate conversion, which is the standard nonlinear state space estimation problem.

In the following subsection, the target measurement model and motion model for the state space estimator are defined.

A. Target Measurement Model

For the implementation of the state space estimator, the measurement $\mathbf{y}_k$ is assumed to be of the following form

$$\mathbf{y}_k = \mathbf{h}(\mathbf{s}_k) + \mathbf{w}_k, \tag{1}$$

where $\mathbf{h}(\cdot)$ is the measurement function, $\mathbf{s}_k \in \mathbb{R}^{n\times 1} = [x_k, v_{xk}, a_{xk}, y_k, v_{y_k}, a_{y_k}, z_k, v_{zk}, a_{zk}]^T$ is the target states for continuous acceleration (CA) target motion model, and $\mathbf{w}_k \in \mathbb{R}^{m\times 1}$ is an error in modeling $\mathbf{y}_k$ considering the estimation and surrounding noises with statistical mean and covariance as $\mathbb{E}[\mathbf{w}_k] = \boldsymbol{\mu}_m \in \mathbb{R}^{m\times 1}$ and $\mathbb{E}[(\mathbf{w}_k - \boldsymbol{\mu}_m)(\mathbf{w}_k - \boldsymbol{\mu}_m)^T] = \mathbf{R}_m \in \mathbb{R}^{m\times m}$, respectively.

B. Target Motion Model

In (1), the target states $\mathbf{s}_k$ is continuously evolving with k and the CA motion model can be represented as follows

$$\mathbf{s}_k = \mathbf{f}(\mathbf{s}_{k-1}) + \mathbf{u}_k, \tag{2}$$

where, $\mathbf{s}_{k-1} \in \mathbb{R}^{n\times 1}$ is the target's state vector at $k-1^{st}$ instant, $\mathbf{f}(\cdot) \in \mathbb{R}^{n\times 1}$ governs how $\mathbf{s}_{k-1}$ evolves with k, and $\mathbf{u}_k \in \mathbb{R}^{n\times 1}$ modeling the error in the evolution of $\mathbf{s}_k$.

In (2), $\mathbf{f}(s_{k-1})$ is given by

$$\mathbf{f}(\mathbf{s}_{k-1}) = [\mathbf{I}_3 \otimes \mathcal{F}]\mathbf{s}_{k-1}, \tag{3}$$

where $\mathcal{F} = \begin{bmatrix} 1 & T_{\text{cpi}} & 0.5T_{\text{cpi}}^2 \\ 0 & 1 & T_{\text{cpi}} \\ 0 & 0 & 1 \end{bmatrix}$, and T_{cpi} is the CPI or filter update time.

In this work, $\mathbf{u}_k$ is considered to be Gaussian distributed with zero mean vector and known covariance matrix $\mathbf{Q}_u = \mathbb{E}[\mathbf{u}_k {\mathbf{u}_k}^T]$, s.t. $\mathbf{u}_k \sim \mathcal{N}_{\mathbb{R}}(\mathbf{0}, \mathbf{Q}_u)$

$$\mathbf{Q}_u = [\mathbf{I}_3 \otimes \mathcal{T}]\sigma_a^2,$$

where σ_a^2 is the acceleration variance, and

$$\mathcal{T} = \begin{bmatrix} \frac{T_{\text{cpi}}^4}{4} & \frac{T_{\text{cpi}}^3}{2} & \frac{T_{\text{cpi}}^2}{2} \\ \frac{T_{\text{cpi}}^3}{2} & T_{\text{cpi}}^2 & T_{\text{cpi}} \\ \frac{T_{\text{cpi}}^2}{2} & T_{\text{cpi}} & 1 \end{bmatrix}.$$

Further, referring to (1) and (2), it is worth noting that $\mathbf{s}_k$ is in Cartesian coordinates, and $\mathbf{y}_k$ is in spherical coordinates;

consequently, the $\mathbf{h}(\cdot)$ is nonlinear. Therefore, to estimate the hidden $\mathbf{s}_k$ from observable $\mathbf{y}_k$, a nonlinear extension of the Kalman filter (EKF) and its adaptive version (EKF-RKHS) could be used. However, EKF and EKF-RKHS both use the MMSE criterion, which considers only second-order error statistics and is suitable only when $\mathbf{w}_k$ is assumed to be Gaussian. In the context of the above statement, as $\mathbf{w}_k$ models the estimation and surrounding noises, the Gaussianity assumption is not certain. Therefore, in practice, the EKF may perform worse when used to estimate $\mathbf{s}_k$ from non-Gaussian $\mathbf{y}_k$.

