A Dynamic Logic of the Right to Know

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Abstract

Different meanings of the "right to know" can be distinguished based on the theory of normative positions. In this paper, we focus on one of them: the power to know. Intuitively, in a sender-receiver setting, the receiver's power to know whether φ is the case means that the sender is obliged to (truthfully) announce the answer if the receiver asks the question φ?. Therefore, we develop a logic called LRK for reasoning about the power to know, the obligatory announcements, and the dynamics of questions and public announcements.

1 Introduction

The aim of this paper is to provide a logical framework to reason about the notion of the right to know and its interaction with other related notions. As a theoretical-conceptual background, we rely on the theory of normative positions [22] based on the theory of Hohfeld [12, 19]. The American legal theorist Wesley Newcomb Hohfeld, after finding that the word "right" is overused in the law for meaning actually differing legal concepts, published a—later becoming very influential—paper about what the possible atomic positions are what lawyers (and lay people) tend to refer to as a right and what are the corresponding "duty positions" meaning each a legally (and logically) different concept:

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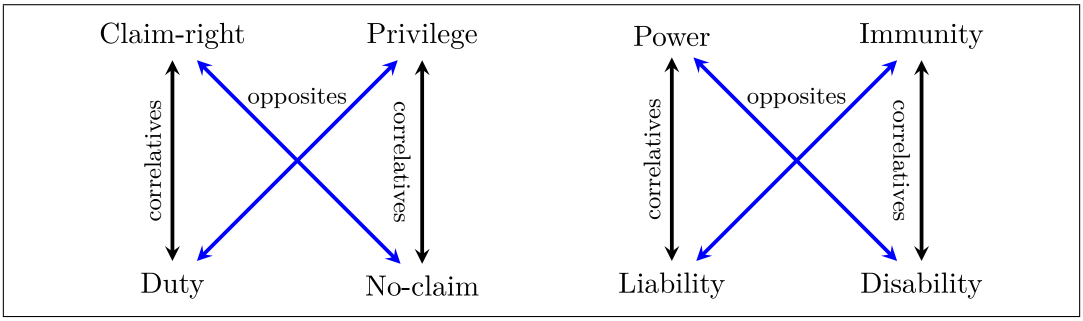

Figure 1: The Hohfeldian atomic types of rights, and their correlatives

The positions in the left square are often referred to as deontic concepts. A claim-right means the right-holder's position when the corresponding agent has a directed obligation, that is, duty, towards her. For example, a lender has a claim-right that requires the debtor to repay the borrowed money on time. The Hohfeldian privilege to φ is actually a directed version of what is referred to as weak permission in deontic logic: it means that the corresponding agent has no claim-right against us to $\neg\varphi$. The positions in the right-hand square are usually referred to as capacitive (or capacitative) concepts: a power(-right) means a position in which the power-holder agent can change the corresponding—in Hohfeldian terms—liable agent's normative positions, like imposing a duty on him (with executing an action, see details in [19]). An immunity hence means the position when the corresponding agent lacks such a capacity as being disabled to execute such an action resulting in a change in the right-holder agent's normative positions.

Using this analysis, we can differentiate between four different meanings of the right to know: the privilege/claim-right/power/immunity to know. The difference between the four meanings of the right to know can be illustrated by the following example [27]: suppose that an agent i has been tested for some disease by his doctor. It is usually mentioned in the healthcare acts of the different legal systems that i has the right to know the test result. However, this can be interpreted in four different ways. First, i has a *privilege-right to know* the result, which means that i has no duty not to know them. Second, i has a *claim-right to know* the result, meaning that his doctor has a duty to inform him about the result. Third, i has a *power-right to know* the result, meaning that his doctor has a duty to inform them if i requests it. Last, i has an *immunity-right to know* the results, which protects him from his doctor taking away or altering his claim-right to know the results.

In the logical literature to date, only the privilege- and claim-right to know have received explicit attention, see, e.g., [20]. In this paper, we will investigate the right to know as a power. Power is characterized in [19] as a potential: the agent having the power is able to execute an action resulting in the counterparty's normative positions changing, e.g. a duty arising. Thus in this case: the patient's asking about the test results creates the doctor's duty to tell.

The notion of the power to know can also be found extensively in database theory under the name of "the right/permission to access" (see, e.g., [7]). The logical characterization of "the right/permission to access" is crucial for practical problems like maintaining security in databases. The point of the problem is that the database has to answer the users' queries while complying with certain security policies (e.g., privacy policies). Many factors affecting the solutions to the problem have been identified in the literature, e.g., the representation of the database, the expressiveness of the query language, the space of admissible responses, the initial knowledge of the user, etc [5]. However, the expressiveness of the language for specifying security policies has somehow been overlooked. To the best of our knowledge, all existing works on the topic consider the permitted, forbidden, and obligatory knowledge/belief of users as the only components of a security policy, see, e.g., [5, 7, 1]. However, as a counter-example, it is stated in the General Data Protection Regulation of Europe (GDPR) that "A data subject should have *the right of access* to personal data which have been collected concerning him or her".

In this paper, we develop a Logic of the Right to Know, LRK, to reason about the power-type of right to know whether something is the case (we often will refer to it as a "power to know" in what follows). As suggested by the previous examples, the notion of the power to know is closely intertwined with other notions such as the obligation to inform, public announcements, and the dynamics of questions. Therefore, LRK is devised such that these notions can also be expressed.

This paper is an extension to [16]. The main improvement is a sound and complete axiomatization of LRK. The paper is structured as follows. In the next section, we introduce the language and semantics of LRK. Section 3 and Section 4 are devoted to some semantic results and the expressive power of fragments of LRK, respectively. We propose an axiomatization of LRK in Section 5 and show its completeness in Section 6. We discuss related literature in Section 7 and conclude with Section 8.

2 Language and Semantics

In this section, we introduce the language and semantics of LRK and illustrate them with some examples. The scenarios that LRK is intended to characterize are communications between two agents[1] where the information can only be transmitted from one agent (the sender/speaker, indicated by s) to the other (the receiver/addressee, indicated by r). The sender is further subject to some security policies such as privacy policies. These scenarios are common in our lives, e.g., communications between a database and its users, conversations between a doctor and their patients, etc. We fix the role of the sender making only one of the agents able to make announcements. The restriction may seem to be unnatural. But, for simplicity, we will only consider the restricted scenarios. We also assume that the sender can only make *truthful* announcements. The following will serve as the running example of this paper:

Example 1. Two patients a and b have been tested for some diseases by the same doctor. The policy consists of the following three clauses: (1) The doctor is obliged to inform the patient a about his test result; (2) It is forbidden that patient a know the test result of patient b; (3) The patient a has the power to know whether the cheaper medicine is as good as the expensive one. Suppose that the results for both a and b are positive and the cheaper medicine is as effective as the expensive one. Let the sender be the doctor and the receiver the patient a. The doctor needs to decide which information should be informed.

Let PROP be a countable infinite set of propositional variables (or atoms).

Definition 2. The language $\mathcal{L}$ is given by the following BNF grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid U\varphi \mid Q\varphi \mid K_r\varphi \mid \mathbb{O}_s\varphi \mid [\varphi?]\varphi \mid [\varphi!]\varphi$$

where $p \in$ PROP. Other boolean connectives are defined as usual. In particular, $\varphi \leftrightarrow \psi := \neg((\varphi \to \psi) \to \neg(\psi \to \varphi))$. We also define the following abbreviations: $R_r\varphi := U(Q\varphi \vee Q\neg\varphi)$, $[r\!:\!\varphi?]\psi := (\neg R_r\varphi \wedge \psi) \vee (R_r\varphi \wedge [\varphi?]\psi)$, and $[s\!:\!\varphi!]\psi := (\varphi \to [\varphi!]\psi)$. Let $\mathcal{PL}$ be the language of propositional logic.

Throughout the paper, the following conventions are adopted. We use $\varphi, \psi, \chi, \ldots$ for arbitrary formulas (in $\mathcal{L}$), and $\pi, \pi', \pi_1, \pi_2, \ldots$ for propositional formulas. $p, q, \ldots$ range over PROP. $\downarrow, \downarrow_1, \downarrow_2, \ldots$ will denote either the symbol ! or the symbol ?. Given a finite set of formulas $\Gamma \subset \mathcal{L}$, $\bigwedge\Gamma$ is the conjunction of all formulas in Γ and likewise

[1]It is crucial that in the formalism to-be-introduced we have exactly two agents. As visible from the introduction and often emphasized (e.g. [18]), the Hohfeldian notions are inherently relational. This relationality is tacitly "enforced" in our setting where we have exactly two agents engaging in communicative actions (that might result in the change of the other agent's normative position). We leave the considerations of how the logic should be changed in order to accommodate a generalized, multi-agent setting to future research.

for $\bigvee\Gamma$. We note that $\bigwedge\varnothing := \top$ and $\bigvee\varnothing := \bot$. Finally, for all formulas φ, $\textsc{prop}(\varphi)$ is the set of all atoms occurring in φ. Given some $A \subseteq \textsc{prop}$, $\mathcal{L}(A)$ is the sublanguage of $\mathcal{L}$ that is restricted to atoms in A.

In our language, $U\varphi$ is the familiar universal modality expressing that "φ is true in all worlds", while $Q\varphi$ is a technical modality which has been introduced in [25] (see Section 7). The core notion that "The receiver has the power to know the answer to the question φ?" is expressed by the formula $R_r\varphi := U(Q\varphi \vee Q\neg\varphi)$. $K_r\varphi$ reads "The receiver knows φ" and $\mathbb{O}_s\varphi$ "The sender is obliged to announce φ". The notion that "After the receiver asked the question φ?, it holds that ψ" is expressed by the formula $[r:\varphi?]\psi := (\neg R_r\varphi \wedge \psi) \vee (R_r\varphi \wedge [\varphi?]\psi)$. Finally, to express the usual notion of truthful announcements, we use $[s:\varphi!]\psi := (\varphi \to [\varphi!]\psi)$, i.e., "After the sender (truthfully) announced φ, it holds that ψ".[2]

Below we define the notion of "degree" of formulas, which will be used later.

Definition 3. For all formulas φ, the degree of φ, notation $d(\varphi)$, is a positive integer inductively defined as follows: (1) $d(p) = 1$. (2) $d(\neg\varphi) = d(K_r\varphi) = d(U\varphi) = d(Q\varphi) = 1 + d(\varphi)$. (3) $d(\varphi \to \psi) = 1 + \max(d(\varphi), d(\psi))$. (4) $d(\mathbb{O}_s\varphi) = 6 + d(\varphi)$. (5) $d([\varphi!]\psi) = d([\varphi?]\psi) = 6 + d(\varphi) + d(\psi)$.

Proposition 4. *The following holds for all formulas φ, ψ and atoms p:*

(1) $d(\psi) < d(\varphi)$ if ψ is a subformula of φ and $\psi \neq \varphi$.

(2) $d(U(p \leftrightarrow \varphi)) < d(\mathbb{O}_s\varphi)$.

(3) $d(U(p \leftrightarrow \psi)) < d([\psi\downarrow]\varphi)$.

Next, we introduce the models for LRK. Our models are essentially the combination of the "neighbourhood epistemic models" introduced in [15] and the "epistemic issue models" in [25], see Section 7.

Definition 5. A model is a tuple $M = (W, \sim, \approx, N, V)$ where:

- W is a non-empty set of possible worlds or states;
- $\sim$ and $\approx$ are two equivalence relations on W;[3]
- $N : W \to \wp(\wp(W))$ is such that $w \in X$ for all $w \in W$ and $X \in N(w)$;
- $V : \textsc{prop} \to \wp(W)$ is a valuation.

