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Optimal Control of Diffusion Systems with State Constraints: Theory and Application to Land Restoration *

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Abstract

We develop an optimal control framework for infinite-dimensional systems with inequality state constraints, extending the Pontryagin Maximum Principle to diffusion-driven dynamics with bounded states. The resulting conditions feature Radon-measure multipliers that characterize boundary behavior in distributed environments. As an illustration, we apply the framework to a model of land fertility evolving through reversible pollution and spatial diffusion. We show how discounting shapes optimal consumption, the activation of state constraints, and long-run spatial patterns. In the homogeneous case, explicit solutions identify conditions for full restoration or persistent degradation, while heterogeneous settings generate hybrid finite-horizon and long-run regimes. The framework provides general analytical tools for dynamic optimization problems with diffusion and bounded state variables.

Keywords: Economic growth, Diffusion, Soil Pollution, Optimal Control, Limited resources.

Journal of Economic Literature: C61, O44, Q15, Q56, R11.

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1 Introduction

Dynamic optimization with state constraints plays a central role in economic theory, yet most existing results are confined to finite-dimensional systems or settings without spatial interactions. Many economic processes, ranging from resource management and technological diffusion to spatial growth and distributed production, are inherently infinite-dimensional and subject to strict feasibility limits. Incorporating these constraints into spatiotemporal models raises fundamental questions: How can optimal control be characterized when the state evolves according to a partial differential equation? What mathematical tools allow us to handle inequality constraints in distributed systems? And how do these constraints shape long-run economic outcomes?

This paper develops a rigorous optimal control framework for infinite-dimensional systems with ecological or physical bounds, focusing on a model where the state variable evolves according to a diffusion equation and is subject to upper and lower constraints. Our contribution is methodological: we extend the Pontryagin Maximum Principle to a setting with spatial diffusion and inequality state constraints, and we derive generalized Kuhn-Tucker conditions involving Radon measures. These measure-valued multipliers capture the shadow value of relaxing the state constraint at each point in space and time, providing a transparent economic interpretation of boundary behavior in distributed systems.

To illustrate the applicability of the framework, we study a model of land fertility and pollution dynamics in which fertile land is the sole bounded production input. Fertility evolves according to a reversible pollution process with spatial diffusion, and consumption is chosen to maximize discounted welfare. This setting, while motivated by environmental concerns, serves as a natural and tractable example of a broader class of problems in which a bounded state variable interacts with diffusion and control. The model highlights how time preferences, spatial heterogeneity, and state constraints jointly determine optimal trajectories and long-run outcomes.

Classical optimal control theory (Pontryagin, 1962) and its extensions to PDE systems (Li and Yong, 1991; Fattorini, 1999) provide necessary conditions for unconstrained problems, but do not address inequality constraints in spatiotemporal models. Recent work in spatial economics (Boucekkine et al., 2013; 2018; 2021) studies diffusion-driven dynamics but assumes unbounded inputs. Similarly, sustainability models under discounting (Chichilnisky et al., 1995; Stern, 2008) abstract from spatial heterogeneity and ecological limits. To our knowledge, no existing framework combines (i) spatial diffusion, (ii) bounded state variables, and (iii) rigorous Kuhn-Tucker conditions for infinite-dimensional systems.

Using the new technical tools developed in this paper, we systematically analyze the optimal control of land fertility and consumption under different discounting regimes. In the low discount rate case, the system exhibits sustainable dynamics, with a full restoration of fertile land in the long run. When space is homogeneous, we derive explicit solutions showing a two-phase structure: an initial growth phase followed by a steady-state regime. In contrast, high discount rates can lead to resource depletion, boundary behavior, and complex spatial dynamics, depending on feasibility and critical thresholds.

Our contribution to the literature is threefold. First, we develop a rigorous spatiotemporal model of soil pollution and land use that incorporates ecological state constraints and intertemporal preferences. By extending the Pontryagin Maximum Principle to a setting with spatial diffusion and inequality constraints, we provide a novel analytical framework for optimal environmental policy in bounded-resource economies. This complements and extends the spatial AK literature (e.g., Boucekkine et al., 2013, 2025) by introducing ecological limits and reversible pollution dynamics.

Second, we characterize optimal policy under different discounting regimes: low, intermediate, and high, and show how time preferences fundamentally alter the long-term spatial distribution of fertile land. In particular, we demonstrate that under low discounting, full restoration is achievable at least at some locations, while high discounting leads to persistent spatial heterogeneity or irreversible degradation. These results connect to and enrich the literature on sustainability under discounting (e.g., Chichilnisky et al., 1995; Stern, 2008), offering new insights into the spatial consequences of impatience.

Third, we construct hybrid solutions in which the system transitions from a finite-horizon control problem to a structured long-run regime. This hybrid structure, governed by adjoint dynamics and Radon measures, offers a new lens through which to understand transitional environmental policy. Thus, our results bridge the gap between theoretical optimal control, spatial economic dynamics, and practical policy design for land restoration and pollution abatement.

The remaining work is organized as follows: Section 2 formalizes the optimization problem and develops the extended maximum principle and associated Kuhn-Tucker conditions. Section 3 illustrates the soil pollution model. Section 4 details the maximum principle and Kuhn-Tucker conditions for soil pollution cases. Section 5 characterizes optimal solutions under different discounting regimes. Section 6 provides numerical illustrations and discusses transitional dynamics. Section 7 concludes with implications for theory and policy.

2 A maximum principle and extended Kuhn-Tucker condition

We present a general version of Pontryagin's Maximum Principle that fits our model. For any $T > 0$, let $\mathcal{S}_T = [0, T] \times \mathcal{S}$ and denote

$$Q = \{y \in C(\mathcal{S}_T) : 0 \leq y(t, \theta) \leq 1 \quad \text{for all } (t, \theta) \in \mathcal{S}_T\}. \tag{1}$$

We consider a more general optimization problem for which the state variable is the solution to the parabolic partial differential equation

$$\begin{cases} y_t - D y_{\theta\theta} = ay + bc & \text{for } (t, \theta) \in (0, T) \times \mathcal{S}, \\ y(0, \theta) = y_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \tag{2}$$

where $T > 0$ is a constant, y is the state variable, $c \in \mathcal{B}(y)$ is the control, D is a positive constant, and a, b, and y_0 given functions. The system dynamics is subject to the state constraint

$$0 \leq y(t, \theta) \leq 1 \quad \text{for } (t, \theta) \in \mathcal{S}_T. \tag{3}$$

Given the initial state y_0, the welfare functional to be maximized is

$$J(y_0, c) = \int_0^T \int_{\mathcal{S}} g(t, \theta, c(t, \theta))\, d\theta dt + h(T, y(T, \cdot)), \tag{4}$$

where the functional $h : \mathbb{R}^+ \times C(\mathcal{S}) \longmapsto \mathbb{R}$ has a Fréchet derivative with respect to y.

Define the Hamiltonian

$$H(t, \theta, y, c, \mu, \psi) = \mu g(t, \theta, c) + \psi f(t, \theta, y, c) \quad \text{in } \mathcal{S}_T \times [0, 1] \times \mathcal{B}(y),$$

where

$$f(t, \theta, y, c) = a(t, \theta)\, y + b(t, \theta)\, c.$$

We make the following assumptions.

Assumption 1. *(1) Functions $a, b : \mathcal{S}_T \longmapsto \mathbb{R}$ are continuously differentiable.*

(2) Function $g : \mathcal{S}_T \times \mathbb{R}^+ \longmapsto \mathbb{R}^+$ is continuously differentiable and for any $B > 0$ there is

a constant β such that

$$|g(t,\theta,c)| \leq \beta \quad \textit{for } (t,\theta,c) \in \mathcal{S}_T \times [0,B]$$

and $h : \mathbb{R}^+ \times [0,1] \longmapsto \mathbb{R}^+$ is continuously differentiable.

A Pontryagin's maximum principle and extended Kuhn-Tucker conditions for inequality constraint of the above optimal control problem (4), subject to (2) and (3) can be obtained by using Ekeland's variational principle (Theorem 3.2.2 in H. Fattorini, 1999) together with a spike perturbation.

Theorem 1. *Let Assumption 1 hold and let $\{c^*, y^*\}$ be an optimal pair. Then there exists a constant $\nu \geq 0$, a function $\psi \in L^q\left(0,T; W^{1,q}(\mathcal{S})\right)$ $(1 < q < 3/2)$ and a Radon measure m such that*

$$\nu + |m|_{\mathcal{M}(\mathcal{S}_T)} > 0, \tag{5}$$

$$\langle m, z - y^* \rangle \leq 0, \quad \forall z \in Q, \tag{6}$$

$$\begin{aligned} &\psi_t + D\psi_{\theta\theta} = -\left[a + bc^*/y^*\right]\psi - \nu g_y^* - m|_{(0,T)\times\mathcal{S}}, \\ &\psi(T,\cdot) = \nu h_y\left(T, y^*(T,\cdot)\right) + m|_{\{T\}\times\mathcal{S}}, \end{aligned} \tag{7}$$

and

$$H\left(t,\theta,y^*(t,\theta), c^*(t,\theta), \nu, \psi(t,\theta)\right) = \max_{0\leq c\leq \mathcal{B}(y^*)} H\left(t,\theta,y^*(t,\theta), c, \nu, \psi(t,\theta)\right) \tag{8}$$

where

$$g_y^* = g_c\left(t,\theta,c^*(t,\theta)\right)c^*(t,\theta)/y^*(t,\theta).$$

A proof is given in Appendix **??**. Boucekkine *et al.* (2025) obtained a similar result but with less constraints than the above Theorem 1. They considered only non-negative constraints: $y(t,\theta) \geq 0$ and $c \geq 0$, thus without the upper-bound constraints.

Furthermore, it is easy to show that the support of the above Radon measure m checks:

Corollary 1.

$$\textit{supp } m \subset \left\{(t,\theta) \in \mathcal{S}_T : y^*(t,\theta) = 0, 1\right\}. \tag{9}$$

To see this, let $\eta \in C(\mathcal{S}_T)$ have the support supp $\eta \subset \mathcal{S}_T \setminus \{(t,\theta) : y^*(t,\theta) = 0, 1\}$. Then, there is $\varepsilon > 0$ such that $z^{\pm} := y^* \pm \varepsilon\eta \in Q$. Hence,

$$\pm\varepsilon\langle m, \eta\rangle = \left\langle m, z^{\pm} - y^*\right\rangle \leq 0.$$

This implies $\langle m, \eta \rangle = 0$. This fact plays an important role in the subsequent analysis.

In the current setting, ν is one Lagrange multiplier, which could be zero, and we shall be more precise when applying this result to the original soil pollution control problem. The Radon measure, m, plays the role of generalized Kuhn-Tucker multipliers for the inequality state constraints, and ψ represents the shadow price, or scarcity value, of relaxing the state constraint at each point in space and time. If m is nonzero at (t, θ), it means that the constraint $y(t, \theta) \leq 1$ or $y(t, \theta) \geq 0$ is binding and relaxing it would improve social welfare. The inequality (6) for all admissible z is analogous to the complementary slackness condition and mirrors the Kuhn-Tucker conditions in finite-dimensional optimization, ensuring that the constraint is only "active" when it binds. Thus, this theorem extends the Pontryagin framework and the Kuhn-Tucker logic to dynamic, spatially distributed systems with PDE constraints.

3 A model of soil pollution

Global soil contamination has escalated to a critical level. The Food and Agriculture Organization (FAO, 2015) estimated that roughly one-third of the world's soils are moderately to severely degraded, primarily due to erosion, nutrient imbalance, salinization, and contamination. More recently, the United Nations Convention to Combat Desertification (UNCCD, 2024) reported that up to 40% of the planet's land is degraded, marked by land abandonment, biodiversity loss, and declining soil health. The latest FAO assessment (2025) further underscores the severity of the crisis, estimating that 2.1 billion hectares—about 23% of global land—are affected by soil pollution, driven largely by human activities: agriculture ($\approx 80\%$) and industrial waste ($\approx 15\%$). Soil pollution not only undermines agricultural productivity and food security, but also poses risks to human health through direct contact, exposure to vapors, or contamination of water supplies. The FAO warns that soil pollution poses "irreversible risks to food security, biodiversity, and human health."

Agricultural pollution arises from the excessive use of pesticides and fertilizers, irrigation with contaminated water, and soil erosion from intensive farming, while industrial pollution arises from improper waste disposal. Crucially, land is a finite resource and pollution is spatially diffusive, affecting surrounding areas. Yet, soils are remediable through methods such as phytoremediation (using plants), bioremediation (using microbes), and soil amendments (adding compost or structural materials).

Given the finiteness of the land, the spatial spread of pollution, and the cost of remediation, the optimal use of land and pollution abatement provide a perfect illustration of our infinite-dimensional optimal control problem with inequality constraint.

We consider a closed economy, where the population is distributed according to a given density N. Following Boucekkine et al. (2013, 2025), we assume that both land and population are distributed over the unit circle on the plane, $\mathcal{S} = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 : \theta \in [-\pi, \pi]\}$.

Let fertile land be the only input that is used to produce crops. The production factor is bounded since land cannot increase beyond the given total land endowment. We further assume that land is composed of both fertile and polluted soil, L_F and L_P. That is, $L = L_P(t,\theta) + L_F(t,\theta)$. All locations produce a unique agricultural good using fertile soil, according to the linear production function $Y(t,\theta) = B(\theta)L_F(t,\theta)$, where $B(\theta)$ is the local production technology in location θ. Hence, a partially polluted location can still produce.