In the literature, to make EKF free of the Gaussianity assumption, an ample amount of research has been done. Notably, in [5], the MCC and in [7], the MEE is suggested as a viable option to MMSE. Later, in [7], it is shown via a simulation study that if $\mathbf{w}_k$ is non-Gaussian distributed, for instance, in a multi-modal distribution, the MEE outperforms MCC.

III. PROPOSED EKF-KRMEE FILTER

In this section, the MEE criterion is described in detail. Subsequently, the implementation of MEE-based EKF is discussed. The estimation of $\mathbf{h}(\cdot)$, using the proposed adaptive version of MEE-based EKF: EKF-KRMEE is discussed next.

A. Minimum Error Entropy Criterion

Unlike the MMSE and MCC, the MEE minimizes the information contained in the error ($e \in \mathbb{R}$). The information contained in e is measured by Reniy's entropy defined as

$$H_\alpha(e) = \frac{1}{(1-\alpha)} \log \mathbb{E}[p^{\alpha-1}(e)], \tag{4}$$

where α is the entropy order, e is the error in the prediction of some random variable $Y \in \mathbb{R}$ as $\hat{f}(X) \in \mathbb{R}$, i.e, $e = Y - \hat{f}(X)$, and $p(e)$ is the distribution function of e.

Instead of minimizing (4), the specialized form of Reniy's entropy defined in (4) is considered for $\alpha = 2$, this results in the famous Shannon's entropy, defined as

$$H_2(e) = -\log \mathbb{E}[p(e)], \tag{5}$$

Subsequently, if the L samples of Y and X are available as $(x_i, y_i) \forall i = 1, \cdots, L$, then $e_i = (y_i - \hat{f}(x_i)) \forall i = 1, \cdots, L$ and $p(e)$ is estimated using kernel $G_\sigma(e - e_i)$ as

$$\hat{p}(e) = \frac{1}{L} \sum_{i=1}^{L} G_\sigma(e - e_i), \tag{6}$$

where $G_\sigma(e - e_i)$ is the most widely used Gaussian kernel function with width σ, i.e, $G_\sigma(e - e_i) = \frac{1}{\sqrt{(2\pi)}\sigma} \exp(-\frac{(e-e_i)^2}{2\sigma^2})$.

Referring to (5), it is evident that minimizing $H_2(e)$ is equivalent to maximizing the argument ($\mathbb{E}[p(e)]$). Consequently, MEE maximizes $\mathbb{E}[p(e)]$, which can be approximated as

$$\mathcal{J}(X) = \frac{1}{L} \sum_{j=1}^{L} p(e_j). \tag{7}$$

Substituting (6) into (7), yields

$$\mathcal{J}(X) = \frac{1}{L^2} \sum_{j=1}^{L} \sum_{i=1}^{L} G_\sigma(e_j - e_i). \tag{8}$$

The MEE maximizes (7) to yield $\hat{f}(X)$ closer to Y as

$$Y \approx \hat{f}(X) = \max_{f(\cdot) \in \mathbb{R}} \frac{1}{L^2} \sum_{j=1}^{L} \sum_{i=1}^{L} G_\sigma(e_j - e_i). \tag{9}$$

In equation (9), the term $G_\sigma(e_j - e_i)$ can be expanded using Taylor's series as $G_\sigma(e_j - e_i) = 1 - \frac{(e_j - e_i)^2}{2\sigma^2} + \frac{(e_j - e_i)^4}{2\sigma^4} - \cdots$. Consequently, equation (9) incorporates higher-order terms of error differences. Specifically, unlike MCC, MEE takes into account higher-order error difference statistics in its estimation process. This consideration of higher-order statistics enhances the robustness of the MEE-based filter against non-Gaussianity compared to MCC and MMSE. Furthermore, MEE employs the error difference $(e_j - e_i)$, providing MEE with an advantage over MCC, which only considers e_i. Beyond these crucial distinctions from MMSE and MCC, the parameter σ prevents the MEE-based filter from becoming unstable for high errors, a situation that commonly arises with multi-modal distributed noises. With these points established, the subsequent step involves describing the MEE-based Extended Kalman Filter (EKF), which utilizes MEE to estimate $\mathbf{s}_k$ from non-Gaussian $\mathbf{y}_k$.

