Strict-Lyapunov-Function-based Design of Distributed Estimators for Consensus on Directed Circular Graphs: An Application to UAVs

Cristiano M. Verrelli, Patrizio Tomei, Mohamed El Arayshi, Ali Youssef, D. M. K. K. Venkateswara Rao, *Member, IEEE*, Hamed Habibi, *Senior Member, IEEE*, and Holger Voos, *Member, IEEE*

***Abstract*— In this letter, a solution to a general distributed identification problem on a directed circular communication graph is proposed. The original use of a new strict Lyapunov function within a persistently excited framework is crucial to prove the consensus achievement for an asymmetric neighbourhood-based decentralized parallel architecture, for which the communication burden is reduced, when compared to the case of symmetrical communication between the neighbours. Experimental results regarding quadrotor unmanned aerial vehicles illustrate the theoretical derivations.**

***Index Terms*— Aerospace, Estimation, Identification, Sensor Networks, Lyapunov methods**

I. INTRODUCTION

RATHER strong attention has been recently paid to the general applicative scenario given by a multisensor network in which estimators have to be designed on the basis of space-distributed sensing [1]–[7] to recover the same parameter vector, *i.e.*, consensus [8]. A similar situation also arises when mobile sensors measure the distribution of an unknown quantity over a field and, depending on the size of the region, visiting every point in the space to collect data [9]–[12] is computationally inefficient or infeasible. The reader is particularly referred to [13] and [14]. The former addresses general graph topologies while using contradiction arguments to prove that the *Cooperative Persistency of Excitation* (PE) *Condition*, namely, the weakest parameter identifiability condition in such a scenario, guarantees exponential consensus [8], [15]. The latter, instead, derives convergence-rate estimates for networks of systems that are interconnected through Persistently Excited (PEd) graphs while recasting the classical consensus paradigm into a problem of stability analysis for systems with PE.

C.M. Verrelli, P. Tomei, M. El Arayshi, and, A. Youssef are with the Electronic Engineering Department of the University of Rome Tor Vergata. D. M. K. K. Venkateswara Rao and H. Voos are with the Automation and Robotics Research Group, Interdisciplinary Centre for Security, Reliability and Trust, University of Luxembourg, Luxembourg. H. Habibi is with the School of Engineering and Energy College of Science, Technology, Engineering and Mathematics. Murdoch University, Perth, WA, Australia. H. Voos is also with the Faculty of Science, Technology, and Medicine (FSTM), Department of Engineering, University of Luxembourg. (Corresponding author: D. M. K. K. Venkateswara Rao.)

In this letter, starting from (and thus expanding) the strict quadratic Lyapunov proposed in [16, Eq. (7)], we do aim at covering the estimation problem on a general directed circular communication graph [17], i.e., a graph consisting of a single cycle with vertices connected in a closed chain and with all the edges being oriented in the same direction – for which an asymmetric neighbourhood-based decentralized parallel architecture has to be designed.

Illustrative application: As an applicative scenario, we shall experimentally explore in Section V, the specific illustrative problem in which a set of p Unmanned Aerial Vehicles (UAVs) moving in the space with (not all zero) positions $(X_i(t), Y_i(t), Z_i(t)) \in \mathbb{R}^3$ at time t, $i = 1, \ldots, p$, face a local identification problem: they have to consistently estimate the space-position (X_o, Y_o, Z_o) of a stationary target object only by engaging communication with the neighbours, circularly and asymetrically, under the condition that each UAV knows its own position in the space, along with its Euclidean distance

$$D_i(t) \quad = \quad \sqrt{(X_i(t) - X_o)^2 + (Y_i(t) - Y_o)^2 + (Z_i(t) - Z_o)^2},$$

from the target object. Now, if the locally measured output[1] $h_i(t) = D_i^2(t) - X_i^2(t) - Y_i^2(t) - Z_i^2(t)$ is put in the form

$$h_i(t) = \overbrace{[-2X_i(t), -2Y_i(t), -2Z_i(t), 1]}^{\phi_i^{\mathrm{T}}(t)} \overbrace{\begin{bmatrix} X_o \\ Y_o \\ Z_o \\ X_o^2 + Y_o^2 + Z_o^2 \end{bmatrix}}^{\Theta} \tag{1}$$

then such an application problem can be framed into a more general consensus framework.

General problem: Define the problem $\mathscr{P}$ as: *Determine p parallel Θ-estimates $\hat{\Theta}^{[i]}(t)$, $i = 1, \ldots, p$ at the nodes v_i of a directed circular communication graph, which all converge exponentially to the unknown constant parameter vector $\Theta \in \mathbb{R}^m$ appearing within the set of equations (1), namely*

$$h_i(t) \quad = \quad \phi_i^{\mathrm{T}}(t)\Theta, \tag{2}$$

[1]It is the difference between the distance of the i^{th} UAV from the target UAV and the distance of the i^{th} UAV from the origin.

where: h_i are the outputs locally measured at each node v_i; $\phi_i(\cdot):\mathbb{R}_0^+\to\mathbb{R}^m$ are the local regressor vectors, $i=1,\ldots,p$, available at each node v_i. According to the above topological structure, each estimation scheme at the node can communicate its own Θ-estimate $\hat{\Theta}^{[i]}(t)$ just with the neighbours over the circular communication graph: agent i sends information to agent $i+1$, and the last agent p communicates with agent 1. Here, measurable and available are meant at any time $t\geq 0$.

