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Capacity Constraints and Semisingular Optimal Controls in a Predator-Prey Fishery *

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Abstract

We study the optimal management of a two-species predator-prey fishery under explicit capacity constraints on harvesting effort. Unlike classical models that assume unconstrained controls, we treat prey and predator capacity limits as bifurcation parameters and show how they reshape both steady states and transition dynamics. Our analysis provides three main contributions. First, we deliver a complete classification of steady state regimes – singular, semisingular, and bang-bang – together with local stability via phase-plane methods. Second, we derive a methodological result: segments of the singular loci, where costates equal marginal revenues, constitute optimal approach paths, and we obtain closed-form feedback expressions for the associated semisingular controls. Third, we uncover sharp policy shifts and path dependence induced by capacity constraints: when prey capacity binds, predator control becomes a compensating strategy; when predator capacity binds, prey harvesting emerges as leverage, and prey closures can arise endogenously. These findings provide valuable extensions to the bioeconomic Gordon–Schaefer model with Lotka–Volterra dynamics.

Keywords: Optimal control; Predator-prey fisheries; Capacity constraints; Singular control; Bang-bang policies; Phase-plane analysis.

JEL Codes: Q22, Q28, C61, C62.

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1 Introduction

Optimal management of renewable resources is a cornerstone of environmental economics (see Clark (2010)). Classical bioeconomic models, such as the Gordon-Schaefer framework (Gordon, 1954; Schaefer, 1957) for fisheries, prescribe bang-bang harvesting policies under centralized control, assuming that the optimal steady state is feasible given technological limits (Clark, 1990). However, a stationary solution can require an effort level that exceeds the maximum capacity of the fleet, making it infeasible. This issue is particularly relevant for interacting species, which often require extreme controls over some populations (e.g. predator control). Ignoring these constraints can lead to unrealistic prescriptions and misaligned management outcomes. Additionally, the literature offers limited guidance on optimal control along transition dynamics, largely due to the linear control structures and high dimensionality making standard results on most rapid approach paths (Spence and Starrett, 1975) inapplicable. This paper studies the economic and ecological implications of fishing capacity constraints and provides new insights into optimal control during the transition paths.

The literature on multi-species fisheries management has long recognized the importance of ecological interactions (Pikitch et al., 2004; Voss et al., 2014). The foundational contributions of Quirk and Smith (1970); Silvert and Smith (1977); May et al. (1979) and later work such as Clark (1976, 1990); Mendelssohn (1980); Hoekstra and Van den Bergh (2005); Wang and Ewald (2010) established dynamic models for interacting populations and highlighted inefficiencies in single-species approaches. Subsequent studies explored predator-prey and competitive systems (Flaaten, 1991, 1998), trophic incentives (Wilen and Wilen, 2012), environmental drivers (Nieminen et al., 2012), and biodiversity values (Bertram and Quaas, 2017). Although these extensions have enriched policy design, they share a common limitation: harvesting capacity constraints are not modeled explicitly, as most frameworks assume that whenever an interior steady state exists, it is reachable through bang–bang control.

This paper addresses this gap. We develop a dynamic predator-prey fishery model with species-specific harvesting technologies and explicit capacity limits on effort. Treating these bounds as bifurcation parameters, we provide the following.

(a) A complete classification of steady states: singular, semisingular, and bang-bang together, with local stability via phase-plane analysis.

(b) An analytical characterization of semisingular feedback controls along singular loci. We show that segments of these loci are themselves optimal approach paths. To our knowledge, we are the first to provide an explicit proof of this result.

(c) Policy insights under constraints of binding capacity: When the prey capacity is binding, predator control serves as a compensatory instrument; when the predator capacity is binding, prey harvesting becomes the operative margin, and endogenous prey closures can emerge.

These results contribute to the literature on natural resource management, particularly in the context of the growing emphasis on ecosystem-based management. We extend the canonical Gordon-Schaefer framework by incorporating Lotka-Volterra dynamics and explicitly accounting for realistic fleet capacity constraints.

Our analysis is motivated by concrete examples in which non-trivial steady states emerge because fishing capacity is limited for some interacting species. A prominent example is the recent northward expansion of octopus populations in the Northeast Atlantic, which has disrupted crustacean fisheries along the coasts of Brittany and southwest England (Arechavala-Lopez et al., 2019; Marine Management Organisation, 2025; Schickele et al., 2021). Effective ecosystem management in this context requires substantial predator control to protect high-value crab and lobster stocks. However, octopus fishing capacity remains severely constrained: few vessels target octopus, reflecting its historically low prevalence in these regions. Moreover, despite the increase in octopus abundance, fishermen are reluctant to invest in specialized gear, given the persistent uncertainty about the long-term viability of the octopus fishery and the opportunity cost of diverting effort from historically lucrative activities such as scallop fishing (Marine Management Organisation, 2025). As a result, the level of octopus harvesting capacity required for optimal ecosystem management may exceed the feasible capacity of the fleet. Such binding constraints makes optimal policy design highly non-trivial and motivate a careful analytical treatment.

Formally, we extend a Gordon-Schaefer model with Lotka-Volterra predator-prey dynamics to analyze the economic and ecological implications of binding capacity constraints on fishing effort. The optimal control problem of a regulator is to choose the harvesting effort of both both prey and the predator to maximize the present value of the profits from exploiting a predator-prey system. We assume that harvesting technologies differ between species, so each is subject to its own capacity constraint. We derive necessary conditions for the existence of singular, semisingular, and bang-bang steady-states. We analyze the local stability via phase-plane methods, and characterize feedback optimal controls along singular loci using the Maximum Principle.

We find that capacity constraints generate sharp, non-linear shifts in optimal policy. In steady-state, limited prey capacity always induces a surplus of prey population which is accompanied by stronger predator control to capture prey indirectly through predators after biomass conversion. In contrast, limited predator capacity leads to a higher predator population, reduced prey stocks, and a lower prey fishing rate. This can even lead to a closure of the prey fishery in the long-run. We analytically derive optimal feedback control on a specific segment of the optimal trajectories, which is a step forward in designing more appropriate public policies. These mechanisms underscore the importance of integrating technological limits into ecosystem-based fishery management.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 classifies steady states and analyzes their stability. Section 4 derives optimal transition paths and semisingular feedback controls. Section 5 investigates the feasibility of steady state and its related trajectories. Section 6 explores capacity-induced policy changes, and Section 7 concludes.

2 The model

Consider an ecological system consisting of two interacting species: prey and predator. The prey population grows according to a logistic growth function and decreases due to predation and fishing. The predator grows according to the number of captured prey and decreases according to a natural mortality rate (see Clark, 2010; Quérou and Tomini, 2013; Begon and Townsend, 2020, for references using this type of predator-prey equations). The population dynamics of the prey, denoted as $x(t)$, and the predator, denoted as $y(t)$, evolve according to the following differential equations:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - axy - ux, \qquad \dot{y} = -\delta y + bxy - vy \tag{1}$$

where $r > 0$ represents the intrinsic growth rate of the prey, $K > 0$ is the carrying capacity, $\delta > 0$ is the intrinsic mortality rate of the predator, $a > 0$ is the predation rate, b/a is the biomass conversion efficiency, and u and v are the fishing efforts for the prey and the predator, respectively[1].

The domain of x and y is $\mathbb{R}^2_+ = \{(x, y) : x, y \geq 0\}$. Denote $\Omega = \{(x, y) : x, y > 0\}$ the interior of the domain. The boundary of the domain consists of the positive x- and y-axes.

We assume that a single social planner controls the fishing of both species. The harvested fish are sold on a competitive market at constant prices, which differs for each species. Fishing also incurs a constant cost per unit of effort. Thus, the profits from fishing for prey and predator are $[p_1 x - c_1]\,u$ and $[p_2 y - c_2]\,v$, respectively, where p_1 and p_2 are the prices of the prey and predator, respectively, and c_1 and c_2 are the cost per unit of effort to fish for prey and predator, respectively. The social planner's objective is to maximize the total discounted

[1]The harvesting function usually depends on a catchability coefficient $q > 0$, we omit these parameters for simplicity.

profit:

$$J = \max_{\substack{0 \le u(\cdot) \le \bar{u} \\ 0 \le v(\cdot) \le \bar{v}}} \int_0^\infty e^{-\rho t} \left\{ \left[p_1 x\left(t\right) - c_1 \right] u\left(t\right) + \left[p_2 y\left(t\right) - c_2 \right] v\left(t\right) \right\} dt$$

subject to (1) and

$$0 \le u \le \bar{u}, \qquad 0 \le v \le \bar{v}, \tag{2}$$

where $\bar{u}$ and $\bar{v}$ are the upper bounds of the prey and predator fishing efforts, respectively.