The dynamics of soil pollution at one location is explained by three factors. First, pollution flows according to Fick's law: pollution diffuses from more polluted locations to less polluted locations and its flux is proportional to the pollution gradient. Based on this law the diffusion of soil pollution is captured by $D\frac{\partial^2 L_P}{\partial\theta^2}(t,\theta)$, where D is the diffusion coefficient. For simplicity reasons, D is assumed both constant in time and homogeneous in space.[1] Second, fertile soil deteriorates locally. Indeed, local production generates some pollutant, which transforms fertile soil into polluted soil. The local effect is measured as $\nu(\theta)Y(t,\theta)$, where $\nu(\theta)$ is the local sensitivity of fertile soil to pollution. $\nu(\theta)$ can be related to more or less polluting technologies, to different levels of biodiversity, etc. And third, we assume for simplicity reasons that soil pollution is reversible.[2] Letting $C(t,\theta)$ denote total consumption at location θ at time t, the amount invested in abatement at location θ is $Y(t,\theta) - C(t,\theta) \geq 0$. Let $\phi(\theta)$ be the local pollution abatement efficiency. Then putting together the three factors behind local pollution, the spatial dynamics of polluted soil can be described as

$$\frac{\partial L_P}{\partial t} = D\frac{\partial^2 L_P}{\partial \theta^2} + \nu B L_F - \phi[BL_F - C].$$

[1] We do not consider any seasonal effect nor heterogeneity in soil porosity, which would lead to study time and space dependent diffusion coefficients. These more general specifications for the diffusion coefficient could be analyzed following Boucekkine et al. (2020), but it remains beyond the scope of this paper.

[2] The existence and outreach of a critical zone for pollution reversibility has been widely studied elsewhere. See for instance Dupouey et al. (2002), Chartier et al. (2006), Gao et al. (2011) and Le Kama et al. (2014), among others. Technically speaking, introducing irreversible pollution damages would lead in our context to impose that above a local pollution threshold concentration, the first partial derivative of fertile soils with respect to time should be negative, that is $\frac{\partial L_F(t,\theta)}{\partial t} \leq 0$.

Assuming that L is a constant, one can write

$$\frac{\partial L_F}{\partial t}(t,\theta) = -\frac{\partial L_P}{\partial t}(t,\theta), \quad \frac{\partial^2 L_F}{\partial \theta^2} = -\frac{\partial^2 L_P}{\partial \theta^2},$$

and writing total consumption $C(t,\theta)$ as the product of per capita consumption, $c(t,\theta)$, and the location's time-independent population $N(\theta)$, the evolution of fertile soil becomes

$$\begin{cases} \frac{\partial L_F}{\partial t} = D\frac{\partial^2 L_F}{\partial \theta^2} + B\left[\phi - \nu\right] L_F - N\phi c & \text{for } t > 0, \quad \theta \in (-\pi, \pi), \\ L_F\left(t, -\pi\right) = L_F\left(t, \pi\right), & \\ \frac{\partial L_F}{\partial \theta}\left(t, -\pi\right) = \frac{\partial L_F}{\partial \theta}\left(t, \pi\right), & \text{for } t > 0, \\ L_F(0,\theta) = L_{F,0}\left(\theta\right), & \text{for } \theta \in (-\pi, \pi). \end{cases} \tag{10}$$

In this economy the policy maker aims at maximizing overall welfare, which is measured as the present value of the spatial aggregate of individuals' utility. Here, utility depends solely on consumption per capita, c, and is measured by a constant intertemporal elasticity of substitution function of parameter $\sigma \in \mathbb{R}$. Knowing that the policy maker discounts time at a constant rate ρ, her problem is written as

$$\max_c \int_0^\infty \left[\int_{-\pi}^{\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta\right] e^{-\rho t} dt, \tag{11}$$

subject to (10) and

$$0 \leq L_F(t,\theta) \ \leq \ L, \quad 0 \leq N\left(\theta\right) c(t,\theta) \leq B\left(\theta\right) L_F\left(t,\theta\right) \tag{12}$$

for any $\theta \in [-\pi, \pi]$, $t \geq 0$. The constraint on c comes from the feasibility constraint $0 \leq C \leq Y$. We assume

Assumption 2. *Parameter σ satisfies* $0 < \sigma < 1$.

To shorten the notation and with an abuse of notation, we denote by A and N the expressions $B\left[\phi - \nu\right]$ and $N\phi$. Next, we normalize variables

$$l\left(t,\theta\right) = \frac{L_F\left(t,\theta\right)}{L}, \quad l_0\left(\theta\right) = \frac{L_{F0}\left(\theta\right)}{L},$$

and rename $c(t,\theta)/L$ as $c(t,\theta)$, thus the system (10) takes the form

$$\begin{aligned} l_t &= Dl_{\theta\theta} + Al - Nc \quad \text{for } t > 0, \qquad \theta \in \mathcal{S}, \\ l(0,\theta) &= l_0(\theta) \qquad\qquad\quad \text{for } \theta \in \mathcal{S}. \end{aligned} \tag{13}$$

The objective is reduced to

$$J(l_0, c) = \max_{c(t,\theta)} \int_0^\infty \int_{\mathcal{S}} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) e^{-\rho t} d\theta dt. \tag{14}$$

The admissible set of controls is

$$\mathcal{B}(l) \equiv \left\{ c \in M\left(\mathbb{R}^+ \times \mathcal{S}; \mathbb{R}^+\right), \qquad 0 \le c \le B(\theta) l \quad \text{a.e.} \right\}, \tag{15}$$

where $M(\mathbb{R}^+ \times \mathcal{S}; S)$ is the set of measurable functions in $\mathbb{R}^+ \times \mathcal{S}$ with range in $\mathcal{S}$. The optimization is subject to the state constraint

$$0 \le l(t,\theta) \le 1 \qquad \text{a.e in } (0,\infty) \times \mathcal{S}. \tag{16}$$

4 Maximum principle for soil pollution control

Building on the mathematical foundations laid by Fattorini (1999) and Li and Yong (1991), we provide in Appendix 2 a general maximum principle for the spatial AK model with state constraints in the most general case, and its results are summarized in Theorem 1. In order to apply these new general results to the particular optimization problem (14)–(16), we need first to rewrite the objective function in (14) as the sum of welfare from 0 to a given time T and a continuation function h, which depends on the final state of land, $l(T,\cdot)$. Note that since we can identify h with welfare from T to ∞, these two writings are equivalent:

$$\max_{c\in\mathcal{B}} J(l_0, c) = \int_0^T \int_{\mathcal{S}} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) e^{-\rho t} d\theta dt + h(T, l(T,\cdot)) \tag{17}$$

subject to

$$\begin{aligned} l_t &= Dk_{\theta\theta} + Al - Nc \quad \text{for } (t,\theta) \in \mathcal{S}_T, \\ l(0,\theta) &= l_0(\theta) \qquad\qquad\quad \text{for } \theta \in \mathcal{S}, \end{aligned} \tag{18}$$

and the state constraints $l \in Q$, where

$$Q = \left\{ y \in C(\mathcal{S}_T) : 0 \le l(t,\theta) \le 1 \quad \text{for all } (t,\theta) \in \mathcal{S}_T \right\}. \tag{19}$$

Let us denote this problem by (I). It is straightforward to obtain

Proposition 1. *Let Assumptions 2 hold, and let* $h : \mathbb{R}^+ \times [0,1] \mapsto \mathbb{R}^+$ *be continuously differentiable. Suppose* $\{c^*, l^*\}$ *is an optimal pair for the optimal control problem (I) with the state constraints* $l \in Q$*. Then, there exists a constant* $\nu \in \{0,1\}$*, a function* $p \in L^q\left(0,T; W^{1,q}(\mathcal{S})\right)$ *with* $1 < q < 3/2$*, and a Radon measure* M *such that* $\nu + |M|_{\mathcal{M}(\mathcal{S}_T)} > 0$*,* $\langle M, z - x^* \rangle \leq 0$ *for any* $z \in Q$*,*

$$\begin{aligned} & p_t + D p_{\theta\theta} + (A - \rho)\, p = -M|_{(0,T)\times\mathcal{S}}, \\ & p(T,\cdot) = \nu e^{\rho T} h_l\left(T, l^*(T,\cdot)\right) + M|_{\{T\}\times\mathcal{S}} \end{aligned} \tag{20}$$

and the optimal pair, $\{c^*, l^*\}$*, satisfies* $c^*(t,\theta) = 0$ *if* $\nu = 0$*; and*

$$c^*(t,\theta) = p(t,\theta)^{-1/\sigma} \qquad \textit{for } (t,\theta) \in \mathcal{S}_T \tag{21}$$

if $\nu = 1$*.*

Proof. See Appendix A.2. □

While Theorem 1 in Appendix 2 extends the classical Pontryagin Maximum Principle to a spatially distributed system with inequality state constraints, Proposition 1 illustrates it with the ecological limits to land use. The introduction of a Radon measure as a generalized Kuhn-Tucker multiplier is particularly relevant in environmental economics, where natural resource stocks are bounded and spatially heterogeneous. In the optimal solution, the adjoint variable (p) represents the shadow value of fertile land, while the new Radom measure (M) captures the marginal value of relaxing the constraint on fertile land. In other words, M measures the welfare increase when at least one location has reached the maximum of fertile land and the policy maker decides to sacrifice full fertility.

Notice that if the output and objective functions are smooth enough and the solution does not hit the bounds (i.e., the state constraints are inactive), then the co-state variable, i.e., the shadow value, p, may be continuously differentiable. However, so far, there is no guarantee that the state constraints are always inactive. Furthermore, we will indeed demonstrate that the constraints are binding depending on the circumstance. Therefore, the adjoint variable, can only be defined in the space given in the proposition.

Since the introduction of Radon measures in optimal control problems is new in this field, let us describe how one can characterize M. Let Ω be the interior of the set

$$\overline{\Omega} = \{(t, \theta) \in (0, \infty) \times \mathcal{S} : l^* (t, \theta) = 1\}.$$

We first observe that by (18), $c^* (t, \theta) = A(\theta)$ in Ω. Thus, by (21),

$$p(t, \theta) = A(\theta)^{-\sigma} N(\theta)^{\sigma} \qquad \text{in } \Omega$$

and by (20)

$$M = -\{\psi_t + \mathcal{L}[\psi]\} = -(\mathcal{L} - \rho)\left[A^{-\sigma} N^{\sigma}\right],$$

where $\mathcal{L}$ is the linear operator defined in $H^2(\mathcal{S})$ by

$$\mathcal{L}[u](\theta) := Du''(\theta) + A(\theta) u(\theta). \tag{22}$$

To further perform a mathematical analysis to problem (I), let us introduce the necessary minimum definitions of eigenvalues and eigenfunctions of the linear operators $\mathcal{L}$ defined in (22) and $\mathcal{M}$ defined by

$$\mathcal{M}[u](\theta) = Du''(\theta) + [A(\theta) - N(\theta) B(\theta)] u(\theta) \tag{23}$$

for $u \in H^2(\mathcal{S})$. A function φ defined on $\mathcal{S}$, regular and non-identically zero, is an eigenfunction of $\mathcal{L}$, with associated eigenvalue $\lambda \in \mathbb{R}$ if $\mathcal{L}[\varphi] = \lambda\varphi$. It can be proven that there exists a countable set of eigenvalues $\{\lambda_n\}_{n \geq 0}$, which can be ordered as a decreasing sequence. The first eigenvalue of $\mathcal{L}$, λ_0, is positive and with multiplicity 1, all other eigenvalues have either multiplicity 1 or 2.[3] The eigenfunction associated to λ_0, φ_0, is strictly positive on the unit circle.[4] Let μ_0 denote the largest (principal) eigenvalue of $\mathcal{M}$. It can be shown that $\mu_0 < \lambda_0$ and an eigenfunction ϕ_0 associated with μ_0 is strictly positive. On the other hand, μ_0 can be negative.

The next proposition shows that the constraint $l(t, \theta) \geq 0$ is not binding:

Proposition 2. *Suppose $\{c^*, l^*\}$ is an optimal pair, then $l^*(t, \theta) > 0$ for all (t, θ). Further-*

[3]The multiplicity of an eigenvalue is the number of times it appears in the sequence $\{\lambda_n\}_{n \geq 0}$.

[4]For further details see Coddinton and Levinson (1955) or Brown et al. (2013).

more, there is an $\varepsilon > 0$ such that

$$\int_{\mathcal{S}} l^* (t, \theta)\, d\theta \geq \varepsilon e^{\mu_0 t} > 0 \qquad \textit{for all } t > 0.$$

Proof. See Appendix A.3. □

Since the constraint $l(t, \theta) \geq 0$ is not binding, from Corollary 1 of Theorem 1 in the Appendix, we can conclude that

$$\text{supp } M \subset \{(t, \theta) \in \mathcal{S}_T : l^* (t, \theta) = 1\}. \tag{24}$$

Proposition 2 shows that the lower bound on fertile land is never binding in the optimal solution, implying that full degradation (i.e., complete pollution) is socially suboptimal. This result aligns with the intuition that even under high pressure for consumption, a forward-looking planner will preserve some fertility to maintain future production capacity.

This finding resonates with the literature on renewable resource management (e.g., Dasgupta and Heal, 1979), where extinction or full depletion is typically avoided under rational planning. In our spatial setting, the result also reflects the self-reinforcing nature of remediation: as long as some fertile land remains, diffusion and abatement can restore degraded areas. The exponential growth of aggregate fertility further supports the idea that patient, coordinated policy can reverse environmental degradation, even in the presence of local pollution spillovers.