B. MEE-based EKF

For the motion and measurement model shown in (1) and (2), respectively, let at k^{th} time instance, the estimate of $\mathbf{s}_{k-1}$ be $\hat{\mathbf{s}}_{k-1}$ and the corresponding error covariance matrix is $\mathbf{P}_{k-1} = \mathbb{E}[(\mathbf{s}_{k-1} - \hat{\mathbf{s}}_{k-1})(\mathbf{s}_{k-1} - \hat{\mathbf{s}}_{k-1})^T]$. Similar to EKF [2], the MEE-based EKF propagates $\hat{\mathbf{s}}_{k-1}$ and $\mathbf{P}_{k-1}$ iteratively and yield the final estimate of $\mathbf{s}_k$ in two steps a) prediction and b) update. In prediction step, the MEE-based EKF, predicts $\mathbf{s}_k$ and $\mathbf{P}_k$ with the help of following equations:

$$\begin{aligned} \mathbf{s}_k^p &= \mathbf{F}_{k-1} \hat{\mathbf{s}}_{k-1}, \\ \mathbf{P}_k^p &= \mathbf{F}_{k-1} \mathbf{P}_{k-1} \mathbf{F}_{k-1}^T + \mathbf{Q}_u, \end{aligned} \tag{10}$$

where $\mathbf{F}_{k-1} \in \mathbb{R}^{n_s \times n_s} = \mathbf{I}_3 \otimes \mathcal{F}$ is the motion model matrix.

After prediction, in the update step, the MEE-based EKF invokes the MEE criterion, and the predictions made in (9) are updated with the help of available measurement from radar sensor ($\mathbf{y}_k$). The following equations give the updated equations of MEE-based EKF:

$$\begin{aligned} \mathbf{y}_k^p &= \mathbf{H}_k \mathbf{s}_k^p, \\ \mathbf{K}_k^{MEE} &= \left[\bar{\mathbf{P}}_{k-1}^p + \mathbf{H}_k^T \mathbf{S}\bar{\mathbf{Y}}_{k-1} + (\bar{\mathbf{Y}}\mathbf{S}_{k-1} + \mathbf{H}^T \bar{\mathbf{R}}_m \mathbf{H})\right]^{-1} \\ &\quad \times ((\bar{\mathbf{Y}}\mathbf{S}_{k-1} + \mathbf{H}^T \bar{\mathbf{R}}_m)), \\ \hat{\mathbf{s}}_k &= \mathbf{s}_k^p + \mathbf{K}_k^{MEE} (\mathbf{y}_k - \mathbf{y}_k^p), \\ \mathbf{P}_k &= \bar{\mathbf{P}}_k^p - \mathbf{K}_k^{MEE} \mathbf{H}_k \bar{\mathbf{P}}_k^p. \end{aligned} \tag{11}$$

where $\mathbf{H}_k = \left.\frac{\partial \mathbf{h}(\mathbf{s}_k)}{\partial \mathbf{s}_k}\right|_{\mathbf{s}_k=\mathbf{s}_k^p} \in \mathbb{R}^{n_m \times n_s}$ is the Jacobain matrix of $\mathbf{h}(\cdot)$ (evaluated at prediction $\mathbf{s}_k^p$), $\mathbf{y}_k^p$ is the predicted measurement, $\mathbf{K}_k^{MEE}$, $\bar{\mathbf{P}}_k^p$, $\mathbf{S}\bar{\mathbf{Y}}_{k-1}$, and $\bar{\mathbf{Y}}\mathbf{S}_{k-1}$ are the Kalman gain, predicted error covariance matrix, cross covariance matrix of $\mathbf{s}_k$ and $\mathbf{y}_k$, cross covariance matrix of $\mathbf{y}_k$ and $\mathbf{s}_k$ of MEE-based EKF, respectively, and $\hat{\mathbf{s}}_k$ and $\mathbf{P}_k$ are the final estimate of $\mathbf{s}_k$ and updated error covaraince matrix, respectively.

From (11), it can be inferred that in comparison to EKF's Kalman gain ($\mathbf{K}_k = \mathbf{P}_k^p \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k^p \mathbf{H}_k^T + \mathbf{R}_m)^{-1}$), the $\mathbf{K}_k^{MEE}$ provides a mean to tackle the effect of non-Gaussianity with the help of σ. In fact, for a suitable value of σ, $\mathbf{K}_k^{MEE}$ differs $\mathbf{K}_k$ and provides robustness against non-Gaussianity. Nevertheless, similar to the EKF, MEE-based EKF also makes a linear approximation of $\mathbf{h}(\cdot)$ via Jacobian. This approximation ceases MEE-based EKF to perform worst, especially when dealing with complex measurement models. Therefore, this work presents a unified approach of clubbing MEE-based EKF with the KRMEE algorithm and proposes a new EKF-KRMEE filter. The EKF-KRMEE first estimates the $\mathbf{h}(\cdot)$ in RKHS using the KRMEE algorithm, subsequently, the $\hat{\mathbf{h}}(\cdot)$ replaces the Jacobian in MEE-based EKF for performing further prediction and update.