Denote the current value Hamiltonian as

$$\begin{aligned} H\left(x, y, \lambda, \mu, u, v\right) = \quad & \left(p_1 x - c_1\right) u + \left(p_2 y - c_2\right) v + \lambda x \left[r\left(1 - x/K\right) - ay - u \right] \\ & + \mu y \left[-\delta + bx - v \right], \end{aligned} \tag{3}$$

where λ and μ are the co-states. The first order conditions with respect to controls yield

$$\lambda x = p_1 x - c_1, \qquad \mu y = p_2 y - c_2 \tag{4}$$

and thus, the optimal controls u and v are, respectively,

$$u \begin{cases} = \bar{u} & \text{if } p_1 x - c_1 > \lambda x, \\ \in [0, \bar{u}] & \text{if } p_1 x - c_1 = \lambda x, \\ = 0 & \text{if } p_1 x - c_1 < \lambda x, \end{cases} \qquad v \begin{cases} = \bar{v} & \text{if } p_2 y - c_2 > \mu y, \\ \in [0, \bar{v}] & \text{if } p_2 y - c_2 = \mu y, \\ = 0 & \text{if } p_2 y - c_2 < \mu y. \end{cases} \tag{5}$$

The dynamics of the co-states are

$$\begin{aligned} \dot{\lambda} &= \rho\lambda - p_1 u - \lambda \left[r\left(1 - \frac{x}{K}\right) - ay - u \right] + \lambda \frac{rx}{K} - \mu b y, \\ \dot{\mu} &= \rho\mu - p_2 v + \lambda a x - \mu \left[-\delta + bx - v \right] \end{aligned} \tag{6}$$

with x and y governed by (1).

We study the existence and stability of interior steady states, which are time-independent solutions to the dynamical system (1) and (6) with positive x- and y-components. Similarly, the associated optimal controls, u and v, that satisfy (5) are also time-independent (constants).

Steady states fall into three categories: bang-bang states, in which $u \in \{0, \bar{u}\}$ and $v \in \{0, \bar{v}\}$; semisingular states, in which $u \in (0, \bar{u})$ or $v \in (0, \bar{v})$; and singular states, in which both controls lie in the interior, $u \in (0, \bar{u})$ and $v \in (0, \bar{v})$.

3 Steady states: classification and local stability

The prey-predator singular loci

$$\Gamma_1 : \ \lambda = p_1 - \frac{c_1}{x}, \qquad \Gamma_2 : \ \mu = p_2 - \frac{c_2}{y} \tag{7}$$

separate the positive quadrant $\Omega = \{(x, y) : x > 0,\, y > 0\}$ into four control regions R_{mn}, $m, n \in \{0, 1\}$, according to the bang–bang policies $(u, v) = (m\bar{u}, n\bar{v})$ through the signs of $p_1 x - c_1 - x\lambda$ and $p_2 y - c_2 - y\mu$. The *isoclines*, where the state velocities change sign inside each region, are

$$l_1^m : \ r\Big(1 - \frac{x}{K}\Big) - ay = m\bar{u}, \qquad l_2^n : \ bx = \delta + n\bar{v}, \qquad m, n \in \{0, 1\}. \tag{8}$$

3.1 Bang-bang equilibria

For $(u, v) = (m\bar{u}, n\bar{v})$, the candidate steady states $(\bar{x}_{mn}, \bar{y}_{mn})$ solve

$$r\Big(1 - \frac{\bar{x}_{mn}}{K}\Big) - a\bar{y}_{mn} = m\bar{u}, \qquad -\delta + b\bar{x}_{mn} = n\bar{v}, \tag{9}$$

with

$$\bar{x}_{mn} = \frac{1}{b}\left[\delta + n\bar{v}\right], \qquad \bar{y}_{mn} = \frac{1}{a}\left[r\left(1 - \frac{\delta + n\bar{v}}{bK}\right) - m\bar{u}\right]. \tag{10}$$

Thus, there are at most four bang-bang steady states: $(\bar{x}_{00}, \bar{y}_{00})$, $(\bar{x}_{01}, \bar{y}_{01})$, $(\bar{x}_{10}, \bar{y}_{10})$, and $(\bar{x}_{11}, \bar{y}_{11})$, corresponding to $(u, v) = (0, 0)$, $(0, \bar{v})$, $(\bar{u}, 0)$, and $(\bar{u}, \bar{v})$, respectively.

Proposition 1. *(Existence and local stability of bang-bang equilibria) For each $m, n \in \{0, 1\}$, the candidate bang-bang equilibrium $(\bar{x}_{mn}, \bar{y}_{mn})$ exist in Ω if and only if*

$$\frac{m\bar{u}}{r} + \frac{\delta + n\bar{v}}{bK} < 1. \tag{11}$$

Furthermore, if the bang–bang controls are optimal in a neighborhood of its definition domain, the steady-state is locally asymptotically stable.

Obviously, $(\bar{x}_{mn}, \bar{y}_{mn})$ may not be in a steady state if the corresponding controls $(m\bar{u}, n\bar{v})$ are not optimal. In other words, a candidate $(\bar{x}_{mn}, \bar{y}_{mn})$ is an actual steady state of the planner only if it lies in its own region R_{mn}, i.e., the sign conditions

$$p_1\bar{x}_{mn} - c_1 \lesseqgtr \lambda\bar{x}_{mn}, \qquad p_2\bar{y}_{mn} - c_2 \lesseqgtr \mu\bar{y}_{mn}$$

hold with appropriate inequalities for m, n. Clearly, $(\bar{x}_{mn}, \bar{y}_{mn})$ is a steady state only if it is located in R_{mn}.

In case $(\bar{x}_{mn}, \bar{y}_{mn})$ is a steady state, its local stability of these steady states can be determined by evaluating the Jacobian matrix at the equilibrium points. The Jacobian is as follows:

$$J(x,y) = \begin{bmatrix} r - \frac{2r}{K}x - ay - m\bar{u} & -ax \\ by & bx - \delta - n\bar{v} \end{bmatrix}.$$

Using (10), the Jacobian simplifies to:

$$J(\bar{x}_{mn}, \bar{y}_{mn}) = \begin{bmatrix} -r\bar{x}_{mn}/K & -a\bar{x}_{mn} \\ b\bar{y}_{mn} & 0 \end{bmatrix}.$$

The trace and determinant of $J\left(\bar{x}_{mn}, \bar{y}_{mn}\right)$ are:

$$\operatorname{tr}\left(J(\bar{x}_{mn}, \bar{y}_{mn})\right) = -\frac{r\bar{x}_{mn}}{K} < 0, \quad \det(J(\bar{x}_{mn}, \bar{y}_{mn})) = ab\bar{x}_{mn}\bar{y}_{mn} > 0$$

provided that (11) holds. Since the determinant is positive and the trace is negative, the steady state is locally asymptotically stable.

In addition, a steady state exhibits a spiral trajectory if and only if

$$(\operatorname{tr}(J(\bar{x}_{mn}, \bar{y}_{mn})))^2 - 4\det(J(\bar{x}_{mn}, \bar{y}_{mn})) < 0$$

which is equivalent to

$$\frac{m\bar{u}}{r} + \frac{n\bar{v}+\delta}{bK} + \frac{r\left(n\bar{v}+\delta\right)}{4\left(bK\right)^2} < 1. \tag{12}$$

This condition ensures that the eigenvalues of the Jacobian have complex conjugate components with negative real parts, leading to oscillatory convergence toward equilibrium.

3.2 Singular and semisingular equilibria

This type of steady state must be on at least one of the two curves Γ_1 or Γ_2. In steady state (x, y) with interior controls, the system satisfies

$$\begin{cases} u = r\left(1 - \dfrac{x}{K}\right) - ay, & v = -\delta + bx, \\ \rho\lambda = p_1 u - \dfrac{\lambda r x}{K} + \mu b y, & \rho\mu = p_2 v + \lambda a x, \end{cases} \tag{13}$$

with singular relations

$$\Gamma_1: \quad \lambda = p_1 - \frac{c_1}{x} \tag{14}$$

and

$$\Gamma_2: \quad \mu = p_2 - \frac{c_2}{y}. \tag{15}$$

Proposition 2. *(Pure singular intersections) If $(x^*, y^*) \in \Gamma_1 \cap \Gamma_2$ is a steady state, it solves*

$$
\begin{aligned}
& p_1 \left[r \left(1 - \frac{x^*}{K} \right) - a y^* \right] - \left[p_1 x^* - c_1 \right] \left[\frac{r}{K} + \frac{\rho}{x^*} \right] + b \left[p_2 y^* - c_2 \right] = 0, \\
& a \left[p_1 x^* - c_1 \right] + p_2 \left(\delta - b x^* \right) + \rho \left[p_2 - \frac{c_2}{y^*} \right] = 0,
\end{aligned} \tag{16}
$$

subject to feasibility

$$
\frac{\delta}{b} \le x^* \le \frac{\delta + \bar{v}}{b}, \qquad y^* \le \frac{r}{a}\Big(1 - \frac{x^*}{K}\Big) \le y^* + \frac{\bar{u}}{a}. \tag{17}
$$

Proof. Suppose (x^*, y^*) is a steady state and let $u^*$and v^* be the corresponding fishing rates. The relations in (17) follow from the first two equations in (13) and the control constraints

$$
0 \le u^* \le \bar{u}, \qquad 0 \le v^* \le \bar{v}. \tag{18}
$$

To show that (x^*, y^*) satisfies (16), we let λ^* and μ^* be the corresponding co-states and use the last two equations in (13). As a result,

$$
\begin{aligned}
\rho\lambda^* - p_1 \left[r \left(1 - \frac{x^*}{K} \right) - a y^* \right] + \lambda^* \frac{r x^*}{K} - \mu^* b y^* &= 0, \\
\rho\mu^* - p_2 \left[-\delta + b x_2^* \right] - \lambda^* a x^* &= 0.
\end{aligned}
$$

Furthermore, since $(x^*, y^*) \in \Gamma_1 \cap \Gamma_2$, by (14) and (15),

$$
\lambda^* = p_1 - \frac{c_1}{x^*}, \qquad \mu^* = p_2 - \frac{c_2}{y^*}.
$$

Substituting them into the above equations, we obtain (16). □

Next, we investigate the existence of steady states with only one control in the interior of the interval, i.e., semisingular steady states.