State of the art and original contributions: The problem above is partial-knowledge-based (just neighbours are asymmetrically involved) to reduce the burden of information that has to be communicated to the various measurement/estimation nodes of the graph. However, differently from the typical scenario in which the nodes are undirectedly connected in series[2], no symmetrical communication between the neighbours is required. In this respect, it is worth recalling that wireless channels performing communication between nodes might be unreliable, often suffering from impairments such as signal fading and additive noise. This leads to degraded tracking accuracy and compromised swarm coordination, so that a reduction in communication channels is desirable [8], [15], especially when the number of agents increases. On the other hand, in contrast to [14], the set of tailored differential equations for the time-dependent vectors $\hat{\Theta}^{[i]}(t)$, $i=1,\ldots,p$, has to be here designed at each node in order to guarantee (and not assume) PE for the error system under the weakest Θ-identifiability condition of [13]. Notice that PE is strictly related to uniform observability properties (the reader is referred to the [16], [18] and references therein). It thus allows for the definition of strong Lyapunov functions with intrinsic robustness properties that come not only from the exponential stability nature of the equilibrium point but also (and mainly) from its uniform-in-time properties. Indeed, the interval excitation condition (see for instance [19] and [18], as well as Remark 2 of [16]) assumes that parameter identification can be performed on the basis of past information, by assuming that the system structure (including the constant uncertain parameters) is somehow represented by a time portion, whereas it is well known that constant parameters in the theoretical framework reproduce slowly-varying or abruptly-changing parameters in practice, with robustness issues playing a crucial role there. Finally, in contrast to [13] and [20], here an original proof of convergence provides an explicit characterization in terms of a new strict Lyapunov function. Indeed, no weaker contradiction arguments [13] or more involved LMIs [20] are used to prove the exponential consensus, while an original constructive proof of convergence is able to provide an explicit characterization of the exponentially achieved consensus in terms of PE. An important application of the proposed method relies in the coordination of UAVs, in tasks such as target point localization, cooperative tracking, and state estimation under GPS denial [21]. In such tasks, reliable information sharing among UAVs is crucial, however, communication constraints often make fully connected or symmetric networks impractical [22]. The circular communication topology, coupled with the strict-Lyapunov-function-based design, provides a feasible framework that enables each UAV to achieve consensus with reduced communication load. This is more significant when a large number of UAVs operate in environments with limited bandwidth, energy, or when mission constraints impose asymmetric communication links [23]. Furthermore, the approach can be applied to formation control of UAV swarms, and collaborative search-and-rescue missions, such that agents must detect and localize survivors, distributed surveillance tasks requiring persistent coverage of an area, and cooperative target interception or escort missions in defense applications [22]. By ensuring stable distributed estimation over a circular communication graph, the proposed approach supports robust cooperative sensing, enhancing the resilience of UAV teams under limited or adversarial conditions [21].

Notation: In the remainder of this letter, $\mathbb{R}^+$, $\mathbb{R}_0^+$ denote the sets of positive and non-negative real numbers, whereas $\mathbb{R}^{n\times m}$ denotes the set of real matrices with n rows and m columns. Moreover, $\mathbb{I}$ and $\mathbb{O}$ represent the identity and the zero matrices of suitable dimensions, respectively, whereas, for symmetric matrices $A,B\in\mathbb{R}^{n\times n}$, $A\succ B$ means that $A-B$ is positive definite and $A\succeq B$ means that $A-B$ is positive semi-definite. $\mathrm{sgn}(\phi)$ is the signum function satisfying $\mathrm{sgn}(\phi)=0$ for $\phi\leq 0$, $\mathrm{sgn}(\phi)=1$ for $\phi>0$. Finally, $\langle x,y\rangle$ denotes the inner product of vectors x and y.

II. Preliminaries

To guarantee that a solution to the problem $\mathscr{P}$ exists, we introduce two standard assumptions as follows.

A1. The components of the regressor vectors $\phi_i(\cdot)$ are assumed to be continuous and uniformly bounded over $[0,+\infty)$ as functions of time.