C. Proposed KRMEE-based adaptive filter

In this subsection, we propose to use the KRMEE algorithm to obtain $\hat{\mathbf{h}}_k(\cdot)$. Subsequently, $\hat{\mathbf{h}}_k(\cdot)$ is used to replace the $\mathbf{H}_k$ in MEE-based EKF and the resulting filter is named EKF-KRMEE. In the realm of RKHS, the hidden state $\mathbf{s}_k$ is mapped to a high dimensional RKHS ($\mathcal{H}$) via an unknown implicit mapping function $\phi(\cdot)$ [8]–[10]. Once the $\mathbf{s}_k$ is mapped in ($\mathcal{H}$), the $\mathbf{y}_k$ would be a linear function of $\mathbf{s}_k$, which originally was the nonlinear function of $\mathbf{s}_k$. Consequently, the estimate of $\mathbf{y}_k$ in $\mathcal{H}$ can be given by the inner product ($\langle \cdot, \cdot \rangle_{\mathcal{H}}$) of $\mathbf{y}_k$ and unknown weight matrix $\mathbf{\Omega}$ as.

$$\hat{\mathbf{y}}_k = \langle \mathbf{\Omega}^T, \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}. \tag{12}$$

To reduce the estimation error ($\mathbf{e}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k$) s.t an optimized value of $\mathbf{\Omega}$ is obtained, the KRMEE algorithm maximizes the cost function defined in (9) as

$$\mathcal{J}(\mathbf{\Omega}) = \max_{\mathbf{\Omega} \in \mathcal{H}} \frac{1}{L^2} \sum_{j=1}^{L} \sum_{i=1}^{L} \gamma^{i+j} G_\sigma(e_j - e_i) - \frac{1}{2} \lambda \left\| \mathbf{\Omega} \right\|^2, \tag{13}$$

where γ is the forgetting factor s.t ($0 < \gamma \leq 1$), introduced in (13) to enhance the effect of latest estimates, and λ is the regularization factor.

Equating the gradient of (13) w.r.t Ω and equating it to zero, yields the following solution

$$\Omega = \mathbf{\Phi}_L (\mathbf{\Phi}_L^T \mathbf{\Phi}_L + \sigma^2 \gamma \mathbf{B}_L^{-1})^{-1} \mathbf{Y}_L, \tag{14}$$

where

$$\begin{cases} \gamma = \frac{L^2 \lambda}{2}, \\ \Phi_L = [\boldsymbol{\phi}(s_1), \boldsymbol{\phi}(s_2), \cdots, \boldsymbol{\phi}(s_L)], \\ \mathbf{Y}_L = [\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_L]^T, \\ \mathbf{B}_L = \mathbf{M}_L - \mathbf{N}_L, \\ \mathbf{M}_L(i,j) = \gamma^{i+j} G_\sigma(e_j - e_i), \forall i,j = 1, \cdots, L, \\ \mathbf{N}_L(i,j) = \begin{cases} \sum_{k=1}^{L} \gamma^{i+k} G_\sigma(e_j - e_k), i = j \\ \mathbf{0}, i \neq j \end{cases} \end{cases}$$

The (14) can be represented as $\Omega = \mathbf{\Phi}_L \mathbf{a}_L$, where $\mathbf{a}_L = (\mathbf{\Phi}_L^T \mathbf{\Phi}_L + \sigma^2 \gamma \mathbf{B}_L^{-1})^{-1} \mathbf{Y}_L$, subsequently, substituting (14) into (12) yields

$$\hat{\mathbf{y}}_k = [\langle \boldsymbol{\phi}(\mathbf{s}_1^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}, \cdots, \langle \boldsymbol{\phi}(\mathbf{s}_L^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}] \mathbf{a}_L^T. \tag{15}$$

Comparing (15) with (1), infer that the $\hat{\mathbf{h}}(\cdot)$ is given by

$$\hat{\mathbf{h}}(\mathbf{s}_k) = [\langle \boldsymbol{\phi}(\mathbf{s}_1^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}, \cdots, \langle \boldsymbol{\phi}(\mathbf{s}_L^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}]. \tag{16}$$