5 Optimal solutions

The optimal solution to the policy maker problem (I) depends on the discount rate and we can fully characterize it in some cases. In all others, we can at least describe the long-run optimal behavior of fertile land. For simplicity of exposition, we divide the spectrum of values for $\rho \in [0, 1]$, in three categories: low, mildly high and high.

5.1 Small time discount.

The time discount is said to be small if

$$0 < \rho < \lambda_0. \tag{25}$$

In this case we prove

Proposition 3. *Let Assumption 2 hold. Suppose that (25) holds and that* $\{c^*, l^*\}$ *is an optimal pair such that*

$$0 < c^*(t,\theta) < B(\theta)\, l^*(t,\theta) \qquad \textit{for all } t > 0, \quad \theta \in \mathcal{S}, \tag{26}$$

that is, there is no time and nowhere at which the decision maker consumes all that is produced. Then, $l^*(t,\theta) = 1$ *for some* $\theta \in \mathcal{S}$*. In other words, fertile land reaches its maximum level somewhere in some time.*

Proof. See Appendix A.4. □

Hence, when future welfare is sufficiently valued, the optimal policy leads to full restoration of fertile land in at least some locations. In this case, consumption is moderated to allow for environmental recovery. This behavior is consistent with the "green golden rule" in Chichilnisky et al. (1995), where sustainability is achieved by balancing current utility with future resource availability. The spatial dimension we introduce here adds a new layer: even if fertility restoration is not full nor uniform, achieving it somewhere prevents the collapse of the entire system. Worth to note, these results echo the concept of "ecological resilience" (Holling, 1973), where partial recovery can stabilize the broader system.

In the special case where the system is spatially homogeneous, more precise results can be obtained. Indeed, if A is a constant, the eigenvalues of $\mathcal{L}$ are

$$\lambda_n = A - Dn^2 \qquad \text{for } n = 0, 1, 2, \ldots \tag{27}$$

with the corresponding normalized eigenfunctions:

$$\varphi_0(\theta) = \frac{1}{\sqrt{2\pi}}, \quad \varphi_{n,1}(\theta) = \frac{\cos n\theta}{\sqrt{\pi}}, \qquad \varphi_{n,2}(\theta) = \frac{\sin n\theta}{\sqrt{\pi}} \qquad \text{for } n \geq 1. \tag{28}$$

In particular, $\lambda_0 = A$.

We have the following long-run outcomes:

Proposition 4. *Let the assumptions of Proposition 3 hold. If* A*,* B *and* N *are constants in* $\mathcal{S}$*, then the unique steady state is* $\bar{l}(\theta) = 1$ *and* $\bar{c}(\theta) = A/N$ *in* $\mathcal{S}$*. As a result, if an optimal* *pair $\{c^*, l^*\}$ converges as $t \to \infty$, then*

$$\lim_{t\to\infty} l^*(t,\theta) = 1, \qquad \lim_{t\to\infty} c^*(t,\theta) = A/N. \tag{29}$$

Proof. See Appendix A.5. □

As in the economy made of a unique location, Proposition 4 proves that a spatially homogeneous economy made of patient agents converges to a steady state with full restoration of fertile land. This mirrors the golden rule in AK-type growth models, where the economy reaches a balanced growth path with maximal sustainable consumption. This result will allow to evaluate the impact of spatial heterogeneity: deviations from full fertility will reflect the cost of uneven land quality, localized pollution, and differential abatement efficiency.

The transitional period. Arguably, the above two propositions are obtained under the assumption that an optimal pair (c^*, l^*) exists, but provide no information on how to obtain this optimal pair. We show next that the entire optimal trajectory can be computed when the optimal solution leads to full fertility in finite time everywhere, i.e., there is $T > 0$ such that $l^*(t,\theta) = 1$ for all $t \geq T$, $\theta \in \mathcal{S}$. Let us compute c^* and l^* for $0 < t < T$ as follows.

Since $A(\theta)/N(\theta) < B(\theta)$ and $l^*(t,\theta) = 1$, it follows that

$$\frac{A(\theta)}{N(\theta)} = \underset{0\leq c\leq \mathcal{B}(k^*)}{\arg\max} \left\{ \frac{\nu e^{-\rho t}}{1-\sigma} c(t,\theta)^{1-\sigma} - c(t,\theta)\, e^{-\rho t} p(t,\theta) \right\}.$$

It is clear that $\nu > 0$. Since ν is either 0 or 1, it must be $\nu = 1$. This leads to

$$A(\theta)/N(\theta) = p(t,\theta)^{-1/\sigma}, \ \text{ or } \ p(t,\theta) = A(\theta)^{-\sigma} N(\theta)^{\sigma} \qquad \text{in } \mathbb{R}^+ \times \mathcal{S}.$$

Hence, $\{c^*, k^*\}$ in $\mathcal{S}_T$ can be found by solving the coupled system

$$p_t + (\mathcal{L} - \rho)[p] = \begin{cases} -M & \text{if } p^{-1/\sigma} < Bl^*, \\ NB\left\{p - [Bk^*]^{-\sigma}\right\} - M & \text{if } p^{-1/\sigma} \geq Bl^*, \end{cases} \quad \text{in } \mathcal{S}_T,$$
$$p(T,\theta) = A(\theta)^{-\sigma} N(\theta)^{\sigma} \qquad \text{in } \mathcal{S},$$

and

$$l^*_t - \mathcal{L}[l^*] = \begin{cases} -Np^{-1/\sigma} & \text{if } p^{-1/\sigma} < Bl^*, \\ -NBl^* & \text{if } p^{-1/\sigma} \geq Bl^*, \end{cases} \quad \text{in } \mathcal{S}_T,$$
$$l^*(0,\theta) = l_0(\theta) \qquad \text{in } \mathcal{S}.$$

Proposition 3 ensures that, provided that the policy maker is sufficiently patient and prioritizes aggregate social welfare, excessive consumption can be avoided at all times. Consequently, there will be at least some regions where fertile land reaches its maximum productive capacity. Furthermore, Proposition 4 shows that in the long run, and at least in the spatially homogeneous case, fertile land can be fully restored in all areas.

5.2 Large time discount.

We consider next the case where the discount rate is relatively large

$$\rho > \lambda_0 \,(> 0)\,. \tag{30}$$

Let us first construct a special pair $\{\hat{c}, \hat{k}\}$ for problem (13)–(14) with

$$\hat{c}\,(t,\theta) = \hat{c}_0 \varphi_0\,(\theta)^{-1/\sigma}\, e^{rt} \qquad \text{for } \,(t,\theta) \in (0,\infty) \tag{31}$$

where $r = \frac{\lambda_0 - \rho}{\sigma} < 0$,

$$\hat{c}_0 = \frac{(\lambda_0 - r)\,\langle k_0, \varphi_0\rangle}{\left\langle N{\varphi_0}^{-1/\sigma}, \varphi_0\right\rangle}, \tag{32}$$

and where $\hat{l}$ solves

$$\begin{aligned} &\hat{l}_t - D\hat{l}_{\theta\theta} = A\hat{l} - N\hat{c}\,(t,\theta) && \text{in } \,(0,\infty)\times\mathcal{S}, \\ &\hat{l}\,(0,\theta) = l_0\,(\theta) && \text{in } \mathcal{S}. \end{aligned} \tag{33}$$

This pair describes a monotonically decreasing trajectory for c at all locations, consumption is forever heterogeneous and it decreases at the same rate everywhere. Note that $\{\hat{c}, \hat{l}\}$ may not be feasible, or even less be optimal, since the control constraint $\hat{c} \in \mathcal{B}(\hat{l})$ and the state constraint (16) may fail. However, the next proposition shows that when the optimal pair $\{c^*, l^*\}$ satisfies the strict control and state constraints, that is, if

$$0 < c^*\,(t,\theta) < B\,(\theta)\, l^*\,(t,\theta)\,, \qquad 0 < l^*\,(t,\theta) < 1 \qquad \text{in } \,(0,\infty)\times\mathcal{S}, \tag{34}$$

then it coincides with $\{\hat{c}, \hat{l}\}$.

Proposition 5. *Let Assumption 2 hold. Suppose that (30) holds and that the pair* $\{\hat{c}, \hat{l}\}$ *defined by (31)–(33) is feasible. If* $\{c^*, l^*\}$ *is an optimal pair such that (34) holds, then* $\{c^*, l^*\} = \{\hat{c}, \hat{l}\}$.

Proof. See Appendix A.6. □

As in the previous subsection, we start by studying the asymptotic behavior of the optimal solution and then, the transitional period.

Proposition 6. *Suppose the assumptions of Proposition 5 are satisfied and* $\{\hat{c}, \hat{l}\}$ *is feasible. Then*

$$\lim_{t\to\infty} \hat{l}\,(t,\theta)\, e^{-rt} = \kappa\,(\theta) \tag{35}$$

where

$$\kappa\,(\theta) = \langle l_0, \varphi_0\rangle\, \varphi_0\,(\theta) + \sum_{j\geq 1}\sum_{i} \frac{\hat{c}_0}{\lambda_j - r} \left\langle N\varphi_0^{-1/\sigma}, \varphi_{j,i}\right\rangle \varphi_{j,i}\,(\theta) \tag{36}$$

if $\lambda_j \neq r$ *for all* j*, and*

$$\begin{aligned}\kappa\,(\theta) = \langle l_0, \varphi_0\rangle\, \varphi_0\,(\theta) &+ \sum_{i=1}^{2} \langle l_0, \varphi_{m,i}\rangle\, \varphi_{m,i}\,(\theta) \\ &+ \sum_{j\neq m}\sum_{i} \frac{\hat{c}_0}{\lambda_j - r} \left\langle N\varphi_0^{-1/\sigma}, \varphi_{j,i}\right\rangle \varphi_{j,i}\,(\theta)\end{aligned} \tag{37}$$

if there is m *such that* $\lambda_m = r$ *and*

$$\left\langle N\varphi_0^{-1/\sigma}, \varphi_{m,i}\right\rangle = 0 \qquad \text{for } i = 1, 2.$$

Proof. See Appendix A.7. □

Hence, under high discounting, fertile land converges to a spatially structured distribution. This result echoes the insights from spatial growth models such as Boucekkine et al. (2013), where long-run spatial patterns emerge with time from initial heterogeneity. Also note that this later result aligns with the idea that impatient societies may underinvest in remediation, leading to persistent inequality in land productivity. This is reminiscent of the literature on spatial poverty traps (e.g., Redding and Rossi-Hansberg, 2017), where local disadvantages are perpetuated due to insufficient forward-looking investment.

Corollary 2. *In the case where* A *is a constant function,*

$$\lim_{t\to\infty} \hat{l}\,(t,\theta)\, e^{-rt} = \frac{1}{2\pi}\int_{\mathcal{S}} l_0\,(\xi)\, d\xi \tag{38}$$

if $r \neq \lambda_j$ *for all* j*, and*

$$\lim_{t\to\infty} \hat{l}\,(t,\theta)\, e^{-rt} = \frac{1}{2\pi}\int_{\mathcal{S}} l_0\,(\xi)\, d\xi + \frac{1}{\pi}\int_{\mathcal{S}} l_0\,(\xi) \left[\frac{1}{2} + \cos\left(m\,(\theta - \xi)\right)\right] d\xi \tag{39}$$

if there is an $m \geq 1$ *such that* $r = \lambda_m$.

Proof. See Appendix A.8. □

The Corollary 2 shows the different spatial pattern of fertility in the long run in the case where A is spatially uniform and $\{\hat{c}, \hat{l}\}$ is feasible. When A is constant, then $\lambda_0 = A$ and we can discuss two particular cases. When

$$A < \rho < (1 - \sigma) A + \sigma D, \tag{40}$$

then $r > \lambda_1$. In this case, we are in case (38) so that the detrended fertile land, $\hat{l}(t, \theta) e^{-rt}$ converges to a constant and the initial spatial variation of the fertility vanishes with time. On the other hand, if

$$A < \rho = (1 - \sigma) A + \sigma D, \tag{41}$$

which is equivalent to $r = \lambda_1$, then according to (39), detrended fertile land converges in general to a non-constant function if l_0 is not constant, and initial fertility inequality will generally persist forever.

Finally, let us consider the case of extreme high impatience, where

$$\rho > \lambda_0 - \sigma\mu_0 = (1 - \sigma) A + \sigma D. \tag{42}$$

Proposition 7. *Let Assumption 2 and (42) hold. Then,* $\{\hat{c}, \hat{l}\}$ *is not an optimal pair for the optimal control problem (13), (14) with* $\hat{c} \in \mathcal{B}(\hat{l})$*. In addition, any optimal pair* $\{c^*, l^*\}$ *has the feature that* $c^*(t, \theta) = B(\theta) l^*(t, \theta)$ *for some* (t, θ).

Proof. See Appendix A.9. □

Proposition 7 demonstrates that when the discount rate exceeds a critical threshold (here $\rho > \lambda_0 - \sigma\mu_0$), the constructed pair $(\hat{c}, \hat{l})$ is no longer optimal, implying that full consumption occurs at some points.

This finding resonates with the classic debate on the social discount rate in climate economics (see Stern, 2008; Dasgupta, 2008), where high discounting undermines sustainability. In our model, the binding of the consumption constraint reflects a shift from an interior solution to a corner solution. Hence under high discounting, the optimal policy may involve full consumption of production, aggressive exploitation of fertile land, and limited or no abatement.

