Deontic Sufficiency in Dyadic Deontic Logic

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Abstract. In this paper, we introduce and study the logics for conditionals of the form "Given φ, it suffices to do ψ", which are known as "deontic sufficiency" in the deontic logic literature and are useful in the decision and game theory contexts. We completely axiomatize the logics under different assumptions about the properties of the preference relations and establish decidability results for each logic.

Keywords: Deontic sufficiency · Dyadic deontic logic · Reasoning about preference.

1 Introduction

Deontic logic is an area of logic that studies normative concepts such as obligation and permission [6]. Among the different approaches to deontic logic, the modal logic approach investigates deontic logic as a branch of modal logic. This is based on an analogy between the deontic modality "It is obligatory that" and the alethic modality "It is necessary that", since the obligatory is "what is necessary for a good person to do", as suggested by Leibniz [11]. As such, the most discussed deontic logic system, standard deontic logic (SDL), is just modal logic of type **KD**. The necessity modality $\Box\varphi$ (or, following the convention in deontic logic, $O\varphi$) in SDL is interpreted as "It is obligatory that φ" and the possibility modality $\Diamond\varphi$ ($P\varphi$) "It is permitted that φ".

Given that obligation can be interpreted as (deontic) necessity, one may wonder whether other modal notions can be applied in deontic logic, e.g. sufficiency. In his paper [20], von Wright noticed the connection between deontic sufficiency and the notion of strong permission, which is deemed different from the notion of weak permission as characterized in SDL. Strong permission is argued to satisfy the free choice principle $P(p \vee q) \leftrightarrow (Pp \wedge Pq)$ (for example, "You may have cake or coffee" implies both "You may have cake" and "You may have coffee"). However, it is not part of SDL (as the $\Diamond$-operator does not distribute over disjunction). Von Wright suggested that interpreting strong permission as deontic sufficiency validates this principle. Van Benthem [2] pursued this idea formally in modal logic. The deontic sufficiency operator $S\varphi$ is interpreted on a Kripke model $M = (W, R, V)$ as follows:

$$M, w \models S\varphi \text{ iff } wRv \text{ for all } v \in W \text{ such that } M, v \models \varphi$$

Since wRv represents that "v is an ideal situation relative to w", the operator $S\varphi$ states that φ is a sufficient condition to achieve ideality.[1] In the deontic logic literature, the deontic sufficiency operator has received far less attention than deontic necessity. However, it has recently regained interest, see [17,19].

This paper follows the above research line. We aim to study a conditional variant of the deontic sufficiency operator, analogous to how monadic obligation in SDL is generalized to conditional obligation in dyadic deontic logic (DDL) [9,18]. In DDL, a new dyadic operator $O(\psi/\varphi)$ is introduced to express the condition obligation that "Given φ, it is obligatory that ψ", which is not representable in SDL when contrary-to-duty scenarios like Chisholm's are considered [5]. This dyadic operator is interpreted on the so-called "preference models", where possible worlds are ranked according to their comparative goodness (instead of divided into either ideal or non-ideal ones), such that $O(\psi/\varphi)$ is true iff all the best φ-worlds are also ψ-worlds.

In this paper, inspired by the semantics of the dyadic obligation in DDL, we introduce and study a dyadic operator $S(\psi/\varphi)$ which is to be understood as "Given φ, ψ is a sufficient condition for achieving ideality". Semantically, this operator states that all $\varphi \wedge \psi$-worlds are best φ-worlds in a preference model.

We could regard $S(\psi/\varphi)$ as a conditional version of strong permission. However, the use of the deontic sufficiency operator in the decision and game theory contexts has already been noted by a series of authors, e.g., [1,4,17]. We illustrate this by the following example:

Example 1 (The umbrella). A professor wakes up in the morning and recalls that there is an important exam today. She must bring the test papers to the classroom. However, she is unsure whether it will rain today. She needs to bring an umbrella to protect the test papers if it rains. Otherwise, there is no need. As a cautious person, the professor decides to bring an umbrella regardless of whether it rains.

The point of the above example is that we cannot explain the professor's decision if we interpreted it as the *necessary* conditions for achieving her goal, i.e., bringing the test papers to the classroom without wetting them. This is because carrying an umbrella is not necessary in case it does not rain. As will be seen, the professor's decision can be explained only if it is interpreted as the *sufficient* conditions for achieving the goal. Our paper presents a series of logic systems to reason about them.

This paper is structured as follows. Following the convention in the modal logic research, we start by introducing the formal language and semantics of our logical systems in Section 2. In Section 3, we analyze the impacts of different properties of the preference relations on the resulting logics. Based on this analysis, three Hilbert-style axiom systems are proposed in Section 4 and the completeness results are established in the next two sections: Sections 5 and 6. Finally, in Section 7, we conclude with some discussions.

[1] In [2], the notation $P\varphi$ is used. But the semantic definition is the same. $S\varphi$ is also known as the "window" modality in the modal logic literature [3].

2 Syntax and Semantics

We start by presenting the basics of our logical systems, including the formal language and semantics.

Let *Prop* be a countable infinite set of propositional variables or atoms. In addition to Boolean connectives, there are two modalities $\Box\varphi$ and $S(\psi/\varphi)$ in our language. $\Box\varphi$ is the alethic necessity modality, which can be read as "φ is necessarily true". The dyadic modality $S(\psi/\varphi)$ is read as "Given φ, it suffices to do ψ".

Definition 1 (Language). *The language $\mathcal{L}$ is given by the following BNF grammar:*

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid S(\varphi/\varphi)$$

where $p \in Prop$. Other Boolean connectives are defined as usual.

The formulas are interpreted on preference models from DDL. In a preference model, possible worlds are ordered according to their comparative goodness.

Definition 2 (Preference model). *A preference model $M = (W, \succeq, V)$ is a tuple where:*

1. *W is a non-empty set of possible worlds (states);*
2. *$\succeq$ is binary relation over W ($s \succeq y$ means that "s is at least as good as y");*
3. *$V : Prop \to \wp(W)$ is a valuation.*

As an example of the preference models, consider the umbrella example in the Introduction. For simplicity, we assume that there are only two atoms r and u in the language: r stands for "It rains" and u for "The professor brings an umbrella". As illustrated in Fig. 1, the most preferred worlds (s_1, s_2, and s_3) are those in which the professor can bring the test papers to the classroom without wetting them, i.e., either it does not rain or she brings an umbrella. The worst is where the professor does not bring an umbrella while it rains.

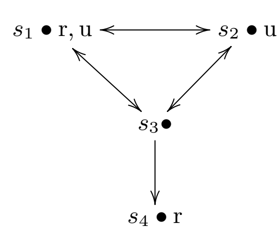

Fig. 1. A preference model M, where $s \to t$ means $s \succeq t$. The reflexive and transitive arrows are omitted.

