Thermodynamics of coherent energy exchanges between lasers and two-level systems

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We study the quantum thermodynamics of a coherent macroscopic electromagnetic field (laser) coupled to a two-level system (qubit) near resonance, from weak- to strong-driving regimes. This combined system is, in turn, weakly coupled to a thermal radiation field and can be described by an autonomous quantum master equation. We show that the laser acts as an autonomous work source and that, in the macroscopic limit, the work produced is independent of the phase of the laser. Using the dressed qubit approach, we show that the variation of energy in the laser is not the work transferred to the dressed qubit, which is instead obtained from the "dressed laser"—a coherent superposition of the laser and the qubit. Using a two-point measurement technique with counting fields, we obtain the full counting statistics for the work of the laser and dressed laser, and show that they satisfy the Crooks fluctuation theorems. We then use these theorems as criteria to investigate the thermodynamic consistency of quantum master equations, first in the autonomous setup for the combined system, then in the nonautonomous setup for the quantum system where the coherent field is eliminated and effectively described by a time-dependent external field. Treating the laser as an external field is known to yield expressions for the work which are in contradiction with quantum thermodynamics predictions in the strong-driving regime. We show that these inconsistencies stem from a confusion between the laser and dressed laser, and show how to correct them. We also derive a generalized Bloch master equation, thermodynamically consistent across all driving regimes, from which the Bloch and Floquet master equations can be obtained using additional approximations (of which we also examine the consistency).

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I. INTRODUCTION

Quantum optics is the study of the interaction of matter (atoms and molecules) with quantized radiation fields [1]. In many cases, the radiation fields can be treated as baths, and their degrees of freedom can be traced out from the equations of motion, leading to a quantum optical master equation describing the dynamics of the system of atoms and molecules. This procedure is well suited, for example, to describe the decay of a two-level system in vacuum, or the resonance fluorescence when an external coherent field is also present [2]. In the context of quantum computing and technologies, many implementations rely on the ability to coherently monitor a two-level system, or qubit, using a laser, while competing with spontaneous emission processes which act as a damping mechanism. The dynamics of the qubit is then described by the optical Bloch equation [3–5], or, in the strong-driving regime, by the Floquet master equation [6–8], derived from the quantum Floquet theory [9].

The optical Bloch equation was primarily used in spectroscopy [10], which motivated early works on its thermodynamic consistency [11]. The rapid and recent development of quantum technologies has revived interest in the thermodynamic consistency of quantum optical master equations [5–7,12–14], with the Bloch and Floquet master equations being increasingly used to study the thermodynamics of driven quantum systems [15–20]. More precisely, the consistency of the Bloch master equation has been studied at the average level [5,11], while the full counting statistics has been done for the steady state of the Floquet master equation [12]. In these approaches, the driving field is described by an external time-dependent field, which interacts with the qubit through a time-dependent Hamiltonian $\hat{V}(t)$. Interestingly, it was found [5] that, in the strong drive regime, the rate of work performed by the driving field on the qubit is *not* equal to the expression predicted by quantum thermodynamics, namely, $\mathrm{Tr}[\hat{\rho}(t)d_t\hat{V}(t)]$ [21,22]. Instead, the work depends on the rates of dissipation to the bath. The qualitative interpretation of this feature is that the work results from a nonconservative force, arising from neglecting the fluctuation of the number of photons [5]. We show that this explanation is not sufficient, and that the complete answer is that the Floquet master equation describes the energy transfers in the dressed qubit basis (the eigenbasis of the joint qubit-coherent field system), in which the work source is not the original coherent field. Moreover, the consistency at the fluctuating level, i.e., whether the master equations preserve fluctuation theorems, has never been addressed.

