FACULTY OF LAW, ECONOMICS AND FINANCE

Department of Economics and Management

Discussion Paper

2026-02

Economics

Department of Economics and Management
University of Luxembourg

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January 2026

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Impulse Control and Multi-Dimensional Differential Games: Optimal Timing of Recycling and Substitution under Strategic Interaction*

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January 20, 2026

Abstract

This paper develops a dynamic game framework for environments in which recycling and substitution technologies emerge endogenously. We formulate the interaction as a Markovian subgame-perfect equilibrium with impulse controls, derive the associated Hamilton–Jacobi–Bellman systems, and establish smooth-pasting conditions governing regime transitions. Departing from classical exhaustible-resource models, our setting introduces recycling as an additional state variable and allows virgin resource prices to depend jointly on substitution and recycling. This structure generates a two-dimensional state space with interconnected regimes, leading to a switching fixed-curve rather than a single threshold and creating new challenges for theoretical characterization. Under broad convex cost functions and CES-type preferences, we characterize the resulting equilibrium and the geometry of the switching regions, thereby providing general insights into multi-dimensional impulse-control problems in dynamic games.

Keywords: Regime-switching, differential game, impulse control, HJB equations, Strategic timing, Multi-dimensional state space.

JEL classification: C61, C73, Q34, D92

*This work is supported by the ANR (France) SCARCYCLET grant (ANR-21-CE03-012). Chen and Ruan acknowledge support from the *Research Excellence Scheme* while visiting at the University of Luxembourg.

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1 Introduction

The strategic management of non-renewable resources under technological transitions has long been studied in dynamic optimization and game theory (Dasgupta and Heal, 1974; Dasgupta *et al.*, 1983; Olsen, 1988; Hung and Quyen, 1993; Acemogul et al, 2012). Recent developments in critical mineral markets, however, introduce multi-dimensional state dynamics and discrete technology activation decisions that challenge classical one-dimensional frameworks. In this paper, we develop a differential game with impulse controls and regime-switching to capture the strategic timing of recycling and substitution technologies.

Motivated by the global energy transition, our framework targets critical minerals (e.g., lithium, cobalt, rare earths) whose supply chains face geopolitical risks and environmental constraints. Unlike fossil fuels, these minerals are not irreversibly consumed; recycling creates an endogenous secondary stock that interacts with extraction decisions. Concurrently, substitution technologies (e.g., sodium-ion batteries, alternative magnetic materials) reshape demand and introduce additional strategic choices. We introduce recycling as an additional state variable alongside extraction and formulate a two-player dynamic game between an exporting country (optimally supplying the virgin resource) and an importing country (optimally managing recycling and substitution). The game evolves through a sequence of regimes–exclusive reliance on extraction, coexistence of extraction and recycling, and eventual dominance of recycling and/or substitution–each governed by distinct Hamilton-Jacobi-Bellman (HJB) systems.

The underlying mathematical structure poses technical challenges: the state space becomes multi-dimensional and technology adoption is governed by impulse controls across interconnected regimes. Regime transitions occur through impulse controls, representing discrete innovation decisions by the importing country, and create smooth-pasting conditions linking value functions across regimes. We meet the challenges by developing a theory of optimal impulse control in a multi-dimensional state space, which leads to a characterization of the optimal switching locus by a fixed-point problem.

Impulse control problems have an extensive history in optimal control and management science (Chahim *et al.*, 2013), but their integration into differential games remains relatively sparse. Early works focused on zero-sum settings (Yong, 1994; El Farouq *et al.*, 2010), while recent contributions address nonzero-sum games with impulse actions (Ruan, 2019, 2021). Two papers are particularly relevant. Sadana *et al.* (2021) analyze a two-player nonzero-sum differential game with one player using piecewise-continuous controls and the other impulses, deriving necessary and sufficient conditions for open-loop Nash equilibria via an extended Pontryagin principle and closed-form impulse timing in linear-state cases. Sadana *et al.* (2022) study feedback Nash equilibria under impulse control using HJB equations coupled with quasi-variational inequalities (QVIs), establishing a verification theorem and bounding the number of impulses. While these works highlight tractability in one-dimensional settings, they do not address multi-dimensional state spaces or regime connectivity.

Our contributions are fourfold. First, we formulate the problem as a Markovian subgame-perfect Nash equilibrium in a multi-dimensional state space, extending classical resource depletion models to incorporate recycling and substitution under strategic interaction. Second, we derive the full system of HJB equations for each regime and establish transition conditions under impulse control. Third, we characterize the geometry of regime-switching in two-dimensional domains and show that the optimal switching curve satisfies a fixed-point condition absent in prior literature. Fourth, we introduce a general price-setting mechanism in which the virgin resource price depends jointly on extraction, recycling, and substitution. This richer specification reflects realistic market interactions.

While our motivation stems from resource economics, the methods developed here have broader applicability to multi-dimensional impulse control problems in technology adoption, infrastructure investment, and environmental policy under uncertainty. By bridging applied resource models and rigorous game-theoretic analysis, this paper contributes to both the theory of differential games and the economics of sustainability.

The remainder of the paper is organized as follows. Section 2 presents the formal model, including state dynamics, control structure, and regime definitions. Section 3 formulates the differential game and characterizes the Markovian subgame-perfect Nash equilibrium under impulse controls; and derives the Hamilton-Jacobi-Bellman systems for each regime and establishes smooth-pasting conditions for regime transitions. Section 4 proves the main results, the regime switching conditions and further characterizing the regime switching-curve. Section 5 provides some comparison studies related to the literature. Section 6 analyzes an example with explicit functional form and provides a geometry of switching curves and discusses potential computational challenges. Finally, Section 7 provides directions for future research and potential extension and Section 8 concludes.

2 Model setup

We consider a two-player differential game between an exporting country (Player j) and an importing country (Player i). The exporter supplies a non-renewable critical mineral, while the importer managing recycling and substitution technologies to reduce dependence on the exporter.

2.1 State variables and controls

Let $X(t) \geq 0$ denote the cumulative extraction of the virgin resource at time t, and $Y(t) \geq 0$ the cumulative amount of recycled material. The state dynamics are given by:

$$\dot{X}(t) = x(t), \qquad \dot{Y}(t) = y(t), \tag{1}$$

where $x(t) \geq 0$ is the extraction rate and $y(t) \geq 0$ is the recycling rate. The substitution technology, when activated, introduces an additional control $z(t) \geq 0$ representing the rate of backstop production.

The state space is defined by:

$$0 \leq X \leq S_0, \qquad 0 \leq Y \leq \eta X,$$

where $S_0 > 0$ is the initial reserve and $0 < \eta < 1$ is the recycling efficiency parameter. Even considering repeated recycling, recyclable resource is always limited (Ruan and Zou, 2024).

2.2 Regimes, modes, cases, and impulse controls

We use "mode" to denote which technology player i uses with or without importing extractions of the natural resources from country j. Thus,

- Mode $m \in \{n, r, s, b\}$: n (no technology),
- Mode r (recycling only), s (substitution only),
- Mode b (both technologies).

Impulse controls occur when Player i activates a technology at times T_r (recycling) or T_s (substitution), incurring lump-sum costs $I_r(X, Y)$ and $I_s(X, Y)$.

We use "case" to denote whether the natural resources and recycled resources are both available or only one of them is. Thus, the following summaries the cases:

- Case 1: Only virgin resource available ($X < S_0$, $Y = 0$).
- Case 2: Virgin resource and recycling coexist ($X < S_0$, $Y > 0$).
- Case 3: Virgin resource exhausted ($X = S_0$).

We warn the reader that the three cases need not be sequential. Each case can last forever. For example, in Case 1, where substitution technology is so efficient and inexpensive, recycling technology may never be activated. That is why we use "cases" instead of "periods."

Note that the state variables in different cases are different. In Case 1, only X is the state variable. In Case 2, both X and Y are state variables, and In Case 3, only Y is the state variable.

Clearly in each case there can be multiple modes, so the game evolves through regimes indexed by (k, m), where k denotes the case (resource availability) and m the mode (technology configuration). For example, Regime $(1, s)$ is in Case 1 when Player i uses only substitution but not recycling, and Regime $(2, b)$ is in Case 2 when Player i uses both the recycling and substitution.[1]

2.3 Objective Functions

Player j maximizes discounted monopoly profit:

$$W_j = \int_0^\infty e^{-rt} \left[P(x + y, z)x - C(X, x) \right] dt,$$

where $P(\cdot)$ is the inverse demand function, which may depend on the existence of the substitution, and $C(X, x)$ is the convex extraction cost.

Player i maximizes discounted social welfare net of import and technology costs:

$$W_i = \int_0^\infty e^{-rt} \left[U(x + y, z) - P(x + y, z)x - R(y) - Z(z) \right] dt - I_r - I_s,$$

where $U(\cdot)$ is a CES-type utility function, and $R(\cdot)$, $Z(\cdot)$ are convex recycling and substitution cost functions.

3 Mathematical formulation

Denote $V_l^{k,m}$ the value function of Player l in Regime (k, m). Similarly, denote $H_l^{k,m}$ the Hamiltonian for Player l in Regime (k, m). In addition, regimes are connected. For example, when the importing country activates recycling in the regime $(1, s)$, the regime is changed to

[1]Impulse timing conditions in our one-dimensional regime switches (e.g., $(1, n) \to (1, s)$, $(3, r) \to (3, b)$) parallel the Hamiltonian continuity condition used to characterize open-loop equilibria with impulses; see Sadana et al. (2021).

$(2, b)$. As a result, value functions for different regimes are connected.

An abrupt action of one player that causes a regime change is called an "impulse control," and the connections between the value functions in different regimes at the transition point are called "transition conditions."

3.1 HJB equations in Case 1

The player j maximizes the profit of the monopoly by exporting $x(t)$ the critical mineral to the player i. Thus, the exporter's utility in Case 1 is

$$G_j^{1,m}(X, x) = xP(x, z) - C(X, x) \qquad \text{for } m = n, s. \tag{2}$$

The instantaneous utility of the player i in Case 1 is represented by the function $U(x, z)$. In addition, she suffers a penalty if not meeting the minimum demand, $x_{\min}$. Thus, the utility for the player i in Case 1 is

$$\begin{aligned} G_i^{1,n}(x) &= U(x, 0) - xP(x, 0) - \gamma(x) && \text{in Mode } n, \\ G_i^{1,s}(x, z) &= U(x, z) - xP(x, z) - Z(z) && \text{in Mode } s, \end{aligned} \tag{3}$$

where $\gamma(x)$ represents the penalty to the importing country for not meeting the minimum demand, $x_{\min}$, for imported and recycled critical mineral when there is no substitution. It has the properties that $\gamma(x) > 0$ if $x < x_{\min}$ and $\gamma(x) = 0$ if $x \geq x_{\min}$.