[2] These notions are introduced as complex formulas rather than primitive operators because it simplifies the completeness proof.

[3] In what follows, $\approx$ will also denote the partition generated by the equivalence classes of $\approx$.

A pointed model is a pair M, w such that w is a state of M. For every state $w \in W$, $\sim (w)$ denotes the set $\{v \in W \mid w \sim v\}$, and similarly for $\approx (w)$.

In the above definition, $\sim$ is the familiar epistemic indistinguishability relation (of the receiver). The partition $\approx$ is intended to encode the set of questions to which the receiver has the power to know the answers. The set of questions may be stipulated by some information security policies such as privacy policies. The idea to represent a set of questions by a partition can be found in, e.g., [10], [6], and [25]. Finally, each subset $X \in N(w)$ is an *ideal epistemic state* for the receiver at w, i.e., the epistemic state X is compliant with the given information security policies (for the receiver and specified in the situation w). Let us illustrate the definition of the models by the running example:

Example 6. Let p_a, p_b, and g be the propositions that "The result for a is positive", "The result for b is positive", and "The cheaper medicine is as good as the expensive one", respectively. The case in Example 1 can be characterized by the model $M = (W, \sim, \approx, N, V)$ illustrated in Figure 2.

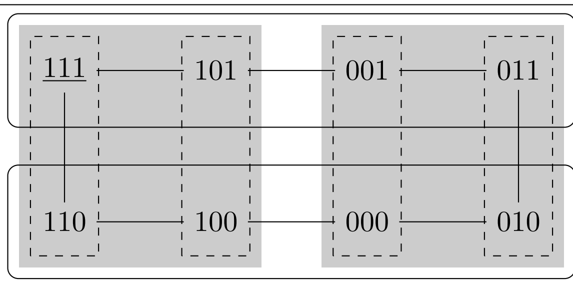

Figure 2: The model $M = (W, \sim, \approx, N, V)$. States are binary numbers 000, 001, ... and the actual state is 111. For each state xyz, $x = 1$ ($y = 1$, $z = 1$, respectively) iff $xyz \in V(p_a)$ ($xyz \in V(p_b)$, $xyz \in V(g)$, respectively). The indistinguishability relation $\sim$ is indicated by the straight line (with the reflexive and transitive arrows omitted). $\approx$ is the equivalence relation (or partition) indicated by the rectangles with rounded corners (this captures the patient's power to know whether g). Finally, for every $xyz \in W$, $N(xyz)$ consists of the subsets such that: (1) it contains xyz itself; (2) it is contained in one of the shaded areas (this corresponds to the doctor's obligation to inform about p_a); (3) it is not contained in one of the dashed rectangles (the patient is prohibited to know p_b).

The next step is to provide the semantics for $\mathcal{L}$, especially for the formulas $R_r\varphi$ and $[r : \varphi?]\psi$. The semantics for $R_r\varphi$ is relatively straightforward: the receiver has

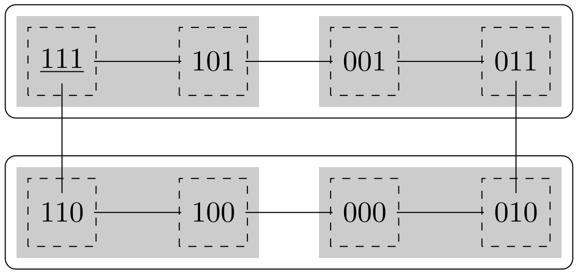

Figure 3: The updated model $M_{g?}$. The same convention is adopted as in Figure 2.

the power to know the answer to φ? iff the question φ? is "settled" by the partition $\approx$, in the sense that every cell in the partition $\approx$ is either a subset of the truth set of φ, or a subset of the truth set of $\neg\varphi$. Interpreting $Q\varphi$ as a normal modality corresponding to the accessibility relation $\approx$, this intuition is exactly captured by the formula $U(Q\varphi \vee Q\neg\varphi)$.

$[r:\varphi?]\psi$ is intended to express that "after the receiver asked the question φ?, it holds that ψ". If the receiver has no power to know the answer to φ?, then nothing will change after the action $[r:\varphi?]$. On the contrary, if the receiver indeed has the power, the sender is then forced to answer the question φ?. This is captured by the model updating $M_{\varphi?}$ such that, in the updated model $M_{\varphi?}$, all epistemic states not answering the question φ? become no longer ideal. The dynamic operator $[\varphi?]\psi$ describes what holds after this kind of model update. Thus, the notion "after the receiver asked the question φ?, it holds that ψ" can be expressed by the complex formula $[r:\varphi?]\psi = (\neg R_r\varphi \wedge \psi) \vee (R_r\varphi \wedge [\varphi?]\psi)$.

Definition 7. Given a model $M = (W, \sim, \approx, N, V)$, for all $w \in W$ and $\varphi \in \mathcal{L}$, the satisfaction relation $M, w \models \varphi$ is inductively defined as follows:

$$
\begin{array}{rll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \neg\varphi & \text{iff} & M, w \not\models \varphi \\
M, w \models (\varphi \to \psi) & \text{iff} & M, w \not\models \varphi \text{ or } M, w \models \psi \\
M, w \models U\varphi & \text{iff} & \text{for all } v \in W,\ M, v \models \varphi \\
M, w \models Q\varphi & \text{iff} & \text{for all } v \in W,\ w \approx v \text{ implies } M, v \models \varphi \\
M, w \models K_r\varphi & \text{iff} & \text{for all } v \in W,\ w \sim v \text{ implies } M, v \models \varphi \\
M, w \models \mathbb{O}_s\varphi & \text{iff} & \text{for all } X \in N(w),\ X \subseteq\sim (w) \text{ implies } X \subseteq [\![\varphi]\!]_M \\
M, w \models [\varphi?]\psi & \text{iff} & M_{\varphi?}, w \models \psi \\
M, w \models [\varphi!]\psi & \text{iff} & M_{\varphi!}, w \models \psi
\end{array}
$$

where $[\![\varphi]\!]_M = \{x \in W \mid M, x \models \varphi\}$ and $M_{\varphi?}$ and $M_{\varphi!}$ are defined as follows:

$M_{\varphi?} = (W_{\varphi?}, \sim_{\varphi?}, \approx_{\varphi?}, N_{\varphi?}, V_{\varphi?})$ where:

- $W_{\varphi?} = W$, $\sim_{\varphi?}=\sim$, $\approx_{\varphi?}=\approx$, $V_{\varphi?} = V$,
- $N_{\varphi?}(x) = \{X \in N(x) \mid X \subseteq [\![\varphi]\!]_M \text{ or } X \subseteq [\![\neg\varphi]\!]_M\}$ for all $x \in W$.

$M_{\varphi!} = (W_{\varphi!}, \sim_{\varphi!}, \approx_{\varphi!}, N_{\varphi!}, V_{\varphi!})$ where:

- $W_{\varphi!} = W$, $\approx_{\varphi!}=\approx$, $N_{\varphi!} = N$, $V_{\varphi!} = V$,
- $\sim_{\varphi!}= \{(u, v) \in \sim \mid M, u \models \varphi \text{ iff } M, v \models \varphi\}$.

The notion of validity is defined as usual.

The semantics for $K_r\varphi$ and $[s:\varphi!]\psi$ is standard, except that we choose to delete the links between the φ-states and $\neg\varphi$-states in the definition of $M_{\varphi!}$, instead of removing all $\neg\varphi$-states from the model. Those $\neg\varphi$-states are reserved for further reference.[4] This kind of model updating can be found in, e.g., [24, 25]. As for $\mathbb{O}_s\varphi$, it reflects the intuition that the sender is obliged to announce φ if φ is "known" in all ideal epistemic states (for the receiver) that are achievable by further announcements (i.e., φ is a piece of necessary information for achieving ideality).

Let us illustrate the semantics with some examples:

Example 8. In the model M, we have, e.g., $M, 111 \models \mathbb{O}_s p_a$, $M, 111 \models R_r g$, and $M, 111 \not\models \mathbb{O}_s p_b$. After the receiver (the patient a) asked "Is the cheaper medicine as good as the expensive one?" ($g?$), the updated pointed model $M_{g?}$ is depicted in Figure 3. We have, e.g., $M_{g?}, 111 \models \mathbb{O}_s p_a$, $M_{g?}, 111 \models R_r g$, and $M_{g?}, 111 \models \mathbb{O}_s g$.

Example 9 ([1])**.** Suppose the sender is communicating classified information to the receiver. Since the receiver is permitted to know some information p, the only constraint for the sender is that it is forbidden for the receiver to know p while not knowing that it is classified (c). Furthermore, suppose that the receiver currently knows (is informed about) p, but she does not know c. The scenario can be represented by the following model $M = (W, \sim, \approx, N, V)$ where:

- $W = \{w, u, v, x\}$;
- $\sim(w) = \sim(u) = \{w, u\}$, $\sim(v) = \{v\}$, and $\sim(x) = \{x\}$;
- $\approx = W \times W$;
- $N(w) = \{Y \subseteq W \mid w \in Y \text{ and } Y \neq \{w, u\}\}$ and $N(u) = N(v) = N(x) = \varnothing$;[5]
- $V(p) = \{w, u\}$ and $V(c) = \{w, v\}$.

[4]This deviates from the classical public announcement logic. But, in our case, deleting all $\neg\varphi$-states may change the truth value of formulas like $R_r\varphi$, which is unreasonable.

[5]Actually, $N(u)$, $N(v)$, and $N(x)$ can be defined arbitrarily as we only intend to model the sender's obligatory announcements at the current world w.

Intuitively, we will agree that it is obligatory for the sender to announce c. This is consistent with our semantics since $M, w \models \mathbb{O}_s c$.

Remark 1. A question may arise regarding the semantics for $\mathbb{O}_s\varphi$: why do we just consider ideal epistemic states achievable by further announcements instead of all? Technically, one may propose the following alternative semantic definition for $\mathbb{O}_s\varphi$:

$$M, w \models \mathbb{O}_s\varphi \quad \text{iff} \quad \text{for all } X \in N(w),\ X \subseteq [\![\varphi]\!]_M. \tag{†}$$

However, in situations like Example 9, (†) does not work, since it predicts that the sender is not obliged to announce c. The reason is that there are also ideal epistemic states (for the receiver) at w in which the receiver knows neither p nor c. However, these ideal epistemic states are not achievable by further announcements.

3 Semantic Results

In this section, we list some (in)validities of LRK. The first group of validities is about the notion of the power to know.

Proposition 10. *The following hold for all formulas φ and ψ:*

(1) $\models R_r\top$.

(2) $\models R_r\varphi \land R_r\psi \to R_r(\varphi \land \psi)$.

(3) $\models R_r\varphi \to R_r\neg\varphi$.

(4) $\models R_r\varphi \leftrightarrow UR_r\varphi$

The proofs are omitted because they are all straightforward. From (1) – (3), we can see that the fragment of LRK on the notion of the power to know is nothing but the answer entailment relation between questions. That is to say, if the answer to a question φ? can be derived from that of a set of questions to which the receiver has the power to know the answers, then the receiver also has the power to know the answer to φ?. [6] The last one states that the notion of the power to know in LRK is global (i.e., its truth does not depend on the evaluating states).[7]

[6]One may argue that the notion of the power to know characterized in LRK is rather weak. For example, in most scenarios, the receiver has only the power to know what they are permitted to know (otherwise there would be conflict in the security policy). To model these scenarios, we can impose extra constraints on the models, e.g., for all $w \in W$, there is $X \in N(w)$ such that $\approx (w) \subseteq X$. Since the focus of this paper is to provide a general framework formalizing the power to know, we leave this for future work.