Next, we present the semantics. Recall that the operator $S(\psi/\varphi)$ is intended to express that ψ is a sufficient condition for achieving ideality in the context of φ. In a preference model, it means that all ψ-worlds (in the context of φ) are the best φ-worlds. This gives rise to the following semantic definition:

Definition 3 (Satisfaction). *Given a preference model $M = (W, \succeq, V)$, for all $w \in W$ and formulas φ, the satisfaction relation $M, s \models \varphi$ is inductively defined as follows (where the clauses for atoms and Boolean connectives are as usual and thus omitted):*

- $M, s \models \Box\varphi$ *iff, for all* $t \in W$, $M, t \models \varphi$
- $M, s \models S(\psi/\varphi)$ *iff* $[\![\psi \wedge \varphi]\!]_M \subseteq opt_{\succeq}([\![\varphi]\!]_M)$

where $[\![\varphi]\!]_M = \{s \in W \mid M, s \models \varphi\}$ *is the truth set of* φ *(and same for* $[\![\psi \wedge \varphi]\!]_M$*), and* $opt_{\succeq}([\![\varphi]\!]_M) = \{s \in [\![\varphi]\!]_M \mid s \succeq t \text{ for all } t \in [\![\varphi]\!]_M\}$*. The notion of validity is defined as usual.*

Remark 1. In DDL [18], the truth definition for conditional obligation $O(\psi/\varphi)$ is given by the expression "$opt_{\succeq}([\![\varphi]\!]_M) \subseteq [\![\psi]\!]_M$". Thus, the truth definition for $S(\psi/\varphi)$ is roughly the inverted version of that for $O(\psi/\varphi)$. Note also that, like in DDL, the truth of $\Box\varphi$ and $S(\psi/\varphi)$ does not depend on the evaluating states.

Remark 2. In the truth definition for $S(\psi/\varphi)$, $opt_{\succeq}([\![\varphi]\!]_M)$ could have been replaced by the set of all $\succeq$-maximal φ-worlds. The choice between optimality and maximality is a long-known problem in the DDL literature, see [13,15]. Here we follow [16] and [18]. We plan to study the maximality version in the future.

Consider again the umbrella example. In the preference model M depicted in Fig. 1, $M, s_1 \models S(\mathrm{u}/\top)$, which means that bringing an umbrella is sufficient to achieve the goal. In contrast, given that it will not rain, whether or not bringing an umbrella will suffice ($M, s_1 \models S(\mathrm{u}/\neg\mathrm{r}) \wedge S(\neg\mathrm{u}/\neg\mathrm{r})$). Note that in both contexts of $\top$ and $\neg$r, bringing an umbrella is not necessary (to achieve the goal): $M, s_1 \not\models O(\mathrm{u}/\top)$ and $M, s_1 \not\models O(\mathrm{u}/\neg\mathrm{r})$.

A key feature of the operator $S(\psi/\varphi)$ is that it satisfies the principle of strengthening the antecedent, which distinguishes it from the conditional obligation $O(\psi/\varphi)$ in DDL. The operator $S(\psi/\varphi)$ also validates the principle of strengthening the consequent: $S(\psi/\varphi) \to S(\psi \wedge \chi/\varphi)$.[2] Moreover, $S(\psi/\varphi)$ validates a conditional version of the free choice principle:

Proposition 1. *The following hold for all formulas* φ, ψ, χ*:*

(1) $\models S(\psi/\varphi) \to S(\psi/\varphi \wedge \chi)$.
(2) $\models S(\psi/\varphi) \to S(\psi \wedge \chi/\varphi)$.
(3) $\models S(\psi \vee \chi/\varphi) \leftrightarrow S(\psi/\varphi) \wedge S(\chi/\varphi)$.

3 Properties of Preference Relations

In this section, we examine the impacts of different properties of preference relations on the resulting logics. We will consider four familiar properties in the DDL literature [18]. Other interesting properties (such as antisymmetry) are left for future investigation.

[2] We thank an anonymous referee for pointing out this. The validity of strengthening the consequent is considered to be problematic if we interpret $S(\psi/\varphi)$ as (conditional) permission, see [10].

Definition 4. *A preference model* $M = (W, \succeq, V)$ *is*

- reflexive *if, for all* $s \in W$, $s \succeq s$*;*
- total *(or* connected*) if, for all* $s, t \in W$, $s \succeq t$ *or* $t \succeq s$*;*
- transitive *if, for all* $s, t, w \in W$, $s \succeq t$ *and* $t \succeq w$ *implies* $s \succeq w$*;*
- limited *if, for all formulas* φ, $[\![\varphi]\!]_M \neq \varnothing$ *implies* $opt_{\succeq}([\![\varphi]\!]_M) \neq \varnothing$*.*

The class of all reflexive (total, transitive, limited, respectively) preference models is denoted by $\mathbb{R}$ *(*$\mathbb{C}$*,* $\mathbb{T}$*,* $\mathbb{L}$*, respectively). A nonempty subset of* $\{\mathbb{R}, \mathbb{C}, \mathbb{T}, \mathbb{L}\}$ *will denote the intersection of all its members, e.g.,* $\mathbb{RTL}$ *denotes the class of all reflexive, transitive, and limited preference models. For each nonempty subset* $X \subseteq \{\mathbb{R}, \mathbb{C}, \mathbb{T}, \mathbb{L}\}$*,* L_X *is the set of all validities on the model class* X*. Finally,* L *is the set of validities on the class of all models.*

We completely studied the logics generated by the 16 (possibly equivalent) classes of models. The results are summarized in Fig. 2. The main observation is that, unlike transitivity, imposing only totalness or limitedness on models has no import to the logic. Only when totalness and limitedness are combined with transitivity can we obtain stronger logics.

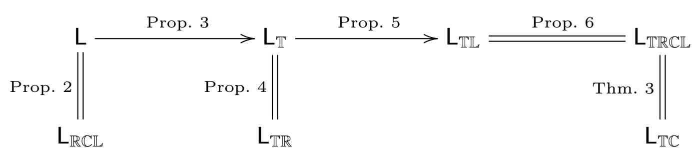

Fig. 2. The logics generated by different model classes, where $\mathsf{L}_X == \mathsf{L}_Y$ means that the two logics are the same and $\mathsf{L}_X \longrightarrow \mathsf{L}_Y$ means that L_X is a proper subset of L_Y.

The first proposition in the section shows that the assumptions of reflexivity, totalness, and limitedness have no import to the logic. [3]

Proposition 2. *For every preference model* $M = (W, \succeq, V)$ *and* $s \in W$*, there is a reflexive, total, and limited preference model* $M' = (W', \succeq', V')$ *and* $s' \in W'$ *such that for all formulas* φ*,* $M, s \models \varphi$ *iff* $M', s' \models \varphi$*.*

Proof. We construct $M' = (W', \succeq', V')$ as follows:

- $W' = \{(a, n) \mid a \in W, n \in \{0, 1\}\}$
- $(a, n) \succeq' (b, m)$ iff $a \succeq b$ or $n \geq m$
- $V'(p) = \{(a, n) \mid a \in V(p)\}$

Let $s' = (s, 0)$. We first show that

$$\text{for all } a \in W \text{ and } n \in \{0, 1\},\ a \models \varphi \text{ iff } (a, n) \models \varphi \tag{$*$}$$

[3] This result is similar to [14, Theorem 3.3], but, of course, we work on different formal languages. We thank an anonymous referee for pointing out this.