Note that in Case 1 the only state variable is X, which is governed by the first equation in (1). The Hamiltonian of Player i and j in Regime $(1, m)$ is

$$\begin{aligned} &H_i^{1,n}(x, \lambda) = G_i^{1,n}(x) + x\lambda, \qquad H_i^{1,s}(x, z, \lambda) = G_i^{1,s}(x, z) + x\lambda, \\ &H_j^{1,m}(X, x, \lambda) = G_j^{1,m}(X, x) + x\lambda. \end{aligned}$$

The value functions, $V_l^{1,m}(X)$ of Player l in Regime $(1, m)$ satisfies the HJB equation

$$rV_j^{1,m}(X) = x^*P(x^*, z^*) - C(X, x^*) + x^*\left(V_j^{1,m}\right)'(X) \qquad \text{for} \qquad m = n, s,$$

$$\begin{aligned} rV_i^{1,n}(X) &= U(x^*, 0) - x^*P(x^*, 0) - \gamma(x^*) + x^*\left(V_i^{1,n}\right)'(X), \\ rV_i^{1,s}(X) &= U(x^*, z^*) - x^*P(x^*, z^*) - Z(z^*) + x^*\left(V_i^{1,s}\right)'(X), \end{aligned}$$

where

$$x^* = \arg\max_{x\geq 0} \left\{ xP(x, z^*) - C(X, x) + x \left(V_j^{1,m} \right)' (X) \right\} \qquad \text{for } m = n, s,$$
$$z^* = \arg\max_{z\geq 0} \{ U(x^*, z) - Z(z) \} \qquad \text{if } m = s.$$

3.2 HJB equations in Case 2

In Case 2 the exporting country still sells the extracted material, but at a price $P(x+y, z)$. Thus, her utility is

$$G_j^{2,m}(X, x, y) = xP(x+y, z) - C(X, x), \qquad m = r, b. \tag{4}$$

That for country i is

$$G_i^{2,m}(x, y, z) = \begin{cases} U(x+y, 0) - xP(x+y, 0) - R(y) - \gamma(x+y) & \text{if } m = r, \\ U(x+y, z) - xP(x+y, z) - R(y) - Z(z) & \text{if } m = b. \end{cases} \tag{5}$$

The constraints are

$$\begin{cases} \dot{X} = x, \ \dot{Y} = y, \\ 0 \leq X < S_0, \qquad 0 \leq Y \leq \eta X, \\ x \geq 0, \qquad y \geq 0. \end{cases} \tag{6}$$

The state variables in Case 2 are X and Y, and the law of motions is given by (1). The Hamiltonians are

$$\begin{aligned} H_j^{2,m}(X, x, y, \lambda, \theta) &= G_j^{2,m}(X, x, y) + x\lambda + y\theta, \\ H_i^{2,m}(x, y, \lambda, \theta) &= G_i^{2,m}(x, y, z) + x\lambda + y\theta. \end{aligned}$$

The HJB equations are

$$\begin{aligned} &rV_j^{2,m}(X, Y) = x^*P(x^*+y^*, z^*) - C(X, x^*) + x^* \left(V_j^{2,m} \right)_X (X, Y) + y^* \left(V_j^{2,m} \right)_Y (X, Y), \\ &rV_i^{2,r}(X, Y) = U(x^*+y^*, 0) - x^*P(x^*+y^*, 0) - R(y^*) - \gamma(x^*+y^*), \\ &\qquad\qquad +x^* \left(V_i^{2,r} \right)_X (X, Y) + y^* \left(V_i^{2,r} \right)_Y (X, Y), \\ &rV_i^{2,b}(X, Y) = U(x^*+y^*, z^*) - x^*P(x^*+y^*, z^*) - R(y^*, z^*) - Z(z^*) + x^* \left(V_i^{2,b} \right)_X (X, Y) \\ &\qquad\qquad +y^* \left(V_i^{2,b} \right)_Y (X, Y), \end{aligned} \tag{7}$$

where $m = r, b$, and x^*, y^*, and z^* are

$$
\begin{aligned}
x^* &= \arg\max_{x \geq 0} \left\{ xP\left(x + y^*, z^*\right) - C\left(X, x\right) + x \left(V_j^{2,m}\right)_X \left(X, Y\right) \right\}, \\
y^* &= \begin{cases} \arg\max_{y \geq 0} \left\{ U\left(x^* + y, 0\right) - x^* P\left(x^* + y, 0\right) - R\left(y\right) - \gamma\left(x^* + y\right) + y \left(V_i^{2,r}\right)_Y \right\} & \text{if } m = r, \\ \arg\max_{y \geq 0} \left\{ U\left(x^* + y, z^*\right) - x^* P\left(x^* + y, z^*\right) - R\left(y\right) + y \left(V_i^{2,b}\right)_Y \right\} & \text{if } m = b, \end{cases} \\
z^* &= \arg\max_{z \geq 0} \left\{ U\left(x^* + y^*, z\right) - Z\left(z^*\right) \right\} \qquad \text{if } m = b.
\end{aligned}
$$

3.3 HJB equations in Case 3

In Case 3 the state variable X remains at S_0 is a constant, and the exporting country is out of the game. Thus, only the importing country needs to solve her optimal control problem. The utility function for the importing country is

$$
\begin{array}{ll} G_i^{3,r}\left(y\right) = U\left(y, 0\right) - R\left(y\right) - \gamma\left(y\right) & \text{if } m = r, \\ G_i^{3,b}\left(y, z\right) = U\left(y, z\right) - R\left(y\right) - Z\left(z\right) & \text{if } m = b, \end{array} \tag{8}
$$

The constraints are

$$
\begin{cases} \dot{Y} = y, & 0 \leq Y \leq \eta S_0, \\ y \geq 0, & \text{if } m = r, b, \\ z \geq 0, & \text{if } m = b. \end{cases} \tag{9}
$$

Note that in Case 3, $X = S_0$ and $Y = \eta S_0$ are both constant in Mode s. Thus, it suffices to consider only Modes r and b. The Hamiltonians are

$$
H_i^{3,r}\left(y, \theta\right) = G_i^{3,r}\left(y\right) + y\theta, \qquad H_i^{3,b}\left(y, z, \theta\right) = G_i^{3,b}\left(y, z\right) + y\theta,
$$

and the value function $V_i^{3,m}\left(Y\right)$ satisfies

$$
rV_i^{3,m}\left(Y\right) = y\left(V_i^{3,m}\right)'\left(Y\right) + \begin{cases} U\left(y^*, 0\right) - R\left(y^*\right) - \gamma\left(y^*\right) & \text{if } m = r, \\ U\left(y^*, z^*\right) - R\left(y^*\right) - Z\left(z^*\right) & \text{if } m = b. \end{cases}
$$

The optimizers y^* and z^* satisfy

$$y^* = \begin{cases} \arg\max_{y\geq 0} \left\{ U(y,0) - R(y) - \gamma(y) + y\left(V_i^{3,r}\right)'(Y) \right\} & \text{if } m = r, \\ \arg\max_{y\geq 0} \left\{ U(y,z^*) - R(y^*) + y\left(V_i^{3,b}\right)'(Y) \right\} & \text{if } m = b, \end{cases}$$

$$z^* = \arg\max_{z\geq 0} \{U(y^*,z) - Z(z)\} \qquad \text{if } m = b.$$

3.4 The overall structure of the differential game

The regimes are connected and the value functions for each player in different regimes are related at the transition points.

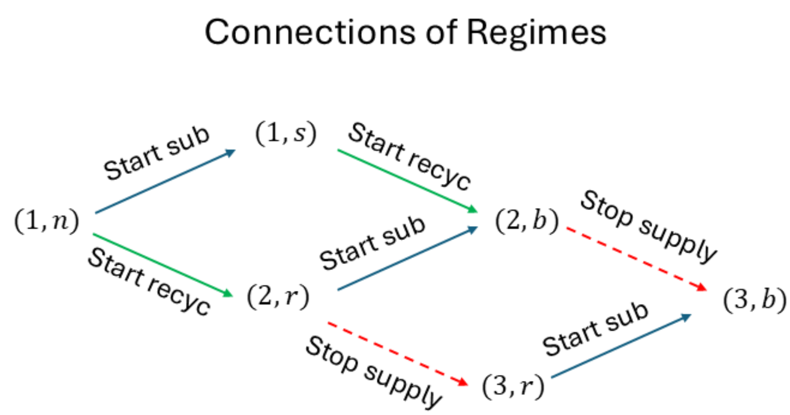

Figure 1: Overall structure of the differential game

Between two regimes which are connected in Fig. 1, the value functions are related by a transition equation.

We also observe that regimes divide the state domain differently depending on whether recycling is activated before or after substitution. Fig. 2 shows the subdomains of regimes in the state domain.

In the case recycling is activated after the natural resources are exhausted, state variables X, and Y are not connected in the X, Y-plane. See Fig. 3

3.4.1 Transition between regimes

Regimes are related by player i activating a technology and by the exhaustion of the natural resources.

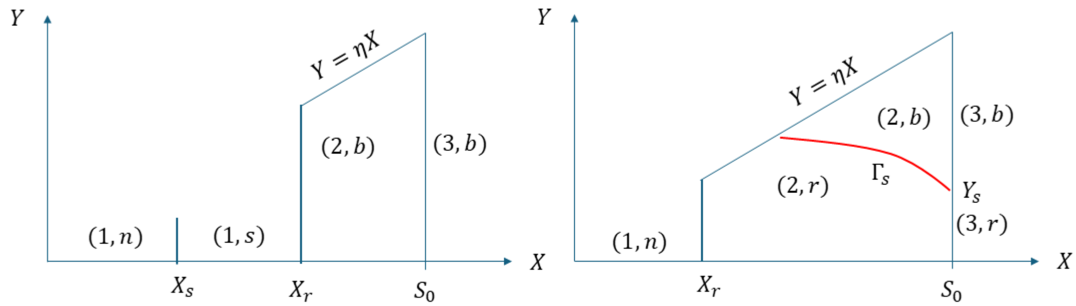

Figure 2: Connections of regimes in the state domain. Left: substitution is activated first. Right: recycling is activated first.

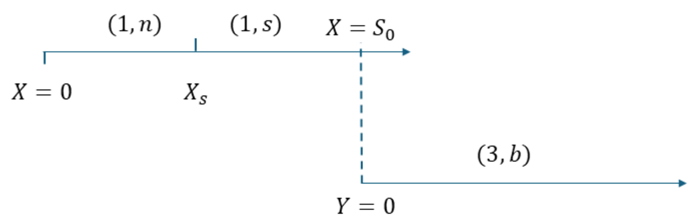

Figure 3: Connections of regimes in the state domain if recycling is activated without natural resources.

When player i activates a technology, either recycling (r) or backstop substitution (s), she pays the expenses of developing that technology. Let I_l, $l = r, s$, be the lump sum cost for the development of technology l. The cost may depend on the state (X, Y). Then, value function for player i changes according to

$$\begin{aligned} &V_i^{1,n}(X) = V_i^{1,s}(X) - I_s^1(X), \quad V_i^{1,n}(X) = V_i^{2,r}(X,Y) - I_r(X), \\ &V_i^{1,s}(X) = V_i^{2,b}(X,Y) - I_r(X), \quad V_i^{2,r}(X,Y) = V_i^{2,b}(X,Y) - I_s^2(X,Y), \\ &V_i^{3,r}(Y) = V_i^{3,b}(Y) - I_s^3(Y). \end{aligned} \tag{10}$$

(Note that the cost function activating the substitution technology depends on the case. We use I_s^k for $k = 1, 2, 3$ to denote the different functions.) In addition, the transition point between Cases 2 and 3 is $X = S_0$. Thus, optimal social welfare for country i satisfies

$$V_i^{3,r}(Y) = V_i^{2,r}(S_0, Y), \qquad V_i^{3,b}(Y) = V_i^{2,b}(S_0, Y). \tag{11}$$

On the other hand, the activation of technologies does not change the exporter's value. Thus, for player j, the transition conditions at the moment of a regime change are

$$\begin{aligned}&V_j^{1,n}(X) = V_j^{1,s}(X), \quad V_j^{1,n}(X) = V_j^{2,r}(X), \quad V_j^{1,s}(X) = V_j^{2,b}(X), \\ &V_j^{2,r}(X,Y) = V_j^{2,b}(X,Y).\end{aligned} \tag{12}$$

Note that country j is out of the game in Case 3. Thus, there is no transition conditions for the value functions for Player j involving Case 3. [2]

3.4.2 Terminal values

Each mode on each stage can be terminal, meaning that the system can remain in the same mode and on the same stage forever. We derive the terminal values at $X = S_0$ and/or $Y = \eta S_0$ as follows.

At $X = S_0$, the natural resources has been all extracted. Thus

$$\begin{aligned}&V_j^{1,m}(S_0) = 0 \qquad \text{for } m = n, s, \quad \text{and} \\ &V_j^{2,m}(S_0, Y) = 0 \quad \text{for } m = r, b, \quad 0 \le Y \le \eta S_0.\end{aligned} \tag{13}$$

In Case 1, depending on whether or not the backstop technology has be activated, the importer has the terminal values

$$V_i^{1,m}(S_0) = \begin{cases} -\gamma(0)/r & \text{if } m = n, \\ \frac{1}{r}\{U(0, \bar{z}^*) - Z(\bar{z}^*)\} & \text{if } m = s. \end{cases} \tag{14}$$

where

$$\bar{z}^* = \arg\max_{z \ge 0} \{U(0, z) - Z(z)\}. \tag{15}$$

In Case 2, we have

$$V_i^{2,m}(S_0, \eta S_0) = \begin{cases} -\gamma(0)/r & \text{if } m = r, \\ \frac{1}{r}\{U(0, \bar{z}^*) - Z(\bar{z}^*)\} & \text{if } m = b. \end{cases} \tag{16}$$

Similarly, in Case 3

$$V_i^{3,m}(\eta S_0) = \begin{cases} -\gamma(0)/r & \text{if } m = r, \\ \frac{1}{r}\{U(0, \bar{z}^*) - Z(\bar{z}^*)\} & \text{if } m = s. \end{cases} \tag{17}$$

[2] Related feedback formulations couple HJB with QVIs for the impulse player, providing continuation/intervention sets and verification theorems; see Sadana et al. (2021b).