Transitional period. In the case of high discounting, the pair $\{\hat{c}, \hat{l}\}$ is not feasible in the long run, which means that along the optimal trajectory there will be times and locations with full consumption, and the associated zero abatement. However, even if full consumption happens in at least one location during some time, we can prove that there exists a solution which will converge at time T to an structured solution $\{c^*, l^*\}$ with positive abatement and fertile land everywhere, and this despite the lack of abatement in some locations before that. Let us define this solution:

$$0 < c^*(t,\theta) < B(\theta)\, l^*(t,\theta), \qquad 0 < l^*(t,\theta) < 1 \qquad \text{in } (T,\infty)\times\mathcal{S} \tag{43}$$

for some $T > 0$. In this case, by a time-translation, $\{c^*, l^*\} = \{\hat{c}_T, \hat{l}_T\}$ where

$$\hat{c}_T(t,\theta) = \hat{c}_{T,0}^{-1/\sigma}\varphi_0(\theta)^{-1/\sigma} e^{r(t-T)} \qquad \text{for } t \geq T, \quad \theta\in\mathcal{S} \tag{44}$$

with

$$\hat{c}_{T,0} = \frac{(\lambda_0 - r)\langle l^*(T,\cdot), \varphi_0\rangle}{\langle N\varphi_0{}^{-1/\sigma}, \varphi_0\rangle} \tag{45}$$

and $\hat{l}_T$ is the solution to the initial value problem

$$\begin{aligned} &l_t - Dx_{\theta\theta} = Al - N\hat{c}_T(t,\theta) && \text{in } (T,\infty)\times\mathcal{S}, \\ &l(T,\theta) = l^*(T,\theta) && \text{in } \mathcal{S} \end{aligned} \tag{46}$$

provided that $\{\hat{c}_T, \hat{l}_T\}$ is feasible in $(T,\infty)\times\mathcal{S}$.

In this case, the solution $\{c^*, l^*\}$ in the initial period $0 < t < T$ can be solved by a finite-horizon optimal control problem. Specifically, we prove

Proposition 8. *Let Assumptions 2 and 1 hold. Suppose that $\rho > \lambda_0$ and that $\{c^*, l^*\}$ is an optimal pair such that (43) for some $T > 0$ holds. Suppose also $\left\{\hat{c}_T, \hat{l}_T\right\}$ defined by (44)–(46) is feasible in $(T,\infty)\times\mathcal{S}$. Then $p(t,\theta)$ satisfies the terminal value problem*

$$\begin{aligned} &p_t + (\mathcal{L} - \rho)[p] = -M|_{(0,T)\times\mathcal{S}} && \textit{for } (t,\theta)\in\mathcal{S}_T, \\ &p(T,\theta) = \hat{c}_{T,0}^{-\sigma}\varphi_0(\theta) && \textit{for } \theta\in\mathcal{S} \end{aligned} \tag{47}$$

where M is a Radon measure such that and $\langle M, z - l^*\rangle \leq 0$ for any $z\in Q$, and v^* satisfies

$$c^*(t,\theta) = \underset{0\leq c\leq\mathcal{B}(l^*)}{\arg\max}\left\{\frac{c(t,\theta)^{1-\sigma}}{1-\sigma} - c(t,\theta)\,p(t,\theta)\right\}.$$

Finally, l^* in $\mathcal{S}_T$ is the solution of (18) with $c = c^*$.

Proof. See Appendix A.10. □

Proposition 8 introduces a hybrid regime where the system transitions from a finite-horizon control problem to a structured long-run regime. This mirrors the concept of "policy switching" in dynamic economic models, where optimal strategies evolve over time in response to changing feasibility conditions. The use of a Radon measure to capture binding constraints during the initial phase is particularly novel, extending the Kuhn-Tucker logic to spatiotemporal systems.

Economically, this result suggests that even under high discounting, a well-designed policy can steer the system toward sustainability, provided that initial sacrifices are made. This echoes the literature on transitional dynamics in environmental economics (e.g., Acemoglu et al., 2012), where short-term costs are justified by long-term gains. Our model formalizes this intuition by showing how the adjoint system governs the optimal trajectory, and how the decision maker's patience (or lack thereof) determines whether the system converges to a viable regime.

5.3 Summary of different time discount

Discount Regime	**Key Conditions**	**Optimal Behavior**	**Long-Run Outcome**	**Analytical Tractability**
Low	$0 < \rho < \lambda_0$	Policymaker avoids excessive consumption; the optimal path remains interior.	Fertile land reaches full capacity at some locations; full restoration in the homogeneous case.	Analytical solution is difficult; explicit solution possible in the homogeneous case.
Intermediate (feasible after T)	$\rho > \lambda_0$, after $T > 0$	Initial phase governed by finite-horizon control, followed by a transition to a structured regime.	Detrended fertility converges to a stable spatial distribution.	Hybrid approach combining finite-horizon analysis with an explicit long-run solution.
High	$\rho > \lambda_0 - \sigma\mu_0$	Full consumption occurs at some points; the constructed pair $\{\hat{c}, \hat{k}\}$ is not optima.l	No convergence to a structured regime; the optimal path exhibits boundary behavior.	Explicit analytical solution is not feasible; the optimal path must be computed numerically.

Table 1: Comparison of optimal behavior across discount rate regimes.

Table 1 synthesizes the core findings of our analysis, highlighting how the discount rate fundamentally shapes the trajectory and feasibility of optimal land use and consumption strategies. In the low discount rate regime, patient policymaking enables interior solutions that avoid excessive consumption and allow for full or partial restoration of fertile land. This regime supports long-term sustainability, especially in homogeneous settings where explicit solutions are attainable.

The intermediate regime, where feasibility is achieved only after a finite time, introduces a hybrid structure: an initial adjustment phase governed by finite-horizon control, followed by convergence to a structured long-run regime. This underscores the importance of transitional dynamics and the role of initial conditions in shaping outcomes.

In contrast, the high discount rate regime reveals the limits of sustainability. Here, short-term optimization may lead to boundary behavior, including full resource exploitation in some regions. The constructed benchmark pair becomes infeasible, and numerical methods are required to characterize the optimal path. This regime illustrates the risk of irreversible degradation when time preferences heavily favor the present.

6 Numerical experiments

We develop next some numerical exercises to illustrate our results and shed light on some of the remaining open questions. In particular, we illustrate how taking into account that the economy operates with bounded production factors does change the optimal dynamics of the economy.

We choose choose the following parameter values

A	Technological level	0.04
B	Maximum consumption coefficient	0.12
σ	Household preference	0.5
D	Diffusion coefficient	0.1
N	Population density	1.0

Table 2: Parameter values.

We also choose

$$l_0 = \begin{cases} 0 & \text{if } -\pi/2 < x < \pi/2, \\ 1 & \text{elsewhere.} \end{cases}$$

The choice of the time discount rate has always been a delicate issue in the literature (see Stern, 2008, or Fleurbaey and Zuber, 2012, among others). We have additionally shown throughout the paper that the choice of the time discount rate not only determines quantitatively optimal consumption, but most importantly, it drives the economy dynamics qualitatively as well. Under the calibration in Table 2

$$(1-\sigma)\,A+\sigma D=0.07<0.08=(1-\sigma)\,A+\sigma NB.$$

Hence, the upper bound of the moderate time discount rate is 7%. In our exercises, we choose ρ to be 3%, 5%, and 7%. $\rho=0.03$ satisfies (25) and as such, represents the low discount regime. The second, $\rho=0.05$, satisfies (40) and is a high intermediate value. Finally, $\rho=0.07$ is a high value for the discount since it satisfies (41).

6.1 Small time discount, $\rho = 3\%$

In this case, $r=\frac{A-\rho}{\sigma}=0.02$. Figure 1 shows our results for fertile land distribution and consumption per capita. Our results reveal that when the time discount is small, the optimal trajectory for consumption allows fertile land to reach the maximum level at all locations in finite time. Once the maximum level for fertility is reached everywhere, consumption remains at the level that ensures maximum fertility forever.

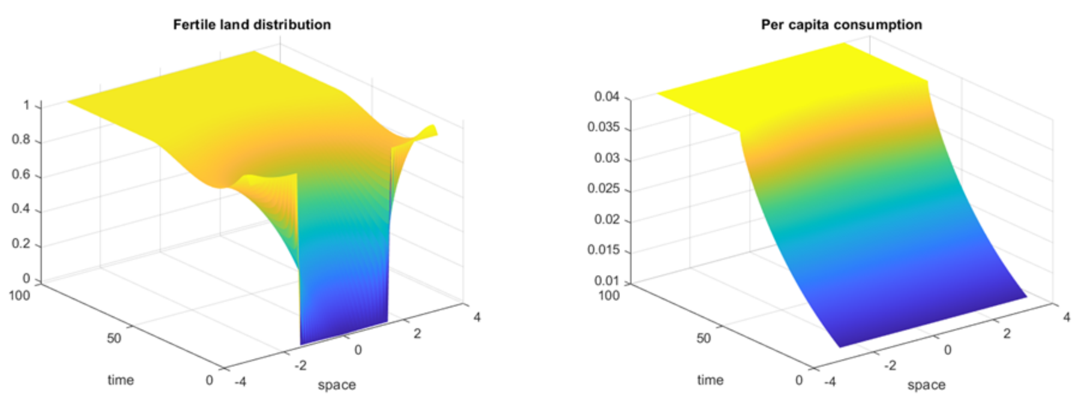

Figure 1: Low discount rate, $\rho = 3\%$. Left: Fertile land. Right: Consumption per capita.

6.2 Moderate time discount, $\rho = 5\%$

In this case (30) holds and $r=-0.02$. When we construct the pair $\{\hat{c},\hat{k}\}$ defined by (31)–(33), we find that fertile land does become negative for some (t,θ). This means that $\{\hat{c},\hat{l}\}$ is not feasible. Instead of using this non feasible solution, we construct a pair $\{c^*, l^*\}$ with

$$c^*(t,\theta) = \begin{cases} 0.035e^{rt} & \text{if } 0.035e^{rt} < Bl^*(t,\theta), \\ Bl^*(t,\theta) & \text{elsewhere.} \end{cases}$$

Then we obtain l^* by solving (13) with c replaced by c^*. The resulting c^* is feasible and it belongs to $\mathcal{B}(l^*)$. Besides, fertile land is always and everywhere positiive, that is, $l^*(t,\theta) > 0$ for all (t,θ). In this case l^* decreases exponentially at rate r. Figure 2 shows the evolution of fertile land and consumption per capita.

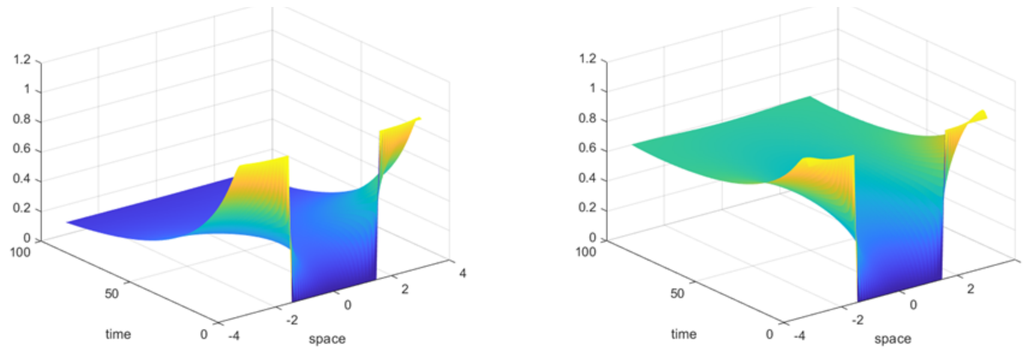

Figure 2: Moderate time discount rate with $\rho = 5\%$. Left: Fertile land. Right: Consumption per capita.

6.3 Large time discount, $\rho = 7\%$

In this case $r = -0.06 = \lambda_1$. By Proposition 6, fertile land converges to a spatially heterogeneous distribution (see Corollary 2) as shown in Figure 3.

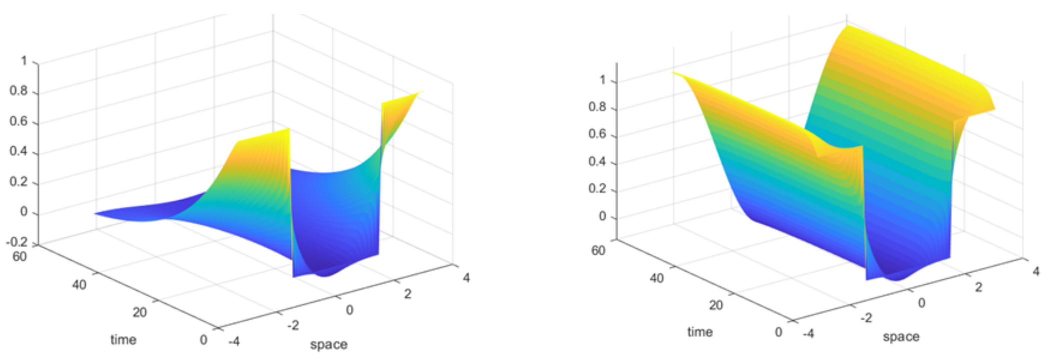

Figure 3: Large time discount rate, $\rho = 7\%$. Left: Fertile land. Right: Consumption per capita.

7 Conclusion

This paper develops a general optimal control framework for diffusion-driven systems with inequality state constraints. By extending the Pontryagin Maximum Principle to incorporate Radon-measure multipliers, we provide a rigorous characterization of optimality in infinite-dimensional environments where the state variable is bounded and evolves according to a partial differential equation. The resulting necessary conditions unify classical optimal control with modern tools from functional analysis and offer a transparent interpretation of boundary behavior in distributed systems.