Proposition 3. *(Semisingular with $v \in (0, \bar{v})$) Suppose $(\hat{x}_m, \hat{y}_m)$ is a semisingular steady state for $m \in \{0, 1\}$ with controls $u = m\bar{u}$ and $v \in (0, \bar{v})$. Then:*

$$
\hat{y}_m = \frac{r}{a}\Big(1 - \frac{\hat{x}_m}{K}\Big) - \frac{m\bar{u}}{a},
$$

and

$$
p_2(\rho + \delta - b\hat{x}_m) - \rho c_2 \hat{y}_m = a\hat{x}_m \rho + \frac{r\hat{x}_m}{K}\Big(p_1 m\bar{u} + b p_2 \hat{y}_m - b c_2\Big),
$$

with feasibility constraint:

$$
\frac{\delta}{b} < \hat{x}_m < \frac{\delta + \bar{v}}{b}.
$$

Proof. Suppose $(\hat{x}_m, \hat{y}_m)$ for $m = 0, 1$ is a steady state with controls $\hat{u}_m = m\bar{u}$ and $\hat{v}_m \in (0, \bar{v})$.

Let $\hat{\lambda}_m$ and $\hat{\mu}_m$ be the corresponding co-states. The five unknowns $\hat{x}_m$, $\hat{y}_m$, $\hat{v}_m$, $\hat{\lambda}_m$, and $\hat{\mu}_m$ are determined by (13) and (15). Eliminating $\hat{v}_m$, $\hat{\lambda}_m$ and $\hat{\mu}_m$, we derive the two equations for $\hat{x}_m$ and $\hat{y}_m$ as

$$\begin{aligned} \hat{y}_m &= \frac{r}{a}\left(1-\frac{\hat{x}_m}{K}\right)-\frac{m\bar{u}}{a}, \\ p_2\left[\rho+\delta-b\hat{x}_m\right]-\frac{\rho c_2}{\hat{y}_m} &= \frac{a\hat{x}_m}{\rho+r\hat{x}_m/K}\left[p_1 m\bar{u}+bp_2\hat{y}_m-bc_2\right]. \end{aligned}$$

In view of the control constraints, $\hat{v}_m \in (0,\bar{v})$, a solution of the above equations is steady state if

$$\frac{\delta}{b} < \hat{x}_m < \frac{\delta+\bar{v}}{b}.$$

□

Similarly, if the steady state lies only in Γ_2 (predator singular), v can be given by the steady state equation and $u \in \{0,\bar{u}\}$ by regions and solve the reduced system. The following results can be shown analogously to the last proposition.

Proposition 4. *(Semisingular with $u \in (0,\bar{u})$) Suppose $(\tilde{x}_n,\tilde{y}_n)$ is a semisingular steady state for $n \in \{0,1\}$ with controls $v = n\bar{v}$ and $u \in (0,\bar{u})$. Then:*

$$\tilde{x}_n = \frac{\delta+n\bar{v}}{b},$$

and

$$\tilde{y}_n = \frac{p_1 r\left(1-\frac{\tilde{x}_n}{K}\right)-(p_1-c_1\tilde{x}_n)(\rho+\frac{r\tilde{x}_n}{K})}{ap_1-b\rho\left(p_2 n\bar{v}+(p_1-c_1\tilde{x}_n)a\tilde{x}_n\right)},$$

with constraint:

$$0 < r\left(1-\frac{\tilde{x}_n}{K}\right)-a\tilde{y}_n < \bar{u}.$$

Proof. Suppose $(\tilde{x}_n,\tilde{y}_n)$ for $n = 0,1$ is a steady state with controls $\tilde{u}_n \in (0,\bar{u})$ and $\tilde{v}_n = n\bar{v}$. Let $\tilde{\lambda}_n$ and $\tilde{\mu}_n$ be the corresponding co-states. Then, the five unknowns, $\tilde{x}_n$, $\tilde{y}_n$, $\tilde{u}_n$, $\tilde{\lambda}_n$, and $\tilde{\mu}_n$ are determined by Eqs. (13) and (14). We first find

$$\tilde{x}_n = \frac{n\bar{v}+\delta}{b}, \qquad \tilde{\lambda}_n = p_1-\frac{c_1}{\tilde{x}_n}, \qquad \tilde{\mu}_n = \frac{1}{\rho}\left[p_2 n\bar{v}+\tilde{\lambda}_n a\tilde{x}_n\right],$$

and then solve the linear equations

$$\tilde{u}_n + a\tilde{y}_n = r\left(1-\frac{\tilde{x}_n}{K}\right), \qquad p_1\tilde{u}_n+\tilde{\mu}_n b\tilde{y}_n = \tilde{\lambda}_n\left[\rho+\frac{r\tilde{x}_n}{K}\right]$$

for $\tilde{u}_n$ and $\tilde{y}_n$. The solution is a steady state if

$$0 < r\left(1 - \frac{\tilde{x}_n}{K}\right) - a\tilde{y}_n < \bar{u}.$$

□

4 Optimal trajectories and semisingular feedback

We show that segments of singular loci, Γ_1 and Γ_2, are optimal approach paths within capacity strips bounded by l_1^0, l_1^1, l_2^0, and l_2^1. The resulting semisingular feedback rules (Props. 6–7) provide a sharp characterization of transition dynamics in multispecies bioeconomic models. Explicit derivations of feedback controls along singular curves are extremely rare in the literature; in this respect, our analysis offers a methodological contribution with direct implications for the design of fisheries policies. To our knowledge, this is the first paper to formally prove these results, which were conjectured but were not proven by Clark (1990).

Proposition 5. *For $j = 1, 2$, the portion of Γ_j lying between straight lines, l_j^0 and l_j^1, defined in (8), attracts optimal trajectories in its neighborhood and is itself part of an optimal trajectory.*

Proof. By Pontryagin's Maximum Principle and the current value Hamiltonian given by (3), the system dynamics of the state variables, x and y, and the costate variables, λ and μ, is governed by (1) and (6) with u and v specified in (5). Let $V(x,y)$ denote the value function. It is well-known that

$$\lambda = V_x(x,y), \qquad \mu = V_y(x,y).$$

Thus, the singular loci Γ_1 and Γ_2 defined by (7) are curves in the xy-plane. Furthermore, the relations in (5) indicate that $u = 0$ on one side of Γ_1, and $u = \bar{u}$ on the other side of Γ_1, and similarly, $v = 0$ on one side of Γ_2, and $v = \bar{v}$ on the other side of Γ_2.

We first consider the case where $j = 1$. That is, we show that the trajectories on the two sides of Γ_1 within l_1^0 and l_1^1 are attracted to Γ_1. In this portion of Γ_1, the control v is either 0 or $\bar{v}$ according to whether Γ_1 is below or above Γ_2, respectively. Hence, in a neighborhood of that portion of Γ_1, the velocity fields with $v = n\bar{v}$ and on the two sides of Γ_1 are

$$F^{0n}(x,y) = \begin{pmatrix} x\big(r(1-\frac{x}{K}) - ay\big) \\ y\big(-\delta + bx - n\bar{v}\big) \end{pmatrix}$$

for (x,y) to the left of Γ_1, and

$$F^{1n}(x,y) = \begin{pmatrix} x\big(r(1-\frac{x}{K}) - ay - \bar{u}\big) \\ y\big(-\delta + bx - n\bar{v}\big) \end{pmatrix}$$

for (x,y) to the right of Γ_1. The state velocity field F^{mn} gives for every (x,y) the instantaneous direction of motion when the bang-bang controls are applied $(m\bar{u}, n\bar{v})$. Note that within the strip bounded by l_1^0 and l_1^1,

$$0 < r\left(1 - \frac{x}{K}\right) - ay < \bar{u}.$$

Hence, the x-components of the two vectors $F^{0n}(x,y)$ and $F^{1n}(x,y)$ have the opposite signs. Specifically, the x-component is positive on the left side of Γ_1, and negative on the right side of Γ_1.