In this paper, we study thermodynamics at the average and fluctuating level of a qubit driven by a coherent monochromatic radiation (further on called the laser) and dissipating to a thermal cavity (the bath). Importantly, everything is derived starting from a microscopic and autonomous description

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of the combined qubit-laser-bath system where the laser is modeled as a monochromatic mode in a macroscopic (large number of photons) coherent state. In Sec. II, we provide a general description of lasers as work sources and show that the work transferred by a single laser source to a generic quantum system is independent of the phase of the laser in the macroscopic limit (large number of photons). In Sec. III, we introduce the qubit-laser-bath model, starting from the full unitary level. We study the model's dynamics, first in the autonomous description using the dressed qubit approach [3], and we introduce the concept of dressed laser. Then, we study the dynamics in the equivalent nonautonomous description for the qubit after a frame rotation. Using the two-point measurement method with counting fields [23], we then derive and compare the laws of thermodynamics and work fluctuation theorems in both descriptions. In Sec. IV, we identify the thermodynamic consistency conditions implied by the previous results at the level of quantum master equations in both the autonomous and nonautonomous pictures. In Sec. V, we use the formalism of quantum maps to derive an alternative master equation, called generalized Bloch equation, valid at all driving strengths and which is thermodynamically consistent. We then examine the Bloch and Floquet quantum master equations, which are obtained from the generalized Bloch equation using additional approximations. We find that the Floquet equation is fully consistent (the fluctuation theorems are preserved and the first and second law hold at the average and fluctuating levels) and that the work performed on the dressed qubit is indeed not expressed in the canonical form $\mathrm{Tr}[\hat{\rho}(t)d_t\hat{V}(t)]$, but has become a nonconservative force of which we explain the origin. The Bloch equation instead satisfies the fluctuation theorems, but the first law of thermodynamics is only satisfied at the average level. In Sec. VI, we comment on an alternative derivation of the Bloch master equation, commonly used in the literature, which relies on the Redfield equation, see, for instance, Refs. [2,5]. We show that the Bloch master equation, dressed with counting fields, obtained from the Redfield equation, is different from the one obtained using the quantum maps, although both approaches yield the same Bloch master equation once the counting fields are set to zero. The Redfield approach breaks the fluctuation theorems, which implies that the thermodynamics at the fluctuating level should be examined using master equations derived from the quantum maps. In Sec. VII, we compare the steady-state heat and work flows predicted by the Bloch and Floquet master equations in their common regime of validity. A summary of results is given in Sec. VIII, while conclusions are drawn in Sec. IX. Throughout the paper, we set $\hbar = 1$ and $k_B = 1$.

II. WORK FROM A LASER SOURCE

In this section, we discuss the properties of a laser coupled to a generic quantum system. The total system-laser is a closed system, described by a density matrix $\hat{\rho}(t)$ evolving according to a unitary operator $\hat{U}$, $\hat{\rho}(t) = \hat{U}\hat{\rho}(0)\hat{U}^\dagger$. We further assume that initially

$$\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_L(0)\,. \tag{1}$$

In later sections, the system will be a qubit.

A. Lasers: Autonomous work sources

Quantum mechanically, a monochromatic radiation field of frequency ω_L is described by the Hamiltonian

$$\hat{H}_L = \omega_L(\hat{a}^\dagger\hat{a} + 1/2), \tag{2}$$

where $\hat{a}^\dagger$, $\hat{a}$ are bosonic creation and annihilation operators. A laser can in turn be modeled as a field in a pure coherent state, described by the density matrix

$$\hat{\rho}_L^{\mathrm{coh}} = |\alpha\rangle\langle\alpha|, \quad \text{with}$$
$$|\alpha\rangle \equiv e^{-|\alpha|^2/2}\sum_{N\geqslant 0}\frac{\alpha^N}{\sqrt{N!}}|N\rangle, \tag{3}$$

where $\alpha = |\alpha|e^{i\phi}$ and $|\alpha|$ and ϕ are respectively the amplitude and phase. The corresponding average number of photons is $\langle N\rangle = |\alpha|^2$, and the standard deviation is $\sigma(N) \equiv (\langle N^2\rangle - \langle N\rangle^2)^{1/2} = |\alpha|$.