In (16), the computation of $\langle \boldsymbol{\phi}(\mathbf{s}_L^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}$, requires the knowledge of $\phi(\cdot)$. However, $\phi(\cdot)$ is an implicit mapping in $\mathcal{H}$ whose exact form is unknown. Therefore, to compute (16), the Mercer's theorem is evoked, which states that $\langle \boldsymbol{\phi}(\mathbf{s}_L^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}}$ could be effectively computed in the Euclidean space via Gaussian kernel $\kappa(\cdot,\cdot)$ as

$$\hat{\mathbf{h}}(\mathbf{s}_k) = [\kappa(\mathbf{s}_1, \mathbf{s}_k), \cdots, \kappa(\mathbf{s}_l, \mathbf{s}_k), \cdots, \kappa(\mathbf{s}_L, \mathbf{s}_k)], \tag{17}$$

where

$$\langle \boldsymbol{\phi}(\mathbf{s}_l^T), \boldsymbol{\phi}(\mathbf{s}_k) \rangle_{\mathcal{H}} = \kappa(\mathbf{s}_l, \mathbf{s}_k) = \frac{1}{\sqrt{(2\pi)}\sigma_1} \exp\left(-\frac{\left\| \mathbf{s}_l - \mathbf{s}_k \right\|^2}{2\sigma_1^2} \right)$$

From (17), it can be easily inferred that, contrary to MEE-based EKF, the exact form of $\mathbf{h}(\cdot)$ is no longer required. Instead, $\mathbf{h}(\cdot)$ can be iteratively estimated using a KRMEE-based adaptive algorithm. Additionally, $\mathbf{h}(\cdot)$ can be calculated along with the prediction and update steps of MEE-based EKF. Therefore, the combination of MEE-based EKF and KRMEE adaptive algorithm results in the proposed EKF-KRMEE adaptive filter.

The EKF-KRMEE is beneficial over MEE-based EKF and EKF for the following reasons: (1) the implementation of EKF-KRMEE is no longer restricted to knowing the exact form of $\mathbf{h}(\cdot)$, (2) using the estimate of $\mathbf{h}(\cdot)$ in place of the Jacobian makes the EKF-KRMEE a suitable choice to estimate $\mathbf{s}_k$, and (3) lastly, since the MEE criterion is evoked, the EKF-KRMEE is suitable to use in non-Gaussian scenarios and consequently provides better estimation accuracy than MEE-based EKF and MCC-based EKF.

The pseudo algorithm for EKF-KRMEE is given in Algorithm 1, where $\mathbf{z}_k$, r_k, $\mathbf{Q}_k$, $\boldsymbol{\theta}_k$, $\mathbf{e}_k$, and $\mathbf{a}_k$ are as per [11].

Algorithm 1: Implementation of EKF-KRMEE

```
Intialization:
$\mathbf{s}_0$, $\mathbf{y}(\mathbf{s}_0)$, $\hat{\mathbf{s}}_0$, $\mathbf{P}_0$, $\hat{\mathbf{h}}_0 = [\kappa(\hat{\mathbf{s}}_0, \hat{\mathbf{s}}_0)]$,
    $\mathbf{Q}_0 = (\lambda\sigma^2 + \kappa(\hat{\mathbf{s}}_0, \hat{\mathbf{s}}_0))^{-1}$, $\mathbf{a}_0 = \mathbf{Q}_0\mathbf{y}_0^T$
for $k = 1, 2, 3, \cdots, K$ do
    EKF-MEE
        Compute $\mathbf{s}_k^p$ and $\mathbf{P}_k^p$ using (6)
        $\mathbf{y}_k^p = \mathbf{a}_{k-1}^T \hat{\mathbf{h}}_{k-1}^T$
        $\hat{\mathbf{H}}_k = \left\{ \left( \frac{\partial \hat{\mathbf{h}}_{k-1}}{\partial \hat{\mathbf{s}}_{k-1}} \Big|_{\hat{\mathbf{s}}_{k-1} = \mathbf{s}_k^p} \right) \mathbf{a}_{k-1} \right\}^T$
        Compute $\mathbf{K}_k^{MEE}$, $\hat{\mathbf{s}}_k$, and $\mathbf{P}_k$ using (7)
    KRMEE
        $\hat{\mathbf{h}}_k = [\kappa(\hat{\mathbf{s}}_0, \hat{\mathbf{s}}_k), \ldots, \kappa(\hat{\mathbf{s}}_L, \hat{\mathbf{s}}_k)]$
        $\mathbf{e}_k = \mathbf{y}_k - \mathbf{a}_{k-1}^T \hat{\mathbf{h}}_k^T$
        $\mathbf{z}_k = \mathbf{Q}_{k-1} \hat{\mathbf{h}}_{k-1}^T$
        $\phi_k = \sum_{i=1}^{L} \gamma^{i+L} G_\sigma(e_k - e_i)$,
        $r_k = \gamma_2 \sigma^2 \boldsymbol{\phi}_k + \kappa(\hat{\mathbf{s}}_k, \hat{\mathbf{s}}_k) - \mathbf{z}_k^T \hat{\mathbf{h}}_{k-1}^T$
        $\mathbf{Q}_k = r_k^{-1} \begin{bmatrix} \mathbf{Q}_{k-1} r_k + \mathbf{z}_k \mathbf{z}_k^T & -\mathbf{z}_k \\ -\mathbf{z}_k^T & 1 \end{bmatrix}$
        $\mathbf{a}_k = \begin{bmatrix} \mathbf{a}_{k-1} - r_k^{-1} \mathbf{z}_k \mathbf{e}_k^T \\ r_k^{-1} \mathbf{e}_k^T \end{bmatrix}$
end
```