4 Main results

The transition points between regimes related to the player i activating a technology are determined by the player i choosing the optimal moment to perform the activation, i.e., taking the optimal impulse control. We determine those transition points below.

With a slight abuse of notation, we also use $H_l^{k,m}$ to denote the optimized Hamiltonians for the player l in the regime (k,m). Note that x^* and z^* are functions of $\left(X, \lambda_j^{1,m}\right)$ for $m = n, s$ in Case 1, x^*, y^*, and z^* are functions of $\left(X, \lambda_i^{2,m}, \lambda_j^{2,m}, \theta_i^{2,m}\right)$ for $m = r, b$, in Case 2, and y^* and z^* are functions of $\theta_i^{3,m}$ for $m = r, b$ in Case 3. Hence, we can write the following.

$$H_i^{1,m} = H_i^{1,m}\left(X, \lambda_i^{1,m}, \lambda_j^{1,m}\right), \quad H_j^{1,m} = H_j^{1,m}\left(X, \lambda_j^{1,m}\right)$$

in Case 1;

$$H_i^{2,m} = H_i^{2,m}\left(X, \lambda_i^{2,m}, \theta_i^{2,m}, \lambda_j^{2,m}\right), \quad H_j^{2,m} = H_j^{2,m}\left(X, \lambda_j^{2,m}, \theta_j^{2,m}, \theta_i^{2,m}\right),$$

in Case 2; and

$$H_i^{3,r} = H_i^{3,m}\left(\theta_i^{3,r}\right)$$

in Case 3. The following theorem gives a criterion for the transition point when player i activates a technology. The detailed proof of this theorem is given in Appendix A.1.

Theorem 1. *Suppose player i activates technology l to cause the regime change $(k,m) \longmapsto (k',m')$.*

(1) In the case where $k = 1$, $l = n, s$, suppose that the change occurs at X_l^1 that satisfies $0 < X_l^1 < S_0$, and that the limit

$$\hat{\Lambda}_j^{1,m} = \lim_{X \uparrow X_l^1} \left(V_j^{1,m}\right)'(X) \tag{18}$$

exists. Then, X_l^1 and $\hat{\Lambda}_j^{1,m}$ satisfies

$$\begin{aligned} r\left[V_i^{k',m'} - I_l^1\right](X_l^1) &= H_i^{k,m}\left(X_l^1, \left[V_i^{k',m'} - I_l^1\right]'(X_l^1), \hat{\Lambda}_j^{1,m}\right), \\ rV_j^{k',m'}(X_l^1) &= H_j^{k,m}\left(X_l^1, \hat{\Lambda}_j^{1,m}\right), \end{aligned} \tag{19}$$

where $I_l^1 = I_r$ if $l = r$ and $I_l^1 = I_s^1$ if $l = s$.

(2) In the case where $k = 3$ and $l = r$. Suppose that the change occurs at Y_s that satisfies

$0 < Y_s < \eta S_0$, *and that the function* $q \mapsto H_i^{3,r}(q)$ *is invertible. Then,* Y_s *satisfies*

$$r\left[V_i^{3,b} - I_s^3\right](Y_s) = H_i^{3,r}\left(\left[V_i^{3,b} - I_s^3\right]'(Y_s)\right). \quad (20)$$

Theorem 1 leads to identification of the state when player i activates recycling or substitution in Case 1 and activates substitution in Case 3. In these cases the impulse controls are taken when there is one state variable. Thus, the state at which the transition occurs is a single point. Specifically, Theorem 1 can be applied to the following regime changes

$$(1,n) \mapsto (1,s), \quad (1,n) \mapsto (2,r), \quad (1,s) \mapsto (2,b), \quad (3,r) \mapsto (3,b).$$

The key condition is given by the first equation (19), and Eq. (20), where the left-hand side represents the instantaneous value function *after* the mode change and the right-hand side represents the Hamiltonian *before* the mode change, but with marginal value functions already evaluated *after* the transition.

Remark. This condition is analogous to the Hamiltonian continuity used to pin down impulse times in open-loop impulse games (Sadana et al., 2021a).

Unlike one-dimensional impulse timing, the transition $(2,r) \to (2,b)$ occurs along a switching curve Γ_s. While feedback impulse games typically identify threshold-type intervention sets via QVIs (Sadana et al., 2021b), here the geometry is multi-dimensional and determined by a fixed-point map. Curve Γ_s separates the subdomains of (X,Y) so that on one side of Γ_s the regime is $(2,r)$, and on the other side $(2,b)$ (see Fig. 2).

The following results can be obtained.

Theorem 2 (Characterization of Switching Curve in Two-Dimensional State Space)**.** *Consider the regime change from* $(2,r)$ *(extraction + recycling) to* $(2,b)$ *(extraction + recycling + substitution) in Case 2, where the state space is* $\Omega = \{(X,Y) : 0 \le X \le S_0,\ 0 \le Y \le \eta X\}$. *Let* $V_i^{2,r}(X,Y)$ *and* $V_i^{2,b}(X,Y)$ *denote the value functions of Player* i *in regimes* $(2,r)$ *and* $(2,b)$, *respectively. Assume:*

(1) $V_i^{2,r}$ *and* $V_i^{2,b}$ *are continuously differentiable on* Ω.

(2) The lump-sum cost $I_s^2(X,Y)$ *is continuously differentiable.*

Then, the optimal switching curve $\Gamma_s \subset \Omega$ *separating regimes* $(2,r)$ *and* $(2,b)$ *satisfies the* *equation:*

$$r\left[V_i^{2,b}-I_s^2\right](X,Y) = G_i^{2,r}(X,x^*(X,Y),y^*(X,Y)) + x^*(X,Y)\left[V_i^{2,b}-I_s^2\right]_X(X,Y) + y^*(X,Y)\left[V_i^{2,b}-I_s^2\right]_Y(X,Y), \tag{21}$$

where (x^*, y^*) *are the optimal controls in regime* $(2,r)$ *given* (X,Y)*. That is,*

$$\begin{aligned} x^* &= \arg\max_{x\geq 0}\left\{G_j^{2,r}(X,x,y^*) + x\left(V_j^{2,r}\right)_X(X,Y)\right\}, \\ y^* &= \arg\max_{y\geq 0}\left\{G_i^{2,r}(x^*,y) + y\left(V_i^{2,r}\right)_Y(X,Y)\right\}. \end{aligned} \tag{22}$$

Proof. The value function of Player i at a point (X_0, Y_0) before activation is:

$$\begin{aligned} V_i^{2,r}(X_0,Y_0) &= \int_0^T e^{-rt} G_i^{2,r}(X(t), x^*(X(t),Y(t)), y^*(X(t),Y(t)))\,dt \\ &\quad + e^{-rT}\left[V_i^{2,b}-I_s^2\right](X(T),Y(T)), \end{aligned}$$

where $(X(t),(Y(t)))$ is the trajectory in Regime $(2,r)$ starting at (X_0,Y_0), and T is the optimal activation time. Differentiating with respect to T and applying the envelope theorem yields:

$$\begin{aligned} r\left[V_i^{2,b}-I_s^2\right](X(T),Y(T)) &= G_i^{2,r}(X(T),x^*(T),y^*(T)) + x^*(T)\left[V_i^{2,b}-I_s^2\right]_X(X(T),Y(T)) \\ &\quad + Y^*(T)\left[V_i^{2,b}-I_s^2\right]_Y(X(T),Y(T)) \end{aligned}$$

where

$$x^*(T) = x^*(X(T),Y(T)), \qquad y^*(T) = y^*(X(T),Y(T)).$$

Note that $(X(T),Y(T))$ is the point at which Player i activates substitution. Hence, state $(X(T),Y(T)) \in \Gamma_s$. This condition must hold for all (X,Y) on the switching curve Γ_s. Thus, (21) must hold.

This completes the proof.

In general, locating the curve Γ_s is difficult. It amounts to solve a fixed point problem in a metric space. Let $\mathcal{M}$ denote the metric space of continuous curves in Ω endowed with the distance of the minimum Euclidean distance between two curves. Starting with a curve $\Gamma \in \mathcal{M}$ that separates Ω into two connected subdomains. The curve Γ determines the value functions $V_i^{2,r}$ and $V_j^{2,r}$ on one side of Γ with the boundary conditions

$$
\begin{aligned}
V_i^{2,r}(X,Y;\Gamma) &= V_i^{2,b}(X,Y) - I_s^2(X,Y), \\
V_j^{2,r}(X,Y;\Gamma) &= V_j^{2,b}(X,Y)
\end{aligned}
\qquad \text{on } \Gamma
$$

and the HJB equations in (7). These value functions, in turn, determine the optimal controls, $x^*(X,Y;\Gamma)$ and $y^*(X,Y;\Gamma)$, by (22). The optimal controls, then, determine another curve $\Gamma' \in \mathcal{M}$ by Eq. (21). This establishes a mapping $\Gamma \mapsto \Gamma'$ in $\mathcal{M}$. The optimal switching curve Γ_s is a fixed point under this mapping.

A typical trajectory starts in Regime $(1,n)$ and after passing a number of regimes, moves toward the point $(S_0, \eta S_0)$. See Fig. 4

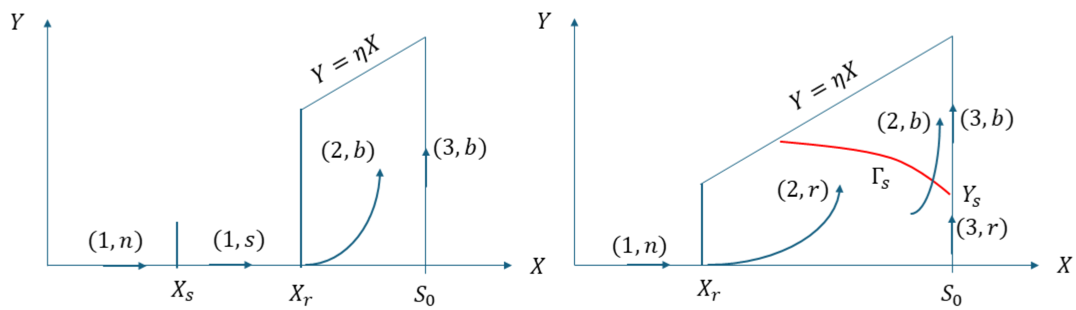

Figure 4: Connections of regimes in the state domain. Left: substitution is activated first. Right: recycling is activated first.

These results have the following strategic implications. In this game, player j optimally supplies x^* to the market. However, whether a mode change occurs depends not only on the accumulation of the state but also on an impulse control decision by player i, who weighs both the transaction cost for activating a technology, and their own social welfare at different levels of accumulated supply. This condition provides the optimal switching criterion for intertemporal decision making: initially, importing may be more or less cost-effective than investing in new technologies or infrastructure for recycling and substitution. Investing in recycling and substitution technologies reduces dependence on foreign suppliers and enhances the resilience of the supply chain. Although upfront investment is high, these technologies offer long-term cost savings and foster a circular economy. Additionally, R&D in substitution technologies may yield innovative solutions that not only replace critical minerals but also improve performance and sustainability.

It is worth noting that the limiting condition requires some degree of continuity in player j's marginal value, thereby reinforcing their strategic actions between the players. In other words, player j is not completely passive in leading up to mode changes. Instead, player j can adjust her supply strategy to keep her monopoly position while simultaneously influencing the accumulation of recyclable resources.