From a thermodynamics viewpoint, a laser is an autonomous work source, because its change in von Neumann entropy, $S \equiv \mathrm{Tr}[\hat{\rho}_L \ln \hat{\rho}_L]$, while interacting with the system, is negligible compared with its corresponding change in energy $E_L \equiv \mathrm{Tr}[\hat{H}_L\hat{\rho}_L]$ [24],

$$\frac{\Delta S_L}{\Delta E_L} \to 0. \tag{4}$$

This remains true when averaging over the phase ϕ of the laser, after which the laser source is described by a phase-averaged state, also called a Poisson state,

$$\hat{\rho}_L^{\mathrm{poi}} \equiv e^{-|\alpha|^2}\sum_{N\geqslant 0}\frac{|\alpha|^{2N}}{N!}|N\rangle\langle N|$$
$$= \int_0^{2\pi} d\phi\, ||\alpha|e^{i\phi}\rangle\langle|\alpha|e^{i\phi}|, \tag{5}$$

which yields, as for a coherent state, $\langle N\rangle = |\alpha|^2$ and $\sigma(N) = |\alpha|$. It will furthermore be useful to think of a Poisson state as a thermal state at infinite temperature. Indeed, it is known that, in the large-$|\alpha|^2$ limit, a Poisson distribution converges to a Gaussian distribution of average $|\alpha|^2$ and standard deviation $|\alpha|$. In turn, such a Gaussian state is equivalent to a Gibbs state at temperature $\beta_L^{-1} \equiv |\alpha|^2$. See Appendix A for a proof of (4) for both a coherent and Poisson state and Fig. 1 for a numerical check in the case where the system is a qubit.

As a result, we may then identify minus the variation of energy in the laser as work,

$$W_L \equiv -\Delta E_L = -\mathrm{Tr}[\hat{H}_L(\hat{\rho}_L(t) - \hat{\rho}_L(0))]. \tag{6}$$

B. Irrelevance of the phase

The phase ϕ may be difficult to determine in practice. However, as we now show, in the macroscopic limit $|\alpha| \to +\infty$, the work (6) becomes independent of ϕ, and the laser source can equivalently be described by a Poisson state (5).

1. Average work

The expectation value of $\hat{H}_L$ is given by

$$\mathrm{Tr}[\hat{H}_L\hat{\rho}(t)] = \sum_{N,N',N''\geqslant 0}\langle N|\hat{H}_L|N\rangle \mathrm{Tr}_S[\langle N|\hat{U}|N'\rangle$$
$$\times\, \hat{\rho}_S(0)\langle N'|\hat{\rho}_L(0)|N''\rangle\langle N''|\hat{U}^\dagger|N\rangle]. \tag{7}$$

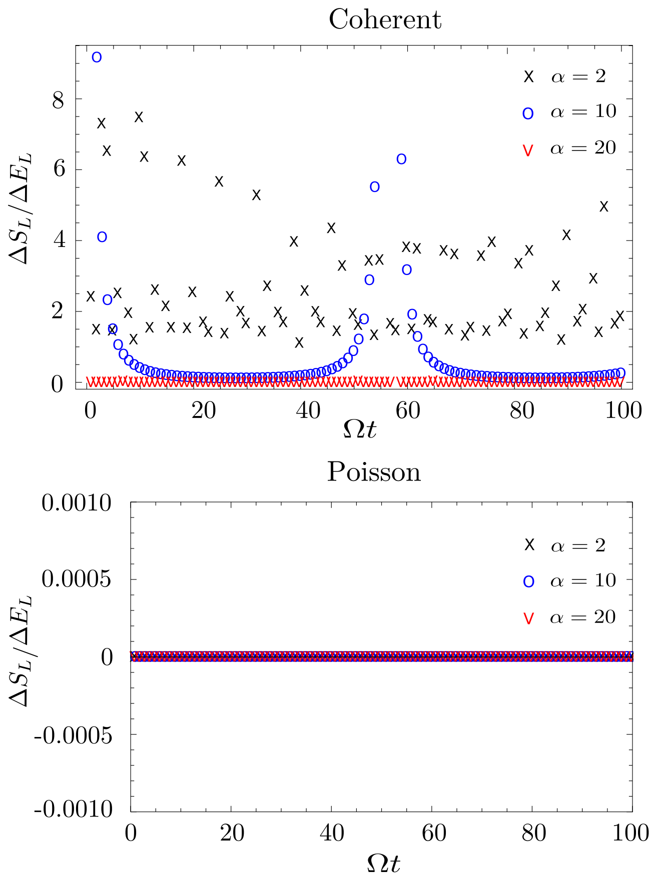


FIG. 1. Ratio of the variation of the von Neumann entropy ΔS_L and of the energy ΔE_L of a laser interacting with a two-level system. The time is in units of the inverse Rabi frequency Ω^{-1}. The coupling Hamiltonian is identical to $\hat{V}_{AL}$, defined in (17), without the $\mathbb{I}_B$ component. (top) Laser in a coherent state. (bottom) Laser in a Poisson state.