To illustrate the applicability of the framework, we examined a model in which fertile land evolves through reversible pollution and spatial diffusion. The example highlights how discounting governs whether the optimal trajectory remains interior, reaches the upper bound of the state constraint, or generates persistent spatial heterogeneity. In the homogeneous case, explicit solutions reveal sharp thresholds separating full restoration from partial degradation, while heterogeneous settings give rise to hybrid dynamics combining finite-horizon control with structured long-run regimes.

Although the environmental application provides a concrete setting, the methodological results apply broadly to dynamic optimization problems with bounded states and diffusion, including models of spatial growth, technological propagation, epidemiological dynamics, and renewable resource management. The framework thus offers analytical tools for studying a wide class of infinite-dimensional control problems where state constraints play a central role.

Future work may extend the analysis to stochastic diffusion, non-reversible state dynamics, or strategic interactions in spatial games. These directions would further expand the scope of the framework and deepen our understanding of constrained dynamics in distributed economic systems.

A Appendix

A.1 Proof of Theorem 1

Since this Theorem is similar to Proposition 1 in Boucekkine *et al.* (2025), the proof is also a simple modification of the corresponding proof.

We introduce the new control, v, such that

$$c(t,\theta) = v(t,\theta)\, l(t,\theta).$$

The control constraint that $c \in \mathcal{B}(kl)$ is then changed to

$$0 \le v(t,\theta) \le B(\theta). \tag{48}$$

That is, $v \in \mathcal{B}(1)$. We denote $\mathcal{B}(1)$ by $\mathcal{B}$. With this control, (2) takes the form

$$\begin{cases} y_t - Dy_{\theta\theta} = [a+bv]\, y & \text{for } (t,\theta) \in (0,T) \times \mathcal{S}, \\ y(0,\theta) = y_0(\theta) & \text{for } \theta \in \mathcal{S} \end{cases} \tag{49}$$

and (4) becomes

$$J(y_0, v) = \int_0^T \int_{\mathcal{S}} g(t,\theta, v(t,\theta)\, y(t,\theta))\, d\theta dt + h(T, y(T,\cdot)).$$

Define the Hamiltonian

$$H(t,\theta,y,v,\nu,\psi) = \nu g(t,\theta,vy) + \psi\, [a(t,\theta) + b(t,\theta)\, v]\, y \qquad \text{in } \mathcal{S}_T \times [0,1] \times \mathcal{B}.$$

The main idea is using Ekeland's variational principle (Theorem 3.2.2 in H. Fattorini, 1999) together with a spike perturbation. For any $\varepsilon > 0$ we define

$$F_\varepsilon(v) = \left\{ [J(v) - J(v^*) + \varepsilon]_+^2 + d_0(y(\cdot; v))^2 \right\}^{1/2} \tag{50}$$

where $y(\cdot; v)$ is the solution to (2) corresponding to the control v and

$$d_0(y) = \operatorname{dist}(y, Q)|_{C(\mathcal{S}_T)} \quad \text{for any } y \in C(\mathcal{S}_T). \tag{51}$$

It is clear that $d_0(y) = 0$ if $y \in \overline{Q}$. In addition, d_0 is the GÃ¢teaux differentiable at every $y \in C(\mathcal{S}_T) \backslash Q$, and its GÃ¢teaux derivative, $\nabla d_0(y)$ is the same as the Clarke's generalized gradient, which is convex and weak*-compact. As a result,

$$|\nabla d_0(y)|_{\mathcal{M}(\mathcal{S}_T)} = 1 \qquad \text{if } y \notin Q \tag{52}$$

and for any $\xi \in \partial d_0(y)$,

$$\langle \xi, z-y\rangle + d_0(y) \leq d_0(z) \qquad \text{for any } z \in C(\mathcal{S}_T),$$

where $\mathcal{M}(\mathcal{S}_T)$ is the set of all Radon measures on $\mathcal{S}_T$. Then,

$$F_\varepsilon(v^*) = \varepsilon \leq \inf F_\varepsilon(v) + \varepsilon.$$

This means v^* is an ε-minimum of F_ε, which is bounded below and semi-lower continuous. Hence, by Ekeland's variational principle, there exists $v^\varepsilon \in \mathcal{U}$ such that

$$F_\varepsilon(v^\varepsilon) \leq F_\varepsilon(v^*), \qquad d(v^\varepsilon, v^*) \leq \sqrt{\varepsilon} \tag{53}$$

where

$$d(u,v) = |\{(t,\theta) \in \mathcal{S}_T | u(t,\theta) \neq v(t,\theta)\}| \tag{54}$$

is the Ekeland distance, and $|\Omega|$ for a Lebesgue measurable set Ω represents its measure. In addition,

$$F_\varepsilon(v^\varepsilon) - F_\varepsilon(v) \leq \sqrt{\varepsilon} d(v^\varepsilon, v) \qquad \text{for any } v \in \mathcal{U}. \tag{55}$$

Let $y^\varepsilon = y(\cdot; v^\varepsilon)$. Fix a $v \in \mathcal{U}$ and an $\varepsilon > 0$. For any $\delta > 0$ and a measurable set $E^\varepsilon_\delta \subset \mathcal{S}_T$ we construct the perturbation

$$v^\varepsilon_\delta(t,\theta) = \begin{cases} v^\varepsilon(t,\theta) & \text{for } (t,\theta) \in \mathcal{S}_T \backslash E^\varepsilon_\delta, \\ v(t,\theta) & \text{for } (t,\theta) \in E^\varepsilon_\delta. \end{cases} \tag{56}$$

It is clear that $v^\varepsilon_\delta \in \mathcal{U}$. Let $y^\varepsilon_\delta = y(\cdot; v^\varepsilon_\delta)$ denote the state corresponding to the perturbation.

We need the following lemma.

Lemma 1. *Let Assumption 1 hold and let $y_0 \in C^\alpha(\mathcal{S})$ for some $\alpha \in (0,1)$. Let $\{\bar{v}, \bar{y}\}$ be a feasible pair and let $v \in \mathcal{U}$ be fixed. Then, for any $\delta \in (0,1)$ there exists a measurable set $E_\delta \subset \mathcal{S}_T$ and the control v_δ defined by*

$$v_\delta(t,\theta) = \begin{cases} \bar{v}(t,\theta) & \textit{if } (t,\theta) \in \mathcal{S}_T \backslash E_\delta, \\ v(t,\theta) & \textit{if } (t,\theta) \in E_\delta, \end{cases} \tag{57}$$

such that $|E_\delta| = \delta\,|\mathcal{S}_T|$ *and the following hold:*

$$y\left(\cdot, v_\delta\right) = \bar{y}\left(\cdot\right) + \delta z\left(\cdot\right) + o\left(\delta\right), \qquad J\left(v_\delta\right) = J\left(\bar{v}\right) + \delta l + o\left(\delta\right) \tag{58}$$

(with the first $o\left(\delta\right)$ *is in space* $C^{\alpha,\alpha/2}\left(\mathcal{S}_T\right)$ *for some* $\alpha \in (0,1)$*) where* z *and* l *satisfy*

$$\begin{aligned} z_t - Dz_{\theta\theta} &= \left[a\left(t,\theta\right) + b\left(t,\theta\right)\bar{v}\left(t,\theta\right)\right]z + \phi\left(t,\theta\right), \\ z\left(0,\theta\right) &= 0 \end{aligned}$$

and

$$\begin{aligned} l &= \int_0^T \int_{\mathcal{S}} \left[g_c\left(t,\theta,\bar{y}\left(t,\theta\right)\bar{v}\left(t,\theta\right)\right)\bar{v}\left(t,\theta\right)z\left(t,\theta\right) + \gamma\left(t,\theta\right)\right]d\theta dt \\ &\quad + \int_{\mathcal{S}} h_y\left(T,\bar{y}\left(T,\theta\right)\right)z\left(T,\theta\right)d\theta, \end{aligned}$$

respectively,

$$\begin{aligned} \phi\left(t,\theta\right) &= b\left(t,\theta\right)\bar{y}\left(t,\theta\right)\left[v\left(t,\theta\right) - \bar{v}\left(t,\theta\right)\right], \\ \gamma\left(t,\theta\right) &= g\left(t,\theta,\bar{y}\left(t,\theta\right)v\left(t,\theta\right)\right) - g\left(t,\theta,\bar{y}\left(t,\theta\right)\bar{v}\left(t,\theta\right)\right). \end{aligned}$$

Proof. By the definition of v_δ in (57), $d\left(v_\delta, \bar{v}\right) \leq |E_\delta|$. Let

$$z_\delta\left(t,\theta\right) = \frac{1}{\delta}\left[y\left(t,\theta;v_\delta\right) - \bar{y}\left(t,\theta\right)\right] \qquad \text{in } \mathcal{S}_T.$$

Then, z_δ satisfies

$$\begin{aligned} \left(z_\delta\right)_t - D\left(z_\delta\right)_{\theta\theta} &= b_\delta\left(t,\theta\right)z_\delta\left(t,\theta\right) + \frac{1}{\delta}\chi_{E_\delta}\left(t,\theta\right)\phi\left(t,\theta\right) \qquad \text{if } t \in (0,T), \theta \in \mathcal{S}, \\ z_\delta\left(0,\theta\right) &= 0 \qquad \text{if } \theta \in \mathcal{S}, \end{aligned}$$

where

$$b_\delta\left(t,\theta\right) = \int_0^1 \left[a\left(t,\theta\right) + b\left(t,\theta\right)v_\delta\left(t,\theta\right)\right]ds,$$

and χ_{E_δ} is the characteristic function of E_δ. From the regularity of the parabolic equation (49) and Assumption 1, we see that b_δ and ϕ are uniformly bounded. Hence, by the HÃ¶lder's estimate there is $\alpha \in (0,1)$ such that

$$\left|y\left(\cdot, v_\delta\right) - \bar{y}\right|_{C^{\alpha,\alpha/2}(\mathcal{S}_T)} \le C \left|\chi_{E_\delta}\right|_{L^p(\mathcal{S}_T)} \to 0 \qquad \text{as } \delta \to 0,$$

where the constant C is independent of E_δ. As a result,

$$b_\delta\left(t,\theta\right) \to a\left(t,\theta\right) + b\left(t,\theta\right)\bar{v}\left(t,\theta\right) \qquad \text{in } L^p\left(\mathcal{S}_T\right) \text{ as } \delta \to 0$$

for any p such that $1 \le p < \infty$. Comparing equations for z_δ and z we derive

$$\begin{aligned}
\left(z_\delta - z\right)_t - D\left(z_\delta - z\right)_{\theta\theta} &= b_\delta\left(t,\theta\right)\left(z_\delta - z\right) + \left\{b_\delta\left(t,\theta\right) - \left[a\left(t,\theta\right) + b\left(t,\theta\right)\bar{v}\left(t,\theta\right)\right]\right\} z \\
&\quad - \left(1 - \frac{1}{\delta}\chi_{E_\delta}\left(t,\theta\right)\right)\phi\left(t,\theta\right), \\
\left(z_\delta - z\right)\left(0,\theta\right) &= 0.
\end{aligned}$$

Using Lemma 3.2 in B. Hu and J. Yong (1995), we see that $\left|z_\delta - z\right|_{C^{\alpha,\alpha/2}(\mathcal{S}_T)} \to 0$ as $\delta \to 0$. This proves the first relation in (58).

To prove the second relation in (58), we let

$$\begin{aligned}
l_\delta &= \frac{1}{\delta}\left[J\left(v_\delta\right) - J\left(\bar{v}\right)\right] \\
&= \frac{1}{\delta}\left\{\int_0^T \int_{\mathcal{S}} \left[g\left(t,\theta,y\left(t,\theta\right)v_\delta\left(t,\theta\right)\right) - g\left(t,\theta,\bar{y}\left(t,\theta\right)\bar{v}\left(t,\theta\right)\right)\right] d\theta dt \right. \\
&\quad \left. + \left[h\left(T, y\left(T,\cdot,v_\delta\left(T,\cdot\right)\right)\right) - h\left(T,\bar{y}\left(T,\cdot\right)\right)\right]\right\}.
\end{aligned}$$

By the definition of c_δ and (58), it follows that

$$\begin{aligned}
l_\delta &= \int_0^T \int_{\mathcal{S}} \left[\beta_\delta\left(t,\theta\right) z_\delta\left(t,\theta\right) + \frac{1}{\delta}\chi_{E_\delta}\left(t,\theta\right)\gamma\left(t,\theta\right)\right] d\theta dt \\
&\quad + \int_{\mathcal{S}} \eta_\delta\left(\theta\right) z_\delta\left(T,\theta\right) d\theta + \frac{1}{\delta}\int_{\mathcal{S}} \chi_{E_\delta}\left(t,\theta\right)\gamma\left(t,\theta\right) d\theta,
\end{aligned}$$

where

$$\begin{aligned}
\beta_\delta\left(t,\theta\right) &= \int_0^1 g_c\left(t,\theta,\bar{y}\left(t,\theta\right)v_\delta\left(t,\theta\right)\right) v_\delta\left(t,\theta\right) ds, \\
\eta_\delta\left(\theta\right) &= \int_0^1 h_y\left(T, \bar{y}\left(T,\theta\right) + s\left(y\left(T,\theta,v_\delta\left(T,\theta\right)\right) - \bar{y}\left(T,\theta\right)\right)\right) ds.
\end{aligned}$$

It is clear that

$$\beta_\delta(t,\theta) \to g_c(t,\theta,\bar{y}(t,\theta)\bar{v}(t,\theta))\bar{v}(t,\theta), \qquad \eta_\delta(\theta) \to h_y(T,\bar{y}(T,\theta))$$

as $\delta \to 0$.