5 Comparative analysis and literature link

Table 1 contrasts our framework with representative models in the dynamic resource and impulse-control literature. While seminal one-dimensional depletion models (Dasgupta and Heal, 1974) and the single-switch substitution timing under uncertainty (Hung and Quyen, 1993) deliver sharp insights in their respective domains, they do not accommodate (i) two state variables that co-evolve, (ii) discrete technology activation with endogenous regime connectivity, nor (iii) geometric features such as switching *curves* in multi-dimensional domains. Our model integrates these elements by formulating a Markovian subgame-perfect Nash equilibrium in which impulse decisions trigger regime transitions, and the optimal switching locus in Case 2 is determined by a fixed-point condition (Theorem 2), yielding existence of the switching curve.[3]

Table 1: Comparison of Our Model with Classical Frameworks

Feature	**Dasgupta–Heal (1974)**	**Hung–Quyen (1993)**	**Our Model**
State Dimension	1D (extraction)	1D (extraction)	2D (extraction + recycling)
Technologies	None	Substitution only	Recycling + substitution
Impulse Control	No	Yes (single switch)	Yes (multi-regime connectivity)
Regime Connectivity	None	Two regimes	Multiple interconnected regimes
Fixed-Point Problem	No	No	Yes (switching curve in 2D)
Equilibrium Concept	Optimal control	Dynamic programming	Markovian subgame-perfect Nash

Positioning within the impulse-game literature. Our analysis bridges the open-loop and feedback strands of impulse differential games. On the open-loop side, Sadana et al. (2021)

[3]See our Theorem 2 and the associated fixed-point characterization on the curve Γ; in one dimension the timing collapses to a point, but in two dimensions the switching geometry must be solved globally.

provide necessary and sufficient conditions for nonzero-sum games with one continuous and one impulse player, using an extended Pontryagin principle and a Hamiltonian continuity condition to pin down impulse timing; their linear-state specialization even yields closed-form timing and level for the single impulse. We adopt an analogous Hamiltonian-based switching condition for the *one-dimensional* regime changes in our model (e.g., $(1, n) \to (1, s)$ and $(3, r) \to (3, b)$), thereby maintaining methodological continuity while generalizing the environment to multi-regime settings.[4]

On the feedback side, Sadana et al. (2022) couple HJB and QVIs to characterize feedback (Markov-perfect) equilibria with impulses, establishing a verification theorem and an *upper bound* on the equilibrium number of interventions. That feedback/QVI framework supports our use of regime-wise HJB systems and boundary (smooth-pasting) conditions: while their continuation/intervention sets are typically threshold-type in a single regime, we identify the switching geometry in a *two-dimensional* state domain and prove existence of the switching curve via a fixed-point map, which is absent in one-dimensional impulse models.

Therefore, relative to the canonical impulse frameworks:

(1) We introduce a *multi-dimensional* state space (X, Y), where the recycling stock interacts with extraction decisions and lump-sum innovation costs. This yields non-trivial geometry in regime transitions and necessitates a fixed-point approach for the switching curve.

(2) We study *multi-regime connectivity* with endogenous impulses that activate recycling and/or substitution, linking regime value functions through transition conditions and technology costs; this inter-regime structure is not present in one-regime impulse games.

(3) We provide *the existence* of the switching curve in Case 2 under general convex cost and CES-type utility assumptions and provide the one-dimensional Hamiltonian timing conditions for point switches in the other cases.

Implications. The comparison in Table 1 clarifies that the novelty lies in combining impulse activation with multi-dimensional dynamics and regime connectivity. Methodologically, our approach retains the familiar Hamiltonian continuity for point impulses (open-loop strand; Sadana et al. (2021)) and inherits well-posedness features such as bounded intervention counts (feedback/QVI strand; Sadana et al. (2022)), while advancing the geometry of impulse activation to multi-dimensional switching curves with fixed-point characterization. This extension

[4] Our one-dimensional regime switches implement Hamiltonian continuity at the activation instant, echoing the timing conditions used by Sadana et al. (2021).

is essential for resource-market applications in which recycling and substitution jointly shape strategic timing and long-run welfare.

6 Illustrative Example with Explicit Functional Forms

To demonstrate the applicability of our theoretical framework, we present an illustrative example using explicit functional forms. This example serves to clarify the structure of the Hamilton-Jacobi-Bellman (HJB) equations and the impulse control conditions derived in Sections 3 and 4.

6.1 The functions

Player i's utility is $U(x+y,z)$ where

$$U(\xi,z) = (a\xi^\rho + bz^\rho)^{\alpha/\rho}, \tag{23}$$

$0<\alpha,\rho<1$, and $(1-\rho)^{-1}$ represents the constant elasticity of substitution between $\xi \equiv x+y$ and z. The penalty function is

$$\gamma(\xi,z) = \begin{cases} \gamma_0 [x_{\min}-\xi]_+^2 & \text{if } z=0, \\ 0 & \text{if } z>0 \end{cases} \tag{24}$$

where $\gamma_0>0$ is a constant and $[x]_+ = \max\{x,0\}$ for any real number x. The inverse demand function is given by:

$$P(\xi,z) = \bar{p}\,(a\xi^\rho + bz^\rho)^{(\alpha-1)/\rho} \tag{25}$$

with $\bar{p}$ being a positive constant. When $z=0$, the above inverse demand function is reduced to

$$P(\xi,0) = p_0\xi^{\alpha-1}$$

with $p_0 = \bar{p}a^{(\alpha-1)/\rho}$.

Player j's optimal control, under mode n to mode r, becomes

$$\max_{x(t)} W^j = \mathbb{E}\left[\int_0^{T_r} [p_0x^\alpha - C(X,x)]\,e^{-rt}dt + \int_{T_r}^{\infty} \left[p_0(x+y)^{\alpha-1}x - C(X,x)\right]e^{-rt}dt\right],$$

and a similar objective holds under mode s.

Taking extraction cost as

$$C(X,x) = c(X)x + \frac{c_2}{2}x^2 = c_1 xX + \frac{c_2}{2}x^2, \tag{26}$$

where parameters, $c_1 > 0$ and $c_2 \geq 0$, the marginal extraction cost, $C_x = c_1 X + c_2 x > 0$, thus decreases in terms of remaining stock.

The cost functions R and Z are defined by

$$R(y) = r_1 y + \frac{r_2}{2}y^2, \qquad Z(z) = s_1 z + \frac{s_2}{2}z^2,$$

where $r_1, s_1 > 0$, and $r_2, s_2 \geq 0$.

In what follows, we use upper case letters, V_m^l, Λ_m^l, etc. to denote functions of X, and use lower case letters, v_m^l, λ_m^l, etc. to denote the respective quantities as functions of t. Thus $\lambda_m^l(t) = \Lambda_m^l(X(t))$, $v_m^l(t) = V_m^l(X(t))$, etc.

6.2 Solutions

In view of the general structure displayed in Fig. 1 and the motion on a typical trajectory illustrated in Fig. 4, we solve the solutions backwards starting in Regime $(3,b)$, followed by Regimes $(3,r)$, $(2,b)$, $(2,r)$, $(1,s)$ and $(1,n)$, in that order.

6.2.1 Solutions in Case 3

In Case 3, the only state variable is Y and the only player is i who needs to solve an optimal control problem in Regimes $(3,b)$ and $(3,r)$, as well as the optimal impulse control of activating substitution in Case 3.

Regime $(3,b)$**.** In Regime $(3,b)$, $V_i^{3,b}(Y)$ satisfies

$$rV_i^{3,b}(Y) = U(y^*(Y), z^*(Y)) - R(y^*(Y)) - Z(z^*(Y)) + y^*(Y)\left(V_i^{3,b}\right)'(Y) \tag{27}$$

for $0 < Y < \eta S_0$, with the terminal condition

$$V_i^{3,b}(\eta S_0) = \frac{1}{r}\left\{U(0,\bar{z}) - Z(\bar{z})\right\}, \tag{28}$$

where

$$\bar{z} = \arg\max_{z \geq 0}\left\{U(0,z) - Z(z)\right\}. \tag{29}$$

$$y^*(Y) = \varphi^{3,b}\left(\left(V_i^{3,b}\right)'(Y)\right), \qquad z^*(Y) = \psi^{3,b}\left(\left(V_i^{3,b}\right)'(Y)\right),$$

and for any fixed θ, $(y^*, z^*) = \left(\varphi^{3,b}(\theta), \psi^{3,b}(\theta)\right)$ is the solution to the optimization problem

$$\begin{aligned} y^* &= \arg\max_{y \geq 0} \{U(y, z^*) - R(y) + y\theta\}, \\ z^* &= \arg\max_{z \geq 0} \{U(y^*, z) - Z(z)\}. \end{aligned} \tag{30}$$

It is easy to see that the derivative of the function

$$U(0, z) - Z(z) = b^{\alpha/\rho} z^\alpha - \left[s_1 z + \frac{s_2}{2} z^2\right]$$

is decreasing in z, approaches ∞ as $z \to 0$ and approaches $-\infty$ as $z \to \infty$. Thus the solution $\bar{z}$ in (29) exists uniquely and is positive. Also, it can be shown that functions $\varphi^{3,b}$ and $\zeta^{3,b}$ are well-defined and are positive for any $\theta \in \mathbb{R}$.

Define $H_i^{3,b}$ by

$$H_i^{3,b}(\theta) = U\left(\varphi^{3.b}(\theta), \psi^{3.b}(\theta)\right) - R\left(\varphi^{3.b}(\theta)\right) - Z\left(\psi^{3,b}(\theta)\right) + \theta\varphi^{3,b}(\theta).$$

Using the FOCs, we can derive

$$(r_1 - \theta)\varphi^{3,b}(\theta) + r_2\varphi^{3,b}(\theta)^2 = s_1\psi^{3,b}(\theta) + s_2\psi^{3,b}(\theta)^2 = \alpha\left[a\varphi^{3,b}(\theta)^\rho + b\psi^{3,b}(\theta)^\rho\right]^{\alpha/\rho}.$$

Hence, $H_i^{3,b}$ can be simplified to

$$H_i^{3,b}(\theta) = (1-\alpha)\left[a\varphi^{3,b}(\theta)^\rho + b\psi^{3,b}(q)^\rho\right]^{\alpha/\rho} + \frac{r_2}{2}\varphi^{3,b}(\theta)^2 + \frac{s_2}{2}\psi^{3,b}(\theta)^2.$$

The HJB equation (27) takes the form

$$rV_i^{3,b}(Y) = H_i^{3,b}\left(\left(V_i^{3,b}\right)'(Y)\right).$$

The equations for the system dynamics of the state $Y(t)$ and the co-state $\theta_i^{3,b}(t) = \left(V_i^{3,b}\right)'(Y(t))$ are

$$\dot{Y} = \varphi^{3,b}\left(\theta_i^{3,b}\right), \qquad \dot{\theta}_i^{3,b} = r\theta_i^{3,b}. \tag{31}$$

Regime $(3,r)$**.** In Regime $(3,r)$ the value function $V_i^{3,r}(Y)$ satisfies the HJB equation

$$rV_i^{3,r}(Y) = a^{\alpha/\rho}\left(y^{3,b}\right)^{\alpha} - R\left(y^{3,b}\right) - \gamma_0\left[x_{\min} - y^{3,b}\right]_+^2 + y^{3,b}\left(V_i^{3,b}\right)'(Y)$$

where

$$y^*(Y) = \varphi^{3,r}\left(\left(V_i^{3,b}\right)'(Y)\right)$$

and

$$\varphi^{3,r}(\theta) = \arg\max_{y\geq 0}\left\{a^{\alpha/\rho}y^{\alpha} - \left[r_1 y + \frac{r_2}{2}y^2\right] - \gamma_0\left[x_{\min} - y\right]_+^2 + y\theta\right\} \tag{32}$$

for any $\theta \in \mathbb{R}$. It is easy to see that the solution to the optimization problem exists for any $\theta \in \mathbb{R}$ and is positive. Thus, $\varphi^{3,r}(\theta)$ is well-defined. Define $H_i^{3,r}(\theta)$ by

$$\begin{aligned} H_i^{3,r}(\theta) = a^{\alpha/\rho}\varphi^{3,r}(\theta)^{\alpha} - \left[r_1\varphi^{3,r}(\theta) + \tfrac{r_2}{2}\varphi^{3,r}(\theta)^2\right] \\ -\gamma_0\left[x_{\min} - \varphi^{3,r}(\theta)\right]_+^2 + \theta\varphi^{3,r}(\theta). \end{aligned} \tag{33}$$

Using FOC

$$\alpha a^{\alpha/\rho}\varphi^{3,r}(\theta)^{\alpha} - r_1\varphi^{3,r}(\theta) - r_2\varphi^{3,r}(\theta)^2 + 2\gamma_0\left[x_{\min} - \varphi^{3,r}(\theta)\right]_+ + \theta\varphi^{3,r}(\theta) = 0,$$

$H_i^{3,r}$ can be simplified to

$$\begin{aligned} H_i^{3,r}(\theta) &= (1-\alpha)\,a^{\alpha/\rho}\varphi^{3,r}(\theta)^{\alpha} + \frac{r_2}{2}\varphi^{3,r}(\theta)^2 \\ &\quad -\gamma_0\left[x_{\min} - \varphi^{3,r}(\theta)\right]_+ \max\left\{x_{\min} + \varphi^{3,r}(\theta), 2\varphi^{3,r}(\theta)\right\}. \end{aligned}$$