A2. (*Cooperative PE Condition* [13]) The corresponding regressor matrix $\Phi(\cdot)\in\mathbb{R}^{m\times p}$

$$\Phi(\cdot)=\left[\phi_1(\cdot),\phi_2(\cdot),\cdots,\phi_p(\cdot)\right], \tag{3}$$

is assumed to be PEd, *i.e.*, there exist known $c_p\in\mathbb{R}^+$ and $T_p\in\mathbb{R}^+$ such that the following condition holds:

$$\int_t^{t+T_p}\Phi(\tau)\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau \quad\succeq\quad c_p\mathbb{I}, \quad \forall\, t\geq 0. \tag{4}$$

Assumptions A1-A2 ensure that the unknown parameter vector Θ is identifiable from the entire set of available measurements, *i.e.*, from all the outputs and all the regressor vector components. Indeed, under Assumptions A1-A2, a solution to the estimation problem exists, at least when all the amount of information coming from the nodes of the considered graph is available for every node of the graph. Apparently, it is provided by the set of differential equations[3]

$$\dot{\hat{\Theta}}(t) \quad=\quad \Phi(t)\left(h(t)-\Phi^{\mathrm{T}}(t)\hat{\Theta}(t)\right), \tag{5}$$

starting from the initial condition $\hat{\Theta}_0=\hat{\Theta}(0)$, where $h(t)=[h_1(t),\ \ldots,\ h_p(t)]^{\mathrm{T}}$. By defining the Θ-estimation error $\tilde{\Theta}=\Theta-\hat{\Theta}$, it satisfies, on the basis of (2), (3) and (5),

$$\dot{\tilde{\Theta}}(t) \quad=\quad -\Phi(t)\Phi^{\mathrm{T}}(t)\tilde{\Theta}(t), \tag{6}$$

[2]In such a scenario, each estimation scheme at the node can symmetrically share its own Θ-estimate with the neighbours, one for the first and last nodes 1 and p, two for the remaining internal nodes.

[3]For the sake of simplicity, no gain matrix multiplies $\Phi(t)$.

with initial condition $\tilde{\Theta}(0) = \Theta(0) - \hat{\Theta}_0$. Equation (6) complies with the structure required by Theorem 1 of [16], which states that the solution of a system in the form (6) tends to zero exponentially for any initial condition $\tilde{\Theta}(0)$ under the related strict quadratic Lyapunov function[4].

$$\mathscr{W}(\tilde{\Theta}) \quad = \quad \frac{1}{2}\tilde{\Theta}^{\mathrm{T}}\left[2Q_*(\cdot) + (1+\mathrm{sgn}(m-1)\gamma_L)\right]\tilde{\Theta}.$$

Within the previous $\mathscr{W}(\tilde{\Theta})$-definition: $Q_*(t)$ is the solution to the matrix-differential equation:

$$\dot{Q}_*(t) \quad = \quad -\theta_c Q_*(t) + \nu\Phi(t)\Phi^{\mathrm{T}}(t), \tag{7}$$

for $\nu = 1$, starting from the symmetric matrix $Q_*(0) \succ 0$ (with sufficiently large minimum eigenvalue) as initial condition and involving $\theta_c \in \mathbb{R}^+$; $\gamma_L \in \mathbb{R}^+$ is sufficiently large. The reader is referred to the Appendix of [16] for related technical details.

III. Distributed Estimation on Circular Communication Graphs

We start by designing the differential equation to be satisfied by the Θ-estimate at node v_i, $i = 1,\dots,p$, denoted by $\hat{\Theta}^{[i]}$, as

$$\dot{\hat{\Theta}}^{[i]}(t) \quad = \quad \phi_i(t)\left(h_i(t) - \phi_i^{\mathrm{T}}(t)\hat{\Theta}^{[i]}(t)\right) - \eta_i(t), \tag{8}$$

starting from $\hat{\Theta}^{[i]}(0) = \hat{\Theta}_{0i}$. Indeed, the Θ-estimator is designed in accordance with the same full-information design process recalled within the previous section but restricted to the node v_i. Now, the main issue regards how the correction terms $\eta_i(t)$ are to be designed to guarantee that all the redundant estimate vectors converge to the (same) unknown parameter vector Θ. In particular, such a consensus has to be achieved through penalization of the mismatch between the parameter estimates [24]. In place of the typical block matrices that come from a topological structure with nodes undirectedly connected in series, we cover the circular communication structure, namely,

$$\begin{aligned} \eta_i \quad &= \quad \hat{\Theta}^{[i]} - \hat{\Theta}^{[i+1]}, \quad i = 1,\dots,p-1 \\ \eta_p \quad &= \quad \hat{\Theta}^{[p]} - \hat{\Theta}^{[1]}, \end{aligned} \tag{9}$$

which can be equivalently rewritten in terms of the Θ-estimation errors $\tilde{\Theta}^{[i]}$ (just add and subtract Θ within any pair of brackets) as follows

$$\begin{aligned} \eta_i \quad &= \quad -\left(\tilde{\Theta}^{[i]} - \tilde{\Theta}^{[i+1]}\right), \quad i = 1,\dots,p-1 \\ \eta_p \quad &= \quad -\left(\tilde{\Theta}^{[p]} - \tilde{\Theta}^{[1]}\right). \end{aligned} \tag{10}$$