In the macroscopic limit $|\alpha| \gg 1$, since the distribution $e^{-|\alpha|^2/2}\frac{\alpha^N}{\sqrt{N!}}$ is peaked around $N = |\alpha|^2$, for both a coherent and a Poisson state, we may approximate

$$\mathrm{Tr}[\hat{H}_L\hat{\rho}(t)] \sim \sum_{N\geqslant 0}\sum_{N'\in\Delta}\langle N|\hat{H}_L|N\rangle \tag{8}$$

$$\times\, \mathrm{Tr}_S[\langle N|\hat{U}|N'\rangle\hat{\rho}_S(0)\langle N'|\hat{\rho}_L(0)|N'\rangle\langle N'|\hat{U}^\dagger|N\rangle], \tag{9}$$

where $\Delta \equiv [|\alpha|^2 - |\alpha|, |\alpha|^2 + |\alpha|]$. This means that for a macroscopic coherent and a Poisson state, (4) and (6) hold. We numerically confirm this expectation in Fig. 1, where the system is a qubit.

2. Work fluctuations

The above discussion is for average quantities. We now show that, at the level of fluctuations, performing a two-point measurement [23] when the laser is initialized in a Poisson state is equivalent to performing a series of measurements on a system initialized in a coherent state and averaging over the initial phase.

Let us first recall the two-point measurement approach with counting fields: given a (possibly time dependent) observable $\hat{A}$, of eigenvalues $\{a_m(\tau)\}$ at time τ, the probability to observe a fluctuation Δa when measuring $\hat{A}$ at times 0 and t is given by

$$p(\Delta a) = \sum_{a_l(0),a_m(t)} P[a_m(t), a_l(0)]\delta[\Delta a - (a_m(t) - a_l(0))], \tag{10}$$

where $P[a_m(t), a_l(0)]$ is the joint probability to measure $a_l(0)$ at time 0 and $a_m(t)$ at time t. The statistics of $p(\Delta a)$ is conveniently described using the moment-generating function, defined as the Fourier transform of $p(\Delta a)$,

$$\mathcal{G}(\lambda, t) \equiv \int_{-\infty}^{+\infty} e^{i\lambda\Delta a} p(\Delta a)\, d\Delta a, \tag{11}$$

where $\lambda \in \mathbb{R}$ is called a *counting field*. The time dependence of $\mathcal{G}(\lambda, t)$ lies in Δa, which corresponds to a fluctuation observed between the times 0 and t. The moment-generating function $\mathcal{G}(\lambda, t)$ is also equal to the trace of the "tilted" density matrix $\hat{\rho}^\lambda(t)$, obtained from the evolution operator $\hat{U}(t, 0) = \mathcal{T}_{\leftarrow}[e^{-i\int_0^t ds\hat{H}(s)}]$ (with $\mathcal{T}_{\leftarrow}$ denoting time-ordering), dressed with the counting field λ:

$$\begin{aligned}\mathcal{G}(\lambda, t) &= \mathrm{Tr}[\hat{\rho}_\lambda(t)],\\ \hat{\rho}_\lambda(t) &\equiv \hat{U}_\lambda(t, 0)\hat{\bar{\rho}}(0)\hat{U}^\dagger_{-\lambda}(t, 0),\\ \hat{U}_\lambda(t, 0) &\equiv e^{i\hat{A}(t)\lambda/2}\hat{U}(t, 0)e^{-i\hat{A}(0)\lambda/2},\end{aligned} \tag{12}$$

where $\hat{\bar{\rho}}(0)$ is the diagonal part of $\hat{\rho}(0)$ in the eigenbasis of $\hat{A}(0)$ chosen for the measurement. From (12), and assuming the initial condition (1), it is clear that the generating function obtained when the laser is initialized in a Poisson state $\mathcal{G}^{\mathrm{poi}}(\lambda, t)$ is related to the generating function $\mathcal{G}^{\mathrm{poi}}(\lambda, t)$ obtained with a coherent state by

$$\mathcal{G}^{\mathrm{poi}}(\lambda, t) = \int_0^{2\pi} d\phi\, \mathcal{G}^{\mathrm{coh}}(\lambda, t). \tag{13}$$