Comparing equations for l_δ and l, we find

$$\begin{aligned} l_\delta - l &= \int_0^T \int_{\mathcal{S}} [\beta_\delta(t,\theta) z_\delta(t,\theta) - g_c(t,\theta,\bar{y}(t,\theta)\bar{v}(t,\theta))\bar{v}(t,\theta) z(t,\theta)]\, d\theta dt \\ &+ \int_{\mathcal{S}} [\eta_\delta(\theta) z_\delta(T,\theta) - h_y(T,\bar{y}(T,\theta)) z(T,\theta)]\, d\theta \\ &- \int_0^T \int_{\mathcal{S}} \left(1 - \frac{1}{\delta}\chi_{E_\delta}(t,\theta)\right) \gamma(t,\theta)\, d\theta dt. \end{aligned}$$

Using again Lemma 3.2 in B. Hu and J. Yong (1995) and the convergence $z_\delta \to z$ in $C^{\alpha,\alpha/2}(\mathcal{S}_T)$, we find $l_\delta - l \to 0$ as $\delta \to 0$.

This completes the proof of the lemma.

Continuation of Proof of the Theorem 1 An application of Lemma 1 leads to

$$y_\delta^\varepsilon = y^\varepsilon + \delta z^\varepsilon + o(\delta), \qquad J(v_\delta^\varepsilon) = J(v^\varepsilon) + \delta l^\varepsilon + o(\delta), \tag{59}$$

where z^ε and $z^{0,\varepsilon}$ satisfy equations

$$z_t^\varepsilon - D z_{\theta\theta}^\varepsilon = [a(t,\theta) + b(t,\theta) v^\varepsilon(t,\theta)] z^\varepsilon(t,\theta) + \phi^\varepsilon(t,\theta), \tag{60}$$

and

$$\begin{aligned} l^\varepsilon = \int_0^T \int_{\mathcal{S}} [g_c(t,\theta,y^\varepsilon(t,\theta) v^\varepsilon(t,\theta)) v^\varepsilon(t,\theta) z^\varepsilon(t,\theta) + \gamma^\varepsilon(t,\theta)]\, d\theta dt \\ + \int_{\mathcal{S}} h_y(T,y^\varepsilon(T,\theta)) z^\varepsilon(T,\theta)\, d\theta \end{aligned} \tag{61}$$

respectively, with

$$\begin{aligned} \phi^\varepsilon(t,\theta) &= b(t,\theta) y^\varepsilon(t,\theta) [v(t,\theta) - v^\varepsilon(t,\theta)], \\ \gamma^\varepsilon(t,\theta) &= g(t,\theta,y^\varepsilon(t,\theta) v(t,\theta)) - g(t,\theta,y^\varepsilon(t,\theta) v^\varepsilon(t,\theta)). \end{aligned}$$

We next choose a E_δ^ε so that $|E_\delta^\varepsilon| = \delta\,|\mathcal{S}_T|$. By (56), $d\left(v_\delta^\varepsilon, v^\varepsilon\right) = |E_\delta^\varepsilon|$. Hence, by (55)

$$\begin{aligned}
\sqrt{\varepsilon}\,|\mathcal{S}_T| &\geq \frac{F_\varepsilon\left(v^\varepsilon\right) - F_\varepsilon\left(v_\delta^\varepsilon\right)}{\delta} \\
&= \frac{1}{\left[F_\varepsilon\left(v^\varepsilon\right) + F_\varepsilon\left(v_\delta^\varepsilon\right)\right]\delta}\left\{\left[J\left(v^\varepsilon\right) - J\left(v^*\right) + \varepsilon\right]_+^2 - \left[J\left(v_\delta^\varepsilon\right) - J\left(v^*\right) + \varepsilon\right]_+^2\right. \\
&\quad \left. + \left[d_0\left(y^\varepsilon\right) - d_0\left(y_\delta^\varepsilon\right)\right]\right\}.
\end{aligned}$$

Taking $\delta \to 0$ and using (59), the right-hand side converges to

$$\frac{\left[J\left(v^\varepsilon\right) - J\left(v^*\right) + \varepsilon\right]_+}{F_\varepsilon\left(v^\varepsilon\right)} l^\varepsilon + \left\langle \frac{d_0\left(y^\varepsilon\right)\xi^\varepsilon}{F_\varepsilon\left(v^\varepsilon\right)}, z^\varepsilon \right\rangle \equiv \nu^\varepsilon l^\varepsilon + \left\langle m^\varepsilon, z^\varepsilon\right\rangle,$$

where

$$\nu^\varepsilon = \frac{\left[J\left(v^\varepsilon\right) - J\left(v^*\right) + \varepsilon\right]_+}{F_\varepsilon\left(v^\varepsilon\right)}, \qquad m^\varepsilon = \frac{d_0\left(y^\varepsilon\right)\xi^\varepsilon}{F_\varepsilon\left(v^\varepsilon\right)}$$

and

$$\xi^\varepsilon\left(y^\varepsilon\right) = \begin{cases} \nabla d_0\left(y^\varepsilon\right), & \text{if } y^\varepsilon \notin Q, \\ 0 & \text{if } y^\varepsilon \in Q. \end{cases}$$

Hence, by (55),

$$\sqrt{\varepsilon}\,|\mathcal{S}_T| \geq \nu^\varepsilon l^\varepsilon + \left\langle m^\varepsilon, z^\varepsilon\right\rangle. \tag{62}$$

Note that by (52),

$$\left|\xi^\varepsilon\left(y^\varepsilon\right)\right|_{C(\mathcal{S}_T)^*} = 1 \qquad \text{if } y^\varepsilon \notin Q.$$

It follows from (50) that $\mu^\varepsilon \geq 0$ and

$$\nu^\varepsilon + \left|m^\varepsilon\right|_{C(\mathcal{S}_T)^*} = 1 \qquad \text{for all } \varepsilon > 0.$$

Also, by Corollary 1,

$$\left\langle m^\varepsilon, z - y^\varepsilon\right\rangle \leq -d_0\left(y^\varepsilon\right) \leq 0 \qquad \text{for any } z \in Q. \tag{63}$$

Next, by (53) and Lemma 1, it follows that

$$y^\varepsilon = y^* + \delta z^* + o(\delta), \qquad J\left(v^\varepsilon\right) = J\left(v^*\right) + \delta l^* + o(\delta), \tag{64}$$

where z^* and l^* satisfy equations

$$\begin{aligned} & z_t^* - Dz_{\theta\theta}^* = [a(t,\theta) + b(t,\theta) v^*(t,\theta)] z^*(t,\theta) + \phi^*(t,\theta), \\ & z^*(0,\theta) = 0 \end{aligned} \tag{65}$$

and

$$\begin{aligned} l^* &= \int_0^T \int_{\mathcal{S}} [g_c(t,\theta,y^*(t,\theta) v^*(t,\theta)) v^*(t,\theta) z^*(t,\theta) + \gamma^*(t,\theta)] \, d\theta dt \\ &\quad + \int_{\mathcal{S}} h_y(T, y^*(T,\theta)) z^*(T,\theta) \, d\theta, \end{aligned}$$

respectively, with

$$\begin{aligned} \phi^*(t,\theta) &= b(t,\theta) y^*(t,\theta) [v(t,\theta) - v^*(t,\theta)], \\ \gamma^*(t,\theta) &= g(t,\theta,y^*(t,\theta) v(t,\theta)) - g(t,\theta,y^*(t,\theta) v^*(t,\theta)). \end{aligned}$$

From (64) we see that $y^\varepsilon \to y^*$ in $C^{\alpha,\alpha/2}(\mathcal{S}_T)$ as $\varepsilon \to 0$. Thus, from (60) and (61) we find $z^\varepsilon \to z^*$ in $C^{\alpha,\alpha/2}$ and $l^\varepsilon \to l^*$ in $\mathbb{R}$ as $\varepsilon \to 0$. Since Q is finite codimensional in $C(\mathcal{S}_T)$, it follows from Lemma 3.2 of X. Li and J. Yong (1991) that the weakly-* limit, (ν, m), of $(\nu^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$ is positive. Taking $\varepsilon \to 0$ in (63) we find $\langle m, z - y^* \rangle \le 0$ for any $z \in Q$. In addition, from (62) we find

$$\nu l^* + \langle m, z^* \rangle \le 0 \qquad \text{for any } v \in \mathcal{B}. \tag{66}$$

We show that the above inequality is equivalent to

$$\begin{aligned} 0 &\le \int_0^T \int_{\mathcal{S}} \{\nu [g(t,\theta,y^*(t,\theta) v^*(t,\theta)) - g(t,\theta,y^*(t,\theta) v(t,\theta))] \\ &\quad + \psi(t,\theta) b(t,\theta) y^*(t,\theta) [v^*(t,\theta) - v(t,\theta)]\} \, d\theta dt \\ &= \int_0^T \int_{\mathcal{S}} [H(t,\theta,y^*(t,\theta), v^*(t,\theta), \nu, \psi(t,\theta)) \\ &\quad - H(t,\theta,y^*(t,\theta), v(t,\theta), \nu, \psi(t,\theta))] \, d\theta dt \end{aligned}$$

for any $v \in \mathcal{B}$. Using (7) and (65), we find

$$\begin{aligned} \int_0^T \int_{\mathcal{S}} [z^* \psi]_t \, d\theta dt &= \int_0^T \int_{\mathcal{S}} [z_t^* \psi + z^* \psi_t] \, d\theta dt \\ &= \int_0^T \int_{\mathcal{S}} \left[\psi \phi^* - \nu z^* g_y^*\right] d\theta dt + \langle m, z^* \rangle_{(0,T)\times\mathcal{S}} \end{aligned}$$

where

$$g_y^* = g_c\left(t,\theta,y^*\left(t,\theta\right)v^*\left(t,\theta\right)\right)v^*\left(t,\theta\right).$$

On the other hand, since

$$z^*\left(0,\theta\right)=0, \qquad \psi\left(T,\cdot\right)=\nu h_y\left(T,y^*\left(T,\cdot\right)\right)+m|_{\{T\}\times\mathcal{S}},$$

it follows that

$$\begin{aligned}\int_0^T\int_{\mathcal{S}}\left[z^*\psi\right]_t d\theta dt &= \int_{\mathcal{S}} z^*\left(T,\theta\right)\psi\left(T,\theta\right)d\theta \\ &= \nu\int_{\mathcal{S}} h_y\left(T,y^*\left(T,\theta\right)\right)z^*\left(T,\theta\right)d\theta+\left\langle m,z^*\left(T,\cdot\right)\right\rangle_{\mathcal{S}}.\end{aligned}$$

As a result,

$$\begin{aligned}\nu\int_{\mathcal{S}} h_y\left(T,y^*\left(T,\theta\right)\right)z^*\left(T,\theta\right)d\theta+\left\langle m,z^*\left(T,\cdot\right)\right\rangle_{\mathcal{S}}+\nu\int_0^T\int_{\mathcal{S}} z^*g_y^* d\theta dt \\ =\int_0^T\int_{\mathcal{S}}\psi\left(t,\theta\right)\phi^*\left(t,\theta\right)d\theta dt-\left\langle m,z^*\right\rangle_{(0,T)\times\mathcal{S}}.\end{aligned} \tag{67}$$

By (66),

$$\nu\int_0^T\int_{\mathcal{S}}\left[z^*g_y^*+\gamma^*\left(t,\theta\right)\right]d\theta dt+\nu\int_{\mathcal{S}} h_y\left(T,y^*\left(T,\theta\right)\right)z^*\left(T,\theta\right)d\theta+\left\langle m,z^*\right\rangle_{\mathcal{S}_T}\leq 0.$$

As a result, by (67),

$$\int_0^T\int_{\mathcal{S}}\left[\nu\gamma^*\left(t,\theta\right)+\psi\left(t,\theta\right)\phi^*\left(t,\theta\right)\right]dtd\theta\leq 0.$$

This is equivalent to

$$\begin{aligned}0 \;\leq\; &\int_0^T\int_{\mathcal{S}}\left\{\nu\left[g\left(t,\theta,y^*\left(t,\theta\right)v^*\left(t,\theta\right)\right)-g\left(t,\theta,y^*\left(t,\theta\right)v\left(t,\theta\right)\right)\right]\right. \\ &\left.+\,\psi\left(t,\theta\right)b\left(t,\theta\right)y^*\left(t,y\right)\left[v^*\left(t,\theta\right)-v\left(t,\theta\right)\right]\right\}d\theta dt\end{aligned}$$

for any $v\in\mathcal{B}$. Since $v\left(t,\theta\right)$ is arbitrary, (8) follows.

The proof is complete.