The terminal condition for $V_i^{3,r}$ depending on whether Regime $(3,r)$ is terminal or it is followed by Regime $(3,b)$. If it is terminal, then the condition is

$$V_i^{3,r}(\eta S_0) = -\frac{\gamma_0 x_{\min}}{r}. \tag{34}$$

Otherwise, if it is followed by Regime $(3,b)$, then the condition is

$$V_i^{3,r}(Y_s) = V_i^{3,b}(Y_s) - I_s^3(Y_s),$$

where Y_s is a solution to the equation

$$r\left[V_i^{3,b} - I_s^3\right](Y_s) = H_i^{3,r}\left(\left[V_i^{3,b} - I_s^3\right]'(Y_s).\right)$$

The state $Y(t)$ and the co-state $\theta_i^{3,r}(t) = \left(V_i^{3,r}\right)'(Y(t))$ are governed by the equations

$$\dot{Y} = \varphi^{3,r}\left(\theta_i^{3,r}\right), \qquad \dot{\theta}_i^{3,r} = r\theta_i^{3,r}. \tag{35}$$

Optimal activation of substitution in Case 3. Comparing (28) and (34) we see that Player i is better off at the end of recycling process if

$$\max_{z\geq 0}\left\{b^{\alpha/\rho}z^{\alpha} - \left[s_1 z + \frac{s_2}{2}z^2\right]\right\} > -\gamma_0 x_{\min}. \tag{36}$$

Hence, whenever (36) holds, Regime change $(3,r)\to(3,b)$ must occur at some $Y_s \in (0,\eta S_0)$. By Theorem 1, Part (2), the optimal point Y_s for activating substitution in Case 3 satisfies the equation

$$r\left[V_i^{3,b} - I_s^3\right](Y_s) = H_i^{3,r}\left(\left[V_i^{3,b} - I_s^3\right]'(Y_s)\right)$$

where $H_i^{3,r}$ is defined by (33). The invertibility of $H_i^{3,r}(\theta)$ follows from the fact that $H_i^{3,r}$ is increasing in $\varphi^{3,r}$, and $\varphi^{3,r}$ is increasing in q.

Time period for recycling in Case 3. We show that recycling does not stop until all recyclable mineral has been exhausted. This justifies exclusion of Regime $(3,s)$, because this regime can only take place after recycling has been completed, and in that case the value function is a constant.

Proposition 1. *Player i will never stop recycling in Regime $(3,m)$ for $m=r,b$. Furthermore, in the long run the recycling rate, $y^*(t)$, is on the order*

$$y^*(t) = O\left(e^{-\frac{r}{1-\alpha}t}\right) \qquad \text{as } t\to\infty. \tag{37}$$

A proof is in Appendix A.2.

6.2.2 Solutions in Case 2

In Case 2 both X and Y are state variables. The domain of the state variable is the triangle

$$\overline{\Omega} = \left\{(X,Y)\in\mathbb{R}^2 : 0\leq X\leq S_0, \quad 0\leq Y\leq \eta X\right\}.$$

(See Fig. 2.) The value functions $V_l^{2,m}(X,Y)$ satisfy HJB equations in Regime $(2,m)$, and boundary and terminal conditions are given by (11), (13) and (16).

Regime $(2,b)$**.** The HJB equations in Regime $(2,b)$ are

$$\begin{aligned} rV_i^{2,b} &= U\left(x^*+y^*,z^*\right)-x^*P\left(x^*+y^*,z^*\right)-R\left(y^*\right)-Z\left(z^*\right) \\ &\quad +x^*\left(V_i^{2,b}\right)_X+y^*\left(V_i^{2,b}\right)_Y, \\ rV_j^{2,b} &= x^*P\left(x^*+y^*,z^*\right)-C\left(X,x^*\right)+x^*\left(V_j^{2,b}\right)_X+y^*\left(V_j^{2,b}\right)_Y. \end{aligned} \tag{38}$$

where

$$\begin{aligned} x^* &= \phi^{2,b}\left(c_1X-\left(V_j^{2,b}\right)_X(X,Y),\left(V_i^{2,b}\right)_Y(X,Y)\right), \\ y^* &= \varphi^{2,b}\left(c_1X-\left(V_j^{2,b}\right)_X(X,Y),\left(V_i^{2,b}\right)_Y(X,Y)\right), \\ z^* &= \psi^{2,b}\left(c_1X-\left(V_j^{2,b}\right)_X(X,Y),\left(V_i^{2,b}\right)_Y(X,Y)\right), \end{aligned}$$

and for any (λ,θ), $(x^*,y^*,z^*)=\left(\phi^{2,b}(\lambda,\theta),\varphi^{2,b}(\lambda,\theta),\psi^{2,b}(\lambda,\theta)\right)$ is the solution to the optimization problem

$$\begin{aligned} x^* &= \arg\max_{x\ge 0}\left\{xP\left(x+y^*,z^*\right)-c_2x^2/2-x\lambda\right\}, \\ y^* &= \arg\max_{y\ge 0}\left\{U\left(x^*+y,z^*\right)-x^*P\left(x^*+y,z^*\right)-R(y)+y\theta\right\} \\ z^* &= \arg\max_{z\ge 0}\left\{U\left(x^*+y^*,z\right)-x^*P\left(x^*+y^*,z\right)-Z(z)\right\} \end{aligned} \tag{39}$$

The terminal conditions are

$$V_i^{2,b}(S_0,Y)=V_i^{3,b}(Y),\quad V_j^{2,b}(S_0,Y)=0.$$

The following proposition lists various types of a Markovian Nash equilibrium.

Proposition 2. *Suppose* $\left(V_i^{2,b}\right)_Y\le 0$ *and* $\left(V_j^{2,b}\right)_X\le 0$ *in* $\overline{\Omega}$*. Then, in Regime* $(2,b)$*, a Markovian subgame perfect Nash equilibrium,* (x^*,y^*,z^*) *at any* $(X,Y)\in\overline{\Omega}$ *must satisfy*

$$x^*+y^*>0,\qquad z^*>0. \tag{40}$$

There are three possible cases:

(a) $x^*, y^*, z^* > 0$ *and solve the nonlinear system*

$$\begin{aligned} P(x+y,z) + xP_\xi(x+y,z) - c_2 x &= c_1 X - \left(V_j^{2,b}\right)_X (X,Y), \\ U_\xi(x+y,z) - xP_\xi(x+y,z) - r_2 y &= r_1 - \left(V_i^{2,b}\right)_Y (X,Y), \\ U_z(x+y,z) - xP_z(x+y,z) - s_2 z &= s_1. \end{aligned} \tag{41}$$

(b) $x^* = 0$, *and* (y^*, z^*) *solves the second and third equations in (41) with* $x = 0$. *This solution is valid only if*

$$P(y^*, z^*) \leq c_1 X - \left(V_j^{2,b}\right)_X (X,Y). \tag{42}$$

(c) $y^* = 0$, *and* (x^*, z^*) *solves the first and third equations in (41) with* $y = 0$. *This solution is valid only if*

$$U_\xi(x^*, z^*) - x^* P_\xi(x^*, z^*) \leq r_1 - \left(V_i^{2,b}\right)_Y (X,Y) \tag{43}$$

A proof is given in Appendix A.3.

Regime $(2, r)$**.** The HJB equations in Regime $(2, r)$ take the following form:

$$\begin{aligned} rV_i^{2,r} &= U(x^*+y^*,0) - x^*P(x^*+y^*,0) - \gamma(x^*+y^*,0) \\ &\qquad -R(y^*) + x^*\left(V_i^{2,r}\right)_X + y^*\left(V_i^{2,r}\right)_Y, \\ rV_j^{2,r} &= x^*P(x^*+y^*,0) - C(X,x^*) + x^*\left(V_j^{2,r}\right)_X + y^*\left(V_j^{2,r}\right)_Y, \end{aligned} \tag{44}$$

where

$$\begin{aligned} x^* &= \phi^{2,r}\left(c_1 X - \left(V_j^{2,r}\right)_X (X,Y), \left(V_i^{2,b}\right)_Y (X,Y)\right), \\ y^* &= \varphi^{2,r}\left(c_1 X - \left(V_j^{2,r}\right)_X (X,Y), \left(V_i^{2,b}\right)_Y (X,Y)\right), \end{aligned}$$

and for any $(\lambda, \theta) \in \mathbb{R}^2$ $(x^*, y^*) = (\phi^{2,r}(\lambda,\theta), \varphi^{2,r}(\lambda,\theta))$ is the solution to the optimization problem

$$\begin{aligned} x^* &= \arg\max_{x \geq 0} \left\{xP(x+y^*,0) - c_2 x^2/2 - x\lambda\right\}, \\ y^* &= \arg\max_{y \geq 0} \{U(x^*+y,0) - x^*P(x^*+y,0) \\ &\qquad -\gamma(x^*+y,0) - R(y) + y\theta\}. \end{aligned} \tag{45}$$

The terminal conditions are

$$V_i^{2,r}\left(S_0,Y\right)=V_i^{3,r}\left(Y\right),\quad V_j^{2,r}\left(S_0,Y\right)=0.$$

The next proposition characterizes the market outcome of optimal supply of the mineral, x^*, and optimal recycling, y^*.

Proposition 3. *Suppose* $\left(V_i^{2,r}\right)_Y\leq 0$ *and* $\left(V_j^{2,r}\right)_X\leq 0$ *in* $\overline{\Omega}$. *Then, a Markovian Nash equilibrium* (x^*,y^*) *in Regime* $(2,r)$ *has the property that* $x^*+y^*>0$ *for all* (X,Y). *Specifically, there are three possible cases.*

(a) $x^*,y^*>0$ *and solves the nonlinear system*

$$\begin{aligned}P\left(x+y,0\right)+xP_\xi\left(x+y,0\right)-c_2x&=c_1X-\left(V_j^{2,r}\right)_X\left(X,Y\right),\\ U_\xi\left(x+y,0\right)-xP_\xi\left(x+y,0\right)+2\gamma_0\left[x_{\min}-x-y\right]_+-r_2y&\\ =r_1-\left(V_i^{2,r}\right)_Y\left(X,Y\right).&\end{aligned}\tag{46}$$

(b) $x^*=0$ *and* y^* *satisfies the second equation in (46) with* $x=0$ *and* $y=y^*$. *This equilibrium is valid only if*

$$P\left(y^*,0\right)\leq c_1X-\left(V_j^{2,r}\right)_X\left(X,Y\right).\tag{47}$$

(c) $y^*=0$ *and* x^* *satisfies the first equation in (46) with* $x=x^*$ *and* $y=0$. *This equilibrium is valid only if*

$$U_\xi\left(x^*,0\right)-x^*P_\xi\left(x^*,0\right)+2\gamma_0\left[x_{\min}-x^*\right]_+\leq r_1-\left(V_i^{2,r}\right)_Y\left(X,Y\right).\tag{48}$$

The detailed proof is given in Appendix A.4.

6.2.3 Solution in Case 1

In Case 1 only X is the state variable.

Regime $(1,s)$. In Regime $(1,s)$, the HJB equations take the form

$$\begin{aligned}rV_i^{1,s}&=U\left(x^*,z^*\right)-x^*P(x^*,z^*)-Z\left(z^*\right)+x^*\left(V_i^{1,s}\right)',\\ rV_j^{1,s}&=x^*P(x^*,z^*)-C\left(X,x^*\right)+x^*\left(V_j^{1,s}\right)'.\end{aligned}\tag{49}$$

and the terminal values depend on whether this regime is terminal or it is followed by another regime. In the case it is terminal, the values $V_i^{1,s}\left(S_0\right)$ and $V_j^{1,s}\left(S_0\right)$ are given by (13) and (14).