IV. SIMULATION RESULTS AND INFERENCES

In this section, we validate the performance of the proposed technique through computer simulations. Firstly, we compare the performance of MEE-based EKF with its counterparts, namely MCC-based EKF and EKF, under measurement non-Gaussianity. Subsequently, based on the results, we draw the conclusion regarding the superiority of MEE-based EKF over MCC-based EKF and EKF. Later, the proposed adaptive EKF-KRMEE filter is compared with its nonadaptive counterpart, MEE-based EKF. Given that MEE-based EKF outperforms EKF and MCC-based EKF, it implies that their adaptive versions, EKF-RKHS and EKF-MCC-RKHS, respectively, will also be surpassed by MEE-based EKF. It is essential to note that the primary goal of this work is to introduce an adaptive algorithm for MEE-based EKF, aiming to enhance the state estimation accuracy of MEE-based EKF in non-Gaussian scenarios. Consequently, in the second part of the simulation, the proposed adaptive EKF-KRMEE filter is compared solely with MEE-based EKF.

Further, to realize the nonGaussianity in radar measurement ($\mathbf{y}_k$), the additive noise ($\mathbf{w}_k$) is modeled as a Gaussian mixture i.e. $\mathbf{w}_k \in p_1 \mathcal{N}_{\mathbb{R}}(\mathbf{0}, \mathbf{R}_w^1) + p_2 \mathcal{N}_{\mathbb{R}}(\mathbf{0}, \mathbf{R}_w^2)$, correspondingly, $\mathbf{R}_w = \sum_{g=1}^{2} p_g \mathbf{R}_w^g$, where $p1 = 0.1$, $p2 = 0.9$, $\mathbf{R}_w^1 = \sigma_{w1}^2 \mathbf{I}_{n_m}$, and $\mathbf{R}_w^2 = \sigma_{w2}^2 \mathbf{I}_{n_m}$, where $\sigma_{w1}^2 = 0.15$ and $\sigma_{w2}^2 = 20$. Out of numerous nonGaussianity, the Gaussian mixture is chosen to check the performance of MEE based filter for multimodal distribution. The simulations are performed for $K = 150$ CPI's i.e., $k = 1, 2, \cdots, 150$, $T_{cpi} = 0.01$ sec, and $\mathbf{Q}_u$ is according to (4), where $\sigma_a = 10^{-2}$. The $\hat{\mathbf{s}}_0$ and $\mathbf{P}_0$ for both scenarios are consider as $[1, 1, \cdots, 1]^{n_s \times 1}$ and $\mathbf{I}_{n_s}$, respectively. The $\mathbf{y}_k$ is assumed to be available in spherical coordinate ($\mathbf{y}_k = [\hat{r}_k, \hat{\theta}_k, \hat{\phi}_k]^T$). Consequently, the associated true form of the non-linear function: $\mathbf{h}(\cdot)$ is given by

$$\mathbf{h}(\mathbf{s}_k) = \left[\sqrt{x_k^2 + y_k^2 + z_k^2}, \tan^{-1}\left(\frac{y_k}{x_k}\right), \tan^{-1}\left(\frac{\sqrt{x_k^2 + y_k^2}}{z_k}\right) \right]^T \tag{18}$$

Please note for EKF and its nonGaussian extensions MCC-based EKF and MEE-based EKF, the exact form of $\mathbf{h}(\cdot)$ described by (18) should be known. However, the proposed adaptive filter EKF-KRMEE, adaptively estimates the $\mathbf{h}(\cdot)$ using a KRMEE algorithm and uses it for further prediction and update steps.