By substituting (10) in (8), the estimation error system can be obtained as

$$\begin{bmatrix} \dot{\tilde{\Theta}}^{[1]}(t) \\ \vdots \\ \dot{\tilde{\Theta}}^{[p]}(t) \end{bmatrix} \quad = \quad -(\Lambda(t) + T)\begin{bmatrix} \tilde{\Theta}^{[1]}(t) \\ \vdots \\ \tilde{\Theta}^{[p]}(t) \end{bmatrix}, \tag{11}$$

where

$$\Lambda(t) \quad = \quad \mathrm{diag}[\phi_1(t)\phi_1^{\mathrm{T}}(t),\dots,\phi_p(t)\phi_p^{\mathrm{T}}(t)] \tag{12}$$

and (blocks are compatible in dimension with $[\tilde{\Theta}^{[1]}(t),\dots,\tilde{\Theta}^{[p]}(t)]^{\mathrm{T}}$)

$$T \quad = \quad \begin{bmatrix} \mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \dots & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \dots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \dots & \dots & \mathbb{I} & -\mathbb{I} \\ -\mathbb{I} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{I} \end{bmatrix}. \tag{13}$$

IV. New Strict Lyapunov Function-Based Proof

In this section, we present the new Theorem 1, whose proof uses stability arguments that move along the direction of finding out a new composite strict Lyapunov function for (11) under PE (4), under the key idea of changing coordinates and resorting to Theorem 1 of [16] (recalled in Section II) once restricted to a suitable m-dimensional subsystem of the tansformed (11) that is strictly compatible with PE (4).

Theorem 1: Consider the estimation error dynamics (11) under Assumptions A1 and A2. Then the n-dimensional extended error vector $\tilde{\Theta}^{[e]}(t) = [\tilde{\Theta}^{[1]\mathrm{T}}(t),\dots,\tilde{\Theta}^{[p]\mathrm{T}}(t)]^{\mathrm{T}}$ ($n = pm$) globally exponentially converges to zero.

Proof: Consider the new variable

$$\alpha_0 \quad = \quad \sum_{i=1}^{p}\tilde{\Theta}^{[i]}, \tag{14}$$

whose dynamics, owing to the zero column-sum T-structure in (13), read

$$\dot{\alpha}_0 \quad = \quad -\sum_{i=1}^{p}\phi_i\phi_i^{\mathrm{T}}\tilde{\Theta}^{[i]}. \tag{15}$$

Then define the new error variables

$$\alpha_i \quad = \quad \tilde{\Theta}^{[i]} - \tilde{\Theta}^{[i+1]}, \quad i = 1,\dots,p-1. \tag{16}$$

The matrix C characterizing the change of coordinates

$$\begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{p-1} \end{bmatrix} \quad = \quad C\begin{bmatrix} \tilde{\Theta}^{[1]} \\ \vdots \\ \tilde{\Theta}^{[p]} \end{bmatrix}, \tag{17}$$

is invertible with the first row of its inverse reading

$$\begin{aligned} t_{C,1}^{\mathrm{T}} \quad &= \quad [1,0,\dots,0]C^{-1} \\ &= \quad [1/p, 1-1/p, 1-2/p,\dots,1/p], \end{aligned} \tag{18}$$

and the other rows being denoted by $t_{C,2}^{\mathrm{T}},\dots,t_{C,p}^{\mathrm{T}}$, respectively. On the other hand, take the Lyapunov function

$$W\left(\tilde{\Theta}^{[1]},\dots,\tilde{\Theta}^{[p]}\right) \quad = \quad \frac{1}{2}\sum_{i=1}^{p}\tilde{\Theta}^{[i]\mathrm{T}}\tilde{\Theta}^{[i]}, \tag{19}$$

whose time derivative along the directions of the error system (11) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}W\left(\tilde{\Theta}^{[1]},\dots,\tilde{\Theta}^{[p]}\right) \quad \leq \quad -\frac{1}{2}\sum_{i=1}^{p-1}\|\alpha_i\|^2. \tag{20}$$

Now recognize that, by definition in (16),

$$\tilde{\Theta}^{[i]} \quad = \quad \tilde{\Theta}^{[1]} - \sum_{k=1}^{i-1}\alpha_k, \quad i,2\dots,p \tag{21}$$

[4]The explicit time dependence is omitted in the remainder of this letter when no clarity issues arise.