In the case where Regime $(1,s)$ is followed by Regime $(2,b)$, there exists a point X_r such that

$$V_i^{1,s}(X_r) = V_i^{2,b}(X_r,0) - I_r(X_r), \qquad V_j^{1,s}(X_r) = V_j^{2,b}(X_r,0).$$

The Hamiltonians $H_i^{1,s}$ and $H_j^{1,s}$ are formulated as follows.
Let

$$x^* = \phi^{1,s}(\lambda), \quad z^* = \psi^{1,s}(\lambda) \tag{50}$$

be the solution of the optimization problem

$$\begin{aligned} x^* &= \arg\max_{x\geq 0}\left\{xP(x,z^*) - c_2x^2/2 - x\lambda\right\}, \\ z^* &= \arg\max_{z\geq 0}\left\{U(x^*,z) - x^*P(x^*,z) - Z(z)\right\}. \end{aligned} \tag{51}$$

In terms of functions ϕ and ψ, the HJB equations (49) can be written as

$$\begin{aligned} rV_i^{1,s} &= H_i^{1,s}\left(c_1X - \left(V_j^{1,s}\right)'(X), \left(V_i^{1,s}\right)'(X)\right), \\ rV_j^{1,s} &= H_j^{1,s}\left(c_1X - \left(V_j^{1,s}\right)'(X)\right) \end{aligned} \tag{52}$$

where $H_i^{1,s}$ and $H_j^{1,s}$ are defined by

$$\begin{aligned} H_i^{1,s}(\lambda,\theta) = {} & U\left(\phi^{1,s}(p), \psi^{1,s}(\lambda)\right) - \phi^{1,s}(\lambda)P(\phi^{1,s}(\lambda), \psi^{1,s}(\lambda)) \\ & -Z\left(\psi^{1,s}(\lambda)\right) + \phi^{1,s}(\lambda)\lambda, \\ H_j^{1,s}(\lambda) = {} & \phi^{1,s}(\lambda)P(\phi^{1,s}(\lambda), \psi^{1,s}(\lambda)) - c_2\phi^{1,s}(\lambda)^2/2 - \phi^{1,s}(\lambda)\lambda. \end{aligned}$$

Observe that the HJB equation for Player j is independent of the value function for Player i. Thus, it can be solved first and the solution can be used to solve the HJB equation for $V_i^{1,s}$. The next proposition shows that in Regime $(1,s)$, Player i never stops producing the substituting substance. However, it is possible that Player j stops supplying the natural resource.

Proposition 4. *In Regime $(1,s)$, a Markovian Nash equilibrium (x^*,z^*) must satisfy $z^*>0$. There are two possibilities.*

(a) $x^*>0$ and $(x,z)=(x^*,z^*)$ is a solution to the system

$$\begin{aligned} P(x,z) + xP_\xi(x,z) - c_2x &= c_1X - \left(V_j^{1,s}\right)'(X), \\ U_z(x,z) - xP_z(x,z) - s_2z &= s_1. \end{aligned} \tag{53}$$

(b) $x^* = 0$ and $z = z^*$ is a solutiojn to the equation

$$U_z(0, z) - s_2 z = s_1. \tag{54}$$

This solution is valid only if

$$P(0, z^*) \leq c_1 X - \left(V_j^{1,s}\right)'(X). \tag{55}$$

A proof is given in Appendix A.5.

Regime $(1, n)$**.** The HJB equations in Regime $(1, n)$ take the form

$$\begin{aligned} rV_i^{1,n} &= \left(a^{\alpha/\rho} - p_0\right)(x^*)^\alpha - \gamma_0 \left[x_{\min} - x^*\right]_+^2 + x^* \left(V_i^{1,n}\right)', \\ rV_j^{1,n} &= p_0 (x^*)^\alpha - \left[c_1 x^* X + \frac{c_2}{2}(x^*)^2\right] + x^* \left(V_j^{1,n}\right)', \end{aligned} \tag{56}$$

where

$$x^* = \arg\max_{x \geq 0} \left\{p_0 x^\alpha - \left[c_1 x X + \frac{c_2}{2} x^2\right] + x \left(V_j^{1,n}\right)'(X)\right\}.$$

We use the notation

$$\phi^{1,n}(\lambda) = \arg\max_{x \geq 0} \left\{p_0 x^\alpha - \frac{c_2}{2} x^2 - x\lambda\right\}.$$

Then,

$$x^* = \phi^{1,n}\left(c_1 X - \left(V_j^{1,n}\right)'(X)\right).$$

The HJB equations in (56) can be written as

$$\begin{aligned} rV_i^{1,n} &= H_i^{1,n}\left(c_1 X - \left(V_j^{1,n}\right)'(X), \left(V_i^{1,n}\right)'(X)\right), \\ rV_j^{1,n} &= H_j^{1,n}\left(c_1 X - \left(V_j^{1,n}\right)'(X)\right) \end{aligned} \tag{57}$$

where

$$\begin{aligned} H_i^{1,n}(\lambda, \theta) &= \left(a^{\alpha/\rho} - p_0\right)\phi^{1,n}(\lambda)^\alpha - \gamma_0 \left[x_{\min} - \phi(\lambda)\right]_+^2 + \phi^{1,n}(\lambda)\,\theta, \\ H_j^{1,n}(\lambda) &= p_0 \phi^{1,n}(\lambda)^\alpha - c_2 \phi^{1,n}(\lambda)^2/2 - \phi^{1,n}(\lambda)\,\lambda. \end{aligned} \tag{58}$$

The terminal conditions depend on whether Regime $(1, n)$ is followed by Regime $(1, s)$, Regime $(2, r)$, or none. If it is followed by Regime $(1, s)$, the terminal conditions are

$$V_i^{1,n}(X_s) = V_i^{1,s}(X_s) - I_s^1(X_s), \qquad V_j^{1,n}(X_s) = V_j^{1,s}(X_s).$$

If it is followed by Regime $(2, r)$, the terminal conditions are

$$V_i^{1,n}(X_r) = V_i^{2,r}(X_r, 0) - I_r(X_r), \qquad V_j^{1,n}(X_r) = V_j^{2,r}(X_r, 0).$$

If there no regime following Regime $(1, n)$, the terminal conditions are

$$V_i^{1,n}(S_0) = -\frac{\gamma_0 x_{\min}}{r}, \qquad V_j^{1,n}(S_0) = 0.$$

We show that in Regime $(1, n)$, Player j will always supply the natural resources.

Proposition 5. *In Regime $(1, n)$ the optimal control x^* for Player j is the unique solution to the equation*

$$\alpha p_0 x^{\alpha-1} - c_2 x - c_1 X + \left(V_j^{1,n}\right)'(X) = 0, \tag{59}$$

which is positive for any X.

The proof is straightforward because

$$\alpha p_0 x^{\alpha-1} - c_2 x \to \begin{cases} +\infty & \text{as } x \to 0^+, \\ -\infty & \text{as } x \to \infty. \end{cases}$$

Hence, Eq. (59) has a unique positive solution for any X.

Optimal activation of technologies in Case 1. Comparing the terminal values $V_i^{1,n}(S_0)$ and $V_i^{1,s}(S_0)$ given in (14), we see that Player i will not remain in Regime $(1, n)$ forever if (36) holds. So she will activate a technology sooner or later. Nevertheless, starting in Regime $(1, n)$, Player i can activate recycling first or activate substitution first, depending on whether $X_s < X_r$ or $X_r < X_s$. In addition, in Regime $(1, s)$, Player i can activate recycling to change the Regime to $(2, b)$. Let

$$\begin{aligned} G_i^{1,n}(\lambda) &= \left(a^{\alpha/\rho} - p_0\right)\phi^{1,n}(\lambda)^{\alpha} - \gamma_0\left[x_{\min} - \phi(\lambda)\right]_+^2, \\ G_i^{1,s}(\lambda) &= U\left(\phi^{1,s}(\lambda), \psi^{1,s}(\lambda)\right) - \phi^{1,s}(\lambda)\lambda(\phi^{1,s}(\lambda), \psi^{1,s}(\lambda) \\ &\quad - Z\left(\psi^{1,s}(\lambda)\right). \end{aligned}$$

Applying Theorem 1, we have the equations that determine these two numbers.

Proposition 6. *Suppose $H_j^{1,m}(\lambda)$ is invertible for $m = n, s$. (1) If Player i activates substitution first at $X_s \in (0, S_0)$, then X_s satisfies*

$$r\left[V_i^{1,s} - I_r\right](X_s) = G_i^{1,n}(c_1 X_s - \Lambda_s) + \phi^{1,n}(c_1 X_s - \Lambda_s)\left[V_j^{1,s} - I_s^1\right]'(X_s) \tag{60}$$

where Λ_s *satisfies*

$$rV_j^{1,s}(X_s) = H_j^{1,n}(c_1 X_s - \Lambda_s). \tag{61}$$

(2) If Player i *activates recycling at* $X_r < S_0$ *after activating substitution, then* X_r *satisfies*

$$r\left[V_i^{2,b}(X_r,0) - I_r(X_r)\right] = G_i^{1,s}(c_1X_r - \Lambda_r) + \phi^{1,s}(c_1X_r - \Lambda_r)\left[\left(V_i^{2,b}\right)_X(X_r,0) - I_r(X_r)\right], \tag{62}$$

where Λ_r *satisfies*

$$rV_j^{2,b}(X_r,0) = H_j^{1,s}(c_1X_r - \Lambda_r). \tag{63}$$

(3) If Player i *activates recycling first at* $X_r \in (0, S_0)$*, then* X_r *satisfies the following equation:*

$$r\left[V_i^{2,r}(X_r,0) - I_r(X_r)\right] = G_i^{1,n}(c_1X_r - \Lambda_r) + \phi^{1,n}(c_1X_r - \Lambda_r)\left[\left(V_j^{2,r}\right)_X(X_r,0) - I_r(X_r)\right] \tag{64}$$

where Λ_r *satisfies*

$$rV_j^{2,r}(X_r,0) = H_j^{1,n}(c_1X_r - \Lambda_r). \tag{65}$$

A proof is given in Appendix A.6.

System dynamics in Regime $(1, n)$. Introducing the quantity

$$\theta_j^{1,n}(t) = \left(V_j^{1,n}\right)'(X(t)), \qquad \lambda_j^{1,n} = c_1X - \theta_j^{1,n}. \tag{66}$$

We obtain equations of system dynamics as follows.

Proposition 7. *Suppose* $H_j^{1,n}$ *is invertible and let* $\left(H_j^{1,n}\right)^{-1}$ *denote its inverse function. Then, in Regime* $(1, n)$ *the evolution of the state* X *and the costate the quantity is governed by the dynamical system*

$$\dot{\theta}_j^{1,n} = r\theta_j^{1,n} + c_1\phi^{1,n}\left(\lambda_j^{1,n}\right), \quad \dot{\lambda}_j^{1,n} = -r\theta_j^{1,n}. \tag{67}$$

The state $X(t)$ *and* $v_j^{1,n}(t)$ *are given by*

$$X(t) = \frac{1}{c_1}\left[\lambda_j^{1,n}(t) + \theta_j^{1,n}(t)\right], \quad v_j^{1,n}(t) = \frac{1}{r}H_j^{1,n}\left(\lambda_j^{1,n}(t)\right)$$

and $v_i^{1,n}(t)$ *solves the terminal value problem*

$$\dot{v}_i^{1,n}(t) = rv_i^{1,n}(t) - G_i^{1,n}\left(\lambda_j^{1,n}\right), \tag{68}$$

The terminal conditions are

$$\begin{aligned}\theta_j^{1,n}(T) &= c_1 X_l - \left(H_j^{1,n}\right)^{-1}\left(rV_j^{k',m'}(X_l)\right),\\ \lambda_j^{1,n}(T) &= \left(H_j^{1,n}\right)^{-1}\left(rV_j^{k',m'}(X_l)\right),\end{aligned} \tag{69}$$

if Regime $(1,n)$ is followed by Regime (k',m') by Player i activating technology $l\,(=r,s)$ at the point X_l. In the case where Regime $(1,n)$ is terminal, the terminal conditions are

$$\lim_{t\to T}\theta_j^{1,n}(t) = -\infty, \qquad \lim_{t\to T} X(t) = S_0.$$

The terminal value of $v_i^{1,n}(T)$ is $V_i^{1,n}(S_0)$ given by (14).

A proof is included in Appendix A.7.

We conclude this example as follows. Proposition 6 highlights the potential computational challenges involved in determining the switching curve Γ_s, as illustrated in Figures 2 and 4. Even with the explicit functional forms provided in this section, which allow for a more precise characterization of the regime switching condition, Proposition 6 shows that identifying a fixed point along the switching curve remains intractable.

As a result, the current framework yields a continuous switching curve that must be approximated within a multi-dimensional state space. This requirement not only increases potential computational complexity, but also raises methodological concerns: numerical outcomes become highly sensitive to the chosen functional specifications. Without careful design and transparent reporting, results across studies may become inconsistent or difficult to compare. Future research should therefore emphasize robust numerical methods and thorough sensitivity analyses to ensure reliability when modeling multi-state switching games.

7 Extensions and future research directions

While the present analysis focuses on a deterministic two-player game with discrete technology activation, several extensions can enrich the framework and broaden its applicability.