The target is assumed to move in an indoor scene. Correspondingly, the maximum distance traveled by the target in x, y, and z directions is limited to 10 m. The performance of the proposed and existing state estimation filters are quantified in terms of root mean square error (RMSE) along the positions (x, y, z), velocities (v_x, v_y, v_z), and accelerations (a_x, a_y, a_z) in the x, y, and z directions. As a result, the RMSEs for the elements of $\hat{\mathbf{s}}_k$ are define as

$$\text{RMSE}(i) = \sqrt{\frac{1}{K} \sum_{k=0}^{K-1} (\mathbf{s}_k(i) - \hat{\mathbf{s}}_k(i))^2}; \quad i = 1, 2, \ldots, n_s,$$

where i denotes the i^{th} element of $\mathbf{s}_k$ and $\hat{\mathbf{s}}_k$.

As shown in Fig. 1, the estimated trajectory obtained by the proposed MEE-based EKF, in comparison to MCC-based EKF and EKF, is closer to the ground truth (GT). The observation made in Fig. 1 is a result of the suitability of the MEE criterion over MCC and MMSE. Furthermore, in the literature, the MCC is also claimed to tackle non-Gaussianity, and the same is inferred from Fig. 1, where the MCC-based EKF yields a trajectory closer to GT compared to EKF. However, as the non-Gaussianity is multimodal, the MEE-based EKF works better than the MCC-based EKF. Additionally, in Fig. 2, the superiority of MEE-based EKF over MCC-based EKF and EKF is shown in terms of RMSEs of different parameters, normalized by their respective unit.

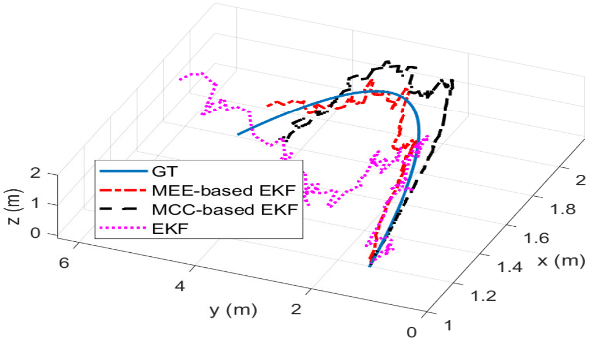

Fig. 1. Evolution of GT and estimated trajectories in 3D space with EKF, MCC-based EKF, and MEE-based EKF under non-Gaussian conditions.

It is already shown in Fig. 1 and Fig. 2, that MEE-based EKF outperforms MCC-based EKF and EKF. Subsequently,

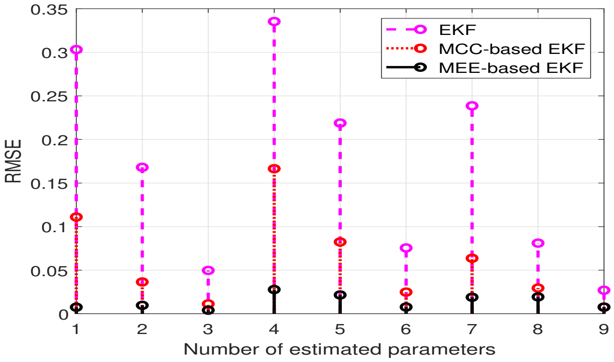

Fig. 2. RMSE in the estimation of positions, velocities, and accelerations using EKF, MCC-based EKF, and MEE-based EKF under non-Gaussian conditions.

the performance of EKF-KRMEE is compared with MEE-based EKF. In the simulations, for MEE-based EKF $\sigma = 0.95$ and for KRMEE $\sigma_1 = 5$, also, β and λ are 1 and 0.004, respectively. Further, it is depicted in Fig. 3 that EKF-KRMEE yields the estimated trajectory close to GT. Also, from Fig. 3, it can be inferred that because of estimating $\mathbf{h}(\cdot)$ and using MEE, EKF-KRMEE performs better than MEE-based EKF. Lastly, Fig. 4 validates the improved performance of EKF-KRMEE over MEE-based EKF in terms of RMSEs of all estimated parameters.

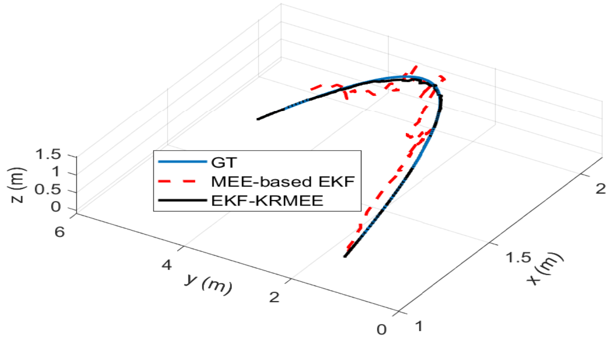

Fig. 3. Evolution of GT and estimated trajectories in 3D space with MEE-based EKF and proposed EKF-KRMEE under non-Gaussian conditions.