We show only the case for $S(\psi/\varphi)$. *From left to right.* Suppose $a \models S(\psi/\varphi)$. Let $(b,m) \models \psi \wedge \varphi$. To show that $(b,m) \in opt_{\succeq'}(\llbracket\varphi\rrbracket_{M'})$, let $(c,l) \models \varphi$ be arbitrary. If $(b,m) \not\succeq' (c,l)$, then $b \not\succeq c$. Note that $b \models \psi \wedge \varphi$ and $c \models \varphi$ by IH. Thus, $\llbracket\psi \wedge \varphi\rrbracket_M \not\subseteq opt_{\succeq}(\llbracket\varphi\rrbracket_M)$, contradicting the assumption. Hence, $(b,m) \succeq' (c,l)$. Thus, $(b,m) \in opt_{\succeq'}(\llbracket\varphi\rrbracket_{M'})$. Therefore, $(a,n) \models S(\psi/\varphi)$. *From right to left.* Suppose $(a,n) \models S(\psi/\varphi)$. Let $b \models \psi \wedge \varphi$. To show that $b \in opt_{\succeq}(\llbracket\varphi\rrbracket_M)$, let $c \models \varphi$ be arbitrary. Suppose, toward a contradiction, that $b \not\succeq c$. Then $(b,0) \not\succeq' (c,1)$. Note that, by IH, $(b,0) \models \psi \wedge \varphi$ and $(c,1) \models \varphi$. Hence, $\llbracket\psi \wedge \varphi\rrbracket_{M'} \not\subseteq opt_{\succeq'}(\llbracket\varphi\rrbracket_{M'})$, contradicting the assumption. Hence, $b \succeq c$ and thus $b \in opt_{\succeq}(\llbracket\varphi\rrbracket_M)$. Therefore, $a \models S(\psi/\varphi)$.

It remains to show that M' satisfies the required properties. Clearly, M' is total (and therefore reflexive). For all formulas φ with $\llbracket\varphi\rrbracket_{M'} \neq \varnothing$, it follows from $(*)$ that $\llbracket\varphi\rrbracket_{M'} \cap (W \times \{1\}) \neq \varnothing$ (since $(s,0) \models \varphi$ iff $(s,1) \models \varphi$ for all $s \in W$). Let $(s,1) \models \varphi$. It is easy to see that $(s,1) \in opt_{\succeq'}(\llbracket\varphi\rrbracket_{M'})$. Therefore, $opt_{\succeq}(\llbracket\varphi\rrbracket_M) \neq \varnothing$. Thus, M' is limited.

The next proposition shows that, by imposing only transitivity on the models, we obtain a stronger logic. Let (Tran) be the following formula:

$$S(\varphi/\varphi \vee \psi) \wedge S(\psi/\psi \vee \chi) \wedge \Diamond\psi \rightarrow S(\varphi/\varphi \vee \chi) \qquad \text{(Tran)}$$

Proposition 3. *The following hold:*

(1) (Tran) *is valid on the class of all transitive preference models.*
(2) (Tran) *is invalid on the class of all preference models.*

Proof. (1) Let $M = (W, \succeq, V)$ be a transitive preference model. Suppose:

(i) $M, s \models \Diamond\psi$,
(ii) $M, s \models S(\varphi/\varphi \vee \psi)$,
(iii) $M, s \models S(\psi/\psi \vee \chi)$,

Let $t_1 \models \varphi \wedge (\varphi \vee \chi)$ (i.e., $t_1 \models \varphi$) and $t_2 \models \varphi \vee \chi$. To show $M, s \models S(\varphi/\varphi \vee \chi)$, it suffices to show that $t_1 \succeq t_2$. If $t_2 \models \varphi$, since $s \models S(\varphi/\varphi \vee \psi)$, $t_1 \succeq t_2$ by semantics. Otherwise, $t_2 \models \chi$, from (i) and (iii) it follows that there must be $t_3 \models \psi$ such that $t_3 \succeq t_2$. Note that $t_1 \succeq t_3$ by (ii). Thus, by transitivity, $t_1 \succeq t_2$.

(2) A counter model is provided in Figure 3 where $s_1 \not\models S(p/p \vee q) \wedge S(q/q \vee r) \wedge \Diamond q \rightarrow S(p/p \vee r)$.

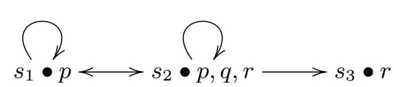

Fig. 3. A preference model M.

Proposition 4. *For every transitive preference model $M = (W, \succeq, V)$ and $s \in W$, there is a reflexive and transitive preference model $M' = (W', \succeq', V')$ and $s' \in W'$ such that, for all formulas φ, $M, s \models \varphi$ iff $M', s' \models \varphi$.*

Proof. We construct M' as follows:

- $W' = W \times \{0, 1\}$ (the elements of W are denoted by $s_0, s_1, t_0, t_1, \dots$)
- $\succeq' = \{(s_i, t_j) \mid s \succeq t\} \cup \{(s_i, s_i) \mid s_i \in W'\}$
- $s_i \in V'(p)$ iff $s \in V(p)$

Obviously, $\succeq'$ is reflexive and transitive. It can be verified that, for all $s \in W$ and $i \in \{0, 1\}$, $M, s \models \varphi$ iff $M', s_i \models \varphi$ for all formulas φ. Here we show only the case for $S(\psi/\varphi)$. The direction from left to right is trivial. From right to left. Suppose $M', s_i \models S(\psi/\varphi)$. Let $M, u \models \psi \wedge \varphi$ and $M, t \models \varphi$. Then, by IH, $M', u_0 \models \psi \wedge \varphi$ and $M', t_1 \models \varphi$. Since $M', s_i \models S(\psi/\varphi)$, $u_0 \succeq' t_1$. By the definition of $\succeq'$, it must be $u \succeq t$. Hence, $M, s \models S(\psi/\varphi)$.

Next we show that, given transitivity, the addition of either limitedness or totalness changes the logic. For this, we need the following lemma. [4]

Lemma 1. *The following holds for every preference model $M = (W, \succeq, V)$ that is either transitive and limited or transitive and total: for all $X, Y \subseteq W$, if $X \subseteq opt_{\succeq}([\![\psi]\!])$ and $Y \subseteq opt_{\succeq}([\![\theta]\!])$, then $X \subseteq opt_{\succeq}([\![\psi \vee \theta]\!])$ or $Y \subseteq opt_{\succeq}([\![\psi \vee \theta]\!])$.*

Proof. Suppose M is transitive and limited. We consider two cases: (1) $X = \varnothing$ or $Y = \varnothing$. Then $X \subseteq opt_{\succeq}([\![\varphi \vee \psi]\!])$ or $Y \subseteq opt_{\succeq}([\![\varphi \vee \psi]\!])$. (2) $X \neq \varnothing$ and $Y \neq \varnothing$. Then $[\![\psi \vee \theta]\!] \neq \varnothing$. By limitedness, $opt_{\succeq}([\![\psi \vee \theta]\!]) \neq \varnothing$. Let $s \in opt_{\succeq}([\![\psi \vee \theta]\!])$. Then $s \in [\![\psi \vee \theta]\!]$. Without loss of generality, we assume $s \in [\![\psi]\!]$. For every $t \in X$, $t \in opt_{\succeq}([\![\psi]\!])$. Thus, $t \succeq s$. By transitivity, $t \in opt_{\succeq}([\![\psi \vee \theta]\!])$. Hence, $X \subseteq opt_{\succeq}([\![\psi \vee \theta]\!])$.

The case where M is transitive and total remains to be considered. We split into the same two subcases as above. We consider only the subcase where both X and Y are nonempty. Let $x \in X$ and $y \in Y$. Since M is total, $x \succeq y$ or $y \succeq x$. (a) If $x \succeq y$, let $x' \in X$ be arbitrary. For all $z \in [\![\psi \vee \theta]\!]$, $z \in [\![\psi]\!]$ or $z \in [\![\theta]\!]$. If $z \in [\![\psi]\!]$, then $x' \succeq z$ since $X \subseteq opt_{\succeq}([\![\psi]\!])$. If $z \in [\![\theta]\!]$, then $y \succeq z$ as $Y \subseteq opt_{\succeq}([\![\theta]\!])$. Since $x' \succeq x$, $x \succeq y$ and $y \succeq z$, $x' \succeq z$ by transitivity. Therefore, $x' \in opt_{\succeq}([\![\psi \vee \theta]\!])$, i.e., $X \subseteq opt_{\succeq}([\![\psi \vee \theta]\!])$. (b) Otherwise, $y \succeq x$. Similarly, we can show that $Y \subseteq opt_{\succeq}([\![\psi \vee \theta]\!])$.