A.2 Proof of Proposition 1

Eq. (7) takes the form

$$\begin{aligned} &\psi_t + D\psi_{\theta\theta} = -\left[A - Nc^*/k^*\right]\psi - \mu N e^{-\rho t}\left(c^*\right)^{1-\sigma}/k^* - m|_{(0,T)\times\mathcal{S}}, \\ &\psi\left(T,\cdot\right) = \mu h_y\left(T, k^*\left(T,\cdot\right)\right) + m|_{\{T\}\times\mathcal{S}}, \end{aligned} \tag{68}$$

where c^* satisfies

$$\begin{aligned} &\frac{\mu e^{-\rho t}}{1-\sigma} c^*\left(t,\theta\right)^{1-\sigma} - c^*\left(t,\theta\right)\psi\left(t,\theta\right) \\ &= \max_{0\le c\le \mathcal{B}(k^*)}\left\{\frac{\mu e^{-\rho t}}{1-\sigma} c\left(t,\theta\right)^{1-\sigma} - c\left(t,\theta\right)\psi\left(t,\theta\right)\right\}. \end{aligned} \tag{69}$$

Either $\mu = 0$ or $\mu > 0$. In the case where $\mu = 0$, (69) implies that $c^* = 0$. Thus, (68) is the same as (20). If $\mu > 0$, then (69) implies

$$\psi\left(t,\theta\right) = \mu e^{-\rho t} c^*\left(t,\theta\right)^{-\sigma}.$$

Substituting the right-hand side for ψ in the right-hand side of the first equation in (68), we again obtain

$$\psi_t + D\psi_{\theta\theta} + A\psi = -m|_{(0,T)\times\mathcal{S}}. \tag{70}$$

In addition, from the proof of Proposition 5 we see that either $c^*\left(t,\theta\right) = 0$ for all $\left(t,\theta\right)$ if $\mu = 0$ or

$$c^*\left(t,\theta\right) = \left[e^{\rho t}\psi\left(t,\theta\right)\right]^{-1/\sigma} \qquad \text{for all } \left(t,\theta\right) \tag{71}$$

if $\mu > 0$ (and therefore is set to be 1). In the former case (26) cannot hold. Thus $\mu = 1$ and (71) holds.

Let Ω be the interior of the set

$$\overline{\Omega} = \left\{\left(t,\theta\right) \in \left(0,\infty\right)\times\mathcal{S} : k^*\left(t,\theta\right) = 1\right\}.$$

We first observe that by (18), $c^*\left(t,\theta\right) = A\left(\theta\right)$ in Ω. Thus, by (71),

$$\psi\left(t,\theta\right) = e^{-\rho t} A\left(\theta\right)^{-\sigma} N\left(\theta\right)^{\sigma} \qquad \text{in } \Omega$$

and by (20)

$$e^{\rho t} m = -e^{\rho t}\left\{\psi_t + \mathcal{L}\left[\psi\right]\right\} = \rho A^{-\sigma} N^{\sigma} - \mathcal{L}\left[A^{-\sigma} N^{\sigma}\right]$$

Let $p(t,\theta) = e^{\rho t}\psi(t,\theta)$. Then (70) is equivalent to

$$p_t + Dp_{\theta\theta} + (A-\rho)\,p = -M|_{(0,T)\times\mathcal{S}} \tag{72}$$

where

$$M = e^{\rho t}m = \rho A^{-\sigma}N^{\sigma} - \mathcal{L}\left[A^{-\sigma}N^{\sigma}\right]. \tag{73}$$

This leads to the first equation in (20). The second equation in (20) and (21) follows directly from the second equation of (68) and (71), respectively. .

This completes the proof.

A.3 Proof of Proposition 2

Let $\underline{k}(t,\theta)$ be the solution to the initial-boundary value problem

$$\begin{aligned} &k_t = Dk_{\theta\theta} + (A-NB)\,k \quad \text{for } t>0, \qquad \theta\in\mathcal{S}, \\ &k(0,\theta) = k_0(\theta) \qquad\qquad\quad \text{for } \theta\in\mathcal{S} \end{aligned} \tag{74}$$

with a nonnegative nontrivial initial function $k_0(\theta)$. By the maximum principle for parabolic partial differential equations, $\underline{k}(t,\theta) > 0$ for all $(t,\theta)\in(0,\infty)\times\mathcal{S}$.

Since $c^*(t,\theta) \geq B(\theta)\,k^*(t,\theta)$ for all (t,θ), it follows that $\underline{k}(t,\theta)$ is a lower solution for (13) with $c=c^*$. It is easy to see that the solution $\tilde{k}$ of the initial-value problem

$$\begin{aligned} &\tilde{k}_t = D\tilde{k}_{\theta\theta} + A\tilde{k} \quad \text{for } t>0, \qquad \theta\in\mathcal{S}, \\ &\tilde{k}(0,\theta) = k_0(\theta) \quad\ \text{for } \theta\in\mathcal{S} \end{aligned}$$

is an upper solution and $\tilde{k}(t,\theta) \geq \underline{k}(t,\theta)$ for all (t,θ). Hence, by the comparison principle (cf, C.V. Pao (1992), Chapter 2, Theorem 4.1), $\underline{k}(t,\theta) \leq k^*(t,\theta) \leq \tilde{k}(t,\theta)$. As a result $k^*(t,\theta) > 0$ for all (t,θ). Furthermore, let ϕ_0 be a positive eigenfunction of the operator $\mathcal{M}$ corresponding to the principal eigenvalue, μ_0. Then

$$\int_{\mathcal{S}} k^*(t,\theta)\,\phi_0(\theta)\,d\theta \geq \int_{\mathcal{S}} \underline{k}(t,\theta)\,\phi_0(\theta)\,d\theta = \langle \underline{k}(t,\cdot),\phi_0\rangle. \tag{75}$$

Multiplying ϕ_0 to the first equation in (74) and integrating over $\mathcal{S}$, we obtain

$$\langle \underline{k}(t,\cdot),\phi_0\rangle' = \langle \mathcal{M}\underline{k}(t,\cdot),\phi_0\rangle = \mu_0\,\langle \underline{k}(t,\cdot),\phi_0\rangle.$$

It follows that

$$\langle \underline{k}(t,\cdot), \phi_0 \rangle = \langle k_0, \phi_0 \rangle e^{\mu_0 t}.$$

Hence, by (75),

$$\int_{\mathcal{S}} k^*(t,\theta)\,\phi_0(\theta)\,d\theta \geq \langle k_0, \phi_0 \rangle e^{\mu_0 t}.$$

Using the positivity and boundedness of ϕ_0 on $\mathcal{S}$, the above relation leads to

$$\int_{\mathcal{S}} k^*(t,\theta)\,d\theta \geq \frac{\langle k_0, \phi_0 \rangle}{\max_{\theta\in\mathcal{S}} \phi_0(\theta)} e^{\mu_0 t}.$$

This completes the proof.

A.4 Proof of Proposition 3

First observe that (26) implies $\mu = 1$ and

$$p(t,\theta) = c^*(t,\theta)^{-\sigma} \qquad \text{for all } (t,\theta) \in (0,\infty)\times\mathcal{S}. \tag{76}$$

satisfies

$$p_t + (\mathcal{L} - \rho)[p] = -M|_{(0,T)\times\mathcal{S}} \tag{77}$$

where M is a Radon measure with a support that satisfies (24).

Suppose for contradiction that $k^*(t,\theta) < 1$ for all $(t,\theta) \in (0,\infty)\times\mathcal{S}$. Then $M = 0$ a.e. As a result, p satisfies

$$p_t + (\mathcal{L} - \rho)[p] = 0$$

everywhere. Using a Fourier expansion

$$p(t,\theta) = p_0(t)\,\varphi_0(\theta) + \sum_{j\geq 1}\sum_{i=1}^{2} p_{j,i}(t)\,\varphi_{j,i}(\theta) \tag{78}$$

we find

$$p_0(t) = p_0(0)\,e^{-(\lambda_0-\rho)t},$$

and

$$p_{j,i}(t) = p_{j,i}(0)\,e^{-(\lambda_j-\rho)t} \qquad \text{for } j\geq 1,\ i=1,2.$$

Observe that (76) implies that $p(t,\theta) > 0$ everywhere. Since $\varphi_{j,i}$ changes the sign in $\mathcal{S}$ and $\lambda_j < \lambda_0$ for all $j \geq 1$, it follows that the only non-zero term on the right-hand side of (78) is

$p_0(t)\varphi_0(\theta)$. Therefore, by (76),

$$c^*(t,\theta) = [p_0(0)\varphi_0(\theta)]^{-1/\sigma} e^{rt} \qquad \text{for all } (t,\theta).$$

Substituting the right-hand side for c in (13), and using Fourier expansions

$$k(t,\theta) = \sum_{j,i} k_{j,i}(t)\varphi_{j,i}(\theta), \qquad [p_0(0)\varphi_0(\theta)]^{-1/\sigma} = \sum_{j,i} \eta_{j,i}\varphi_{j,i}(\theta)$$

we find

$$k'_{j,i}(t) = \lambda_j k_{j,i}(t) - \eta_{j,i}e^{rt},$$

where $r = \frac{\lambda_0 - \rho}{\sigma}$. Solving the equation we find

$$k_{j,i}(t) = \frac{\eta_{j,i}}{\lambda_j - r}e^{rt} + \left[k_{j,i}(0) - \frac{\eta_{j,i}}{\lambda_j - r}\right]e^{\lambda_j t}$$

if $\lambda_j \neq r$ and

$$k_{j,i}(t) = [k_{j,i}(0) - \eta_{j,i}t]\, e^{rt}$$

if $\lambda_j = r$. In particular

$$k_0(t) = \frac{\eta_0}{\lambda_0 - r}e^{rt} + \left[\tilde{k}_0(0) - \frac{\eta_0}{\lambda_0 - r}\right]e^{\lambda_0 t}$$

if $\lambda_0 \neq r$ and

$$k_0(t) = [k_0(0) - \eta_0 t]\, e^{rt}$$

if $\lambda_0 = r$. Since r and λ_0 are both positive, $k_0(t)$ is unbounded. Note that $k_0(t) = \langle k(t,\cdot), \varphi_0\rangle$ and φ_0 is positive in $\mathcal{S}$, the state constraint $0 \leq k(t,\theta) \leq 1$ implies

$$0 \leq k_0(t) \leq \int_{\mathcal{S}} \varphi_0(\theta)\, d\theta.$$

Thus, $k_0(t)$ is bounded. This is a contradiction.

The proof is complete.

A.5 Proof of Proposition 4

Let $\{\bar{k}(\theta), \bar{c}(\theta)\}$ be a steady state, and let $\bar{p}(\theta) = \bar{c}(\theta)^{-\sigma}$. Then, by (13) and (77),

$$\begin{aligned} D\bar{k}''(\theta) + A\bar{k}(\theta) &= N\bar{p}(\theta)^{-1/\sigma}, \\ D\bar{p}''(\theta) + (A-\rho)\bar{p}(\theta) &= -\overline{M}(\theta) \qquad \text{for } \theta \in \mathcal{S} \end{aligned}$$

where

$$\overline{M}(\theta) = \begin{cases} (\rho - A) A^{-\sigma} N^{\sigma} & \text{if } \bar{k}(\theta) = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Suppose for contradiction that there is an interval I on which $\bar{k}(\theta) < 1$. It is not possible that $I = \mathcal{S}$, because if so, $\bar{p}$ would be a positive eigenfunction of the operator $\mathcal{L}$ corresponding to the eigenvalue ρ. However, the only eigenvalue of $\mathcal{L}$ that can have positive eigenfunction is $\alpha > \rho$. This is a contradiction. Hence, I is a proper subinterval of $\mathcal{S}$. Without loss of generality, we may assume that $I = (-a, a)$ is symmetric, with a constant a satisfying $0 < a < \pi$. As a result, $\bar{k}(-a) = \bar{k}(a) = 1$. Also, since $\bar{k}(\theta) \leq 1$ for all θ, it follows that $\bar{k}'(-a) = \bar{k}'(a) = 0$.

We first solve $\bar{p}(\theta)$. Note that $\overline{M}(\theta) = 0$ in $(-a, a)$, $\bar{p}$ satisfies

$$D\bar{p}''(\theta) + (A-\rho)\bar{p}(\theta) = 0 \qquad \text{in } (-a, a).$$

Due to the symmetry, $\bar{p}(-\theta) = \bar{p}(\theta)$. Let b denote the common value of $\bar{p}(-a)$ and $\bar{p}(a)$. By computation

$$\bar{p}(\theta) = b\cos\omega\theta \qquad \text{in } (-a, a)$$

for some $b > 0$, where $\omega = \sqrt{(A-\rho)/D}$. Solving the differential equation for $\bar{x}$, we find

$$\bar{k}(\theta) = c_1 \cos\sqrt{\frac{A}{D}}\theta + c_2 \sin\sqrt{\frac{A}{D}}\theta + Nb^{-1/\sigma}\int_0^{\theta} \frac{\cos^{-1/\sigma}(\omega\xi)}{\sqrt{AD}} \sin\sqrt{\frac{A}{D}}(\theta - \xi)\, d\xi \tag{79}$$

for some constants c_1 and c_2. Using the boundary conditions $\bar{k}(-a) = \bar{k}(a) = 1$, we find $c_2 = 0$. Then, from the conditions $\bar{k}'(-a) = \bar{k}'(a) = 0$ we find

$$\begin{aligned} c_1 \cos\sqrt{\frac{A}{D}}a + Nb^{-1/\sigma}\int_0^{a} \frac{\cos^{-1/\sigma}(\omega\xi)}{\sqrt{AD}} \sin\sqrt{\frac{A}{D}}(a-\xi)\, d\xi &= 1, \\ c_1 \sin\sqrt{\frac{A}{D}}a - Nb^{-1/\sigma}\int_0^{a} \frac{\cos^{-1/\sigma}(\omega\xi)}{\sqrt{AD}} \cos\sqrt{\frac{A}{D}}(a-\xi)\, d\xi &= 0. \end{aligned}$$

Solving the equations, we find

$$c_1 = \int_0^a \cos^{-1/\sigma}(\omega\xi)\cos\sqrt{\frac{A}{D}}(a-\xi)\,d\xi \Big/ \int_0^a \cos^{-1/\sigma}(\omega\xi)\cos\sqrt{\frac{A}{D}}\xi d\xi$$

We show that $c_1 > 1$. It amounts to show that

$$\int_0^a \cos^{-1/\sigma}(\omega\xi)\cos\sqrt{\frac{A}{D}}(a-\xi)\,d\xi > \int_0^a \cos^{-1/\sigma}(\omega\xi)\cos\sqrt{\frac{A}{D}}\xi d\xi. \tag{80}$$

Note that both $\cos^{-1/\sigma}(\omega\xi)$ and $\cos\sqrt{\frac{A}{D}}(a-\xi)$ are positive and increasing in $(0,a)$. Let us use $f(\xi)$ and $g(\xi)$ to denote two positive and strictly increasing functions. It is easy to see that

$$[f(\xi) - f(a-\xi)][g(\xi) - g(a-\xi)] \geq 0 \qquad \text{for } \xi \in (0,a)$$

and the strict inequality holds for all $\xi \in (0,a)$ except for $\xi = a/2$. Thus,

$$\int_0^a [f(\xi) - f(a-\xi)][g(\xi) - g(a-\xi)]\,d\xi > 0.$$

On the other hand, the left-hand side can be written as

$$\int_0^a [f(\xi)g(\xi) + f(a-\xi)g(a-\xi)]\,d\xi - \int_0^a [f(\xi)g(a-\xi) + f(a-\xi)g(\xi)]\,d\xi. \tag{81}$$

Using a change of variable, it is easy to verify that

$$\begin{aligned} \int_0^a f(a-\xi)g(a-\xi)\,d\xi &= \int_0^a f(\xi)g(\xi)\,d\xi, \\ \int_0^a f(\xi)g(a-\xi)\,d\xi &= \int_0^a f(a-\xi)g(\xi)\,d\xi. \end{aligned}$$

Thus, by (81) we find

$$\int_0^a f(\xi)g(\xi)\,d\xi > \int_0^a f(\xi)g(a-\xi)\,d\xi.$$

Apply the above inequality to the function $f(\xi) = \cos^{-1/\sigma}(\omega\xi)$ and $g(\xi) = \cos\sqrt{A/D}(a-\xi)$, (80) follows.