To show that the trajectories near Γ_1 are attracted to Γ_1, we compare the slopes of the trajectories off Γ_1 and the slope of the tangent lines on Γ_1. A trajectory off Γ_1 has the slope

$$S^{0n}(x,y) = \frac{y(-\delta + bx - n\bar{v})}{x\left[r(1 - x/K) - ay\right]}, \qquad n \in \{0,1\},$$

in (x,y) to the left of Γ_1, and the slope

$$S^{1n}(x,y) = \frac{y(-\delta + bx - n\bar{v})}{x\left[r(1 - x/K) - ay - \bar{u}\right]}, \qquad n \in \{0,1\}$$

in (x,y) to the right of Γ_1. For (x,y) on Γ_1, the slope of the tangent line to Γ_1 is

$$S^{un}(x,y) = \frac{y(-\delta + bx - n\bar{v})}{x\left[r(1 - x/K) - ay - u(x,y)\right]}$$

where $u(x,y)$ satisfies

$$0 < u(x,y) < \bar{u}. \tag{19}$$

There are two cases, $y(-\delta + bx - n\bar{v}) < 0$ or $y(-\delta + bx - n\bar{v}) > 0$. In the former case, $S^{0n}(x,y) < 0 < S^{1n}(x,y)$. If $S^{un}(x,y) < 0$, then

$$x\left[r\left(1 - \frac{x}{K}\right) - ay - u(x,y)\right] > 0. \tag{20}$$

By (19),

$$S^{un}(x,y) < S^{0n}(x,y) < 0.$$

This implies that the trajectories on the left side of Γ_1 have less negative slopes than the tangent line of Γ_1. Thus $F^{0n}(x,y)$ points to Γ_1 on the left side. Since $S^{1n}(x,y) > 0$ on (x,y) on the right side of Γ_1, $F^{1n}(x,y)$ points to Γ_1, on the right side. If $S^{un}(x,y) > 0$, then

$$x\left[r\left(1 - \frac{x}{K}\right) - ay - u(x,y)\right] < 0. \tag{21}$$

By (19),

$$S^{un}(x,y) > S^{1n}(x,y) > 0.$$

This means that trajectories on the right side of Γ_1 have less positive slopes than the tangent line of Γ_1. Thus, $F^{1n}(x,y)$ points to Γ_1 on the right side. Since $S^{0n}(x,y) < 0$, $F^{0n}(x,y)$ points to Γ_1 from the left side. This shows that trajectories off Γ_1 are attracted to Γ_1 if $y(-\delta + bx - n\bar{v}) < 0$.

In the latter case where $y(-\delta + bx - n\bar{v}) > 0$, $S^{0n}(x,y) > 0 > S^{1n}(x,y)$. If $S^{un}(x,y) < 0$, then (21) is valid. By (19),

$$S^{un}(x,y) < S^{1n}(x,y) < 0.$$

Hence, $F^{1n}(x,y)$ points toward Γ_1 from the right side. Since $S^{0n}(x,y) > 0$, $F^{0n}(x,y)$ points towards Γ_1 from the left side. If $S^{un}(x,y) > 0$, then (20) holds. By (19)

$$S^{un}(x,y) > S^{0n}(x,y) > 0.$$

So, $F^{0n}(x,y)$ points toward Γ_1 from the left side. Since $S^{1n}(x,y) < 0$, $F^{1n}(x,y)$ points towards Γ_1 from the right side. This proves that the trajectories off the curve Γ_1 are attracted to Γ_1.

The proof in the case where $j = 1$ is complete.

The proof in the case where $j = 2$ is similar.

Since Γ_j between l_j^0 and l_j^1 attracts optimal trajectories in its neighborhood, starting at any point in this part of Γ_j the trajectory would stay in Γ_j until it reaches a steady state or the boundary of the strip, i.e., l_j^0 or l_j^1. Thus, this part of Γ_j is itself part of an optimal trajectory.

The proof of the proposition is complete. □

This constitutes a step forward in characterizing appropriate policy during the transition path toward an optimal solution, something that has been largely overlooked with this type of model. Furthermore, it is also possible to characterize the semisingular optimal feedback controls along this part of the trajectory. To do so, we use (1) and (6), together with the equation for the curve Γ_1 or Γ_2 given by (4). The following results can be stated for the optimal control of the prey and predator.

4.1 Feedback control along Γ_1

Proposition 6. *(Semisingular feedback along Γ_1) Along the trajectory on Γ_1 within the strip bounded by ℓ_1^0 and ℓ_1^1, whenever $v = n\bar{v}$ for $n \in \{0,1\}$, the optimal prey control is:*

$$u = r\left(1 - \frac{x}{K}\right) - ay - \frac{-\delta y + bxy - n\bar{v}y}{\sqrt{g(y,\mu(y))^2 + \frac{8c_1\rho}{p_1 K r}}}\left\{-\frac{1}{r}\left[a - \frac{b\mu}{p_1}\right] + \frac{by}{rp_1}\frac{d\mu}{dy}\right\}, \tag{22}$$

with

$$b\mu y = p_1\left[\rho - r + \frac{2rx}{K} + ay\right] - \frac{\rho c_1}{x} - \frac{c_1 r}{K}, \tag{23}$$

$$g(y,\mu)=\frac{c_1}{p_1K}+1-\frac{\rho}{r}-\frac{y}{r}\left[a-\frac{b\mu}{p_1}\right], \tag{24}$$

$$\frac{d\mu}{dy}=\frac{a\left[p_1x(y,\mu)-c_1\right]+\left[\rho+\delta-bx(y,\mu)\right]\mu+(\mu-p_2)\,n\bar{v}}{-\delta y+byx(y,\mu)-n\bar{v}y}, \tag{25}$$

and,

$$x(y,\mu)=\frac{K}{4}\left\{g(y,\mu)+\sqrt{g(y,\mu)^2+\frac{8c_1\rho}{p_1Kr}}\right\}. \tag{26}$$

Proof. If a part of Γ_1 is a trajectory, we solve (1) and (6) with λ related to x by (14). By differentiating the two sides of Eq. (14), we obtain

$$\dot{\lambda}=\frac{c_1}{x^2}\dot{x}=\frac{c_1}{x^2}\left[rx\left(1-\frac{x}{K}\right)-axy-ux\right].$$

Compared to the first equation in (6), we find

$$\frac{c_1}{x^2}\left[rx\left(1-\frac{x}{K}\right)-axy\right]=\left[\rho-\left(r-\frac{2rx}{K}-ay\right)\right]\lambda-\mu by,$$

which leads to

$$b\mu y=p_1\left[\rho-r+\frac{2rx}{K}+ay\right]-\frac{\rho c_1}{x}-\frac{c_1r}{K}.$$

This is a quadratic equation in x. Solving the above equation for x, we can write

$$x(y,\mu)=\frac{K}{4}\left\{g(y,\mu)+\sqrt{g(y,\mu)^2+\frac{8c_1\rho}{p_1Kr}}\right\} \tag{27}$$

where

$$g(y,\mu)=\frac{c_1}{p_1K}+1-\frac{\rho}{r}-\frac{y}{r}\left[a-\frac{b\mu}{p_1}\right]. \tag{28}$$

For the part of Γ_1 that is above (resp. below) Γ_2, we have $v=\bar{v}$ (resp. $v=0$). Therefore, denoting $v=n\bar{v}$ for $n=0,1$ depending on whether Γ_1 is above or below Γ_2, we use the second equation in (6) and the equation

$$\dot{y}=-\delta y+bxy-n\bar{v}y$$

leads to

$$\frac{d\mu}{dy}=\frac{a\left[p_1x(y,\mu)-c_1\right]+\left[\rho+\delta-bx(y,\mu)\right]\mu+(\mu-p_2)\,n\bar{v}}{-\delta y+byx(y,\mu)-n\bar{v}y}. \tag{29}$$

Given the initial condition

$$\mu(y^*)=p_2-\frac{c_2}{y^*}, \tag{30}$$

it follows by (27),

$$x(y) = \frac{K}{4}\left\{g(y,\mu(y)) + \sqrt{g(y,\mu(y))^2 + \frac{8c_1\rho}{p_1Kr}}\right\}. \tag{31}$$

For $n = 0, 1$, we find

$$\frac{dx}{dy} = \frac{rx(1 - x/K) - axy - ux}{-\delta y + bxy - n\bar{v}y}.$$

As a result,

$$u = \frac{1}{x}\left\{rx\left(1 - \frac{x}{K}\right) - axy - [-\delta y + bxy - n\bar{v}y]\frac{dx}{dy}\right\}.$$

On the other hand, by (31),

$$\frac{dx}{dy} = \frac{x}{\sqrt{g(y,\mu(y))^2 + \frac{8c_1\rho}{p_1Kr}}}\left[g_y(y,\mu(y)) + g_{z_2}(y,\mu(y))\frac{d\mu}{dt}\right].$$

Hence, by (28),

$$u = r\left(1 - \frac{x}{K}\right) - ay - \frac{-\delta y + bxy - n\bar{v}y}{\sqrt{g(y,\mu(y))^2 + \frac{8c_1\rho}{p_1Kr}}}\left\{-\frac{1}{r}\left[a - \frac{b\mu}{p_1}\right] + \frac{by}{rp_1}\frac{d\mu}{dy}\right\}. \tag{32}$$

This completes the proof. □

The previous proposition gives the explicit values of the feedback control for the semisingular trajectories lying on Γ_1. The trajectory starts at a point y_1^0 that satisfies

$$u(y_1^0) = 0, \tag{33}$$

and ends either at y^* satisfying

$$u(y^*) = r\left(1 - \frac{x^*}{K}\right) - ay^*; \tag{34}$$

or at y_1^1, if (x^*, y^*) is not feasible, satisfying

$$u(y_1^1) = \bar{u}. \tag{35}$$

4.2 Feedback control along Γ_2

Similarly, the following proposition gives the results for the optimal predator control.