Stochastic Impulse Control

A natural extension involves introducing uncertainty in technology activation or resource prices. For example: Random innovation times modeled as stopping times in a stochastic control framework; Price volatility incorporated through diffusion processes, leading to stochastic HJB equations.

Such extensions would connect our model to the literature on stochastic differential games and optimal stopping, requiring advanced tools such as viscosity solutions and probabilistic verification.

Multi-Player and Coalition Settings

Another promising direction is to generalize the game to multiple importing countries competing for critical minerals while investing in recycling and substitution. This raises questions about: Coalition formation and cooperative equilibria, strategic spillovers in technology adoption, and distributional effects of recycling capacity across heterogeneous players.

Analyzing these settings would require combining impulse control with network game theory.

Policy Instruments and Market Design

Our framework can also incorporate policy interventions such as Price floors or subsidies for recycling and substitution technologies, and trade restrictions or tariffs affecting extraction incentives. These extensions would allow for welfare analysis under alternative regulatory regimes.

Each of these extensions preserves the core mathematical challenge–multi-dimensional impulse control under regime connectivity, while introducing additional layers of complexity relevant to real-world resource management and technology adoption.

8 Conclusion

This study develops a comprehensive framework for analyzing multi-state switching games, advancing both the theoretical foundations and the computational methodology of dynamic strategic interactions. The model accommodates rich state dependence, endogenous regime changes, and strategic timing, offering a unified structure that extends existing approaches in several directions.

First, the framework establishes a general characterization of equilibrium switching behavior in environments where players face multi-dimensional state variables and non-trivial interdependencies. Second, it provides conditions under which switching incentives can be expressed through tractable value-difference representations, enabling a clearer interpretation of strategic thresholds. Third, the analysis identifies structural features that govern the geometry of switching regions, clarifying how economic primitives shape the boundaries between regimes. Finally, the framework highlights the inherent computational challenges that arise when switching curves are continuous objects embedded in high-dimensional state spaces. The resulting switching boundary must therefore be approximated numerically, and its shape is highly sensitive to functional specifications. This sensitivity underscores a broader methodological point: reliable numerical implementation in multi-state switching games requires careful algorithmic design, transparent reporting, and systematic robustness checks.

Taken together, the contributions of this paper provide a foundation for studying a wide class of dynamic strategic problems in which agents can alter the economic environment through discrete actions. Future research may explore stochastic impulse control, multi-player settings, and policy instruments such as subsidies or price floors, further enriching the analytical framework.

A Appendix

A.1 Proof of Theorem 1

(1) We first consider the case where $m = n$. Starting from any $X < X_l^1$ at $t = 0$, we suppose player i activates substitution at $t = T$. Then

$$V_i^{1,n}(X) = \int_0^T e^{-rt} G_i^{1,n}\left(x^*\left(X(t), \left(V_j^{1,n}\right)'(X(t))\right)\right) dt + e^{-rT}\left[V_i^{k',m'} - I_l^1\right](X(T)).$$

By optimality, player i chooses T so that the derivative of the right-hand side with respect to T vanishes. The derivative has the form

$$\begin{aligned}
&e^{-rT} G_i^{1,n}\left(x^*\left(X(t), \left(V_j^{1,n}\right)'(X(t))\right)\right) - re^{-rT}\left[V_i^{k',m'} - I_l^1\right](X(T)) \\
&+e^{-rT}\left[V_i^{k',m'} - I_l^1\right]'(X(T))\, x^*\left(X(T), \left(V_j^{1,n}\right)'(X(T))\right).
\end{aligned}$$

Let T_l^1 be the optimal timing for the mode change. Taking the limit $T \to T_l^1$, it follows that

$$\begin{aligned}
r\left[V_i^{k',m'} - I_l\right]\left(X_l^1\right) &= G_i^{1,n}\left(x^*\left(X_l^1, \hat{P}_j^{1,n}\right)\right) + \left[V_i^{k',m'} - I_l^1\right]'\left(X_l^1\right) x^*\left(X_l^1, \hat{P}_j^{1,n}\right) \\
&= H_i^{1,n}\left(X_l^1, \left[V_i^{k',m'} - I_l^1\right]'\left(X_l^1\right), \hat{P}_j^{1,n}\right).
\end{aligned}$$

In addition, taking the limit $T \to T_l^1$ in the second HJB equation for $V_j^{1,n}(X)$, and using the transition condition (12), we obtain

$$rV_j^{k',m'}\left(X_l^1\right) = H_j^{1,n}\left(X_l^1, \hat{P}_j^{1,n}\right).$$

This proves (19) with $(k,m) = (1,n)$.

The case where $m = s$ is proved in a similar way. The only difference is $G_i^{1,s}$ has two inputs, x^* and z^*, both are functions of $\left(X, \left(V_j^{1,s}\right)'(X)\right)$.

(2) Since $H_i^{3,r}$ is invertible, the HJB equation

$$rV_i^{3,r}(Y) = H_i^{3,r}\left(\left(V_i^{3,r}\right)'(Y)\right)$$

can be written as

$$\left(V_i^{3,r}\right)'(Y) = F\left(V_i^{3,r}(Y)\right)$$

for some differentiable function F. Thus, for any $Y < Y_s$,

$$V_i^{3,r}(Y) = V_i^{3,b}(Y_s) - I_s^3(Y_s) + \int_{Y_s}^{Y} F\left(V_i^{3,r}(\xi)\right) d\xi.$$

By optimality, the derivative of the right-hand side with respect to Y_s vanishes. Thus

$$\left[V_i^{3,b} - I_s^3\right]'(Y_s) - F\left(V_i^{3,r}(Y_s)\right) = 0.$$

This is equivalent to

$$rV_i^{3,r}(Y_s) = H_i^{3,r}\left(\left[V_i^{3,b} - I_s^3\right]'(Y_s)\right).$$

This proves (20).

The proof is complete.

A.2 Proof of Proposition 1

Proof. We show that $y^{3,m} > 0$ for $m = r, b$.

For $m = r$, $\varphi^{3,r}(\theta)$ is defined by (32). The function

$$y \mapsto a^{\alpha/\rho} y^{\alpha} - \left[r_1 y + \frac{r_2}{2} y^2\right] - \gamma_0 \left[x_{\min} - y\right]_+^2 + y\theta$$

has the derivative

$$\alpha a^{\alpha/\rho} y^{\alpha-1} - r_1 - r_2 y + 2\gamma_0 \left[x_{\min} - y\right]_+ + \theta.$$

It approaches $+\infty$ as $y \to 0^+$, and to $-\infty$ as $y \to \infty$, for any $\theta \in \mathbb{R}$. Hence, $y^*(Y) > 0$ for any $Y < \eta S_0$.

For $m = b$, $\varphi^{3,b}$ is given by (30). Note that the function

$$f_b : y \mapsto \left[ay^\rho + b\left(z^*\right)^\rho\right]^{\alpha/\rho} - r_1 y - \frac{r_2}{2} y^2 + y\theta$$

with fixed z^* and θ has the derivative

$$f_b'(y) = \alpha a \left[ay^\rho + b\left(z^*\right)^\rho\right]^{\alpha/\rho - 1} y^{\rho-1} - r_1 - r_2 y + \theta$$

which approaches to $+\infty$ as $y \to 0^+$ and it approaches to $-\infty$ as $y \to \infty$, for any $z^* \geq 0$ and $\theta \in \mathbb{R}$. Thus $y^*(Y) \equiv \varphi^{3,b}\left(\left(V_i^{3,b}\right)'(Y)\right) > 0$ whenever $Y < \eta S_0$.

Let $\bar{z}$ be defined by (29) and let $\hat{f}_b$ and $\tilde{f}_b$ be functions

$$\begin{aligned} \tilde{f}_b(y) &= a^{\alpha/\rho} y^\alpha - r_1 y - \frac{r_2}{2} y^2 + y\theta, \\ \hat{f}_b(y) &= \left[ay^\rho + b\bar{z}^\rho\right]^{\alpha/\rho} - r_1 y - \frac{r_2}{2} y^2 + y\theta. \end{aligned}$$

It can be shown that $0 < z^*(Y) < \bar{z}$. Hence,

$$\hat{f}_b'(y) < f_b'(y) < \tilde{f}_b'(y) \qquad \text{for any } y.$$

This implies that the critical points $\hat{y}$, $y^*(Y)$, and $\tilde{y}$ of $\hat{f}_b$, f_b and $\tilde{f}_b$ with $\theta = \left(V_i^{3,b}\right)'(Y)$ are ranked as

$$0 < \hat{y} < y^{3,b} < \tilde{y}.$$

In addition, by (31),

$$\theta_i^{3,b}(t) = \left(V_i^{3,b}\right)'(Y(t))$$

is an exponential function $\theta_i^{3,b}(t) = \theta_i^{3,b}(0)\, e^{rt}$. Thus, from the equation

$$\alpha a^{\alpha/\rho} \tilde{y}^{\alpha-1} - r_1 - r_2 \tilde{y} + \theta_i^{3,b}(t) = 0$$

we can approximate $\tilde{y}$ by

$$\tilde{y}(t) = \left[\frac{r_1 + r_2\tilde{y}(t) - \theta_i^{3,b}(t)}{\alpha a^{\alpha/\rho}}\right]^{1/(\alpha-1)} \approx \left[\frac{\alpha a^{\alpha/\rho}}{\left|\theta_i^{3,b}(0)\right|}\right]^{1/(1-\alpha)} e^{-rt/(1-\alpha)}.$$

We can estimate $\hat{y}(t)$ in a similar way to obtain

$$\hat{y}(t) = O\left(e^{-rt/(1-\alpha)}\right) \qquad \text{as } t \to \infty.$$

This completes the proof.

A.3 Proof Proposition 2

The subgame perfect Markovian Nash equilibrium (x^*, y^*, z^*) is a solution to the optimization problem (39). Define functions M, N and L by

$$\begin{aligned} M(x,y,z,X) &= xP(x+y,z) - C(X,x). \\ N(x,y,z) &= U(x+y,z) - xP(x+y,z) - R(y), \\ L(x,y,z) &= U(x+y,z) - xP(x+y,z) - Z(z) \end{aligned} \tag{70}$$

By differentiation,

$$\begin{aligned} M_x &= P(x+y,z) + xP_\xi(x+y,z) - c_1X - c_2x \\ &= \bar{p}\left(a\xi^\rho + bz^\rho\right)^{(\alpha-1)/\rho-1}\left[(\alpha x + y)\,a\xi^{p-1} + bz^\rho\right] \end{aligned}$$

where $\xi = x + y$, and

$$\begin{aligned} M_{xx} = \bar{p}a(\alpha-1)\,\xi^{\rho-2}\left(a\xi^\rho + bz^\rho\right)^{(\alpha-1)/\rho-2}\{&(2y+\alpha x)\,a\xi^\rho \\ &+ (2y + (1+\rho)\,x)\,bz^\rho\} - c_2 < 0. \end{aligned}$$

So, M_x is decreasing in x. Also,

$$\begin{aligned} N_y &= U_\xi(x+y,z) - xP_\xi(x+y,z) - r_1 - r_2y \\ &= \alpha a\xi^{\rho-1}\left(a\xi^\rho + bz^\rho\right)^{\alpha/\rho-1} + \bar{p}(1-\alpha)\,xa\xi^{\rho-1}\left(a\xi^\rho + bz^\rho\right)^{(\alpha-1)/\rho-1} \\ &\quad - r_1 - r_2y. \end{aligned}$$

Hence,

$$\begin{aligned} N_{yy} &= \alpha a\xi^{\rho-2}\left(a\xi^{\rho}+bz^{\rho}\right)^{\alpha/\rho-2}\left[(\alpha-1)\,a\xi^{\rho}+(\rho-1)\,bz^{\rho}\right] \\ &\quad +\bar{p}\,(1-\alpha)\,xa\xi^{\rho-2}\left(a\xi^{\rho}+bz^{\rho}\right)^{(\alpha-1)/\rho-1}\left[(\alpha-2)\,a\xi^{\rho}+(\rho-1)\,bz^{\rho}\right] \\ &\quad -r_2<0. \end{aligned}$$

So, N_y is decreasing in y. Finally,

$$\begin{aligned} L_z &= U_z\,(x+y,z)-xP_z\,(x+y,z)-s_1-s_2z \\ &= \alpha bz^{\rho-1}\left(a\xi^{\rho}+bz^{\rho}\right)^{\alpha/\rho-1}+\bar{p}\,(1-\alpha)\,xbz^{\rho-1}\left(a\xi^{\rho}+bz^{\rho}\right)^{(\alpha-1)/\rho-1} \\ &\quad -s_1-s_2z \end{aligned}$$

and

$$\begin{aligned} L_{zz} &= \alpha bz^{\rho-2}\left(a\xi^{\rho}+bz^{\rho}\right)^{\alpha/\rho-2}\left[(\rho-1)\,a\xi^{\rho}+(\alpha-1)\,bz^{\rho}\right] \\ &\quad +\bar{p}\,(1-\alpha)\,xbz^{\rho-2}\left(a\xi^{\rho}+bz^{\rho}\right)^{(\alpha-1)/\rho-1}\left[(\rho-1)\,a\xi^{\rho}+(\alpha-2)\,bz^{\rho}\right] \\ &\quad -s_2<0. \end{aligned}$$

So, L_z is decreasing in z.