[7]It also makes sense to generalize our models to account for cases where, for example, the receiver is not aware of their rights to know. We leave that for future work.

Proposition 11. *The following hold for all formulas φ and ψ:*

(1) $\models \mathbb{O}_s(\varphi \to \psi) \to (\mathbb{O}_s\varphi \to \mathbb{O}_s\psi)$.

(2) $\models U\varphi \to \mathbb{O}_s\varphi$.

(3) $\models \neg\mathbb{O}_s\bot \to (\mathbb{O}_s\varphi \to \varphi)$.

(4) $\models K_r(\varphi \to \psi) \to (\mathbb{O}_s\varphi \to \mathbb{O}_s\psi)$.

(5) $\models K_r\varphi \to \mathbb{O}_s\varphi$.

The proofs are again omitted for the same reason. A few remarks can be made on the above logical rules governing the behavior of the operator $\mathbb{O}_s$. The first and the second indicate that $\mathbb{O}_s\varphi$ is a normal modality. The third says that the sender is not obliged to lie unless they are obliged to announce the contradiction (i.e., they face a deontic dilemma). The fourth can be understood as follows: it says that if φ is more informative than ψ for the receiver and the sender is obliged to inform the receiver about φ,[8] then the sender is also obliged to inform the receiver about ψ. But we have problems with interpreting the last validity. Literally, it states that the sender is obliged to announce whatever the receiver knows. This seems counterintuitive. To understand (5), note that, if φ is known by the receiver, the announcement of φ is actually less informative than any announcement for the receiver. Thus, from the informational point of view, the announcement of φ is "implied" by any announcement. In this sense, the announcement of φ is inevitable or necessary in our system since we assume that the sender can only make truthful announcements. So, the obligatory announcement of φ simply follows from that the announcement of φ is necessary. [9]

The previous two propositions are about the properties of "the power to know" and "obligatory announcements" separately. However, it is natural to expect that there would be some interaction between them. One candidate is the formula $R_r\varphi \to (\varphi \to [r\!:\!\varphi?]\mathbb{O}_s\varphi)$, expressing that if the receiver has the power to know whether φ and φ is the case, then the sender is obliged to announce φ once the receiver has asked the question $\varphi?$. It is not hard to show the validity of the formula when φ is propositional. But the next proposition shows that this needs not to be the case when φ is a general formula:

Proposition 12. $\not\models R_r\varphi \to (\varphi \to [r\!:\!\varphi?]\mathbb{O}_s\varphi)$ *for some formulas φ.*

[8] We follow [1, Definition 9] for the definition of "informativeness" of formulas.

[9] For readers still against (5), we may define another operator for obligatory announcements in LRK: $\mathbb{O}^*_s\varphi := \neg K_r\varphi \wedge \mathbb{O}_s\varphi$. After all, if φ is known (or informed), the sender does not have the obligation to inform it again.

Proof. We show that $\not\models R_r(p\wedge\neg\mathbb{O}_s p) \to ((p\wedge\neg\mathbb{O}_s p) \to [r:(p\wedge\neg\mathbb{O}_s p)?]\mathbb{O}_s(p\wedge\neg\mathbb{O}_s p))$. Let the model $M = (W, \sim, \approx, N, V)$ be as follows:

- $W = \{w, v\}$, $\sim = W \times W$, $\approx = \{\{v\}, \{w\}\}$, $V(p) = \{w\}$,
- $N(w) = \{\{w\}, \{v, w\}\}$ and $N(v) = \{\{v\}\}$.

We show that $M, w \not\models R_r(p\wedge\neg\mathbb{O}_s p) \to ((p\wedge\neg\mathbb{O}_s p) \to [r:(p\wedge\neg\mathbb{O}_s p)?]\mathbb{O}_s(p\wedge\neg\mathbb{O}_s p))$. First, it is not hard to see that $M, w \models p\wedge\neg\mathbb{O}_s p$ (1) and $M, v \not\models p\wedge\neg\mathbb{O}_s p$. Therefore $M, w \models R_r(p\wedge\neg\mathbb{O}_s p)$ (2). In the updated model $M_{(p\wedge\neg\mathbb{O}_s p)?}$, the only change is that $N_{(p\wedge\neg\mathbb{O}_s p)?}(w) = \{\{w\}\}$. It follows that $M_{(p\wedge\neg\mathbb{O}_s p)?}, w \models \mathbb{O}_s p$. Thus $M_{(p\wedge\neg\mathbb{O}_s p)?}, w \not\models \mathbb{O}_s(p \wedge \neg\mathbb{O}_s p)$. Therefore $M, w \not\models [r:(p \wedge \neg\mathbb{O}_s p)?]\mathbb{O}_s(p \wedge \neg\mathbb{O}_s p)$ (3). By (1), (2), and (3), $M, w \not\models R_r(p \wedge \neg\mathbb{O}_s p) \to ((p \wedge \neg\mathbb{O}_s p) \to [r:(p \wedge \neg\mathbb{O}_s p)?]\mathbb{O}_s(p \wedge \neg\mathbb{O}_s p))$. $\square$

In the above, the formula $p \wedge \neg\mathbb{O}_s p$ is used to show the invalidity of the given schema. The formula has the same structure as the Moore sentence [21], i.e., p is true but I do not believe p. It is well known, in dynamic epistemic logic [26], that the Moore sentence $(p \wedge \neg Kp)$ is an unsuccessful formula, in the sense that the Moore sentence may become false after the announcement of itself. Here we see a similar situation: after the receiver asked the question $p \wedge \neg\mathbb{O}_s p?$, it becomes obligatory for the sender to announce p. Thus the formula $p \wedge \neg\mathbb{O}_s p$ becomes false. However, the operator $\mathbb{O}_s$ satisfies a weak form of the axiom (T): $\neg\mathbb{O}_s\bot \to (\mathbb{O}_s\varphi \to \varphi)$ (Proposition 11(3)). Hence, in the updated model, $\mathbb{O}_s(p \wedge \neg\mathbb{O}_s p)$ does not hold.

At first glance, the invalidity of the axiom schema in Proposition 12 may seem to be counterintuitive. How could it be that the receiver has the power to know something while the sender has no obligation to inform even if the receiver requests it? We will, nevertheless, argue that the phenomenon can be explained if we make explicit the time involved in the axiom schema. The operator $\mathbb{O}_s\varphi$ in LRK expresses veritable obligations ([11], i.e., obligations specific to a particular situation) instead of normative rules. This means that the truth of formulas like $\mathbb{O}_s\varphi$ may flip after the receiver asks some questions because the situation changes. Consider the formula $R_r(p \wedge \neg\mathbb{O}_s p) \to ((p \wedge \neg\mathbb{O}_s p) \to [r : (p \wedge \neg\mathbb{O}_s p)?]\mathbb{O}_s(p \wedge \neg\mathbb{O}_s p))$. The first three occurrences of the operator $\mathbb{O}_s$ really refer to the obligation of the sender *before* the question $(p \wedge \neg\mathbb{O}_s p)?$, whereas the last two occurrences express the sender's obligation *after* the question. Thus, the antecedent $R_r(p \wedge \neg\mathbb{O}_s p)$ just asserts that the receiver has the power to know the sender's deontic status before the question. However, in LRK, there is no way to express the sender's previous obligation in the scope of the dynamic operator $[r:p \wedge \neg\mathbb{O}_s p?]$. This suggests that LRK may be equipped with temporal operators like "Yesterday".

Proposition 13. *The following holds for all formulas φ, ψ and $\downarrow \in \{!, ?\}$:*

(1) $\models [\varphi\downarrow]p \leftrightarrow p.$ (↓Atom)

(2) $\models [\varphi\downarrow]\neg\psi \leftrightarrow \neg[\varphi\downarrow]\psi.$ (↓Neg)

(3) $\models [\varphi\downarrow](\psi \to \chi) \leftrightarrow ([\varphi\downarrow]\psi \to [\varphi\downarrow]\chi).$ (↓Imp)

(4) $\models [\varphi\downarrow]U\psi \leftrightarrow U[\varphi\downarrow]\psi.$ (↓U)

(5) $\models [\varphi\downarrow]Q\psi \leftrightarrow Q[\varphi\downarrow]\psi.$ (↓Q)

(6) $\models [\varphi?]K_r\psi \leftrightarrow K_r[\varphi?]\psi.$ (?K)

(7) $\models [\varphi!]K_r\psi \leftrightarrow (\varphi \wedge K_r(\varphi \to [\varphi!]\psi)) \vee (\neg\varphi \wedge K_r(\neg\varphi \to [\varphi!]\psi)).$ (!K)

4 Expressivity

In this section, we investigate the expressive power of some fragments of $\mathcal{L}$. It is well known that adding the public announcement operator to the language of epistemic logic (without common knowledge) does not increase expressive power. In Proposition 13, we have seen a series of reduction axioms for the two dynamic operators $[\varphi!]$ and $[\varphi?]$, but not for formulas of the form $[\varphi!]\mathbb{O}_s\psi$ and $[\varphi?]\mathbb{O}_s\psi$. Thus it is interesting to know whether the same holds for LRK. To answer this question, let $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ be the sublanguages of $\mathcal{L}$ defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid U\varphi \mid Q\varphi \mid K_r\varphi \mid \mathbb{O}_s\varphi \qquad (\mathcal{L}_0)$$

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid U\varphi \mid Q\varphi \mid K_r\varphi \mid \mathbb{O}_s\varphi \mid [\varphi!]\varphi \qquad (\mathcal{L}_1)$$

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid U\varphi \mid Q\varphi \mid K_r\varphi \mid \mathbb{O}_s\varphi \mid [\varphi?]\varphi \qquad (\mathcal{L}_2)$$

The next theorem shows that adding any of the two operators $[\varphi!]\psi$ and $[\varphi?]\psi$ to the static language $\mathcal{L}_0$ does increase the expressive power. This is in contrast with the situation in public announcement logic. It also follows that no DEL-style reduction axiomatization exists for LRK.

Theorem 1. *Both $\mathcal{L}_1$ and $\mathcal{L}_2$ are strictly more expressive than $\mathcal{L}_0$.*

Proof. We first show that $\mathcal{L}_1$ is strictly more expressive than $\mathcal{L}_0$. It is clear that $\mathcal{L}_1$ is at least as expressive as $\mathcal{L}_0$ since $\mathcal{L}_0$ is a sublanguage of $\mathcal{L}_1$. We show that $\mathcal{L}_0$ is not at least as expressive as $\mathcal{L}_1$. This is done by showing that there is no $\psi \in \mathcal{L}_0$ such that $[p!]\mathbb{O}_s\bot \equiv \psi$ (i.e., they are satisfied at exactly the same pointed models). Consider two models $M_1 = (W, \sim, \approx, N_1, V)$ and $M_2 = (W, \sim, \approx, N_2, V)$ where:

- $W = \{w, u\}$, $\sim = W \times W$, $\approx = \{\{w\}, \{u\}\}$, $V(p) = \{w\}$;
- $N_1(w) = \{\{w, u\}\}$, $N_2(w) = \{\{w\}, \{w, u\}\}$, $N_1(u) = N_2(u) = \{\{w, u\}\}$.