Proposition 7. *(Semisingular feedback along Γ_2) Along the trajectory on Γ_2 within the strip bounded* *by ℓ_2^0 and ℓ_2^1, whenever $u = m\bar{u}$ for $m \in \{0,1\}$, the optimal predator control is:*

$$v = -\delta + bx + \frac{rx\,(1 - x/K) - axy - m\bar{u}x}{(\rho + \delta - bx)\,p_2 + ax\lambda\,(x)} \left[-p_2 b + a\lambda\,(x) + ax\frac{d\lambda}{dx} \right]. \tag{36}$$

with

$$\lambda = -\frac{1}{ax} \left[(\rho + \delta - bx)\,\mu + c_2 \frac{\delta - bx}{y} \right], \tag{37}$$

$$\frac{d\lambda}{dx} = \frac{[\rho - r + 2rx/K + ay\,(x, \lambda)]\,\lambda - b\,(p_2 y\,(x, \lambda) - c_2) + (\lambda - p_1)\,m\bar{u}}{rx\,(1 - x/K) - axy - m\bar{u}x}, \tag{38}$$

and,

$$y = \frac{c_2 \rho}{(\rho + \delta - bx)\,p_2 + ax\lambda} \equiv y\,(x, \lambda)\,. \tag{39}$$

Proof. If the trajectory lies on Γ_2, then

$$\mu = p_2 - \frac{c_2}{y}. \tag{40}$$

Hence,

$$\dot{\mu} = \frac{c_2}{y^2}\dot{y} = \frac{c_2}{y^2}\,[-\delta y + bxy - vy]\,. \tag{41}$$

Thus,

$$\frac{c_2}{y^2}\,(-\delta y + bxy) = (\rho + \delta - bx)\,\mu + \lambda ax. \tag{42}$$

This leads to

$$\lambda = -\frac{1}{ax} \left[(\rho + \delta - bx)\,\mu + c_2 \frac{\delta - bx}{y} \right]. \tag{43}$$

Solving the above equation for y, we find

$$y = \frac{c_2 \rho}{(\rho + \delta - bx)\,p_2 + ax\lambda} \equiv y\,(x, \lambda)\,. \tag{44}$$

For the trajectory lying on Γ_2 below Γ_1 (resp. above), we have $u = 0$ (resp. $u = \bar{u}$). Therefore, whenever $u = m\bar{u}$ for $m = 0, 1$, we have

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - axy - m\bar{u}x.$$

Thus, by the first equation in (6),

$$\frac{d\lambda}{dx} = \frac{[\rho - r + 2rx/K + ay\,(x, \lambda)]\,\lambda - b\,(p_2 y\,(x, \lambda) - c_2) + (\lambda - p_1)\,m\bar{u}}{rx\,(1 - x/K) - axy - m\bar{u}x},$$

together with the initial condition

$$\lambda\left(x^*\right) = p_1 - \frac{c_1}{x^*}.$$

With $\lambda\left(x\right)$ solved, we find $y\left(x\right)$ by (44) as

$$y\left(x\right) = \frac{c_2 \rho}{\left(\rho + \delta - bx\right) p_2 + ax\lambda\left(x\right)}. \tag{45}$$

To find v on Γ_2, we use

$$\frac{dy}{dx} = \frac{-\delta y + bxy - vy}{rx\left(1 - x/K\right) - axy - \bar{u}x}$$

to obtain

$$v = \frac{1}{y}\left\{-\delta y + bxy - \left[rx\left(1 - \frac{x}{K}\right) - axy - \bar{u}x\right]\frac{dy}{dx}\right\}.$$

By (45),

$$\begin{aligned}\frac{dy}{dx} &= -\frac{c_2\rho}{\left[\left(\rho + \delta - bx\right) p_2 + ax\lambda\left(x\right)\right]^2}\left\{-p_2 b + a\lambda\left(x\right) + ax\frac{d\lambda}{dx}\right\} \\ &= -\frac{y}{\left(\rho + \delta - bx\right) p_2 + ax\lambda\left(x\right)}\left\{-p_2 b + a\lambda\left(x\right) + ax\frac{d\lambda}{dx}\right\}.\end{aligned}$$

Thus,

$$v = -\delta + bx + \frac{rx\left(1 - x/K\right) - axy - \bar{u}x}{\left(\rho + \delta - bx\right) p_2 + ax\lambda\left(x\right)}\left[-p_2 b + a\lambda\left(x\right) + ax\frac{d\lambda}{dx}\right].$$

This completes the proof. □

The previous proposition gives an explicit solution for the control value along the semisingular trajectory lying on Γ_2. The trajectory starts at a point x_2^0 that satisfies

$$v\left(x_2^0\right) = 0, \tag{46}$$

and ends either at a point x^* that satisfies

$$v\left(x^*\right) = -\delta + bx^*; \tag{47}$$

or either at x_2^1 if $\left(x^*, y^*\right)$ is not feasible, satisfying :

$$v\left(x_2^1\right) = \bar{v}. \tag{48}$$

4.3 Optimal approach path along singular loci

Concluding the above propositions, we can draw the following policy along singular loci:
On Γ_1:

Step 1. Start at any $y_0^{(1)}$ such that $u(y_0^{(1)}) = 0$.

Step 2. Evolve under feedback (22) until:

(2.1) Reach (x^*, y^*) (feasible singular steady state), or

(2.2) Hit $y_1^{(1)}$ where $u(y_1^{(1)}) = \bar{u}$.

On Γ_2:

Step 1. Start at $x_0^{(2)}$ such that $v(x_0^{(2)}) = 0$.

Step 2. Evolve under feedback (36) until:

(2.1) Reach (x^*, y^*), or

(2.2) Hit $x_1^{(2)}$ where $v(x_1^{(2)}) = \bar{v}$.

5 Feasible intersection

In most existing economic studies of optimal control with interacting populations, it is assumed that (i) fishing capacities $\bar{u}$ and $\bar{v}$ are sufficiently large for (x^*, y^*) to be a *feasible* steady state, and that (ii) there are no initial conditions under which trajectories converge to semisingular or bang-bang steady states. In the next two sections, we use phase-plane analysis to discuss these two assumptions.

This section first focuses on the case in which $(x^*, y^*) \in \Gamma_1 \cap \Gamma_2$ satisfies the feasibility constraints

$$\Delta = \Big\{(x, y) : 0 \le r(1 - x/K) - ay \le \bar{u},\ \ 0 \le bx - \delta \le \bar{v}\Big\}, \tag{49}$$

while allowing for the existence of additional steady states that cannot be ignored in policy recommendations. On the other hand, Section 6 examines the role of capacity constraints $\bar{u}$ and $\bar{v}$ as bifurcation parameters, with particular attention to the cases where $(x^*, y^*) \in \Gamma_1 \cap \Gamma_2$ is outside Δ.

The phase plane is divided by the straight lines l_1^m, l_2^n for $m, n = 0, 1$ defined by (8) into the regions R_{mn} in which $\dot{x}$ and $\dot{y}$ take certain signs. Specifically, in R_{mn}, $\dot{x} < 0$ above the line l_1^m and $\dot{x} > 0$ below it, and $\dot{y} < 0$ to the left of l_2^n and $\dot{y} > 0$ to its right.

It also contains the curves Γ_1 and Γ_2 in (16) that divide the plane into four regions, R_{00}, R_{01}, R_{10} , and R_{11}. These curves have different behaviors depending on the biomass value along the trophic chain, that is, depending on whether $ap_1 - bp_2 \lesseqgtr 0$. This condition compares the relative market prices of the species, p_1/p_2, with the biomass conversion efficiency of the predator, b/a, i.e., to compare the market value of a prey unit versus the same unit after processing by predators. For expositional clarity, in the rest of this study we impose the following:

$$ap_1 > bp_2,$$

that is, the market value of one unit of prey biomass, p_1, exceeds its implicit value when converted to predator biomass, $(b/a)p_2$. Although predators are often high-value species, the low biomass conversion rates typical of marine ecosystems make this condition consistent with most commercial fisheries.

Unique feasible steady state: Global Attractor. A feasible steady state is one of which Γ_1 and Γ_2 intersect in the region that lies between the lines l_1^0 and l_1^1 and the vertical lines l_2^0 and l_2^1. This clearly happens if (x^*, y^*) satisfies the inequalities in (17). Conversely, the intersection is not steady state as long as one of the four inequalities fails. Figure 1 provides an illustration where $(x^*, y^*) \in \Gamma_1 \cap \Gamma_2$ is the unique steady-state and is a global attractor. The long-run outcome does not depend on initial conditions and the optimal control is therefore to switch between $(m\bar{u}, n\bar{v})$ in region R_{mn} until the trajectory hits the part of Γ_1 or Γ_2 where the semisingular controls are as in (22) and (36).