In what follows, let (Lim) be the formula below:

$$S(\varphi/\psi) \wedge S(\chi/\theta) \rightarrow (S(\varphi \wedge \psi/\psi \vee \theta) \vee S(\chi \wedge \theta/\psi \vee \theta)) \qquad \text{(Lim)}$$

Proposition 5. *The following hold:*

(1) (Lim) *is valid on the class of all transitive and limited preference models.*
(2) (Lim) *is valid on the class of all transitive and total preference models.*
(3) (Lim) *is invalid on the class of all transitive models.*

Proof. (1) and (2) follow immediately from Lemma 1. For (3), a counter-model is provided in Figure 4 where $s \not\models S(p/p) \wedge S(\neg p/\neg p) \rightarrow (S(p/p \vee \neg p) \vee S(\neg p/p \vee \neg p))$.

Fig. 4. A counter model M.

The last proposition in the section shows that, given transitivity and limitedness, totalness has no import to the logic. In the proof, we use the same model construction method as in the proof of [13, Prop. 13]:

Proposition 6. *For all transitive and limited preference models $M = (W, \succeq, V)$ and $s \in W$, there is a reflexive, total, transitive, and limited preference model $M' = (W', \succeq', V')$ and $s' \in W'$ such that for all formulas φ, $M, s \models \varphi$ iff $M', s' \models \varphi$.*

Proof. Let $U = \{x \in W \mid \text{there is a formula } \varphi \text{ such that } x \in opt_{\succeq}(\llbracket\varphi\rrbracket_M)\}$. We construct M' as follows:

- $W' = W$ and $V' = V$,
- For all $x, y \in W$:
  - If $x, y \in U$, then $x \succeq' y$ iff $x \succeq y$;
  - If $x \in U$ and $y \notin U$, then $x \succeq' y$;
  - If $x \notin U$ and $y \notin U$, then $x \succeq' y$ and $y \succeq' x$.

We first show that

$$\text{for all } x \in W \text{ and formulas } \varphi,\ M, x \models \varphi \text{ iff } M', x \models \varphi. \qquad (*)$$

We show only the case $S(\psi/\chi)$. *From left to right.* Suppose $M, x \models S(\psi/\chi)$. Then $\llbracket\psi \wedge \chi\rrbracket_M \subseteq opt_{\succeq}(\llbracket\chi\rrbracket_M)$. Let $y \in \llbracket\psi \wedge \chi\rrbracket_{M'}$. Then, by IH, $y \in \llbracket\psi \wedge \chi\rrbracket_M \subseteq opt_{\succeq}(\llbracket\chi\rrbracket_M)$. For every $z \in \llbracket\chi\rrbracket_{M'}$, by IH we have $z \in \llbracket\chi\rrbracket_M$. Thus, $y \succeq z$. If $z \in U$, then $y \succeq' z$ by the definition of $\succeq'$ (note that $y \in U$). Otherwise, $z \notin U$, we also have $y \succeq' z$ by the definition of $\succeq'$. Hence, $y \in opt_{\succeq'}(\llbracket\chi\rrbracket_{M'})$. Therefore, $\llbracket\psi \wedge \chi\rrbracket_{M'} \subseteq opt_{\succeq'}(\llbracket\chi\rrbracket_{M'})$ and thus $M', x \models S(\psi/\chi)$. *From right to left.* Suppose $M', x \models S(\psi/\chi)$. Then $\llbracket\psi \wedge \chi\rrbracket_{M'} \subseteq opt_{\succeq'}(\llbracket\chi\rrbracket_{M'})$. If $\llbracket\chi\rrbracket_M = \varnothing$, then it holds trivially that $M, x \models S(\psi/\chi)$. Otherwise, $\llbracket\chi\rrbracket_M \neq \varnothing$, by the limitedness of $\succeq$ it follows that $opt_{\succeq}(\llbracket\chi\rrbracket_M) \neq \varnothing$. Let $t \in opt_{\succeq}(\llbracket\chi\rrbracket_M)$. Thus, $t \in U$. For each $y \in \llbracket\psi \wedge \chi\rrbracket_M = \llbracket\psi \wedge \chi\rrbracket_{M'}$, since $t \in \llbracket\chi\rrbracket_M = \llbracket\chi\rrbracket_{M'}$, we have $y \succeq' t$ by our assumption that $M', x \models S(\psi/\chi)$. Since $t \in U$, it follows from the definition of $\succeq'$ that $y \succeq t$. As $t \in opt_{\succeq}(\llbracket\chi\rrbracket_M)$, by the transitivity of $\succeq$ we have $y \in opt_{\succeq}(\llbracket\chi\rrbracket_M)$. Therefore, $M, x \models S(\psi/\chi)$.

It remains to show that M' satisfies the required properties. *Transitivity*: Suppose, toward a contradiction, that there are x, y, z such that $x \succeq' y$, $y \succeq' z$, and $x \not\succeq' z$. Since $x \not\succeq' z$, by the definition of $\succeq'$, it can only be the following two cases: (1) $x, z \in U$ and $x \not\succeq z$. Since $x \succeq' y$ and $y \succeq' z$, it must be that $y \in U$, $x \succeq y$, and $y \succeq z$ by the definition of $\succeq'$. Thus, by the transitivity of $\succeq$,

[4] This is a generalization of [13, Lemma 8]. We thank an anonymous referee for pointing out this.

$x \succeq z$ and thus $x \succeq' z$. Contradiction! (2) $x \notin U$ and $z \in U$. Since $x \succeq y$, $y \notin U$. However, it implies that $y \not\succeq z$. Contradiction!

The *limitedness* of M' follows directly from that of M and $(*)$. To show that $\succeq'$ is *total*, let $x, y \in W$. The only non-trivial case is when $x, y \in U$. By the definition of U, $x \in opt_{\succeq}([\![\psi]\!])$ and $y \in opt_{\succeq}([\![\theta]\!])$. By Lemma 1, $x \in opt_{\succeq}([\![\psi \vee \theta]\!])$ or $y \in opt_{\succeq}([\![\psi \vee \theta]\!])$. In either case, $x \succeq y$ or $y \succeq x$. Hence, $x \succeq' y$ or $y \succeq' x$. Thus, $\succeq'$ is total and thus reflexive.

4 Axiomatizations

In this section, we present three Hilbert-style axiom systems for our language $\mathcal{L}$.