so that (15) becomes

$$\dot{\alpha}_0 \;=\; -\sum_{i=1}^{p}\phi_i\phi_i^{\mathrm{T}}\tilde{\Theta}^{[1]} + \sum_{i=2}^{p}\phi_i\phi_i^{\mathrm{T}}\sum_{k=1}^{i-1}\alpha_k, \qquad (22)$$

while, by (18), it reads

$$\begin{aligned}\dot{\alpha}_0 \;&=\; -\frac{1}{p}\sum_{i=1}^{p}\phi_i\phi_i^{\mathrm{T}}\alpha_0 + \sum_{i=2}^{p}\phi_i\phi_i^{\mathrm{T}}\sum_{k=1}^{i-1}\alpha_k - \sum_{i=1}^{p}\phi_i\phi_i^{\mathrm{T}}\pi \\ &\doteq\; -\bar{\Phi}\bar{\Phi}^{\mathrm{T}}\alpha_0 + \Upsilon(\alpha_1,\ldots,\alpha_{p-1}), \qquad (23)\end{aligned}$$

in which $\pi = \langle[1-1/p, 1-2/p, \ldots, 1/p]^{\mathrm{T}}, [\alpha_1, \alpha_2, \ldots, \alpha_{p-1}]^{\mathrm{T}}\rangle$ and $\bar{\Phi}$ is given by $\bar{\Phi} = \Phi/\sqrt{p}$. At this stage, system (23) complies with the structure required by Theorem 1 of [16]. In particular, α_0 plays the role of x, $\bar{\Phi}$ plays the role of Φ, and $\Upsilon(\alpha_1,\ldots,\alpha_{p-1})$ play the role of $R(t, w(t))$ therein. With (19), (20) in mind, consider the composite strict Lyapunov function

$$\mathscr{L}\left(\tilde{\Theta}^{[1]},\ldots,\tilde{\Theta}^{[p]},\alpha_0\right) = W\left(\tilde{\Theta}^{[1]},\ldots,\tilde{\Theta}^{[p]}\right) + \mu\mathscr{W}(\alpha_0), \quad (24)$$

where

$$\mathscr{W}(\alpha_0) = \frac{\mu}{2}\alpha_0^{\mathrm{T}}\left[2Q_*(\cdot) + (1+\mathrm{sgn}(m-1)\gamma_L)\right]\alpha_0, \qquad (25)$$

in which $Q_*(t)$ is given by (7) with $\nu = 1/p$. According to the proof of Theorem 1 of [16], the time derivative of $\mathscr{L}\left(\tilde{\Theta}^{[1]},\ldots,\tilde{\Theta}^{[p]},\alpha_0\right)$, call such derivative $\mathscr{L}_D$, satisfies along the trajectories of the closed loop system the following

$$\begin{aligned}\mathscr{L}_D \;&\leq\; -\mu\frac{\theta_c}{2}\alpha_0^{\mathrm{T}}Q_*(\cdot)\alpha_0 - \frac{1}{2}\sum_{i=1}^{p-1}\|\alpha_i\|^2 \qquad (26)\\ &\quad -\mu\alpha_0^{\mathrm{T}}\left[(1+\gamma_L) + 2Q_*\right]\Upsilon(\alpha_1,\ldots,\alpha_{p-1})\end{aligned}$$

for $\gamma_L \geq 2\beta^2\sup_{t\in[0,+\infty)}\|\bar{\Phi}(t)\|^2\left(\theta_c c_p \mathrm{e}^{-\theta_c T_p}\right)^{-1}$ and $c_p\mathrm{e}^{-\theta_c T_p}\mathbb{I} \leqq Q_*(t) \leqq \beta\mathbb{I}$. Therefore, a sufficiently small μ in (26) leads to (c_W is a suitable positive number)

$$\begin{aligned}\mathscr{L}_D \;&\leq\; -c_W\|[\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{p-1}]\|^2 \\ &\leq\; -c_W\lambda_m(C^{\mathrm{T}}C)\|[\tilde{\Theta}^{[1]}, \tilde{\Theta}^{[2]}, \ldots, \tilde{\Theta}^{[p]}]\|^2, \qquad (27)\end{aligned}$$

where $\lambda_m(C^{\mathrm{T}}C)$ denotes the minimum positive eigenvalue of the symmetric positive-definite $C^{\mathrm{T}}C$. The thesis of the theorem follows by finally recognizing that $\|\alpha_0\|^2 \leq n\sum_{i=1}^{p}\|\tilde{\Theta}^{[i]}\|^2$ in (25). ■

Remark: The previous analysis also applies, with slight modifications, to the case of different graph configurations, provided that, in order to get (20), a path connecting all the nodes of the graph is preserved and the zero column- (zero row-) sum structure for T is kept. As an example, if the last node views the second one as a neighbour, it suffices to modify the second and last row in the matrix T in (13) by $[-\mathbb{I}, 2\mathbb{I}, -\mathbb{I}, \mathbb{O}, \ldots, \ldots, \mathbb{O}]$ and $[\mathbb{O}, -\mathbb{I}, \mathbb{O}, \mathbb{O}, \ldots, \mathbb{O}, \mathbb{I}]$.