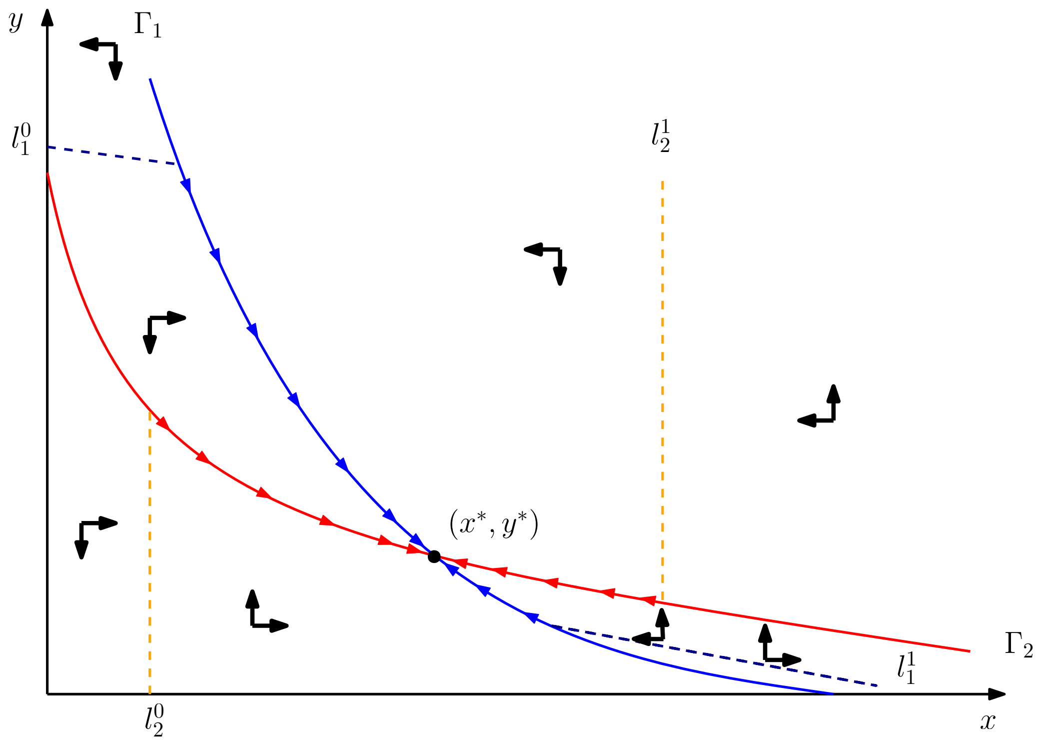

Figure 1: (x^*, y^*) is a global attractor.

Bang-bang, semisingular, and singular steady-state : Basin Partition. However, in addition to a singular steady-state, there may also exist semisingular and bang-bang steady-states. Figure 2 shows an example where, depending on initial conditions, three different steady-state equilibria may coexist.

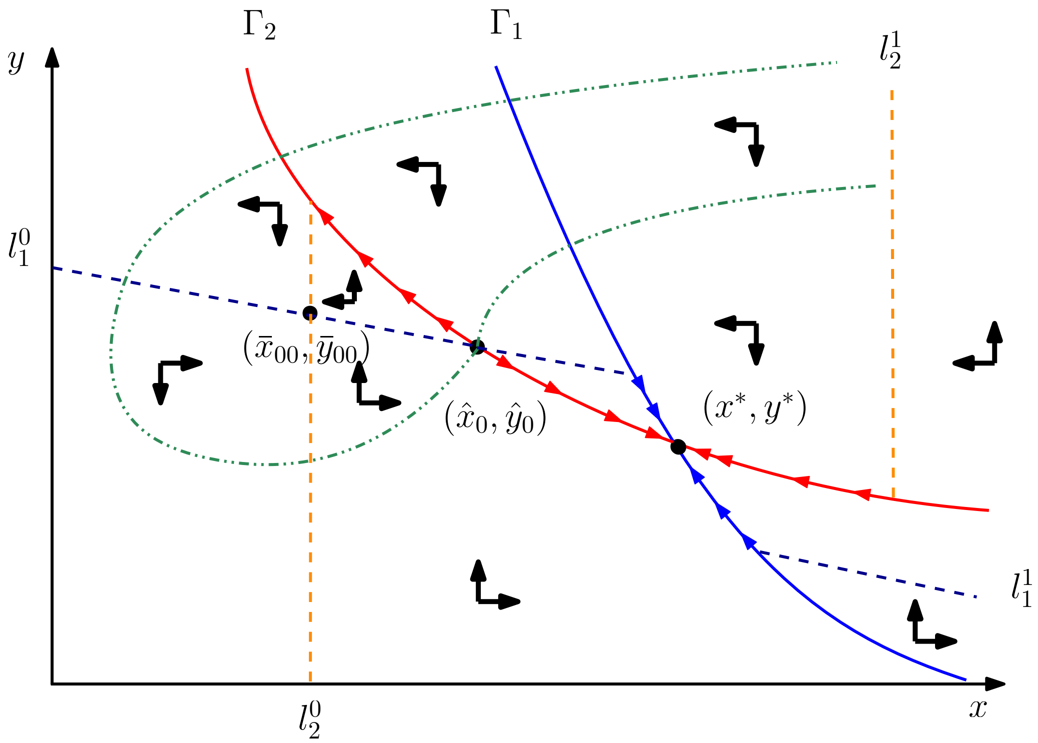

Figure 2: (x^*, y^*) is feasible with both $(\hat{x}_0, \hat{y}_0)$ and $(\bar{x}_{00}, \bar{y}_{00})$ exist.

The equilibrium $(\hat{x}_0, \hat{y}_0)$, corresponding to predator-only fishing, is unstable. Both (x^*, y^*) (joint prey-predator fishing) and $(\bar{x}_{00}, \bar{y}_{00})$ (no fishing) are locally asymptotically stable. Although optimal, closing the fishing industry may not be desirable outcomes. For policy-makers, this underscores the importance of stock rebuilding programs: without adequate biomass, the fishery may collapse into inactivity, even if profitable equilibria exist in theory. Therefore, one may think about ways to deviate, as a second-best policy, from the attractive region of $(\bar{x}_{00}, \bar{y}_{00})$, and move toward a sustainable long run outcome (x^*, y^*). For example, stopping predator fishing for some time in R_{11}, so that the trajectory becomes attracted by (x^*, y^*) instead.

There exist another interesting case where singular, semisingular and bang-bang equilibria may exist. Figure 3 shows an example where $(\hat{x}_1, \hat{y}_1)$ and $(\bar{x}_{11}, \bar{y}_{11})$ are in addition to (x^*, y^*):

This case is particularly interesting, as for a specific initial start of the system, we may converge to a long run result with fishing at maximum capacity $(\bar{u}, \bar{v})$. The path of the approach is also interesting, as the optimal policy is characterized by bang-bang control in addition to singular control when the trajectory hits along Γ_2 between l_2^0 and l_2^1. The range of different outcomes is most of the time not taken into account in the design of policy instruments. This may be one of the reasons for persistent inefficiencies in regulating

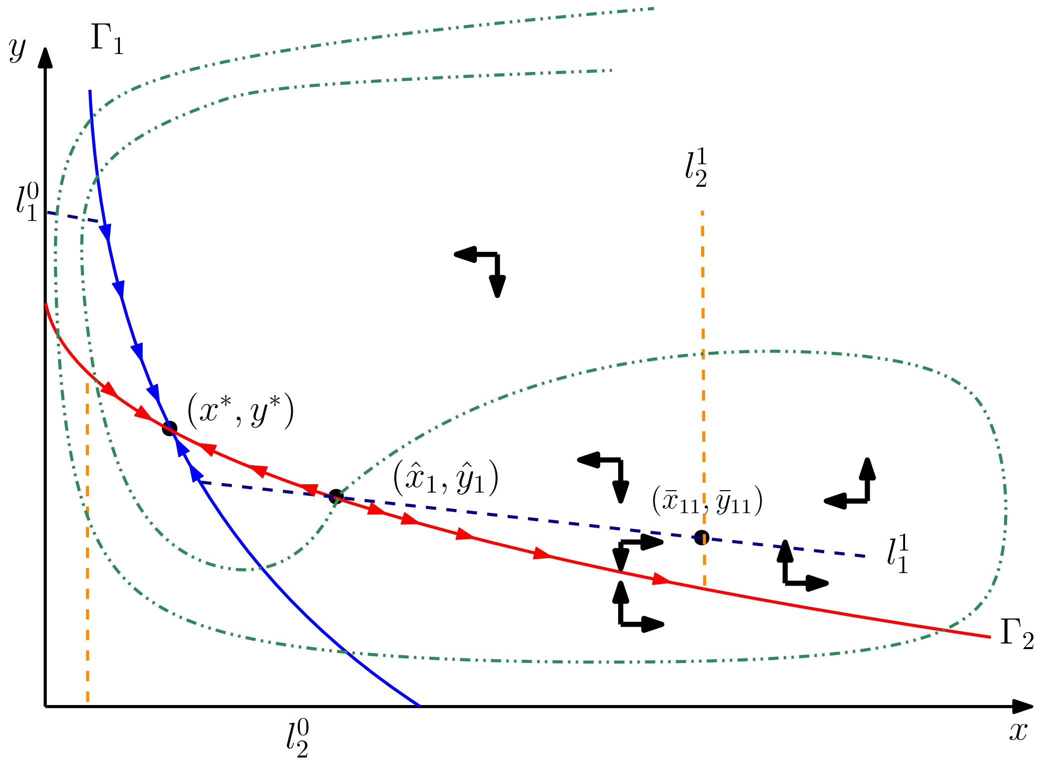

Figure 3: (x^*, y^*) is feasible with both $(\hat{x}_1, \hat{y}_1)$ and $(\bar{x}_{00}, \bar{y}_{00})$ exists.

interacting fisheries. Stock assessment programs are therefore the basis for designing more effective regulations. In practice, the planner must recognize that feasible equilibria are not unique and that the system's trajectory may lock into distinct stable states depending on ecological and economic circumstances.