It is not hard to see that $M_1, w \models [p!]\mathbb{O}_s\bot$ and $M_2, w \not\models [p!]\mathbb{O}_s\bot$. However, by an induction on the structure of ψ, we can show that $M_1, y \models \psi$ iff $M_2, y \models \psi$ for all $\psi \in \mathcal{L}_0$ and $y \in W$. Here we show only the inductive step for $\mathbb{O}_s\chi$. The case for $y = u$ follows directly from that $N_1(u) = N_2(u)$ and the IH. For $y = w$, we have:

	$M_1, w \models \mathbb{O}_s\chi$	
iff	$\forall X \in N_1(w)$, $X \subseteq\sim(w)$ implies $X \subseteq [\![\chi]\!]_{M_1}$	(semantics)
iff	$\forall X \in N_1(w)$, $X \subseteq [\![\chi]\!]_{M_1}$	(def. of $N_1(w)$ and $\sim$)
iff	$\forall X \in N_1(w)$, $X \subseteq [\![\chi]\!]_{M_2}$	(IH)
iff	$\forall X \in N_2(w)$, $X \subseteq [\![\chi]\!]_{M_2}$	($\bigcup N_1(w) = \bigcup N_2(w)$)
iff	$\forall X \in N_2(w)$, $X \subseteq\sim(w)$ implies $X \subseteq [\![\chi]\!]_{M_2}$	(def. of $N_2(w)$ and $\sim$)
iff	$M_2, w \models \mathbb{O}_s\chi$	(semantics)

Finally, to show that $\mathcal{L}_2$ is more expressive than $\mathcal{L}_0$, we replace the formula $[p!]\mathbb{O}_s\bot$ in the proof by $[p?]\mathbb{O}_s\bot$. $\square$

5 Axiomatization

In this section, we give an axiomatization for LRK and show its soundness. To axiomatize LRK, we employ the notion of "necessity forms" which was originally proposed in [9] and adapted to axiomatize arbitrary public announcement logic [3].

Definition 14 (Necessity forms)**.** Let $\sharp$ be a special symbol not occurring in $\mathcal{L}$. The set of *necessity forms* is inductively defined as follows:

$$\xi(\sharp) ::= \sharp \mid (\varphi \to \xi(\sharp)) \mid K_r\xi(\sharp) \mid U\xi(\sharp) \mid Q\xi(\sharp) \mid [\varphi!]\xi(\sharp) \mid [\varphi?]\xi(\sharp)$$

where $\varphi \in \mathcal{L}$. Given a necessity form $\xi(\sharp)$ and a formula $\varphi \in \mathcal{L}$, $\xi(\varphi)$ will denote the result of replacing the unique occurrence of $\sharp$ in $\xi(\sharp)$ by φ.

Next, we present the axiomatization. We recall, from Section 2, that $\pi, \pi', \pi_1, \ldots$ represents any propositional formulas.

Definition 15 (Axiomatization)**.** The axiomatization **LRK** for $\mathcal{L}$ consists of, in addition to all the axiom schemas in Proposition 13, the following axiom schemas and inference rules. Let the set of **LRK**-theorems be the least set of formulas that contains all instances of the axiom schemas and is closed under the inference rules. If φ is an **LRK**-theorem, we write $\vdash \varphi$.

(PL) All propositional tautologies

(S5) S5 axioms for U, Q, and K_r

(U) $U\varphi \to Q\varphi \wedge K_r\varphi$

(K-$\mathbb{O}$) $K_r\varphi \to \mathbb{O}_s\varphi$

($\mathbb{O}$) $\neg\mathbb{O}_s\bot \to (\mathbb{O}_s\varphi \to \varphi)$

(Dist) $\mathbb{O}_s(\varphi \to \psi) \to (\mathbb{O}_s\varphi \to \mathbb{O}_s\psi)$

(U$\downarrow$) $U(\varphi \leftrightarrow \psi) \vee U(\varphi \leftrightarrow \neg\psi) \to ([\varphi\downarrow]\chi \leftrightarrow [\psi\downarrow]\chi)$

(Trans) $[\pi_1?]\mathbb{O}_s\pi_2 \to ([\pi_3?]\mathbb{O}_s\pi_1 \to [\pi_3?]\mathbb{O}_s\pi_2)$

(!?) $[\pi_1\downarrow_1]\cdots[\pi_n\downarrow_n]\mathbb{O}_s\pi \leftrightarrow \bigvee_{P\subseteq\{\pi_1,\ldots,\pi_n\}} \Big(\bigwedge P \wedge \neg\bigvee \bar{P} \wedge [(\bigwedge P \wedge \neg\bigvee \bar{P})?]\mathbb{O}_s\pi\Big)$,
where $n \geq 0$ and $\bar{P} = \{\pi_1, \ldots, \pi_n\} \setminus P$

(MP) from φ and $\varphi \to \psi$, infer ψ

(Nec) from ψ, infer $U\psi$ and $[\varphi\downarrow]\psi$

(R$_U$) from $\xi(\neg U(p \leftrightarrow \varphi))$, infer $\xi(\varphi \wedge \neg\varphi)$, where $p \notin \textsc{prop}(\xi(\varphi))$

We list some theorems of **LRK**, which will be used in the completeness proof.

Proposition 16. *The following hold:*

(1) $\vdash [\varphi\downarrow]\pi \leftrightarrow \pi$.

(2) $\vdash [\varphi\downarrow]\psi \leftrightarrow [\neg\varphi\downarrow]\psi$.

(3) $\vdash [\pi!]\mathbb{O}_s\pi' \leftrightarrow [\pi?]\mathbb{O}_s\pi'$.

(4) $\vdash \mathbb{O}_s\pi \to [\pi'?]\mathbb{O}_s\pi$.

(5) $\vdash \pi \to [\pi?]\mathbb{O}_s\pi$.

Proof. See Appendix A. ☐

In the remainder of this section, we show that **LRK** is sound with respect to the semantics. We need to show that all the axioms in **LRK** are valid and all the inference rules preserve validity. Here we focus on the axioms (Trans), (!?), and the rule (R$_U$), as the validity of other axioms and rules is straightforward or has been shown in the last section.

Proposition 17. *The axiom (Trans) is valid.*

Proof. See Appendix A. ☐

Lemma 18. *The following holds:*

(1) $\models \pi_1 \wedge \pi_2 \to ([\pi_1\downarrow][\pi_2?]\mathbb{O}_s\pi \leftrightarrow [\pi_1 \wedge \pi_2?]\mathbb{O}_s\pi)$.

(2) $\models \neg\pi_1 \wedge \pi_2 \to ([\pi_1{\downarrow}][\pi_2?]\mathbb{O}_s\pi \leftrightarrow [\neg\pi_1 \wedge \pi_2?]\mathbb{O}_s\pi)$.

Proof. See Appendix A. □

Proposition 19. *The axiom (!?) is valid.*

Proof. See Appendix A. □

Given a model M and $p \in \textsc{prop}$, a *p-variant* of M is a model M' such that $M = M'$ or they differ only in the valuation of p.

Lemma 20. *For all models M, M' and atoms p, if M' is a p-variant of M then for all formula $\varphi \in \mathcal{L}(\textsc{prop} \setminus \{p\})$ and $w \in M$, $M, w \models \varphi$ iff $M', w \models \varphi$.*

Proof. An easy induction on the structure of ψ. □

Lemma 21. *For any necessity form $\xi(\sharp)$, formula $\varphi \in \mathcal{L}$ and atom $p \notin \textsc{prop}(\xi(\varphi))$, if $M, w \not\models \xi(\varphi \wedge \neg\varphi)$, then there is a p-variant of M, M', such that $M', w \not\models \xi(\neg U(p \leftrightarrow \varphi))$.*

Proof. Induction on the structure of $\xi(\sharp)$. Here we show only the base. Suppose $M = (W, \sim, \approx, N, V)$ and $M, w \not\models \varphi \wedge \neg\varphi$. Let M' be the p-variant of M such that $V'(p) = [\![\varphi]\!]_M$. Since $[\![\varphi]\!]_{M'} = [\![\varphi]\!]_M$ by Lemma 20, $M', w \models U(p \leftrightarrow \varphi)$. □

Given Lemma 21, the next proposition follows immediately.

Proposition 22. *The inference rule (R_U) preserves validity.*

Theorem 2 (Soundness)**.** *For all formulas $\varphi \in \mathcal{L}$, if $\vdash \varphi$ then $\models \varphi$.*

6 Completeness

In this section, we prove that the axiomatization **LRK** is weakly complete with respect to the semantics. This is done by making a detour. Let $\mathbf{LRK}^\omega$ be the axiomatization obtained by replacing the rule (R_U) in **LRK** by (R_U^ω):

(R_U^ω) from $\xi(\neg U(p \leftrightarrow \varphi))$ for all $p \in \textsc{prop}$, infer $\xi(\varphi \wedge \neg\varphi)$.

It is clear that (R_U^ω) is an admissible rule in **LRK**, i.e., the set of $\mathbf{LRK}^\omega$-theorems is a subset of the set of **LRK**-theorems.[10] Thus the soundness of $\mathbf{LRK}^\omega$ follows from that of **LRK**. Conversely, the (weak-)completeness of **LRK** follows from that of $\mathbf{LRK}^\omega$. In the following, we will focus on the completeness of $\mathbf{LRK}^\omega$.

[10] This is because every application of the infinitary rule (R_U^ω) in a derivation in $\mathbf{LRK}^\omega$ can be replaced by the finitary rule (R_U). For details, see [3, Proposition 4.10]. However, since we prefer to have a finitary axiomatization, we adopt the inference rule (R_U) in **LRK** instead of (R_U^ω).

Definition 23 (Theory)**.** A set $x \subseteq \mathcal{L}$ is called a theory iff it satisfies:

- x contains all $\mathbf{LRK}^\omega$-theorems.
- x is closed under the inference rules (MP) and (R_U^ω).

A set of formulas $x \subseteq \mathcal{L}$ is *consistent* iff there are no $\varphi_1, \ldots, \varphi_n \in x$ such that $\vdash_{\mathbf{LRK}^\omega} \varphi_1 \wedge \cdots \wedge \varphi_n \to \bot$. A set $x \subseteq \mathcal{L}$ is *maximal* iff for all $\varphi \in \mathcal{L}$, $\varphi \in x$ or $\neg\varphi \in x$. The set of all maximal consistent theories will be denoted by MCT.

It is clear that the smallest theory is the set of all $\mathbf{LRK}^\omega$-theorems. If x is a theory, then x is consistent iff $\bot \notin x$.

Definition 24. Let $x \subseteq \mathcal{L}$. For all formulas φ, χ and $\Box \in \{K_r, U, Q, \mathbb{O}_s\}$, let

- $x + \varphi = \{\psi \mid \varphi \to \psi \in x\}$;
- $\Box x = \{\psi \mid \Box\psi \in x\}$;
- $[\chi!]x = \{\psi \mid [\chi!]\psi \in x\}$;
- $[\chi?]x = \{\psi \mid [\chi?]\psi \in x\}$.

Lemma 25. *The following holds for all theories x:*

(1) for all formulas φ, $x + \varphi$ is a theory containing x and φ.

(2) for all $\Box \in \{K_r, U, Q\}$, $\Box x$ is a theory.

Proof. See Appendix A. □

Lemma 26. *The following holds for all theories x and formulas φ:*

(1) $x + \varphi$ is consistent iff $\neg\varphi \notin x$.

(2) if x is a consistent theory, then $x + \varphi$ is consistent or $x + \neg\varphi$ is consistent.

Lemma 27 (Lindenbaum)**.** *For all consistent sets of formulas x, there is $y \in MCT$ such that $x \subseteq y$.*

Proof. Let $\psi_0, \psi_1, \ldots$ be an enumeration of all formulas such that $i > j$ whenever $\psi_i \in x$ and $\psi_j \notin x$. We inductively define a sequence $y_0, y_1, \ldots$ of consistent theories as follows. First, let y_0 be the set of all $\mathbf{LRK}^\omega$-theorems. Note that y_0 is a consistent theory. Second, suppose that, for some $n \geq 0$, y_n is a consistent theory that has been already defined. By Lemma 26(2), either $y_n + \psi_n$ is consistent or $y_n + \neg\psi_n$ is consistent. If $y_n + \psi_n$ is consistent, then we define $y_{n+1} = y_n + \psi_n$. Otherwise, $\neg\psi_n \in y_n$ (Lemma 26(1)) and we consider the following two cases.