V. CONCLUSION

This work presents a comprehensive solution to the state estimation problem, considering arbitrary system non-linearity and perturbation distribution. Specifically, the paper addresses the challenge of handling nonlinearity in the non-Gaussian extension of the EKF, known as the MEE-based EKF. The MEE-based EKF utilizes the MEE criterion for state estimation and has demonstrated superior performance compared to its immediate counterparts, MCC-based EKF, and conventional EKF. To tackle the issue of nonlinearity, the proposed approach employs the RKHS-based adaptive KRMEE algorithm. This algorithm adapts the unknown nonlinearity $\mathbf{h}(\cdot)$ using MEE at each radar frame, eventually converging to an efficient estimate $\hat{\mathbf{h}}(\cdot)$. The resulting filter, named EKF-KRMEE, combines the strengths of MEE-based EKF and KRMEE. With the help of computer simulations, the superior efficiency of the proposed estimator compared to other state estimators is demonstrated. The proposed estimator has proven to be liable in the presence of multi-modal non-Gaussian noise modeled as a Gaussian mixture.

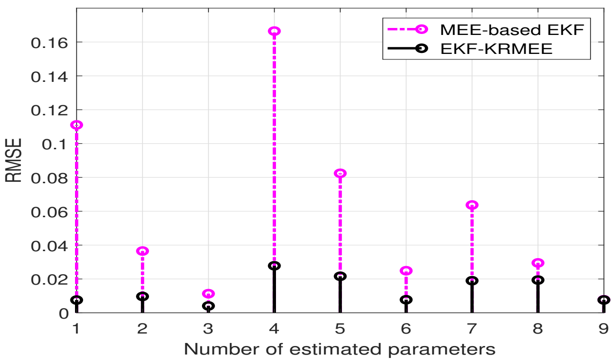

Fig. 4. RMSE in the estimation of positions, velocities, and accelerations using MEE-based EKF, and proposed EKF-KRMEE under non-Gaussian conditions.

ACKNOWLEDGMENT

The authors' work was supported by FNR BRIDGES MASTERS under grant BRIDGES2020/IS/15407066/MASTERS.

REFERENCES

[1] S. S. Blackman, "Multiple-target tracking with radar applications," *Dedham*, 1986.
[2] S. J. Julier and J. K. Uhlmann, "New extension of the Kalman filter to nonlinear systems," in *Signal Proces., sensor fusion, target recognition*, vol. 3068. Int. Soc. Optics Photon., 1997, pp. 182–194.
[3] ——, "Unscented filtering and nonlinear estimation," *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401–422, 2004.
[4] U. K. Singh, M. Alaee–Kerahroodi, and M. R. B. Shankar, "RKHS based State Estimator for Radar Sensor in Indoor Application," in *2022 IEEE Radar Conference (RadarConf22)*, 2022, pp. 01–06.
[5] X. Liu, H. Qu, J. Zhao, and B. Chen, "Extended Kalman filter under maximum correntropy criterion," in *2016 International Joint Conference on Neural Networks (IJCNN)*, 2016, pp. 1733–1737.
[6] U. K. Singh, M. Alaee-Kerahroodi, and M. R. Bhavani Shankar, "RKHS based dynamic state estimator for non-gaussian radar measurements," in *2023 IEEE Radar Conference (RadarConf23)*, 2023, pp. 1–6.
[7] B. Chen, L. Dang, Y. Gu, N. Zheng, and J. C. Príncipe, "Minimum error entropy kalman filter," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, vol. 51, no. 9, pp. 5819–5829, 2021.
[8] W. Liu, P. P. Pokharel, and J. C. Principe, "The kernel least-mean-square algorithm," *IEEE Trans. Signal Process.*, vol. 56, no. 2, pp. 543–554, Jan. 2008.
[9] U. K. Singh, R. Mitra, V. Bhatia, and A. K. Mishra, "Kernel LMS-based estimation techniques for radar systems," *IEEE Trans. Aerosp. and Electron. Sys.*, vol. 55, no. 5, pp. 2501–2515, Oct 2019.
[10] ——, "Range and velocity estimation using kernel maximum correntropy based nonlinear estimators in non-Gaussian clutter," *IEEE Trans. Aerosp. and Electron. Sys.*, vol. 56, no. 3, pp. 1992–2004, 2020.
[11] G. Wang, X. Yang, L. Wu, Z. Fu, X. Ma, Y. He, and B. Peng, "A kernel recursive minimum error entropy adaptive filter," *Signal Processing*, vol. 193, p. 108410, 2022.