So far, we have assumed that the singular steady-state is feasible, that is, it satisfies inequality (17). We now turn to the exploration of the existence and stability of alternative steady-state where the singular solution is not feasible, that is, one of the capacity constraints is binding.

6 Capacity-Induced Policy Shifts

The intersections of the singular curves Γ_1 and Γ_2 identify candidate steady states. However, feasibility depends on whether the implied fishing rates fall within the capacity bounds $[0, \bar{u}]$ and $[0, \bar{v}]$. When either bound is violated, the planner adopts second-best strategies that alter species composition and long run attractors. This distinction is crucial: if the unconstrained optimum requires effort beyond these bounds, the planner must adopt second-best strategies. Such adjustments alter both the composition of the species and the economic outcomes. This section explores the existence and stability of steady-states when intersection(s) point between Γ_1 and Γ_2 are not feasible steady-states. We consider two cases: low prey capacity and low predator capacity.

Binding prey capacity: predator control as compensation. When the prey fishing capacity is low, i.e. $\bar{u} < r(1 - \frac{x^*}{K}) - ay^*$, the unconstrained singular steady state (x^*, y^*) is infeasible. The planner must fix the prey effort at its upper bound $u = \bar{u}$, and adjust the predator effort to restore equilibrium. The resulting steady state satisfies $v^* = bx^* - \delta$, provided $0 \leq v^* \leq \bar{v}$. Economically, this reflects a situation in which prey cannot be harvested directly at optimal levels, so predator control becomes the compensating strategy. By harvesting predators more aggressively, the planner reduces predation pressure and indirectly exploits prey stocks. Thus, limited prey capacity forces managers into second-best strategies with more predator control.

To illustrate this case, we present two examples. Either l_1^1 intersects l_2^1 in R_{11} or not.

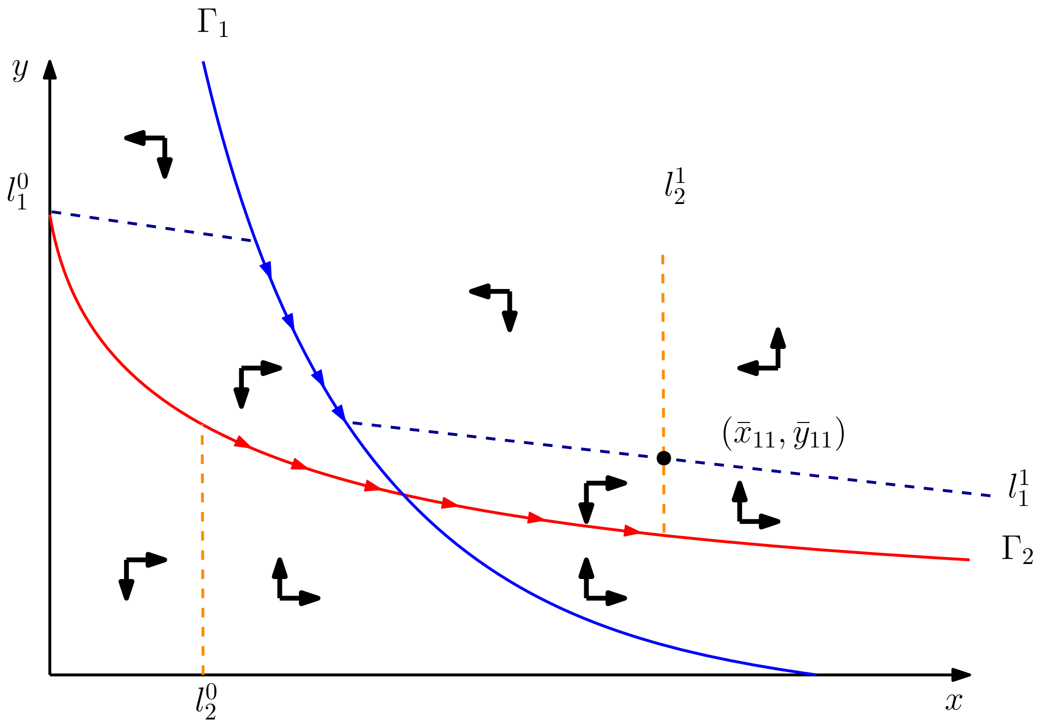

Figure 4: Low prey capacity: Strong predator control.

In the former case, all the trajectories approach $(\bar{x}_{11}, \bar{y}_{11})$ as $t \to \infty$, as shown in Figure 4. This is a bang-bang steady-state where the optimal policy requires fishing at maximum capacity both the prey and the predator.

In the latter case, l_1^1 does not intersect Γ_2 before entering R_{10} at $(\hat{x}_1, \hat{y}_1)$. It turns out that $(\hat{x}_1, \hat{y}_1)$ becomes the only stable steady state, as shown in Figure 5.

Despite prey still being harvested at maximum capacity, predators are harvested at a lower rate. This is a semisingular steady-state which gives different policy recommendations compared to the former case. Differences arise depending on the slope of l_1^1, i.e., if the carrying capacity, K, is sufficiently large, then the optimal policy requires more predator controls and a larger prey population.

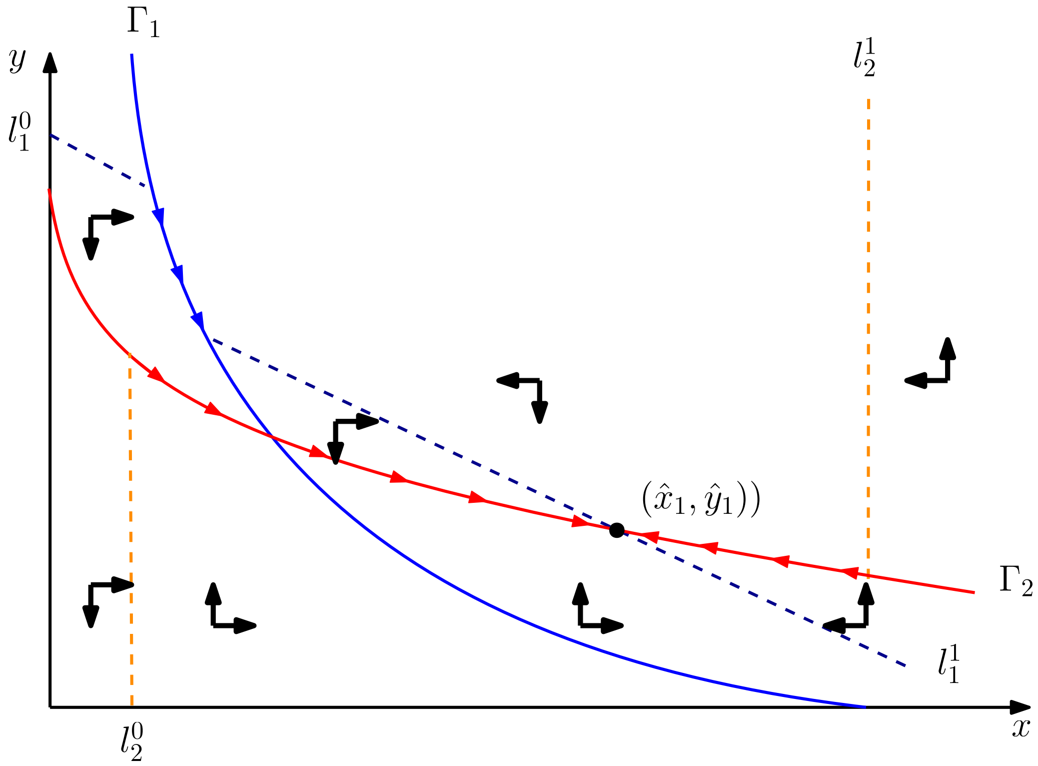

Figure 5: Low prey capacity: Weak predator control.