(1) If ψ_n is not a conclusion of (R^ω_U), we define $y_{n+1} = y_n$.

(2) Otherwise, ψ_n is a conclusion of (R^ω_U). Let $\xi_1(\chi_1 \wedge \neg\chi_1), \ldots, \xi_k(\chi_k \wedge \neg\chi_k)$ be all the representations of ψ_n as a conclusion of (R^ω_U). We define the sequence $y^0_n, \ldots, y^k_n$ of consistent theories as follows. First, let $y^0_n = y_n$. Second, suppose that, for some $i < k$, y^i_n is a consistent theory containing y_n that has been already defined. Then, it contains $\neg\xi_i(\chi_1 \wedge \neg\chi_i)$. Since y^i_n is closed under (R^ω_U), then there is $p \in \textsc{prop}$ such that $\xi_i(\neg U(p \leftrightarrow \varphi)) \notin y^i_n$. In this case, define $y^{i+1}_n = y^i_n + \neg\xi_i(\neg U(p \leftrightarrow \varphi))$. We put $y_{n+1} = y^k_n$.

Let $y = \bigcup_{n\geq 0} y_n$. It can be verified that y is a maximal consistent theory. It remains to show that $x \subseteq y$. We show the following stronger claim:

$$\text{For all } i \geq 0, \text{ if } \psi_i \in x \text{ then } y_{i+1} = y_i + \psi_i.$$

We prove the claim by induction on the value of i. If $i = 0$, it suffices to show that $y_0 + \psi_0$ is consistent. Suppose not, then $\bot \in y_0 + \psi_0$. Hence $\psi_0 \to \bot \in y_0$. Therefore $\vdash_{\mathbf{LRK}^\omega} \psi_0 \to \bot$, contradicting the consistency of x. Suppose the claim holds for all $i \leq n$. Let $i = n+1$ and $\psi_{n+1} \in x$. Then $\psi_j \in x$ for all $j \leq n+1$. To show that $y_{n+2} = y_{n+1} + \psi_{n+1}$, it suffices to show that $y_{n+1} + \psi_{n+1}$ is consistent. Suppose not. Note that, by the IH, $y_{n+1} = ((y_0 + \psi_0) + \cdots) + \psi_n$. Since $y_{n+1} + \psi_{n+1}$ is inconsistent, $\bot \in y_{n+1} + \psi_{n+1}$. Thus $\psi_0 \to (\psi_1 \to \cdots \to (\psi_{n+1} \to \bot)) \in y_0$. Hence $\vdash_{\mathbf{LRK}^\omega} \psi_0 \to (\psi_1 \to \cdots \to (\psi_{n+1} \to \bot))$, contradicting the consistency of x. □

Since $K_r\varphi, U\varphi$ and $Q\varphi$ are S5-modalities, the next Lemma is standard [4].

Lemma 28. *The following hold for all $x, y, z \in MCT$ and $\Box \in \{K_r, U, Q\}$,*

(1) $\Box x \subseteq x$.

(2) if $\Box x \subseteq y$ and $\Box y \subseteq z$, then $\Box x \subseteq z$.

(3) if $\Box x \subseteq y$, then $\Box y \subseteq x$.

Definition 29. For all propositional π, R_π is a binary relation over MCT such that $xR_\pi y$ if $\mathbb{O}_s[\pi?]x \cap \mathcal{PL} \subseteq y$.

Lemma 30. *For all propositional π and $x \in MCT$, if $[\pi?]\neg\mathbb{O}_s\bot \in x$ then $xR_\pi x$.*

Proof. Suppose $[\pi?]\neg\mathbb{O}_s\bot \in x$. Note that $\vdash_{\mathbf{LRK}^\omega} [\pi?]\neg\mathbb{O}_s\bot \to ([\pi?]\mathbb{O}_s\pi' \to [\pi?]\pi')$ by the axiom $(\mathbb{O})$ and that $[\pi?]$ is a normal modality. Hence $[\pi?]\mathbb{O}_s\pi' \to [\pi?]\pi' \in x$ for all propositional π'. Note also that $\vdash_{\mathbf{LRK}^\omega} [\pi?]\pi' \leftrightarrow \pi'$ by Proposition 16(1). Therefore $[\pi?]\mathbb{O}_s\pi' \to \pi' \in x$ for all propositional π'. Hence $xR_\pi x$. □

In the remainder of this section, we fix an $x \in MCT$.

Definition 31 (Canonical model)**.** The canonical model for x is a structure $M(x) = (W, \sim, \approx, N, V)$ where:

- $W = \{y \in MCT \mid Ux \subseteq y\}$.
- for all $w, v \in W$, $w \sim v$ if $K_r w \subseteq v$.
- for all $w, v \in W$, $w \approx v$ if $Qw \subseteq v$.
- for all $w \in W$, $N(w) = \{R_\pi(w) \cap W \mid \pi \in \mathcal{PL} \,\&\, [\pi?]\neg\mathbb{O}_s\bot \in w\}$.
- for all $p \in \textsc{prop}$, $V(p) = \{y \in W \mid p \in y\}$.

Given Lemma 28 and Lemma 30, it is straightforward to verify that $M(x)$ is a model. In the following, for all formulas φ, let $\|\varphi\| = \{w \in W \mid \varphi \in w\}$ and $[\![\varphi]\!] = [\![\varphi]\!]_{M(x)}$. We recall, from Section 2, the following conventions: $\pi, \pi', \pi_1, \pi_2, \ldots$ are matavariables for propositional formulas and $\downarrow, \downarrow_1, \downarrow_2, \ldots$ denote either ! or ?.

Lemma 32. *$\|\pi\| = [\![\pi]\!]$ for all propositional formulas π.*

Proof. Simple induction on the structure of π. □

In what follows, we will use Lemma 32 implicitly.

Lemma 33. *For all $\varphi \in \mathcal{L}$, $w \in W$ and finite sequences $\vec{\pi} = \pi_1\downarrow_1, \ldots, \pi_k\downarrow_k$ $(k \geq 0)$:*

$$[\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]Q\varphi \in w \text{ iff for all } v \in \approx_{\vec{\pi}} (w),\ [\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]\varphi \in v.$$

Proof. We consider two cases. (1) $k = 0$. The direction from left to right follows directly from the definition of $\approx$. From right to left: Suppose $Q\varphi \notin w$, then $\varphi \notin Qw$. By Lemma 25, Qw is a theory, so $\neg\neg\varphi \notin Qw$. Thus $Qw + \neg\varphi$ is a consistent theory containing Qw and $\neg\varphi$ (Lemma 25 and 26(1)). Applying the Lindenbaum lemma, there is a maximal consistent theory v extending $Qw + \neg\varphi$. Since $Qw \subseteq v$, by the axiom (U), it follows that $Uw \subseteq v$. Since $Ux \subseteq w$ and $Uw \subseteq v$, $Ux \subseteq v$ by Lemma 28. That is, $v \in W$. Therefore $w \approx v$.

(2) $k \neq 0$. We have:

$[\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]Q\varphi \in w$

iff $Q[\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]\varphi \in w$ (by the axioms ($\downarrow$Q) and ($\downarrow$Imp) and the rule (Nec))

iff for all $v \in \approx (w)$, $[\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]\varphi \in v$ (from the case $k = 0$)

iff for all $v \in \approx_{\vec{\pi}} (w)$, $[\pi_1\downarrow_1]\cdots[\pi_k\downarrow_k]\varphi \in v$ ($\approx = \approx_{\vec{\pi}}$)

The next lemma can be shown in the same way as the above.

Lemma 34. *For all $\varphi \in \mathcal{L}$, $w \in W$ and finite sequences $\vec{\pi} = \pi_1\downarrow_1, \ldots, \pi_k\downarrow_k$ $(k \geq 0)$:*

$$[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]U\varphi \in w \textit{ iff for all } v \in W,\ [\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\varphi \in v.$$

Lemma 35. *For all $\varphi \in \mathcal{L}$, $w \in W$ and finite sequences $\vec{\pi} = \pi_1\!\downarrow_1, \ldots, \pi_k\!\downarrow_k$ ($k \geq 0$): if $[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]K_r\varphi \in w$ then for all $v \in \sim_{\vec{\pi}} (w)$, $[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\varphi \in v$.*

Proof. Induction on k. The base follows directly from the definition of $\sim$. Suppose $k = n+1$, $[\pi_1\!\downarrow_1]\cdots[\pi_{n+1}\!\downarrow_{n+1}]K_r\varphi \in w$, and $v \in \sim_{\vec{\pi}} (w)$. We consider two cases:

(1) $\downarrow_{n+1}=!$. It follows from the axiom (!K) that

$$[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]\big((\pi_{n+1} \wedge K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \vee (\neg\pi_{n+1} \wedge K_r(\neg\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi))\big) \in w.$$

Then $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\pi_{n+1} \wedge K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \in w$ or $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\neg\pi_{n+1} \wedge K_r(\neg\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \in w$. We again consider two sub-cases:

(a) $\pi_{n+1} \in w$. Then $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]\neg\pi_{n+1} \notin w$ by Proposition 16(1). Thus $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\neg\pi_{n+1} \wedge K_r(\neg\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \notin w$. Therefore $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\pi_{n+1} \wedge K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \in w$. Hence $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi) \in w$. Since we assume that $v \in \sim_{\vec{\pi}} (w)$, $v \in \sim_{\pi_1\downarrow_1,\ldots,\pi_n\downarrow_n} (w)$. Using the IH, $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi) \in v$. Note that, since $v \in \sim_{\vec{\pi}} (w)$ and $w \in [\![\pi_{n+1}]\!]$, $v \in [\![\pi_{n+1}]\!] = \|\pi_{n+1}\|$. Therefore $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]\pi_{n+1} \in v$ by Proposition 16(1). Hence $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n][\pi_{n+1}!]\varphi \in v$.

(b) $\pi_{n+1} \notin w$. Similarly to the above, it holds that $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n][\pi_{n+1}!]\varphi \in v$.

(2) $\downarrow_{n+1}=?$. It follows from the axiom (?K) that $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]K_r[\pi_{n+1}?]\varphi \in w$. Since $v \in \sim_{\vec{\pi}} (w)$ and $\sim_{\vec{\pi}} = \sim_{\pi_1\downarrow_1,\ldots,\pi_n\downarrow_n}$, $v \in \sim_{\pi_1\downarrow_1,\ldots,\pi_n\downarrow_n} (w)$. Applying the IH, we have $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n][\pi_{n+1}?]\varphi \in w$.