V. Experimental results

The indoor experimental setup, including the UAV, onboard components, positioning system, and network architecture, is shown in Figure 1. The UAV is a custom QAV250 quadrotor, equipped with a Pixhwak autopilot running PX4 firmware and a Raspberry Pi serving as the onboard computer. For indoor localization of the UAV, an Optitrack system with Motive

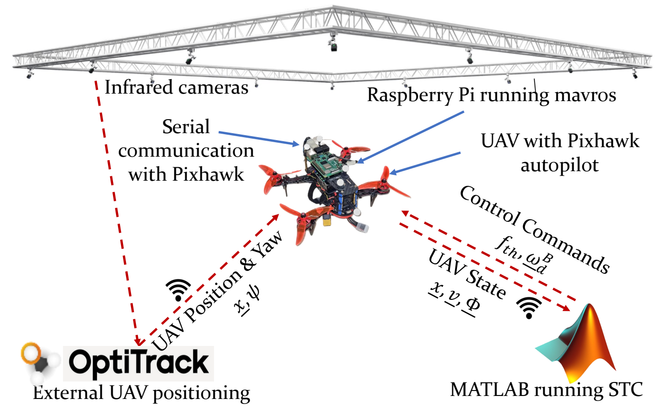

Fig. 1: Illustration of UAV experimental setup.

software is used. Infrared cameras track reflective markers on the UAV and send images to Motive software, which estimates its position and orientation in a predefined Front-Left-Up frame at 120 Hz. The onboard computer runs a Robot Operating System, `rosmaster` and `mavros`. It i) acts as a point of interface between all the components in the network, ii) receives the UAV's estimated position and orientation from the Motive software over WiFi, and iii) relays them to the autopilot via `mavros`. The autopilot fuses them with its acceleration and gyroscope measurements to estimate the full state of the UAV, including position, velocity, and orientation and relays it back to the onboard computer. The Super-Twisting Controller (STC) in [25] (see Appendix I) running at a frequency of 250 Hz in Matlab on a ground computer is used to control the UAV. It has real-time access to the full state of the UAV and communicates with autopilot via `mavros` and onboard computer. Viewing UAVs 1,2,3 as sentinel units and $\mathrm{UAV_T}$ as target object, the control algorithm controls such four UAVs, with desired positions set to (in SI units): $X_{d1} = -1.5$, $Y_{d1} = 1.5$, $Z_{d1} = 0.75$; $X_{d2}(t) = \sin(2\pi/10t)$, $Y_{d2} = 0$, $Z_{d2} = 1$; $X_{d3}(t) = \cos(2\pi/10t)$, $Y_{d3} = 0$, $Z_{d3} = -1$, $X_{d\mathrm{T}} = 0$, $Y_{d\mathrm{T}} = 0$, $Z_{d\mathrm{T}} = 1$ (so that (4) is satisfied by direct computation). The reader is referred to Figure 2 for the (x, y, z)-coordinates of the three sentinel UAVs 1, 2, 3. The computed control inputs (normalized thrust and angular rates) are sent to the onboard computer, which forwards them to the autopilot via `mavros`. The autopilot's low-level controller then computes and sends the PWM motor commands. The performance of the original estimator presented in this paper, based on the real experimental data, is obtained by simulating its learning-like (see technical remark 3 of [26]), non-implicit node-based form (just use the explicit expression for η_i and rearrange terms to resemble the Jacobi method for positive definite matrices)

$$\begin{aligned}(2\mathbb{I} + K\phi_i\phi_i^{\mathrm{T}})\hat{\Theta}^{[i]}(t) \;&=\; \hat{\Theta}^{[i]}(t,\alpha) + K\phi_i y_i(t) - [\eta_i - \hat{\Theta}^{[i]}](t) \\ \hat{\Theta}^{[i]}(t,\alpha) \;&=\; [\hat{\theta}_1^{[i]}(t-\alpha), \ldots, \hat{\theta}_m^{[i]}(t-\alpha)], \qquad (28)\end{aligned}$$

where $i = 1, 2, 3$ and $K = \mathrm{diag}[1.1, 1.1, 25, 50]$ is the matrix of gains, used to shape the promptness of response and the robustness of the scheme, while $\alpha = 0.1$ s. Indeed, system (28) is solved in discrete-time with a $0.01(s)$ step. The results are shown in Figures 3 and 4. Satisfactory estimation and consensus are definitely achieved. In particular, the 3 Θ-estimates $\hat{\Theta}^{[i]}(t)$, $i = 1,2,3$ at the nodes (UAVs 1, 2, 3) in Figure 3 all converge exponentially to the unknown constant parameter vector $\Theta \in \mathbb{R}^4$, in spite of no symmetrical communication between the UAVs occurring. Accordingly, the corresponding estimation errors in Figure 4 tend exponentially to zero.