The main message from this subsection is that when the prey fleet is technologically or economically limited (few vessels, gear restrictions, etc.), direct exploitation of prey cannot reach the optimal level. In this case, the planner compensates by harvesting predators more aggressively and, therefore, exploiting preys in terms of predators. This shifts the equilibrium toward higher prey but lower predator populations, even though the prey effort remains capped.

Binding predator capacity: prey harvesting as leverage. The intersection(s) between Γ_1 and Γ_2 can also be unstable steady-states due to low predator fishing capacity, that is, when $\bar{v} < -\delta + bx^*$. Similarly to the previous subsection, we assume that the prey fishing industry has a sufficient fishing capacity, so that the inequality $y^* \leq \frac{r}{a}\left(1 - \frac{x^*}{K}\right) \leq y^* + \frac{\bar{u}}{a}$ holds.

When predator fishing capacity is low, prey harvesting becomes the primary lever to reduce predation pressure and generate revenue. Again, there are two different cases. Either l_1^0 intersects with l_2^1 in R_{01} or not.

As a first illustration, Figure 6 shows a case where l_1^0 intersects with l_2^1 in R_{01}.

In this case, the only steady-state is bang-bang type, characterized by full predator fishing and no prey fishing. This case is likely to arise when the intrinsic growth rate of the prey, or the carrying capacity, is sufficiently low. Excessive predation induced by the predator population depletes prey so that the social planner has an incentive to close the fishery in the long-run. However, when l_1^0 does not intersect with l_2^1 in R_{01}, then some prey fishing can be optimal to help reduce the predator population. This is therefore a semisingular steady-state,

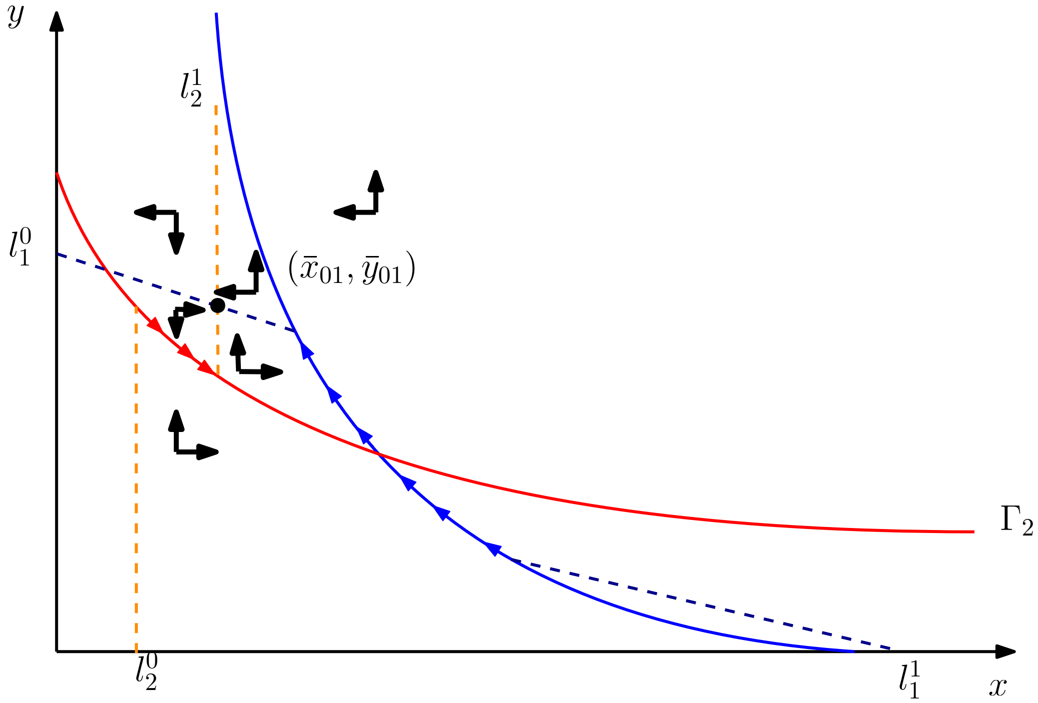

Figure 6: Low predator capacity: no prey fishing.

and Figure 7 provides an illustration.

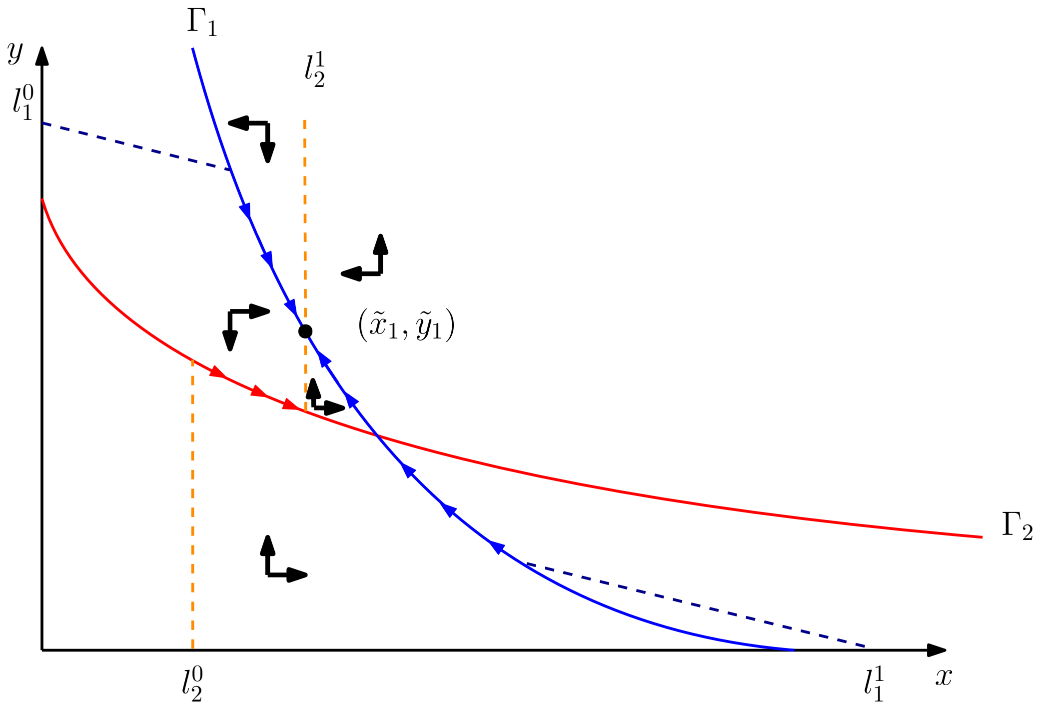

Figure 7: Low predator capacity: some prey fishing.

The general message of these two cases is the following. Without any restriction on fishing rates, the optimal policy requires strong predator control to reduce the stock of predators and release valuable prey to generate economic benefits. However, the predator fishing rate needed to reduce the predator population and increase the prey population to the level represented by the intersection between Γ_1 and Γ_2 exceed the maximum capacity $\bar{v}$. The resulting situation is too much prey killed by the predator and an increase in predator stock. In both cases, the optimal level of prey is $x = \delta + \bar{v}/aK$. It does not require prey fishing (Figure 7) if the carrying capacity, K, is low enough or if the predation rate a is high enough. In contrast, some prey fishing (Figure 6) is required to contribute to the reduction of the predator population if K is large or if a is low.

7 Conclusion

This paper has examined the optimal management of a predator-prey fishery under limited fishing capacity, extending the Gordon-Schaefer framework to account for multi-species interactions and capacity constraints. Our analysis of steady states (Section 3), optimal feedback strategies (Section 4), feasibility conditions (Section 5), and capacity-induced policy shifts (Section 6) yields several important conclusions.

First, we establish a new result on the optimal approach paths in the context of interacting species. Specifically, we show that segments of the singular loci constitute optimal approach paths along which semisingular controls can be derived analytically. This constitutes an important contribution, as it yields new insights into the optimal management of interacting species during transition dynamics.

Second, we investigate the role of fishing capacity constraints in shaping the existence and stability of steady states. When the singular steady state is feasible, bang-bang and semisingular steady states may also arise, with convergence depending on initial conditions. The feasibility analysis shows that when the singular solution is not feasible, there is a unique alternative steady state, which is of the bang-bang or semisingular type. When the unconstrained optimum requires effort beyond fleet capacity, the planner must rely on second-best strategies. This distinction is crucial: ignoring capacity constraints can lead to unrealistic policy prescriptions and misaligned management outcomes. In practice, feasible equilibria are jointly determined by biological dynamics and technological limits.

Third, capacity constraints induce sharp policy adjustments. When prey capacity is low, predator control becomes the compensating strategy: the planner harvests predators more aggressively to reduce predation pressure and indirectly exploit prey stocks. When predator capacity is low, the opposite pattern emerges: predators overpopulate, prey stocks collapse, and the social planner reduces the prey fishing effort.

Taken together, these findings provide both methodological and analytical contributions. Our analysis contributes to the relatively small literature on ecosystem-based fisheries management by showing that technological constraints are key determinants of long run optimal outcomes and, consequently, of the transition paths leading to these outcomes. Designing appropriate incentives for ecosystem-level resource management is therefore of first-order importance, and this paper directly contributes to that objective.

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