Lemma 36. *For all $\varphi \in \mathcal{L}$, $w \in W$ and finite sequences $\vec{\pi} = \pi_1\!\downarrow_1, \ldots, \pi_k\!\downarrow_k$ ($k \geq 0$): if $[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]K_r\varphi \notin w$ then there is $v \in \sim_{\vec{\pi}} (w)$ such that $[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\varphi \notin v$.*

Proof. Induction on k. The base can be shown as in the proof of Lemma 33. Suppose $k = n+1$. Suppose $[\pi_1\!\downarrow_1]\cdots[\pi_{n+1}\!\downarrow_{n+1}]K_r\varphi \notin w$. We split into two cases:

(1) $\downarrow_{n+1}=!$. It follows from the axiom (!K) that

$$[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]\big((\pi_{n+1} \wedge K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \vee (\neg\pi_{n+1} \wedge K_r(\neg\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi))\big) \notin w.$$

Thus $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\pi_{n+1} \wedge K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \notin w$ and $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\neg\pi_{n+1} \wedge K_r(\neg\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi)) \notin w$. We consider two sub-cases:

(a) $\pi_{n+1} \in w$. Then $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n]K_r(\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi) \notin w$. Using the IH, there is $v \in \sim_{\pi_1\downarrow_1,\ldots,\pi_n\downarrow_n} (w)$ such that $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n](\pi_{n+1} \rightarrow [\pi_{n+1}!]\varphi) \notin v$. Hence $\pi_{n+1} \in v$ and $[\pi_1\!\downarrow_1]\cdots[\pi_n\!\downarrow_n][\pi_{n+1}!]\varphi \notin v$. Note that from the former it follows that $w \sim_{\vec{\pi}} v$ since $\|\pi_{n+1}\| = [\![\pi_{n+1}]\!]$.

(b) $\pi_{n+1} \notin w$. Similar to the above.

(2) $\downarrow_{n+1} = ?$. It follows that $[\pi_1 \downarrow_1] \cdots [\pi_n \downarrow_n] K_r [\pi_{n+1}?]\varphi \notin w$ by the axiom (?K). Using the IH, there is $v \in \sim_{\pi_1 \downarrow_1, \dots, \pi_n \downarrow_n} (w)$ such that $[\pi_1 \downarrow_1] \cdots [\pi_n \downarrow_n][\pi_{n+1}?]\varphi \notin v$. Since $\sim_{\pi_1 \downarrow_1, \dots, \pi_n \downarrow_n} = \sim_{\pi_1 \downarrow_1, \dots, \pi_n \downarrow_n, \pi_{n+1}?}$, we have $v \in \sim_{\pi_1 \downarrow_1, \dots, \pi_n \downarrow_n, \pi_{n+1}?} (w)$.

Lemma 37. *For all propositional formulas π and π' and $w \in W$, if $R_{\pi'}(w) \cap W \subseteq \|\pi\|$ then $[\pi'?]\mathbb{O}_s \pi \in w$.*

Proof. Suppose $[\pi'?]\mathbb{O}_s \pi \notin w$. We show that there is $y \in R_{\pi'}(w) \cap W$ and $y \notin \|\pi\|$. We first show that $Uw \cup \{\neg\pi\} \cup (\mathbb{O}_s[\pi'?]w \cap \mathcal{PL})$ is consistent. Suppose, toward to a contradiction, that $Uw \cup \{\neg\pi\} \cup (\mathbb{O}_s[\pi'?]w \cap \mathcal{PL})$ is inconsistent. Then there must be $\pi_1, \dots, \pi_n \in \mathbb{O}_s[\pi'?]w \cap \mathcal{PL}$ and $\psi_1, \dots, \psi_m \in Uw$ such that

$$\vdash_{\mathbf{LRK}^\omega} \psi_1 \wedge \cdots \psi_m \to (\pi_1 \wedge \cdots \wedge \pi_n \to \pi)$$

Since, for each ψ_i, $U\psi_i \in w$ and U is a normal modality, $U(\pi_1 \wedge \cdots \wedge \pi_n \to \pi) \in w$. Hence $\mathbb{O}_s(\pi_1 \wedge \cdots \wedge \pi_n \to \pi) \in w$ by the axioms (U) and (K-$\mathbb{O}$). Therefore $[\pi'?]\mathbb{O}_s(\pi_1 \wedge \cdots \wedge \pi_n \to \pi) \in w$ by Proposition 16(4). Note that $[\pi'?]\mathbb{O}_s \pi_j \in w$ for all $1 \leq j \leq n$ and both $\mathbb{O}_s$ and $[\pi'?]$ are normal modalities. Thus $[\pi'?]\mathbb{O}_s \pi \in w$, contradicting our assumption. Hence $Uw \cup \{\neg\pi\} \cup (\mathbb{O}_s[\pi'?]w \cap \mathcal{PL})$ is consistent.

By the Lindenbaum lemma, there is $y \in MCT$ such that $Uw \cup \{\neg\pi\} \cup (\mathbb{O}_s[\pi'?]w \cap \mathcal{PL}) \subseteq y$. Since $w \in W$, by Lemma 28 it follows that $y \in W$. By the definition of $R_{\pi'}$, we also have $y \in R_{\pi'}(w)$. □

Lemma 38. *For all $w \in W$, if $[\pi\downarrow]\mathbb{O}_s p \in w$ then $M(x), w \models [\pi\downarrow]\mathbb{O}_s p$.*

Proof. Note that $[\pi!]\mathbb{O}_s p \leftrightarrow [\pi?]\mathbb{O}_s p$ is an $\mathbf{LRK}^\omega$-theorem (by Proposition 16(3)) and a validity (by Theorem 2). Therefore, it suffices to show that $[\pi?]\mathbb{O}_s p \in w$ implies $M(x), w \models [\pi?]\mathbb{O}_s p$. Suppose $[\pi?]\mathbb{O}_s p \in w$, we need to show that for all $X \in N_{\pi?}(w)$, $X \subseteq \sim (w)$ implies $X \subseteq V(p)$. We consider two cases:

(1) $\pi \in w$. Since $X \in N_{\pi?}(w)$, $X \in N(w)$ and $X \subseteq [\![\pi]\!] = \|\pi\|$. From the former it follow that there is π' with $X = R_{\pi'}(w) \cap W$. Thus, by Lemma 37, $[\pi'?]\mathbb{O}_s \pi \in w$. Since we assume that $[\pi?]\mathbb{O}_s p \in w$, by the axiom (Trans) it follows that $[\pi'?]\mathbb{O}_s p \in w$. Thus $X = R_{\pi'}(w) \cap W \subseteq V(p)$.

(2) $\pi \notin w$. Since $X \in N_{\pi?}(w)$, $X \in N(w)$ and $X \subseteq [\![\neg\pi]\!] = \|\neg\pi\|$. From the former it follow that there is π' with $X = R_{\pi'}(w) \cap W$. Thus, by Lemma 37, $[\pi'?]\mathbb{O}_s \neg\pi \in w$. Since we assume that $[\pi?]\mathbb{O}_s p \in w$, $[\neg\pi?]\mathbb{O}_s p \in w$ by Proposition 16(2). From the axiom (Trans) it then follows that $[\pi'?]\mathbb{O}_s p \in w$. Thus $X = R_{\pi'}(w) \cap W \subseteq V(p)$.

Lemma 39. *For all $w \in W$, if $[\pi\downarrow]\mathbb{O}_s p \notin w$ then $M(x), w \not\models [\pi\downarrow]\mathbb{O}_s p$.*

Proof. Since $[\pi!]\mathbb{O}_s p \leftrightarrow [\pi?]\mathbb{O}_s p$ is an $\mathbf{LRK}^\omega$-theorem (by Proposition 16(3)) and a validity (by Theorem 2), it suffices to show that $[\pi?]\mathbb{O}_s p \notin w$ implies $M(x), w \not\models [\pi?]\mathbb{O}_s p$. Suppose $[\pi?]\mathbb{O}_s p \notin w$ and $\pi \in w$ (the case $\pi \notin w$ can be shown similarly). It suffices to show $R_\pi(w) \cap W$ is the desired X. That is to show:

(1) $R_\pi(w) \cap W \in N_{\pi?}(w)$,

(2) $R_\pi(w) \cap W \subseteq\sim (w)$, and

(3) $R_\pi(w) \cap W \not\subseteq V(p)$.

For (1), it suffices to show that $R_\pi(w) \cap W \in N(w)$ and $R_\pi(w) \cap W \subseteq [\![\pi]\!] = \|\pi\|$. For the latter, note that, for all $v \in R_\pi(w)$, $\mathbb{O}_s[\pi?]w \cap \mathcal{PL} \subseteq v$. Because we assume $\pi \in w$, $[\pi?]\mathbb{O}_s\pi \in w$ by Proposition 16(5). Thus $\pi \in v$. For the former, we show that $[\pi?]\neg\mathbb{O}_s\bot \in w$. Since $[\pi?]\mathbb{O}_s p \notin w$, we have $[\pi?]\mathbb{O}_s\bot \notin w$ because $\vdash_{\mathbf{LRK}^\omega} [\pi?]\mathbb{O}_s\bot \to [\pi?]\mathbb{O}_s p$. Thus $\neg[\pi?]\mathbb{O}_s\bot \in w$. It follows from the axiom ($\downarrow$Neg) that $[\pi?]\neg\mathbb{O}_s\bot \in w$.

For (2), it suffices to show that $w \sim v$ for all $v \in R_\pi(w) \cap W$. Suppose $K_r\varphi \in w$. Then $\mathbb{O}_s\varphi \in w$ by (K-$\mathbb{O}$). Since w is closed under (R^ω_U), there must be q such that $U(q \leftrightarrow \varphi) \in w$. Hence $\mathbb{O}_s q \in w$. By Proposition 16(4), $[\pi?]\mathbb{O}_s q \in w$. Thus $q \in v$. Besides, since $v \in W$, $Uw \subseteq v$ by Lemma 28. Hence $q \leftrightarrow \varphi \in v$. Thus $\varphi \in v$.

Finally, (3) follows directly from Lemma 37.

Lemma 40. *For all $w \in W$ and all finite sequences $\vec{\pi} = \pi_1\!\downarrow_1, \ldots, \pi_k\!\downarrow_k$ ($k \geq 0$):*

$$[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\mathbb{O}_s p \in w \text{ iff } M(x), w \models [\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\mathbb{O}_s p.$$

Proof. Let P be the set of propositional formulas, among π_1 to π_k, that are contained in w and $\bar{P} = \{\pi_1, \ldots, \pi_k\} \setminus P$. Then $\bigwedge P \wedge \neg \bigvee \bar{P} \in w$ and $M(x), w \models \bigwedge P \wedge \neg \bigvee \bar{P}$.

$[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\mathbb{O}_s p \in w$

iff $[(\bigwedge P \wedge \neg \bigvee \bar{P})?]\mathbb{O}_s p \in w$ (by the axiom (!?))

iff $M(x), w \models [(\bigwedge P \wedge \neg \bigvee \bar{P})?]\mathbb{O}_s p$ (Lemma 38 and 39)

iff $M(x), w \models [\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\mathbb{O}_s p$ (by the validity of the axiom (!?))

Lemma 41 (Truth)**.** *Let $\varphi \in \mathcal{L}$. For all $w \in W$ and for all finite sequences $\vec{\pi} = \pi_1\!\downarrow_1, \ldots, \pi_k\!\downarrow_k$ ($k \geq 0$):*

$$[\pi_1\!\downarrow_1]\cdots[\pi_k\!\downarrow_k]\varphi \in w \text{ iff } M(x)_{\vec{\pi}}, w \models \varphi.$$

Proof. Induction on $d(\varphi)$. The 8 cases below cover the base and the inductive step.
Cases p, $\neg\varphi$, and $\varphi \to \psi$. Trivial.
Case $K_r\varphi$. This follows from Lemmas 35 and 36 and the IH.

Case $U\varphi$. This follows from Lemma 34 and the IH.
Case $Q\varphi$. This follows from Lemma 33 and the IH.
Case $\mathbb{O}_s\varphi$. We first note the following facts:

(1) $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]U(p\leftrightarrow\varphi)\to([\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]\mathbb{O}_s p\leftrightarrow[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]\mathbb{O}_s\varphi)$ is an $\mathbf{LRK}^\omega$-theorem and a validity.