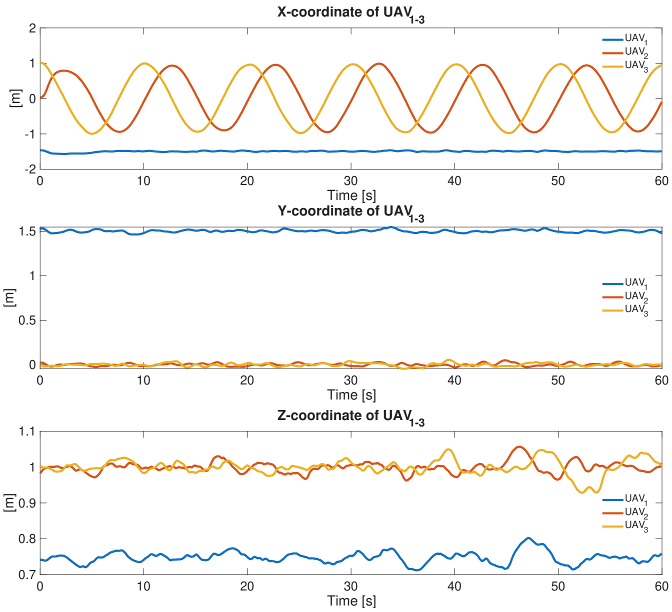

Fig. 2: (x,y,z)-coordinates of the three sentinel UAVs characterizing the three regressor components ϕ_i^{T}, $i = 1,2,3$, in (1) for UAVs 1, 2, 3.

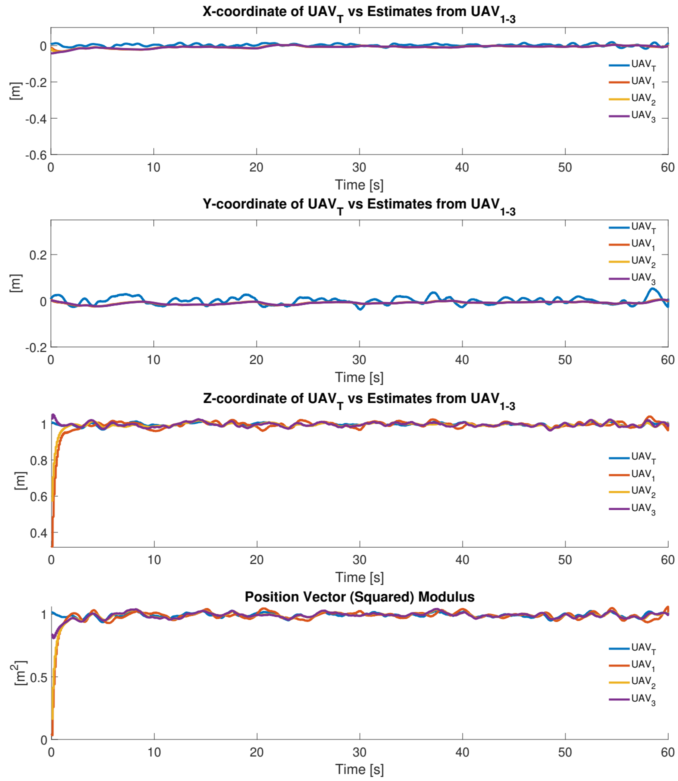

Fig. 3: Four $\hat{\Theta}$-components in (1) (UAV_{T}) and corresponding distributed estimates from UAVs 1, 2, 3.

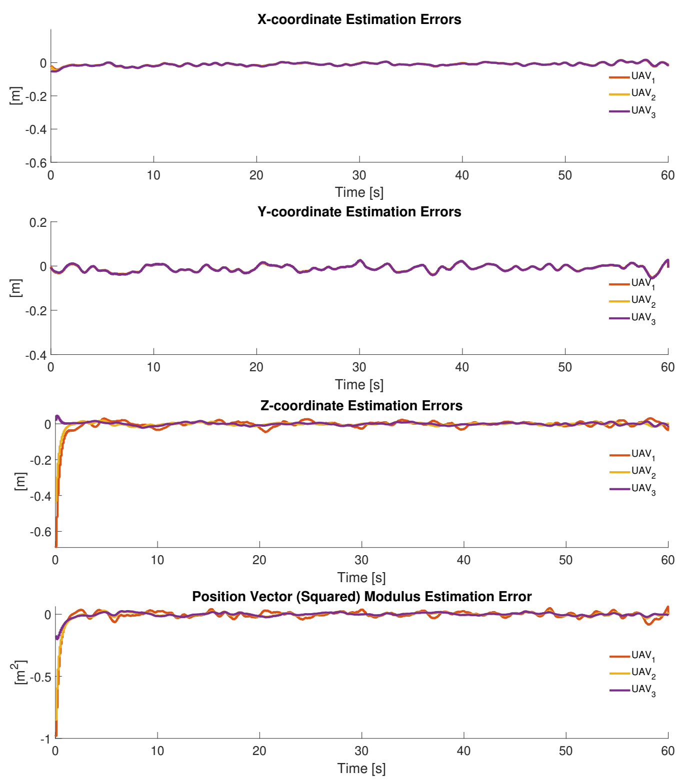

Fig. 4: Four $\tilde{\Theta}$-components in (1) ($\mathrm{UAV_T}$) and corresponding distributed estimates from UAVs 1, 2, 3.

VI. Conclusions

An innovative solution to a distributed identification problem on a circular communication graph was presented. A strict Lyapunov function within a PE framework was originally used to prove the consensus achievement for an asymmetric neighbourhood-based decentralized parallel architecture. Experimental results involving quadrotor UAVs finally illustrated all the theoretical derivations. Different asymmetrical graph structures can be covered by the proposed approach, provided that the main features of the proof are kept.