By (79), $\bar{k}(0) = c_1 > 1$. This contradicts the constraint $\bar{k}(\theta) \leq 1$ in $\mathcal{S}$.

If an optimal pair $\{k^*, c^*\}$ converges as $t \to \infty$, then the limit must be a steady state. Thus, (29) must hold.

This completes the proof.

A.6 Proof of Proposition 5

By (34), $M = 0$ and $0 < c^*(t,\theta) < B(\theta) k^*(t,\theta)$. Thus, p satisfies

$$p_t + (\mathcal{L} - \rho)[p] = 0 \qquad \text{in } \mathcal{S}_T$$

for any $T > 0$. Since $p > 0$, it follows that

$$p(t,\theta) = p_0(0) e^{-(\lambda_0 - \rho)t} \varphi_0(\theta) \qquad \text{for all } (t,\theta).$$

As a result,

$$c^*(t,\theta) = p(t,\theta)^{-1/\sigma} = c_0 \varphi_0(\theta)^{-1/\sigma} e^{rt}$$

where $c_0 = p_0(0)^{-1/\sigma}$ is a constant.

Consider the initial value problem (13) with

$$c = c^* \equiv c_0 \varphi_0(\theta)^{-1/\sigma} e^{rt},$$

i.e.,

$$\begin{aligned} k_t &= \mathcal{L}[k] - N c_0 \varphi_0^{-1/\sigma} e^{rt} && \text{for } t > 0, \quad \theta \in \mathcal{S}, \\ k(0,\theta) &= k_0(\theta) && \text{for } \theta \in \mathcal{S}. \end{aligned} \tag{82}$$

Let $a_0(t) = \langle k(t,\cdot), \varphi_0 \rangle$. Clearly for $k(t,\theta) > 0$ in $(0,\infty) \times \mathcal{S}$, it is necessary that $a_0(t) > 0$ for all t. By multiplying φ_0 to the both sides of (82) and integrating the results over $\mathcal{S}$, we obtain

$$\begin{aligned} a_0' &= \lambda_0 a_0 - c_0 e^{rt} \left\langle N \varphi_0^{-1/\sigma}, \varphi_0 \right\rangle \qquad \text{for } t > 0, \\ a_0(0) &= \langle k_0, \varphi_0 \rangle. \end{aligned}$$

The solution is

$$a_0(t) = \frac{c_0 \left\langle N \varphi_0^{-1/\sigma}, \varphi_0 \right\rangle}{\lambda_0 - r} e^{rt} + \left[\langle k_0, \varphi_0 \rangle - \frac{c_0 \left\langle N \varphi_0^{-1/\sigma}, \varphi_0 \right\rangle}{\lambda_0 - r} \right] e^{\lambda_0 t}.$$

Note that since $\rho > \lambda_0 \geq (1-\sigma)\lambda_0$ it follows that

$$\lambda_0 > \frac{\lambda_0 - \rho}{\sigma} = r.$$

Hence, for $a_0(t) > 0$ for all t, it is necessary that

$$c_0 \leq \frac{(\lambda_0 - r)\langle k_0, \varphi_0 \rangle}{\left\langle N\varphi_0^{-1/\sigma}, \varphi_0 \right\rangle} \equiv \hat{c}_0$$

Note that by (14),

$$J(k_0, c^*) = \frac{c_0^{1-\sigma}}{1-\sigma} \int_0^\infty e^{[-\rho+(1-\sigma)r]t} dt \int_{\mathcal{S}} N(\theta)\, \varphi_0(\theta)^{1-1/\sigma}\, d\theta$$

which is increasing in c_0. Hence, since $\left\{\hat{c}, \hat{k}\right\}$ is feasible, it follows that

$$c^*(t,\theta) = \hat{c}_0 \varphi_0(\theta)^{-1/\sigma} e^{rt} = \hat{c}(t,\theta) \qquad \text{in } (0,\infty) \times \mathcal{S}.$$

This completes the proof.

A.7 Proof of Proposition 6

We use the initial value problem (82) with $c_0 = \hat{c}_0$ using the Fourier series expansion

$$\hat{k}(t,\theta) = a_0(t)\, \varphi_0(\theta) + \sum_{j\geq 1} \sum_i a_{j,i}(t)\, \varphi_{j,i}(\theta).$$

Substituting the right-hand side for k in (82), we obtain

$$a'_{j,i}(t) = \lambda_j a_{j,i}(t) - \hat{c}_0 e^{rt} \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle, \qquad a_{j,i}(0) = \langle k_0, \varphi_{j,i} \rangle.$$

The solution is

$$a_{j,i}(t) = \frac{\hat{c}_0 \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle}{\lambda_j - r} e^{rt} + \left[\langle k_0, \varphi_{j,i} \rangle - \frac{\hat{c}_0 \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle}{\lambda_j - r} \right] e^{\lambda_j t} \tag{83}$$

if $\lambda_j \neq r$ and

$$a_{j,i}(t) = \left[\langle k_0, \varphi_{j,i} \rangle - t\hat{c}_0 \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle \right] e^{rt} \tag{84}$$

if $\lambda_j = r$ for all integers j and i.

Since $\left\{\hat{c}, \hat{k}\right\}$ is feasible, $0 \leq \hat{k}(t,\theta) \leq 1$ for all (t,θ). In particular,

$$\hat{k}(t,\theta)\, e^{-rt} \equiv \langle k_0, \varphi_0 \rangle\, \varphi_0(\theta) + \sum_{j\geq 1} \sum_i a_{j,i}(t)\, e^{-rt} \varphi_{j,i}(\theta) \geq 0.$$

Note that $\varphi_{j,i}$ changes the sign in $\mathcal{S}$ for any $j \geq 1$, it follows that each $a_{j,i}(t) e^{-rt}$ is bounded. Hence, by (83) and (84), $\lambda_j \neq r$ if $\left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle \neq 0$ and

$$\frac{\hat{c}_0 \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle}{\lambda_j - r} = \langle k_0, \varphi_{j,i} \rangle$$

if $\lambda_j > r$. As a result,

$$\lim_{t \to \infty} a_{j,i}(t) e^{-rt} = \frac{\hat{c}_0 \left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{j,i} \right\rangle}{\lambda_j - r}$$

if $\lambda_j \neq r$ for all j and

$$a_{m,i}(t) e^{-rt} = \langle k_0, \varphi_{m,i} \rangle$$

if $\lambda_m = r$ and $\left\langle N\varphi_0{}^{-1/\sigma}, \varphi_{m,i} \right\rangle = 0$ for $i = 1, 2$.

The proof is complete.

A.8 Proof of Corollary 2

Since A is a constant function, the eigenvalues and eigenfunctions are given by (27) and (28), respectively. In this case,

$$\left\langle N\varphi_0^{-1/\sigma}, \varphi_{j,i} \right\rangle = 0 \qquad \text{for all } j \geq 1, i = 1, 2.$$

Hence, (36) become

$$\bar{y}(\theta) = \langle l_0, \varphi_0 \rangle \varphi_0 = \frac{1}{2\pi} \int_{\mathcal{S}} l_0(\xi) \, d\xi$$

if $r \neq \lambda_j$ for all j. This leads to (38).

In the case where $r = \lambda_m$ for some m, by (37) and (28)

$$\begin{aligned} \bar{y}(\theta) &= \langle l_0, \varphi_0 \rangle \varphi_0 + \langle l_0, \varphi_{m,1} \rangle \varphi_{m,1}(\theta) + \langle l_0, \varphi_{m,2} \rangle \varphi_{m,2} \\ &= \frac{1}{2\pi} \int_{\mathcal{S}} l_0(\xi) \, d\xi + \frac{1}{\pi} \int_{\mathcal{S}} l_0(\xi) \left[\cos m\theta \cos m\xi + \sin m\theta \sin m\xi\right] d\xi. \end{aligned}$$

This leads to (39).

The proof is complete.

A.9 Proof of Proposition 7

Suppose by contradiction that $\left\{\hat{c}, \hat{k}\right\}$ is feasible and $\hat{c} \in \mathcal{B}\left(\hat{k}\right)$. Then $\hat{c}(t,\theta) \leq B(\theta)\,\hat{k}(t,\theta)$ for all $(t,\theta) \in [0,\infty) \times \mathcal{S}$. This implies that

$$\hat{k}_t \geq D\hat{k}_{\theta\theta} + (A - NB)\,\hat{k} \qquad \text{for all } t > 0, \quad \theta \in \mathcal{S}.$$

Let ϕ_0 be a positive eigenfunction of $\mathcal{M}$ corresponding to μ_0, and let $\tilde{k}_0(t) = \left\langle \hat{k}(t,\cdot), \phi_0 \right\rangle$, we obtain

$$\tilde{k}_0'(t) \geq \eta_0 \tilde{k}_0, \qquad \tilde{k}_0(0) = \left\langle \hat{k}(0,\cdot), \phi_0 \right\rangle.$$

Hence,

$$\tilde{k}_0(t) \geq \left\langle \hat{k}(0,\cdot), \phi_0 \right\rangle e^{\eta_0 t}.$$

That is

$$\int_{\mathcal{S}} \hat{k}(t,\theta)\,\phi_0(\theta)\,d\theta \geq e^{\eta_0 t} \int_{\mathcal{S}} k_0(\theta)\,\phi_0(\theta)\,d\theta. \tag{85}$$

On the other hand, by (31) and (33), $\hat{k}(t,\theta)$ satisfies

$$\int_{\mathcal{S}} \hat{k}(t,\theta)\,\varphi_0(\theta)\,d\theta = e^{rt} \int_{\mathcal{S}} k_0(\theta)\,\varphi_0(\theta)\,d\theta.$$

In view of (85), it follows that

$$\eta_0 \leq r = \frac{\lambda_0 - \rho}{\sigma}.$$

This contradicts (42).

Suppose $\{c^*, k^*\}$ is an optimal pair such that $c^*(t,\theta) < B(\theta)\,k^*(t,\theta)$ for all (t,θ). Then, by Proposition 3, $\left\{\hat{c}, \hat{k}\right\} = \{c^*, k^*\}$ is feasible. This is a contradiction.

This completes the proof.

A.10 Proof of Proposition 8

Since, by (43), $k^* < 1$ at $t = T$, it follows that $m = 0$ at $t = T$. Also, since $\left\{\hat{c}_T, \hat{k}_T\right\}$ is feasible, by Proposition 3, $\{c^*, k^*\} = \left\{\hat{c}_T, \hat{k}_T\right\}$ in $(T,\infty) \times \mathcal{S}$. Thus by (44),

$$c^*(t,\theta) = \hat{c}_T(t,\theta) \equiv \hat{c}_{T,0}\varphi_0(\theta)^{-1/\sigma}\, e^{r(t-T)}$$

for $(t,\theta) \in (T,\infty) \times \mathcal{S}$. In view of (43) and (70),

$$c^*(t,\theta) = \underset{0 \le c \le \mathcal{B}(k^*)}{\arg\max} \left\{ \frac{\mu c(t,\theta)^{1-\sigma}}{1-\sigma} - c(t,\theta)\, p(t,\theta) \right\} = \left[\frac{p(t,\theta)}{\mu} \right]^{-1/\sigma}$$

for $(t,\theta) \in (T,\infty) \times \mathcal{S}$. Clearly, $\mu > 0$. So we assume $\mu = 1$. Therefore,

$$p(t,\theta) = c^*(t,\theta)^{-\sigma} = \hat{c}_{T,0}^{-\sigma} \varphi_0(\theta)\, e^{-\sigma r (t-T)} \qquad \text{for } t > T.$$

It follows that

$$p(T,\theta) = \hat{c}_{T,0}^{-\sigma} \varphi_0(\theta).$$

The result of the proposition then follows from Proposition 5. The statement regarding k^* is obviously true. That completes the proof.

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