(2) For all formulas φ, since $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}](\varphi\wedge\neg\varphi)\notin w$ by the consistency of w, there must be an atom p such that $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}](\neg U(p\leftrightarrow\varphi))\notin w$ as w is closed under the rule (R_U^ω). That is, $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}](U(p\leftrightarrow\varphi))\in w$.

Then we have:

$[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]\mathbb{O}_s\varphi\in w$

iff for all p, if $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]U(p\leftrightarrow\varphi)\in w$ then $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]\mathbb{O}_s p\in w$ (down by (1) and up by (2) and (1))

iff for all p, if $M(x)_{\vec{\pi}},w\models U(p\leftrightarrow\varphi)$ then $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]\mathbb{O}_s p\in w$ (Proposition 4(2) and IH)

iff for all p, if $M(x)_{\vec{\pi}},w\models U(p\leftrightarrow\varphi)$ then $M(x)_{\vec{\pi}},w\models\mathbb{O}_s p$ (Lemma 40)

iff $M(x)_{\vec{\pi}},w\models\mathbb{O}_s\varphi$ (up by (1), down by (2), IH, and (1))

Case $[\psi{\downarrow}]\varphi$. We first note the following fact:

(3) $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]U(p\leftrightarrow\psi)\to([\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}][p{\downarrow}]\varphi\leftrightarrow[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}][\psi{\downarrow}]\varphi)$ is an $\mathbf{LRK}^\omega$-theorem and a validity.

Then we have:

$[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}][\psi{\downarrow}]\varphi\in w$

iff for all p, if $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}]U(p\leftrightarrow\psi)\in w$ then $[\pi_1{\downarrow_1}]\cdots[\pi_k{\downarrow_k}][p{\downarrow}]\varphi\in w$ (down by (3), up by (2) and (3))

iff for all p, if $M(x)_{\vec{\pi}},w\models U(p\leftrightarrow\psi)$ then $M(x)_{\vec{\pi},p\downarrow},w\models\varphi$ (Proposition 4(3) and IH)

iff $M(x)_{\vec{\pi}},w\models[\psi{\downarrow}]\varphi$ (up by (3), down by (2), IH, and (3))

Theorem 3 (Completeness)**.** *The axiomatization* $\boldsymbol{LRK}^\omega$ *is weakly complete with respect to the semantics. Thus, the axiomatization* ***LRK*** *is also weakly complete with respect to the semantics.*

Proof. Suppose $\nvdash_{\mathbf{LRK}^\omega}\varphi$. Then $\{\neg\varphi\}$ is consistent. By the Lindenbaum lemma, there is a maximal consistent theory x containing $\neg\varphi$. By the Truth lemma, we have $M(x),x\models\neg\varphi$. Hence $\not\models\varphi$. □

7 Related Work

Deontic logic for epistemic actions. In general, LRK is a deontic logic for actions in the epistemic context. Several attempts at developing deontic logic for epistemic actions can be found in the literature, e.g., [2], [1], and [15].

A logic of permitted announcements is developed in [15]. The logic is based on the idea that a piece of information φ is permitted to be announced ($\mathbb{P}\varphi$) if the *epistemic state* after the announcement is ideal. The notion of epistemic state can be understood either syntactically or semantically. Syntactically, an epistemic state is just a set of epistemic formulas representing the knowledge of an agent. In contrast, from the semantic perspective, an epistemic state is a set of indistinguishable possible worlds by an agent. In [15], two semantic definitions of permitted announcements have been proposed based on the two understandings of epistemic states. Interestingly, the two definitions are shown to be equivalent, in the sense that they give the same set of logical validities. In this paper, we consider epistemic states as a semantic notion and apply the "neighbourhood epistemic model" introduced in [15]. Formally, a neighbourhood epistemic model is a structure $M = (W, \sim, N, V)$ where W, $\sim$, and V are the same as in the standard epistemic models (S5 models) and $N : W \to \wp(\wp(W))$ is a neighbourhood function assigning a set of *ideal* epistemic states to each possible world. Note that the public announcement of a proposition φ may restrict the current epistemic state to only those possible worlds satisfying φ. Hence, in [15], the formula $\mathbb{P}\varphi$ is interpreted such that it is true at a possible world w iff the epistemic state after the announcement of φ is contained in $N(w)$. In this paper, we employ a similar idea to provide semantics for obligatory announcements.

In [2], two binary operators $P(\psi, \varphi)$ and $O(\psi, \varphi)$ are introduced to express the notions that "after announcing ψ, it is permitted/obligatory to announce φ". It is clear that our operator $\mathbb{O}_s\varphi$ can be expressed in their framework as $O(\top, \varphi)$. Conversely, the operator $O(\psi, \varphi)$ can be expressed as $[\psi!]\mathbb{O}_s\varphi$ in LRK. In [2], a ternary relation $\mathcal{P} \subseteq S \times \wp(S) \times \wp(S)$ is used to provide the semantics for $O(\psi, \varphi)$ in such a way that $\mathcal{M}, s \models O(\psi, \varphi)$ iff for all $(s, [\![\psi]\!]_M, S'') \in \mathcal{P}, S'' \subseteq [\![\langle\psi\rangle\varphi]\!]_M$, where S is the domain of the model. The major difference between the semantics for $\mathbb{O}_s\varphi$ and $O(\psi, \varphi)$ is that our operator $\mathbb{O}_s\varphi$ is specific to the receiver's knowledge. This is reflected in the fact that the formula $K_r(\varphi \to \psi) \to (\mathbb{O}_s\varphi \to \mathbb{O}_s\psi)$ is valid in LRK whereas not in the logic of [2]. We think that LRK is more suitable for reasoning about obligatory announcements in the context of, e.g., database security because the receiver's initial knowledge is crucial for the sender's decision on which information should be disclosed, as suggested by Example 9.

An alternative definition of obligatory announcements has also been proposed in [1, Definition 10]. Aucher et al. [1] define the obligatory message of a security monitor as the minimal informative message such that, by sending it, the privacy policy compliance is restored. Clearly, the notion of "obligatory message" is different from the notion of obligatory announcements in our paper and [2]. However, a detailed conceptual analysis of the difference is beyond the scope of the paper.

Logic of questions. The semantics of questions or interrogatives has received much attention in logic, see [10]. The basic idea is that the meaning of a question is what counts as an answer to that question. For example, the question "Is it raining in Guangzhou?" has two possible answers: "It is raining in Guangzhou" and "It is not raining in Guangzhou". Observe that they are both propositions and, furthermore, they logically exclude each other and are jointly exhaustive. Hence, some logicians propose that questions can be represented semantically as a partition over the set of all possible worlds (or the logical space), e.g., [10], [6], and [25]. The semantics for questions proposed in [10], [6], and [25] are different from each other. In this paper, we follow the approach in [25], because it is a conservative extension of the standard epistemic logic and we want to focus on formalizing the power to know.

In [25], the so-called *epistemic issue models* are used to provide the semantics for questions. Formally, they are structures $M = (W, \sim, \approx, V)$ where only the equivalence relation $\approx$ on W (or, equivalently, a partition of W) is novel. In addition to the modality $K\varphi$ for knowledge, there are two new modalities $U\varphi$ and $Q\varphi$ in [25] with the same interpretations as in our paper. The notion that "the question whether φ is one of the current issues" is then expressed as the formula $U(Q\varphi \vee Q\neg\varphi)$.

As mentioned above, the idea behind the logic of question in [25] is that a set of questions can be represented as a partition over the logical space. But the logic remains neutral about where the partition or the set of questions is induced. For example, it can be induced either by a conversation, or by a game, or even by a research program [25]. In this paper, we assume that the partition is induced by the part of a normative system, such as a privacy policy, stipulating the questions to which the receiver has the power to know the answers.

Our work is closely related to [25] as the fragment of LRK without the operators $\mathbb{O}_s\varphi$, $[\varphi?]\varphi$ and $[\varphi!]\varphi$ is roughly a reinterpretation of the static logic $\mathrm{EL_Q}$ in [25]. However, there is also an important difference between our paper and [25]. In [25], there is also a dynamic operator $[\varphi?]\psi$ expressing that ψ holds after asking the question whether φ. But the effect of $[\varphi?]$ is to add the question $\varphi?$ to the set of current issues. In contrast, the operator $[r:\varphi?]\psi$ in LRK captures the deontic aspect of asking questions. It seems more appropriate to interpret the operator $[r:\varphi?]$ in LRK as the action that *commands* the sender to inform whether φ.

Logic of legal rights. The right to know or epistemic rights is a form of right. Works on the logical analysis of legal rights include, e.g., [14, 17, 18, 13]. Recent works on the topic investigating explicitly the power type of right include [19, 23, 8]. Given that these works are on general (power-)rights, one may wonder why there is a need to develop a separate logic for epistemic rights. One reason is that there are some valid reasoning patterns for epistemic rights that do not hold for general rights. For example, the formula $R_r p \to R_r \neg p$ is valid in LRK. However, it would be counterintuitive if we interpret the operator $R_r p$ as general power-rights: a registrar has the power to declare a marriage, but the registrar has no power to declare a divorce.

In our paper, we treat the power to know as an independent notion (from obligatory announcements). As pointed out by [8], there exist two different approaches formalizing legal power: an earlier tradition reduces power to (a combination of) obligations, permissions, and actions ([14, 17]); whereas the other (e.g.[18, 13]) holds that power is not reducible to static normative positions, which follows the original separation of Hohfeld (see [19]). Thus, our work adopts the second approach.

8 Conclusion and Future Work

In this paper, a logic LRK was introduced for reasoning about the power to know, the obligatory announcements, and the dynamics of questions and public announcements. We explored some (in)validities of LRK, where the interaction between the power to know and the obligatory announcements has been highlighted. We showed that the incorporation of the two dynamic operators in LRK increases the expressive power, thus no DEL-style reduction axiomatization exists for LRK. Our main technical contribution is a sound and complete axiomatization for LRK.

There are many directions for future research. A natural task is to investigate the computational complexity of LRK. We can also consider extensions to LRK. E.g., it is interesting to reason about the ability of the receiver in LRK since the receiver may use their power to know by asking (a sequence of) questions. Other extensions to LRK include "the power not to know" [20], the actions of adding or removing the receiver's power to know, and extensions to the multi-agent case.

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A Technical Appendices (Proofs)

Proof of Proposition 16.

(1) Simple induction on the structure of π.

(2) Since $\varphi \leftrightarrow \neg\neg\varphi$ is a propositional tautology, by the rule (Nec) it follows that $\vdash U(\varphi \leftrightarrow \neg\neg\varphi)$. Hence, by (U↓) and propositional logic, $\vdash [\varphi\downarrow]\psi \leftrightarrow [\neg\varphi\downarrow]\psi$.

(3) As an instance of the axiom (!?), we have:

$$\vdash [\pi!]\mathbb{O}_s\pi' \leftrightarrow (\top \wedge \neg\pi \wedge [(\top \wedge \neg\pi)?]\mathbb{O}_s\pi') \vee (\pi \wedge \top \wedge [(\pi \wedge \top)?]\mathbb{O}_s\pi').$$

Since $\vdash [\pi?]\mathbb{O}_s\pi' \leftrightarrow [(\pi\wedge\top)?]\mathbb{O}_s\pi'$ and $\vdash [\pi?]\mathbb{O}_s\pi' \leftrightarrow [(\top\wedge\neg\pi)?]\mathbb{O}_s\pi'$ (by the axiom (U↓)), from propositional logic it follows that $\vdash [\pi!]\mathbb{O}_s\pi' \leftrightarrow [\pi?]\mathbb{O}_s\pi'$.