Appendix I
Super-Twisting Controller for UAV

The super-twisting controller for UAV is recalled here for the sake of exhaustiveness. Let $\underline{\Phi}(t) = [\phi(t), \theta(t), \psi(t)]^T \in \mathbb{R}^3$ denote the Euler angle vector, with $\phi(t)$, $\theta(t)$, and $\psi(t)$ being roll, pitch, and yaw angles. Let g denote the gravity acceleration. Let the transformation matrix $R_q(t)$ be given as:

$$R_q(t) = \begin{bmatrix} 1 & 0 & -S_\theta \\ 0 & C_\phi & C_\theta S_\phi \\ 0 & -S_\phi & C_\theta C_\phi \end{bmatrix}, \tag{29a}$$

where, C_α and S_α denote $\cos(\alpha(t))$ and $\sin(\alpha(t))$, respectively, for an angle $\alpha(t)$. Let the sliding functions for tracking a three-dimensional trajectory be defined as

$$\sigma_x(t) = (\dot{X}(t) - \dot{X}_d(t)) + \lambda_x (X(t) - X_d(t)), \tag{30a}$$

$$\sigma_y(t) = (\dot{Y}(t) - \dot{Y}_d(t)) + \lambda_y (Y(t) - Y_d(t)), \tag{30b}$$

$$\sigma_z(t) = (\dot{Z}(t) - \dot{Z}_d(t)) + \lambda_z (Z(t) - Z_d(t)), \tag{30c}$$

where $[X(t), Y(t), Z(t)]^T \in \mathbb{R}^3$ is the position vector of center of gravity of the UAV in the inertial frame of reference; $[X_d(t), Y_d(t), Z_d(t)]^T \in \mathbb{R}^3$ is the desired position and vector in the inertial frame of reference, respectively; λ_x, λ_y, and λ_z are positive design parameters. The control laws for the acceleration inputs, $a_{X_d}(t)$, $a_{Y_d}(t)$, and $a_{Z_d}(t)$, in the inertial frame are given by

$$\begin{aligned} a_{X_d}(t) = {} & \ddot{X}_d(t) - \lambda_x(\dot{X}(t) - \dot{X}_d(t)) \\ & - \alpha_x \operatorname{sign}(\sigma_x(t))\sqrt{|\sigma_x(t)|} - \int_0^t \beta_x \operatorname{sign}(\sigma_x(t)), \end{aligned} \tag{31a}$$

$$\begin{aligned} a_{Y_d}(t) = {} & \ddot{Y}_d(t) - \lambda_y(\dot{Y}(t) - \dot{Y}_d(t)) \\ & - \alpha_y \operatorname{sign}(\sigma_y(t))\sqrt{|\sigma_y(t)|} - \int_0^t \beta_y \operatorname{sign}(\sigma_y(t)), \end{aligned} \tag{31b}$$

$$\begin{aligned} a_{Z_d}(t) = {} & \ddot{Z}_d(t) - \lambda_z(\dot{Z}(t) - \dot{Z}_d(t)) + g \\ & - \alpha_z \operatorname{sign}(\sigma_z(t))\sqrt{|\sigma_z(t)|} - \int_0^t \beta_z \operatorname{sign}(\sigma_z(t)), \end{aligned} \tag{31c}$$

where α_x, α_y, α_z, β_x, β_y, and β_z are positive design parameters. The total thrust acceleration input $T_d(t)$, distributed across all UAV motors, is computed as $T_d(t) = \sqrt{a_{X_d}(t)^2 + a_{Y_d}(t)^2 + a_{Z_d}(t)^2}$, whereas the desired roll $\phi_d(t)$ and pitch $\theta_d(t)$ for chosen yaw $\psi_d(t)$ are determined from the acceleration inputs as

$$\phi_d(t) = \arcsin\left(\frac{-a_{X_d}(t) S_{\psi_d} + a_{Y_d}(t) C_{\psi_d}}{T_d(t)}\right), \tag{32a}$$

$$\theta_d(t) = \arcsin\left(\frac{a_{X_d}(t) C_{\psi_d} + a_{Y_d}(t) S_{\psi_d}}{T_d(t) C_{\psi_d}}\right). \tag{32b}$$

The commanding angular velocity vector $\underline{\omega}_c(t) \in \mathbb{R}^3$ is determined as $\underline{\omega}_c(t) = -\lambda_\omega R_q(t) \underline{e}_\Phi(t)$, where $\underline{e}_\Phi(t) = [\phi(t) - \phi_d(t), \theta(t) - \theta_d(t), \psi(t) - \psi_d(t)]^T \in \mathbb{R}^3$ and λ_ω is a positive design parameter [27]. Finally, the commanding normalized thrust, $T_c(t)$, is determined as $T_c(t) = T_d(t) T_{ref}/g$, where T_{ref} is the experimentally determined hover thrust.

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