Definition 5 (Axiomatizations). *The axiomatization* $\mathbf{DLDS}_0$ *consists of the axioms and rules listed below. The axiomatization* $\mathbf{DLDS}_1$ *is obtained by supplementing* $\mathbf{DLDS}_0$ *with the axiom* (Tran)*. The axiomatization* $\mathbf{DLDS}_2$ *is obtained by supplementing* $\mathbf{DLDS}_1$ *with the axiom* (Lim)*. For each* $x \in \{0, 1, 2\}$*, let the set of* $\mathbf{DLDS}_x$*-theorems be the least set of formulas that contains all instances of the axiom schemas and is closed under the inference rules in* $\mathbf{DLDS}_x$*. If a formula* φ *is a* $\mathbf{DLDS}_x$*-theorem, we write* $\vdash_{\mathbf{DLDS}_x} \varphi$*.*

PL	*All instances of propositional tautologies*
□-K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
□-T	$\Box\varphi \to \varphi$
□-5	$\neg\Box\varphi \to \Box\neg\Box\varphi$
A1	$S(\psi/\varphi) \to \Box S(\psi/\varphi)$
A2	$S(\psi/\varphi) \wedge S(\chi/\varphi) \to S(\psi \vee \chi/\varphi)$
A3	$S(\varphi/\varphi \vee \psi) \to (S(\varphi/\varphi \vee \chi) \to S(\varphi/\varphi \vee \psi \vee \chi))$
A4	$\Box(\psi \to \chi) \to (S(\chi/\varphi) \to S(\psi/\varphi))$
A5	$\Box(\psi \to \varphi) \to (S(\chi/\varphi) \to S(\chi/\psi))$
A6	$\Box\neg(\psi \wedge \varphi) \to S(\psi/\varphi)$
MP	*From* φ *and* $\varphi \to \psi$*, infer* ψ
Nec	*From* φ*, infer* $\Box\varphi$

The following result on $\mathbf{DLDS}_2$ will be used in the completeness proof.

Proposition 7. *The following holds for all integers* $n \geq 1$*:*

$$\vdash_{\mathbf{DLDS}_2} \left(\bigwedge_{1 \leq i \leq n} \Box(\varphi_i \to \psi_i) \wedge S(\varphi_i/\psi_i) \right) \to \bigvee_{1 \leq i \leq n} S(\varphi_i / \bigvee_{1 \leq i \leq n} \psi_i).$$

Proof. Induction on n. The axiom (Lim) is used in the inductive step.

It is not hard to verify that all the axioms and rules in $\mathbf{DLDS}_0$ are valid or preserve validity on the class of all preference models. Therefore,

Proposition 8. *The axiomatization* $\mathbf{DLDS}_0$ *is sound with respect to the class of all preference models.*

5 Canonical Model

In this and the next sections, we establish the completeness results for $\mathbf{DLDS}_0$–$\mathbf{DLDS}_2$ with respect to the intended model classes. We will focus on the *weak* completeness of the three systems. Following the classical canonical model method, we construct the canonical model using maximal consistent sets (MCS). However, since we aim only to prove weak completeness, we assume a finite number of propositional variables in the language. The main novelty of our proof lies in using a propositional formula as the "name" of each MCS t, which we denote by $Nom(t)$. This allows us to reduce the truth of formulas $S(\psi/\varphi)$ to that of formulas like $S(Nom(t)/\varphi)$ (the latter can be semantically regarded as a monadic modality).

Throughout this section, let $A \subset Prop$ be a finite nonempty set of atoms and $\mathcal{L}_A$ be the sublanguage of $\mathcal{L}$ restricted to atoms in A. Let Λ range over $\{\mathbf{DLDS}_0, \mathbf{DLDS}_1, \mathbf{DLDS}_2\}$. A set Γ of formulas in $\mathcal{L}_A$ is said to be Λ-*consistent* if there are no $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_\Lambda \varphi_1 \wedge \cdots \wedge \varphi_n \to \bot$. Γ is *maximal* Λ-*consistent* if Γ is Λ-consistent, and any set of formulas in $\mathcal{L}_A$ properly containing Γ is Λ-inconsistent. The standard properties of maximal consistent sets are taken as granted, i.e., for all Λ-maximal consistent sets Γ and formulas $\varphi, \psi \in \mathcal{L}_A$:

- $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$, and
- $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.

The Lindenbaum lemma also holds, which claims that every Λ-consistent set of formulas in $\mathcal{L}_A$ can be extended to a maximal Λ-consistent one. The set of all Λ-maximal consistent sets in $\mathcal{L}_A$ will be denoted by MCS_Λ.

In what follows, we fix a $w \in MCS_\Lambda$ and let $\Box w = \{\varphi \in \mathcal{L}_A \mid \Box\varphi \in w\}$. For each $s \in MCS_\Lambda$, let $Nom(s)$ be the formula $\bigwedge_{p \in s} p \wedge \bigwedge_{p \in A \setminus s} \neg p$. Note that $Nom(s) \in \mathcal{L}_A$ and $Nom(s) \in s$ (by the properties of maximal consistent sets).

Lemma 2. *For all $s, t \in MCS_\Lambda$, if $Nom(t) \in s$ then $Nom(s) = Nom(t)$.*

Proof. It suffices to show that for all $p \in A$, $p \in s$ iff $p \in t$. For all $p \in A$, if $p \in t$ then $Nom(t) \to p$ is a propositional tautology. Since $Nom(t) \in s$, $p \in s$. Conversely, if $p \notin t$ then $Nom(t) \to \neg p$ is a propositional tautology. Since $Nom(t) \in s$, $\neg p \in s$. Thus, $p \notin s$.

Now we are ready to define the canonical model.

Definition 6 (Canonical model). *The canonical model for $w \in MCS_\Lambda$ is a structure $M(w) = (W, \succeq, V)$ such that*

- *$W = \{s \in MCS_\Lambda \mid \Box w \subseteq s\}$.*
- *$s \succeq t$ iff there is $\varphi \in \mathcal{L}_A$ such that $\varphi \in t \cap s$ and $S(Nom(s)/\varphi) \in w$.*
- *For all $p \in Prop$, $s \in V(p)$ iff $p \in s$.*

It is clear that $M(w)$ is a preference model and $w \in W$. For each formula $\psi \in \mathcal{L}_A$, let $\|\psi\| = \{s \in W \mid \psi \in s\}$ and $[\![\psi]\!] = [\![\psi]\!]_{M(w)}$.

Lemma 3. *The following holds for all* $s \in W$*:*

(1) for all formulas $\Box\varphi \in \mathcal{L}_A$*,* $\Box\varphi \in s$ *iff* $\Box\varphi \in w$*;*
(2) for all formulas $S(\psi/\varphi) \in \mathcal{L}_A$*,* $S(\psi/\varphi) \in s$ *iff* $S(\psi/\varphi) \in w$*.*

Proof. Since $\Box\varphi$ is an $S5$-modality, (1) follows from the standard argument.

(2): The direction from right to left follows directly from the axiom A1 and the definition of W. For the converse, if $S(\psi/\varphi) \in s$ then $\Box S(\psi/\varphi) \in s$ by A1. From (1) it follows that $\Box S(\psi/\varphi) \in w$. Hence, $S(\psi/\varphi) \in w$ by $\Box$-T.

Lemma 4. *For all* $s, t \in W$*, if* $Nom(s) = Nom(t)$ *then* $s = t$*. Thus, by Lemma 2, if* $Nom(t) \in s$ *then* $s = t$*.*

Proof. Suppose $Nom(s) = Nom(t)$. We show that for all formulas $\varphi \in \mathcal{L}_A$, $\varphi \in s$ iff $\varphi \in t$. The cases for atoms and Boolean connectives are trivial. The cases $\Box\varphi$ and $S(\psi/\varphi)$ follow directly from Lemma 3.

Note that Lemma 4 implies that W is finite. The next lemma is standard.