(4) As an instance of the axiom (Trans), we have $\vdash [\top?]\mathbb{O}_s\pi \rightarrow ([\pi'?]\mathbb{O}_s\top \rightarrow [\pi'?]\mathbb{O}_s\pi)$. Note that, as an instance of the axiom (!?), we have $\vdash \mathbb{O}_s\pi \leftrightarrow [\top?]\mathbb{O}_s\pi$. Hence $\vdash \mathbb{O}_s\pi \rightarrow ([\pi'?]\mathbb{O}_s\top \rightarrow [\pi'?]\mathbb{O}_s\pi)$. Note also that $\vdash [\pi'?]\mathbb{O}_s\top$ since both $[\pi'?]$ and $\mathbb{O}_s$ are normal modalities . Therefore, $\vdash \mathbb{O}_s\pi \rightarrow [\pi'?]\mathbb{O}_s\pi$.

(5) By (1), $\vdash \pi \to [\pi!]\pi$ and thus $\vdash K_r(\pi \to [\pi!]\pi)$ by (Nec) and (U). By the axiom (!K) and propositional logic, we have $\vdash \pi \to ([\pi!]K_r\pi \leftrightarrow K_r(\pi \to [\pi!]\pi))$. Therefore, $\vdash \pi \to [\pi!]K_r\pi$. Note that $\vdash K_r\pi \to \mathbb{O}_s\pi$ by the axiom (K-$\mathbb{O}$). Hence $\vdash [\pi!]K_r\pi \to [\pi!]\mathbb{O}_s\pi$ since $[\pi!]$ is a normal modality. Hence $\vdash \pi \to [\pi!]\mathbb{O}_s\pi$. By item (3), it follows that $\vdash \pi \to [\pi?]\mathbb{O}_s\pi$.

Proof of Proposition 17.

Let $M = (W, \sim, \approx, N, V)$ be an arbitrary model and w a state of M. Suppose $M, w \models [\pi_1?]\mathbb{O}_s\pi_2$ and $M, w \models [\pi_3?]\mathbb{O}_s\pi_1$. We consider two cases: (1) $M, w \not\models \pi_1$. Then $M_{\pi_3?}, w \not\models \pi_1$ since π_1 is propositional. Since $M_{\pi_3?}, w \models \mathbb{O}_s\pi_1$, it follows that there is no $X \in N_{\pi_3?}(w)$ with $X \subseteq\sim_{\pi_3?}(w)$. Therefore $M_{\pi_3?}, w \models \mathbb{O}_s\pi_2$. Namely $M, w \models [\pi_3?]\mathbb{O}_s\pi_2$. (2) $M, w \models \pi_1$. Since $M, w \models [\pi_1?]\mathbb{O}_s\pi_2$, it follows that for all $X \subseteq N_{\pi_1?}(w)$, $X \subseteq\sim_{\pi_1?}(w)$ implies that $X \subseteq [\![\pi_2]\!]_M$. Since $M, w \models \pi_1$, we have

$$\text{for all } X \in N(w),\ X \subseteq [\![\pi_1]\!]_M \text{ and } X \subseteq\sim(w) \text{ implies that } X \subseteq [\![\pi_2]\!]_M \qquad \text{(A.1)}$$

Since we also assume that $M, w \models [\pi_3?]\mathbb{O}_s\pi_1$, we have

$$\text{for all } X \subseteq N_{\pi_3?}(w),\ X \subseteq\sim(w) \text{ implies that } X \subseteq [\![\pi_1]\!]_M \qquad \text{(A.2)}$$

Thus, by (A.1) and (A.2), we have

$$\text{for all } X \subseteq N_{\pi_3?}(w),\ X \subseteq\sim(w) \text{ implies that } X \subseteq [\![\pi_2]\!]_M \qquad \text{(A.3)}$$

Hence $M, w \models [\pi_3?]\mathbb{O}_s\pi_2$.

Proof of Lemma 18.

We show only (1). Let $M = (W, \sim, \approx, N, V)$ be an arbitrary model and w a state of M. Suppose $M, w \models \pi_1 \wedge \pi_2$, we show that $M, w \models [\pi_1\downarrow][\pi_2?]\mathbb{O}_s\pi \leftrightarrow [\pi_1 \wedge \pi_2?]\mathbb{O}_s\pi$. If $\downarrow=?$, we have:

$M, w \models [\pi_1?][\pi_2?]\mathbb{O}_s\pi$

iff $(M_{\pi_1?})_{\pi_2?}, w \models \mathbb{O}_s\pi$

iff $\forall X \in (N_{\pi_1?})_{\pi_2?}(w)$, $X \subseteq\sim(w)$ implies $X \subseteq [\![\pi]\!]_{(M_{\pi_1?})_{\pi_2?}} = [\![\pi]\!]_M$

iff $\forall X \in N(w)$ with $X \subseteq [\![\pi_1 \wedge \pi_2]\!]_M$, $X \subseteq\sim(w)$ implies $X \subseteq [\![\pi]\!]_M$
(π_1 and π_2 are propositional and they are true at M, w)

iff $M_{\pi_1 \wedge \pi_2?}, w \models \mathbb{O}_s\pi$

iff $M, w \models [\pi_1 \wedge \pi_2?]\mathbb{O}_s\pi$

Otherwise $\downarrow =!$, we have:

$$
\begin{aligned}
& M, w \models [\pi_1 !][\pi_2 ?]\mathbb{O}_s \pi \\
\text{iff } & (M_{\pi_1 !})_{\pi_2 ?}, w \models \mathbb{O}_s \pi \\
\text{iff } & \forall X \in N_{\pi_2 ?}(w),\ X \subseteq \sim_{\pi_1 !} (w) \text{ implies } X \subseteq [\![\pi]\!]_{(M_{\pi_1 ?})_{\pi_2 ?}} = [\![\pi]\!]_M \\
\text{iff } & \forall X \in N(w) \text{ with } X \subseteq [\![\pi_2]\!]_M, \text{ if } X \subseteq \sim (w) \text{ and } X \subseteq [\![\pi_1]\!]_M, \text{ then } X \subseteq [\![\pi]\!]_M \\
\text{iff } & \forall X \in N(w) \text{ with } X \subseteq [\![\pi_1 \wedge \pi_2]\!]_M,\ X \subseteq \sim (w) \text{ implies } X \subseteq [\![\pi]\!]_M \\
& (\pi_1 \text{ and } \pi_2 \text{ are propositional}) \\
\text{iff } & M_{\pi_1 \wedge \pi_2 ?}, w \models \mathbb{O}_s \pi \\
\text{iff } & M, w \models [\pi_1 \wedge \pi_2 ?]\mathbb{O}_s \pi
\end{aligned}
$$

Proof of Proposition 19.

Induction on the value of n. The base is to show that $\models \mathbb{O}_s\pi \leftrightarrow [\top ?]\mathbb{O}_s\pi$, which is obvious as $M_{\top ?} = M$ for all models M. Suppose that (!?) is valid when $n = k$. We show that it also holds when $n = k + 1$. Let $M = (W, \sim, \approx, N, V)$ be an arbitrary model and w a state of M. Let P be the set of propositional formulas, among $\pi_1, \dots, \pi_{k+1}$, that are true at w. It suffices to show that

$$M, w \models [\pi_1 \downarrow_1] \cdots [\pi_{k+1} \downarrow_{k+1}]\mathbb{O}_s\pi \leftrightarrow [(\bigwedge P \wedge \neg \bigvee(\{\pi_1, \dots, \pi_{k+1}\} \setminus P))?]\mathbb{O}_s\pi \quad \text{(A.4)}$$

We have the following equivalent statements:

$$
\begin{aligned}
& M, w \models [\pi_1 \downarrow_1][\pi_2 \downarrow_2] \cdots [\pi_{k+1} \downarrow_{k+1}]\mathbb{O}_s\pi \\
\text{iff } & M_{\pi_1 \downarrow_1}, w \models [\pi_2 \downarrow_2] \cdots [\pi_{k+1} \downarrow_{k+1}]\mathbb{O}_s\pi \\
\text{iff } & M_{\pi_1 \downarrow_1}, w \models [(\bigwedge(\{\pi_2, \dots, \pi_{k+1}\} \cap P) \wedge \neg \bigvee(\{\pi_2, \dots, \pi_{k+1}\} \setminus P))?]\mathbb{O}_s\pi \\
& (\text{IH and } M_{\pi_1 \downarrow_1}, w \models \bigwedge(\{\pi_2, \dots, \pi_{k+1}\} \cap P) \wedge \neg \bigvee(\{\pi_2, \dots, \pi_{k+1}\} \setminus P)) \\
\text{iff } & M, w \models [\pi_1 \downarrow_1][(\bigwedge(\{\pi_2, \dots, \pi_{k+1}\} \cap P) \wedge \neg \bigvee(\{\pi_2, \dots, \pi_{k+1}\} \setminus P))?]\mathbb{O}_s\pi
\end{aligned}
$$

Note that $M, w \models \bigwedge(\{\pi_2, \dots, \pi_{k+1}\} \cap P) \wedge \neg \bigvee(\{\pi_2, \dots, \pi_{k+1}\} \setminus P)$. If $\pi_1 \in P$ (namely, $M, w \models \pi_1$), by Lemma 18 it follows that

$$
\begin{aligned}
M, w \models [\pi_1 \downarrow_1][(\bigwedge(\{\pi_2, \dots, \pi_{k+1}\} \cap P) \wedge \neg \bigvee(\{\pi_2, \dots, \pi_{k+1}\} \setminus P))?]\mathbb{O}_s\pi \leftrightarrow \\
[(\bigwedge P \wedge \neg \bigvee(\{\pi_1, \dots, \pi_{k+1}\} \setminus P))?]\mathbb{O}_s\pi \quad \text{(A.5)}
\end{aligned}
$$

If $\pi_1 \notin P$, by Lemma 18 it also holds that (A.5). Therefore (A.4) holds.

Proof of Lemma 25.

We show only (2): We first show that $\Box x$ contains all $\mathbf{LRK}^{\omega}$-theorems. Let $\vdash_{\mathbf{LRK}^{\omega}} \psi$. Then $\vdash_{\mathbf{LRK}^{\omega}} \Box\psi$ by the rules (Nec) and the axiom (U). Since x contains all $\mathbf{LRK}^{\omega}$-theorems, $\Box\psi \in x$. Thus $\psi \in \Box x$. We then show that $\Box x$ is closed under the rule (MP). Suppose $\psi, \psi \to \chi \in \Box x$. It follows that $\Box\psi, \Box(\psi \to \chi) \in x$. Note that $\vdash_{\mathbf{LRK}^{\omega}} \Box(\psi \to \chi) \to (\Box\psi \to \Box\chi)$ by the axiom (S5). Thus $\Box(\psi \to \chi) \to (\Box\psi \to \Box\chi) \in x$ since x is a theory. It then follows that $\Box\chi \in x$ since x is closed under (MP). Hence $\chi \in \Box x$. To show that $\Box x$ is also closed under $(\mathrm{R}^{\omega}_{U})$, suppose $\xi(\neg U(p \leftrightarrow \varphi)) \in \Box x$ for all $p \in \textsc{prop}$. Then $\Box\xi(\neg U(p \leftrightarrow \varphi)) \in x$ for all $p \in \textsc{prop}$. Note that $\Box\xi(\sharp)$ is a necessity form. Thus $\Box\xi(\varphi \wedge \neg\varphi) \in x$ since x is closed under $(\mathrm{R}^{\omega}_{U})$. Therefore $\xi(\varphi \wedge \neg\varphi) \in \Box x$.

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