Proof of Part (a). An interior solution (x^*,y^*,z^*) satisfies

$$\begin{aligned} x^* &= \arg\max_{x\geq 0}\left\{M\,(x,y^*,X)+x\left(V_j^{2,b}\right)_X(X,Y)\right\}, \\ y^* &= \arg\max_{y\geq 0}\left\{N\,(x^*,y,z^*)+y\left(V_i^{2,b}\right)_Y(X,Y)\right\}, \\ z^* &= \arg\max_{z\geq 0} L\,(x^*,y^*,z)\,. \end{aligned}$$

For all components to be positive, it is necessary that

$$\begin{aligned} &M_x\,(x^*,y^*,z^*,X)+\left(V_j^{2,b}\right)_X(X,Y)=0, \\ &N_y\,(x^*,y^*,z^*)+\left(V_i^{2,b}\right)_Y(X,Y)=0, \\ &L_z\,(x^*,y^*,z^*)=0. \end{aligned}$$

These equations are equivalent to the system (41).

Proof Part (b). In the case where $x^*=0$, it is necessary that

$$0=\arg\max_{x\geq 0}\left\{M\,(x,y^*,z^*,X)+x\left(V_j^{2,b}\right)_X(X,Y)\right\},$$

and y^* and z^* satisfies the second and third equations in (41) with $x = 0$, $y = y^*$, and $z = z^*$. The above relation is equivalent to

$$M(x, y^*, z^*, X) + x\left(V_j^{2,b}\right)_X (X, Y) \le 0 \qquad \text{for all } x > 0.$$

Note that $M(0, y^*, z^*, X) = 0$ and $M_x(x, y^*, z^*, X)$ is decreasing in x for $x > 0$, it follows that $M(x, y^*, z^*, X) + x\left(V_j^{2.b}\right)_X (X, Y) \le 0$ only if

$$M_x(0, y^*, z^*, X) + \left(V_j^{2,b}\right)_X (X, Y) \le 0.$$

This inequality is the same as (42).

Proof of Part (c). In the case where $y^* = 0$, it is necessary that

$$0 = \arg\max_{y \ge 0} \left\{ N(x^*, y, z^*) + y\left(V_i^{2,b}\right)_Y (X, Y) \right\}$$

and x^* and z^* satisfies the first and third equation in (41) with $x = x^*$, $y = 0$ and $z = z^*$. The above relation is equivalent to

$$N(x^*, y, z^*) + y\left(V_i^{2,b}\right)_Y (X, Y) \le N(x^*, 0, z^*) \qquad \text{for all } y \ge 0.$$

Since N_y is decreasing in y, the above inequality holds if and only if

$$N_y(x^*, 0, z^*) + \left(V_i^{2,b}\right)_Y (X, Y) \le 0.$$

This is the same as (43).

Proof of (40). Note that if $y^* = 0$ then (x^*, z^*) satisfies (43) which takes the form

$$\begin{aligned} & \alpha a (x^*)^{\rho-1} (a (x^*)^\rho + b z^\rho)^{\alpha/\rho-1} + \bar{p}(1-\alpha) a (x^*)^\rho (a (x^*)^\rho + b z^\rho)^{(\alpha-1)/\rho-1} \\ \le\ & r_1 - \left(V_i^{2,b}\right)_Y (X, Y). \end{aligned}$$

The left-hand-side of the above inequality tends to $+\infty$ as $x^* \to 0^+$. Therefore, x^* and y^* cannot be both zero.

Observe that $L_z(x^*, y^*, z)$ is decreasing in z, $L_z(x^*, y^*, z) \to \infty$ as $z \to 0^+$ and $L_z(x^*, y^*, z) \to -\infty$ as $z \to \infty$, it follows that the zero of $L_z(x^*, y^*, z)$ is positive. Thus, $z^* > 0$.

The proof is complete.

A.4 Proof of Proposition 3

The subgame perfect Markovian Nash equilibrium (x^*, y^*) in Regime $(2, r)$ is a solution to the optimization problem (45). Let the function M be defined in (70) and define function K by

$$K(x,y) = U(x+y,0) - xP(x+y,0) - \gamma(x+y,0) - R(y).$$

This function has the partial derivatives

$$\begin{aligned} K_y &= U_\xi(x+y,0) - xP_\xi(x+y,0) + 2\gamma_0[x_{\min} - x - y]_+ - r_1 - r_2 y \\ &= (x+y)^{\alpha-2}\left[\alpha a^{\alpha/\rho}(x+y) + (1-\alpha)p_0 x\right] + 2\gamma_0[x_{\min} - x - y]_+ \\ &\quad - r_1 - r_2 y. \end{aligned}$$

It is easy to see that K_y is decreasing in y.

Proof of Part (a). An interior solution (x^*, y^*) satisfies

$$\begin{aligned} x^* &= \arg\max_{x \ge 0} \left\{ M(x,y,0,X) + x\left(V_j^{2,r}\right)_X(X,Y) \right\}, \\ y^* &= \arg\max_{y \ge 0} \left\{ K(x,y) + y\left(V_i^{2,r}\right)_Y(X,Y) \right\}. \end{aligned}$$

For both x^* and y^* to be positive, it is necessarily that

$$\begin{aligned} M_x(x^*,y^*,0,X) + \left(V_j^{2,r}\right)_X(X,Y) &= 0, \\ K_y(x^*,y^*) + \left(V_i^{2,r}\right)_Y(X,Y) &= 0. \end{aligned}$$

These equations are equivalent to the system (46).

Proof of Part (b). In the case where $x^* = 0$, it is necessary that

$$0 = \arg\max_{x \ge 0} \left\{ M(x,y^*,0,X) + x\left(V_j^{2,r}\right)_X(X,Y) \right\} \tag{71}$$

and y^* satisfies the second equation in (46) with $x = 0$ and $y = y^*$. Eq. (71) is equivalent to

$$M(x,y^*,0,X) + x\left(V_j^{2,r}\right)_X(X,Y) \le 0 \qquad \text{for all } x > 0.$$

Since M_x is decreasing in x for $x > 0$, it follows that the above inequality is equivalent to

$$M_x(0, y^*, 0, X) + \left(V_j^{2,r}\right)_X (X, Y) \leq 0.$$

This proves the necessity of (47).

Proof of Part (c). In the case where $y^* = 0$, it is necessary that

$$0 = \arg\max_{y \geq 0} \left\{K(x^*, y) + y\left(V_i^{2,r}\right)_Y (X, Y)\right\}$$

and x^* satisfies the first equation in (46) with $x = x^*$ and $y = 0$. The above relation is equivalent to

$$K(x^*, y) + y\left(V_i^{2,r}\right)_Y (X, Y) \leq R(x^*, 0).$$

Since K_y is decreasing in y, the above inequality is equivalent to

$$K_y(x^*, 0) + \left(V_i^{2,r}\right)_Y (X, Y) \leq 0.$$

This is the same as (48).

Proof of $x^* + y^* > 0$. If $y^* = 0$ then x^* satisfies (48) which is the same as

$$\left[\alpha a^{\alpha/\rho} + \bar{p}(1-\alpha) a^{(\alpha-1)/\rho}\right](x^*)^{\alpha-1} + 2\gamma_0 \left[x_{\min} - x^*\right] \leq r_1 - \left(V_i^{2,b}\right)_Y (X, Y).$$

This inequality does not hold at $x^* = 0$. Thus, $x^* > 0$.

The proof is complete.

A.5 Proof of Proposition 4

From the proof of Proposition 2, we see that $M_x(x, 0, z, X)$ is decreasing in x, and $L_z(x, 0, z)$ is decreasing in z.

Proof of Part (a). An interior solution (x^*, z^*) satisfies

$$\begin{aligned} x^* &= \arg\max_{x \geq 0} \left\{M(x, 0, z^*, X) + x\left(V_j^{1,s}\right)'(X)\right\}, \\ z^* &= \arg\max_{z \geq 0} L(x^*, 0, z). \end{aligned}$$

For both components are positive, it is necessary that

$$M_x(x^*, 0, z^*, X) + \left(V_j^{1,s}\right)'(X) = 0,$$
$$L_z(x^*, 0, z^*) = 0.$$

These equations are equivalent to (53).

Proof of Part (b). In the case where $x^* = 0$, it is necessary that

$$0 = \arg\max_{x \geq 0} \left\{ M(x, 0, z^*, X) + x\left(V_j^{1,s}\right)'(X) \right\}$$

and $z = z^*$ satisfies the second equation in (53) with $x = 0$, which is (54). The above relation is equivalent to

$$M(x, 0, z^*, X) + x\left(V_j^{1,s}\right)'(X) \leq 0 \qquad \text{for all } x > 0.$$

Note that the left-hand side vanishes if $x = 0$, the inequality follows if

$$M_x(0, 0, z^*, X) + \left(V_j^{1,s}\right)'(X) \leq 0.$$

This inequality is the same as (55).

Proof of $z^* > 0$. Note that z^* satisfies

$$U_z(x^*, z^*) - x^* P_z(x^*, z^*) - s_2 z^* = s_1$$

whether $x^* > 0$ or $x^* = 0$. By differentiation,

$$U_z(x, z) = \alpha b z^{\rho-1}(ax^\rho + bz^\rho)^{\alpha/\rho-1}.$$

Since $0 < \alpha, \rho < 1$, for any fixed $x \geq 0$, $U_z(x, z) \to +\infty$ as $z \to 0^+$. On the other hand,

$$xP_z(x, z) - s_2 z = \bar{p}(\alpha - 1) b z^{\rho-1}(ax^\rho + bz^\rho)^{(\alpha-1)/\rho-1} - s_2 z \to -\infty$$

a $z \to \infty$. Thus, $z^* > 0$ in any case.

The proof is complete.

A.6 Proof of Proposition 6

We use Part (1) of Theorem 1. If Player i activates substitution before recycling, at X_s, the regime change $(1,n) \mapsto (1,s)$ occurs. Using Part (1) of Theorem 1 with $(k,l) = (1,n)$ and $(k',m') = (1,s)$, Eqs. in (19) lead to (60) and (61) provided that

$$\Lambda_s \equiv \hat{\Lambda}_j^{1,n} = \lim_{X \uparrow X_s} \left(V_j^{1,n}\right)' (X)$$

exists. The existence follows from the continuity of $V_j^{1,n}(X)$ and the continuity and invertibility of the Hamiltonian $H_j^{1,n}$.

The other cases are similar.

The proof is complete.

A.7 Proof of Proposition 7

Differentiating the both sides of the second equations in (56) with respect to X, and using the FOC

$$p_0 \alpha (x^*)^{\alpha-1} - [c_1 X + c_2 x^*] + \left(V_j^{1,n}\right)' (X) = 0,$$

we obtain

$$r \left(V_j^{1,n}\right)' (X) = -c_1 x^* + x^* \left(V_j^{1,n}\right)'' (X). \tag{72}$$

In view of the equation of the dynamics (1), we find

$$x^* \left(V_j^{1,n}\right)'' (X) = \frac{d}{dt} \left(V_j^{1,n}\right)' (X(t)).$$

Thus, Eq. (72) becomes

$$r\theta_j^{1,n} = -c_1 x^* + \dot{\theta}_j^{1,n}.$$

Note that by (50), $x^*(t) = \phi^{1,n}\left(\lambda_j^{1,n}(t)\right)$, the above equation is the same as the first equation in (67). The second equation in (67) follows from differentiating the both sides of Eq. (66) with respect to t and using the first equation.

Eqs. in (68) follow from HJB equations in (52) and (57).

This completes the proof.

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