Lemma 5. *The following hold:*

(1) for all propositional formulas $\pi \in \mathcal{L}_A$ *and* $s \in W$*,* $\pi \in s$ *iff* $M(w), s \models \pi$*;*
(2) for all formulas $\Box\varphi \in \mathcal{L}_A$*,* $\Box\varphi \in w$ *iff* $\varphi \in t$ *for all* $t \in W$*.*

Lemma 6. *For all formulas* $\psi \in \mathcal{L}_A$*,* $\Box(\psi \leftrightarrow \bigvee_{t \in \|\psi\|} Nom(t)) \in w$*.*

Proof. By Lemma 5(2), it suffices to show that $\psi \leftrightarrow \bigvee_{t \in \|\psi\|} Nom(t) \in s$ for all $s \in W$. That is, $\psi \in s$ iff $\bigvee_{t \in \|\psi\|} Nom(t) \in s$. The direction from left to right is straightforward (as $Nom(s) \in s$). For the converse, let $s \in W$ be such that $\bigvee_{t \in \|\psi\|} Nom(t) \in s$. Then there must be $t \in \|\psi\|$ such that $Nom(t) \in s$. Thus, by Lemma 4, $s = t \in \|\psi\|$.

Lemma 7. *For all formulas* $S(\psi/\varphi) \in \mathcal{L}_A$*,* $S(\psi/\varphi) \in w$ *iff for all* $t \in \|\psi\|$*,* $S(Nom(t)/\varphi) \in w$*.*

Proof. From left to right. Let $t \in \|\psi\|$. By Lemma 6, $\Box(Nom(t) \to \psi) \in w$. Since $S(\psi/\varphi) \in w$, $S(Nom(t)/\varphi) \in w$ by A4. From right to left. Suppose that for all $t \in \|\psi\|$, $S(Nom(t)/\varphi) \in w$. Then, by A2 and A6, $S(\bigvee_{t \in \|\psi\|} Nom(t)/\varphi) \in w$. Since $\Box(\psi \to \bigvee_{t \in \|\psi\|} Nom(t)) \in w$ by Lemma 6, $S(\psi/\varphi) \in w$ by A4.

Lemma 8. *For all formulas* $S(\psi/\varphi) \in \mathcal{L}_A$*,* $w \models S(\psi/\varphi)$ *iff for all* $t \in [\![\psi]\!]$*,* $w \models S(Nom(t)/\varphi)$*.*

Proof. From left to right. Suppose $w \models S(\psi/\varphi)$. Let $t \in [\![\psi]\!]$ and $s \models Nom(t) \wedge \varphi$. Since $s \models Nom(t)$, $Nom(t) \in s$ by Lemma 5(1). Therefore, $s = t$ by Lemma 4. Since $w \models S(\psi/\varphi)$ and $s \models \psi \wedge \varphi$, $s \in opt_{\succeq}([\![\varphi]\!])$. Thus, $w \models S(Nom(t)/\varphi)$. From right to left. Suppose that for all $t \in [\![\psi]\!]$, $w \models S(Nom(t)/\varphi)$. Let $s \in [\![\psi \wedge \varphi]\!]$. Since $w \models S(Nom(s)/\varphi)$ and $s \models Nom(s) \wedge \varphi$, $s \in opt_{\succeq}([\![\varphi]\!])$. Therefore, $w \models S(\psi/\varphi)$.

Lemma 9 (Truth). *For all $s \in W$ and $\alpha \in \mathcal{L}_A$, $\alpha \in s$ iff $M(w), s \models \alpha$.*

Proof. Induction on the structure of α. We show only the case for $S(\psi/\varphi)$. We first show the following claim:

Claim. For all $s \in W$ and $\varphi \in \mathcal{L}_A$, $S(Nom(s)/\varphi) \in w$ iff $M(w), w \models S(Nom(s)/\varphi)$.

Proof. From left to right. Suppose $S(Nom(s)/\varphi) \in w$. Let $t \models Nom(s) \wedge \varphi$. By Lemma 5(1), $Nom(s) \in t$. Thus, by Lemma 4, $t = s$. It then suffices to show that $s \succeq s'$ for all $s' \models \varphi$. Since $s' \models \varphi$ and $s \models \varphi$, $\varphi \in s \cap s'$ by IH. As we assume $S(Nom(s)/\varphi) \in w$, $s \succeq s'$ by the construction of $\succeq$.

From right to left. Suppose $S(Nom(s)/\varphi) \notin w$. We need to show that there is $s' \in W$ such that $s' \models Nom(s) \wedge \varphi$ and $s' \notin opt_{\succeq}([\![\varphi]\!])$. Since $s \models Nom(s)$, it suffices to show that $s \models \varphi$ and $s \notin opt_{\succeq}([\![\varphi]\!]) = opt_{\succeq}(\|\varphi\|)$. Since $S(Nom(s)/\varphi) \notin w$, by A6, $\Box\neg(Nom(s) \wedge \varphi) \notin w$. Thus, by Lemma 5(2), there must be $t \in W$ with $Nom(s) \wedge \varphi \in t$. Thus, $s = t$ (by Lemma 4). Since $\varphi \in t = s$, $s \models \varphi$ by IH.

It remains to show that $s \notin opt_{\succeq}(\|\varphi\|)$, i.e., there is $t \in W$ such that $\varphi \in t$ and $s \not\succeq t$. We consider two cases:

(1) For all $S(Nom(s)/\psi) \in w$, $\psi \notin s$. Then $s \not\succeq s$ by the construction of $\succeq$.
(2) Otherwise, by the Lindenbaum lemma and the construction of $\succeq$, it suffices to show that the following set of formulas is consistent:

$$\Gamma = \Box w \cup \{\varphi\} \cup \{\neg\psi \mid S(Nom(s)/\psi) \in w \,\&\, \psi \in s\}$$

Suppose, toward a contradiction, that Γ is inconsistent. Then there must be $\Box\chi_1, \ldots, \Box\chi_m \in w$ and $S(Nom(s)/\psi_1), \ldots, S(Nom(s)/\psi_n) \in w$ (with $n \geq 1$ and each $\psi_i \in s$) such that

$$\vdash_\Lambda \chi_1 \wedge \cdots \wedge \chi_m \to (\varphi \to (\psi_1 \vee \cdots \vee \psi_n))$$

Since $\Box$ is a normal modality, $\Box(\varphi \to \psi_1 \vee \cdots \vee \psi_n) \in w$ $(*)$. For each ψ_i, since $s \in \|\psi_i\|$, it follows from Lemma 6 that $\Box(Nom(s) \to \psi_i) \in w$. Thus, $\Box(Nom(s) \vee \psi_i \to \psi_i) \in w$. Since $S(Nom(s)/\psi_i) \in w$, by A5 we have $S(Nom(s)/Nom(s) \vee \psi_i) \in w$ for all $1 \leq i \leq n$. Using A3, we derive that $S(Nom(s)/Nom(s) \vee \psi_1 \vee \cdots \vee \psi_n) \in w$. From $(*)$ and A5, it follows that $S(Nom(s)/\varphi) \in w$, contradicting our assumption.

Given the above claim, we are ready to show the inductive case $S(\psi/\varphi)$:

$S(\psi/\varphi) \in s$
iff $S(\psi/\varphi) \in w$ (Lemma 3)
iff for all $t \in \|\psi\|$, $S(Nom(t)/\varphi) \in w$ (Lemma 7)
iff for all $t \in [\![\psi]\!]$, $w \models S(Nom(t)/\varphi)$ (IH and the above claim)
iff $w \models S(\psi/\varphi)$ (Lemma 8)
iff $s \models S(\psi/\varphi)$ (by semantics)

6 Completeness

Given the canonical model and the truth lemma in the previous section, we are ready to present our completeness results for the three systems $\mathbf{DLDS}_0$, $\mathbf{DLDS}_1$, and $\mathbf{DLDS}_2$.

6.1 $\mathbf{DLDS_0}$

The completeness of $\mathbf{DLDS}_0$ with respect to the class of all preference models follows from the routine argument (the soundness has been established in Proposition 8). Together with Proposition 2, it implies that $\mathbf{DLDS}_0$ is also sound and complete with respect to the model class $\mathbb{RCT}$.

Theorem 1 (Completeness of $\mathbf{DLDS}_0$). *The axiomatization $\mathbf{DLDS}_0$ is sound and weakly complete with respect to the class of all preference models and the class of all reflexive, total, and limited preference models.*

Since the canonical model is finite, the decidability of $\mathbf{DLDS}_0$ is straightforward:

Proposition 9. *The theoremhood problem in $\mathbf{DLDS}_0$ is decidable.*

6.2 $\mathbf{DLDS_1}$

To establish the completeness of $\mathbf{DLDS}_1$ with respect to the class of all transitive preference models, we need the following lemma (which is called the canonicity lemma in the literature):

Lemma 10. *If $w \in MCS_{\mathbf{DLDS_1}}$, then $M(w)$ is a transitive preference model.*

Proof. Suppose $w \in MCS_{\mathbf{DLDS_1}}$. Suppose $s \succeq t$ and $t \succeq u$. Then there must be $\varphi, \psi \in \mathcal{L}_A$ such that $\varphi \in s \cap t$, $S(Nom(s)/\varphi) \in w$, $\psi \in t \cap u$, and $S(Nom(t)/\psi) \in w$. We need to show that $s \succeq u$.

Since $s, t \in \|\varphi\|$, it follows from Lemma 6 that $\Box(Nom(s) \vee Nom(t) \to \varphi) \in w$. Since $S(Nom(s)/\varphi) \in w$, by A5, $S(Nom(s)/Nom(s) \vee Nom(t)) \in w$ $(*)$. On the other hand, since $t \in \|\psi\|$, $\Box(Nom(t) \to \psi) \in w$ and thus $\Box(Nom(t) \vee \psi \to \psi) \in w$. As $S(Nom(t)/\psi) \in w$, $S(Nom(t)/Nom(t) \vee \psi) \in w$ by A5 $(**)$. Note that since $Nom(t) \in t$, $\Diamond Nom(t) \in w$ by Lemma 5(2). By $(*)$ and $(**)$, it follows from (Tran) that $S(Nom(s)/Nom(s) \vee \psi) \in w$. Since $Nom(s) \vee \psi \in s \cap u$, by the definition of $\succeq$ it follows that $s \succeq u$.

Theorem 2. *The axiomatization $\mathbf{DLDS}_1$ is sound and weakly complete with respect to the class of all transitive preference models and the class of all reflexive and transitive preference models.*

Proof. The soundness of $\mathbf{DLDS}_1$ with respect to the class of all transitive preference models follows from Proposition 8 and Proposition 3(1). For completeness, let $\varphi \in \mathcal{L}$ be a $\mathbf{DLDS}_1$-consistent formula, and let A be the set of atoms occurred in φ. By the Lindenbaum lemma, there is $w \in MCS_{\mathbf{DLDS_1}}$ such that $\varphi \in w$.

By the truth lemma, $M(w), w \models \varphi$. Note that $M(w)$ is transitive by Lemma 10. Hence, φ is satisfiable on the class of all transitive preference models.

The soundness and completeness of $\mathbf{DLDS}_1$ with respect to the class of all reflexive and transitive models follows from the above and Proposition 4.

Proposition 10. *The theoremhood problem in* $\mathbf{DLDS}_1$ *is decidable.*

6.3 $\mathbf{DLDS_2}$

The completeness of $\mathbf{DLDS}_2$ is more involved. We aim to show that $\mathbf{DLDS}_2$ is complete with respect to the class of all transitive and limited preference models. However, there is no guarantee that the canonical model for $\mathbf{DLDS}_2$ is limited. Fortunately, we can transform the canonical model for $\mathbf{DLDS}_2$ into an equivalent limited preference model. In the following, we describe the transformation.

Definition 7. *Given a preference model* $M = (W, \succeq, V)$*, the* transformed model of M*, notation* $\tau(M)$*, is a preference model* $\tau(M) = (W', \succeq', V')$ *where:*

(1) $W' = W \times \{0, 1\}$ *(the elements of* W' *are denoted by* $s_0, s_1, t_0, t_1, \ldots$*);*
(2) $\succeq' = \{(s_i, t_j) \mid s \succeq t\} \cup \{(s_1, t_0), (s_1, t_1) \mid$ *for all* $u \in W$*,* $u \succeq s$ *implies* $u \succeq t\}$*;*
(3) $s_i \in V(p)$ *iff* $s \in V(p)$*.*

It is essential that the above transformation preserves the truth of formulas:

Proposition 11. *For every preference model* $M = (W, \succeq, V)$ *and its transformed model* $\tau(M) = (W', \succeq', V')$*, it holds that, for all* $s \in W$*,* $i \in \{0, 1\}$*, and formulas* φ*,* $M, s \models \varphi$ *iff* $\tau(M), s_i \models \varphi$*.*

Proof. Induction on the structure of φ. We consider only the inductive case $S(\psi/\varphi)$. *From left to right.* Suppose $M, s \models S(\psi/\varphi)$. Let $x_i \in [\![\psi \wedge \varphi]\!]_{\tau(M)}$ and $y_j \in [\![\varphi]\!]_{\tau(M)}$. By IH, $x \in [\![\psi \wedge \varphi]\!]_M$ and $y \in [\![\varphi]\!]_M$. Since $M, s \models S(\psi/\varphi)$, $x \succeq y$. Thus, $x_i \succeq' y_j$. Therefore, $\tau(M), s_i \models S(\psi/\varphi)$. *From right to left.* Suppose $M, s \not\models S(\psi/\varphi)$. Then there must be $x \in [\![\psi \wedge \varphi]\!]_M$ and $y \in [\![\varphi]\!]_M$ such that $x \not\succeq y$. By the definition of $\succeq'$, $x_0 \not\succeq' y_1$. Note that $\tau(M), x_0 \models \psi \wedge \varphi$ and $\tau(M), y_1 \models \varphi$ by IH. Hence, $\tau(M), s_i \not\models S(\psi/\varphi)$.

For the transformed model to be limited, the original model needs to satisfy a property. We call a preference model $M = (W, \succeq, V)$ *almost-limited* if, for all formulas φ, $[\![\varphi]\!]_M \neq \varnothing$ implies that there is $s \in [\![\varphi]\!]_M$ such that, for all $t \in W$, if $t \succeq s$ then $t \succeq u$ for all $u \in [\![\varphi]\!]_M$.

Proposition 12. *Given a transitive and almost-limited preference model* $M = (W, \succeq, V)$*, the transformed model of* M *is a transitive and limited preference model.*

Proof. We first show that $\succeq'$ is transitive. Let $s_i \succeq' t_j$ and $t_j \succeq' u_k$. We consider the following cases:

(1) $s \succeq t$ and $t \succeq u$. Then $s \succeq u$ by the transitivity of $\succeq$. Hence, $s_i \succeq' u_k$.

(2) $s \not\succeq t$ and $t \succeq u$. Then $i = 1$. To see $s_1 \succeq' u_k$, it suffices to show that $v \succeq s$ implies $v \succeq u$ for all $v \in W$. Let $v \succeq s$. Since $s_1 \succeq' t_j$ and $s \not\succeq t$, by the definition of $\succeq'$, $v \succeq t$. Thus, $v \succeq u$ by the transitivity of $\succeq$.
(3) $s \succeq t$ and $t \not\succeq u$. Since $t_j \succeq' u_k$ and $t \not\succeq u$, by definition it follows that $s \succeq t$ implies $s \succeq u$. Thus, $s \succeq u$. Thus, $s_i \succeq u_k$.
(4) $s \not\succeq t$ and $t \not\succeq u$. Then $i = j = 1$. To see $s_1 \succeq' u_k$, it suffices to show that $v \succeq s$ implies $v \succeq u$ for all $v \in W$. Suppose $v \succeq s$. Since $s_1 \succeq' t_1$ and $s \not\succeq t$, $v \succeq t$. Since $t_1 \succeq u_k$ and $t \not\succeq u$, $v \succeq u$.

It remains to show that $\tau(M)$ is limited. Let $[\![\varphi]\!]_{\tau(M)} \neq \varnothing$. By Proposition 11, $[\![\varphi]\!]_{\tau(M)} = [\![\varphi]\!]_M \times \{0,1\}$. Thus, $[\![\varphi]\!]_M \neq \varnothing$. Since M is almost-limited, there is $s \in [\![\varphi]\!]_M$ such that, for all $t \in W$, if $t \succeq s$ then $t \succeq u$ for all $u \in [\![\varphi]\!]_M$. Thus, by the definition of $\succeq'$, $s_1 \succeq' u_0$ and $s_1 \succeq' u_1$ for all $u \in [\![\varphi]\!]_M$. Since $[\![\varphi]\!]_{\tau(M)} = [\![\varphi]\!]_M \times \{0,1\}$, we conclude that $s_1 \in opt_{\succeq'}([\![\varphi]\!]_{\tau(M)})$.

Now we are ready to show the completeness of $\mathbf{DLDS}_2$. Given the previous two propositions, it suffices to show that the canonical model for $\mathbf{DLDS}_2$ is transitive and almost-limited.

Lemma 11. *If w is a $\mathbf{DLDS}_2$-maximal consistent set, then the canonical model for w, $M(w) = (W, \succeq, V)$, is transitive and almost-limited.*

Proof. The transitivity follows from Proposition 10. To show that $M(w)$ is almost-limited, it suffices to show that, for all $\varnothing \neq U \subseteq W$, there is $s \in U$ such that, for all $t \in W$, if $t \succeq s$ then $t \succeq u$ for all $u \in U$. Suppose, toward a contradiction, that for all $s \in U$, there are $s^* \in W$ and $s^\dagger \in U$ such that $s^* \succeq s$ and $s^* \not\succeq s^\dagger$. We are going to show that for some $t \in U$, $t^* \succeq y$ for all $y \in U$ (which contradicts that $t^* \not\succeq t^\dagger$).

For each $s \in U$, since $s^* \succeq s$, by the definition of the canonical model, there is $\varphi_s \in s^* \cap s$ such that $S(Nom(s^*)/\varphi_s) \in w$. Note also that, since $\varphi_s \in s^*$, $\Box(Nom(s^*) \to \varphi_s) \in w$ by Lemma 6. Thus, it follows from Proposition 7 that

$$\bigwedge_{s\in U} S\big(Nom(s^*)/\varphi_s\big) \to \bigvee_{t\in U} S\Big(Nom(t^*)/\bigvee_{s\in U}\varphi_s\Big) \in w$$

Thus, $\bigvee_{t\in U} S\Big(Nom(t^*)/\bigvee_{s\in U}\varphi_s\Big) \in w$. This implies that there is $t \in U$ such that $S(Nom(t^*)/\bigvee_{s\in U}\varphi_s) \in w$. For each $y \in U$, since $\bigvee_{s\in U}\varphi_s \in t^* \cap y$, $t^* \succeq y$ by the definition of the canonical model. This contradicts the fact that $t^* \not\succeq t^\dagger$.

Theorem 3. $\mathbf{DLDS}_2$ *is sound and weakly complete with respect to*

(1) the class of all transitive and limited preference models;
(2) the class of all transitive, limited, and total (thus reflexive) preference models;
(3) the class of all transitive and total preference models.

Proof. (1) Straightforward. (2) follows from (1) and Proposition 6.

(3) The soundness follows from Propositions 8, 3(1), and 5(2). For completeness, note that $\mathsf{L}_{\mathbb{TC}} \subseteq \mathsf{L}_{\mathbb{TRCL}}$. Thus, the completeness of $\mathbf{DLDS}_2$ with respect to the class of all transitive and total preference models follows from (2).

Proposition 13. *The theoremhood problem in* $\mathbf{DLDS}_2$ *is decidable.*

$$\begin{array}{ccccccc} \mathsf{L}(=\mathbf{DLDS}_0) & \longrightarrow & \mathsf{L}_{\mathbb{T}}(=\mathbf{DLDS}_1) & \longrightarrow & \mathsf{L}_{\mathbb{TL}}(=\mathbf{DLDS}_2) & = & \mathsf{L}_{\mathbb{TRCL}} \\ \| & & \| & & & & \| \\ \mathsf{L}_{\mathbb{RCL}} & & \mathsf{L}_{\mathbb{TR}} & & & & \mathsf{L}_{\mathbb{TC}} \end{array}$$

Fig. 5. Summary of the soundness and completeness results, where the same convention is adopted as in Fig. 2.

7 Discussion and Conclusion

In this paper, following the previous work on deontic sufficiency in deontic logic, we introduced and studied the logics of conditionals $S(\psi/\varphi)$ which state that ψ is a sufficient condition for achieving ideality in the context of φ. For the semantics of the operator $S(\psi/\varphi)$, we adopted the preference semantics used in DDL. Like in DDL, preference relations may exhibit different properties. In this paper, we considered four such properties, i.e., reflexivity, totalness, transitivity, and limitedness, and we comprehensively analyzed the logics generated by different combinations of the four properties. Our results are summarized in Fig. 5. In general, we obtained three logics $\mathbf{DLDS}_0$ – $\mathbf{DLDS}_2$ with increasing deductive power, which is analogous to the situation in DDL (where we also have three systems [18]: **E**, **F**, and **G**). However, the main difference is that the operator $S(\psi/\varphi)$ is monotonic, which may make it less interesting than the conditional obligation operator $O(\psi/\varphi)$ as the latter can be used for nonmonotonic reasoning. Another difference is that our language $\mathcal{L}$ does not distinguish between the class of all preference models and the class of limited preference models. Thus, in a certain sense, our language $\mathcal{L}$ can be said to be "weaker" than the language of DDL.

The method used in our completeness proof is, to the best of our knowledge, new in the DDL literature. The general idea is to reduce the truth of the dyadic operator $S(\psi/\varphi)$ to that of a set of monadic operators. Our method has some advantages. First, the construction of the canonical model is relatively straightforward. Second, it allows us to obtain completeness results for different model classes in a modular way. Last, since the canonical model is finite, we automatically derive the decidability of our logics. However, this method seems to be hard to be adapted to obtain strong completeness results. For this, we may need more involved constructions of the canonical model, as we have seen in the literature [7,8,12,13].

For future work, one direction is to incorporate the conditional obligation in the language, which will enable us to reason about both deontic sufficiency and deontic necessity. One can also explore different truth definitions for the operator $S(\psi/\varphi)$, e.g., employing the maximality condition instead of optimality.

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