The work W_L defined in (6) is then obtained by measuring $-\hat{H}_L$. Since the Poisson state (5) commutes with $\hat{H}_L$, the initial projective measurement does not modify the density matrix, $\hat{\bar{\rho}}(0) = \hat{\rho}(0)$, hence does not modify the dynamics, and the work fluctuations can be rigorously computed. This is not the case for a coherent state. However, from (13), we see that we may obtain the statistics of the work during a series of measurements performed on coherent states by performing a single measurement on a Poisson state. Therefore, for all practical purposes, when dealing with measurements of thermodynamic observables, we always assume that the laser is initialized in a Poisson state.

To summarize, we showed that, from a thermodynamics viewpoint, a laser can equivalently be described as a pure coherent state or as a Poisson state. The advantage of the Poisson state description is that it allows a rigorous investigation of the work statistics using the two-point measurement scheme with counting fields. However, in many applications, a laser is described as an external, time-dependent field. We refer to the time-dependent field description as the nonautonomous case, and we investigate the thermodynamics in this case as well.

As a final remark, we point out that knowing the initial phase only becomes important if other lasers with different phases were used later on, since they would induce a dephasing. In the case of multiple coherent sources, one could resort instead to the recently developed photon-resolved Floquet theory [25], which is consistent with full counting statistics methods [26]. The results presented in this work could in principle be extended to multiple light sources using the framework of Ref. [26] for the counting statistics of the laser's photons. However, for the sake of clarity and without loss of generality, we focus on the case of a single laser.

We now proceed to analyzing the thermodynamics of a qubit driven by a laser.

III. UNITARY DESCRIPTION

A. Model

We denote by X the system consisting of qubit A and laser L. The total qubit-laser-bath system evolves in the product Hilbert space

$$\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_B, \quad \mathcal{H}_X = \mathcal{H}_A \otimes \mathcal{H}_L, \tag{14}$$

where $\mathcal{H}_A$, $\mathcal{H}_L$, $\mathcal{H}_B$ are respectively the Hilbert spaces of the qubit, laser, and bath, and where $\mathcal{H}_X$ is the Hilbert space of the qubit-laser system. The qubit is characterized by a ground state $|a\rangle$ and an excited state $|b\rangle$, separated by ω_A. The total Hamiltonian is

$$\hat{H} = \hat{H}_X \otimes \mathbb{I}_B + \hat{V}_{AB} + \mathbb{I}_X \otimes \hat{H}_B, \tag{15}$$

where

$$\hat{H}_X = \hat{H}_A \otimes \mathbb{I}_L + \hat{V}_{AL} + \mathbb{I}_A \otimes \hat{H}_L. \tag{16}$$

The laser Hamiltonian $\hat{H}_L$ was defined in (2), while the qubit and bath Hamiltonians are respectively $\hat{H}_A = \frac{\omega_A}{2}\hat{\sigma}_z$ and $\hat{H}_B = \sum_k \omega_k(\hat{b}_k^\dagger \hat{b}_k + \frac{1}{2})$, where $\hat{b}_k^\dagger$, $\hat{b}_k$ are bosonic creation and annihilation operators. The qubit-laser and qubit-bath interaction Hamiltonians are respectively

$$\hat{V}_{AL} = \frac{g_0}{2}(\hat{\sigma}_+ + \hat{\sigma}_-) \otimes (\hat{a} + \hat{a}^\dagger) \otimes \mathbb{I}_B,$$
$$\hat{V}_{AB} = (\hat{\sigma}_+ + \hat{\sigma}_-) \otimes \mathbb{I}_L \otimes \left(\hat{B} + \hat{B}^\dagger\right), \tag{17}$$

with

$$\hat{\sigma}_+ = |b\rangle\langle a|, \quad \hat{\sigma}_- = |a\rangle\langle b|,$$
$$\hat{\sigma}_z = (|b\rangle\langle b| - |a\rangle\langle a|), \tag{18}$$

and $\hat{B} = \sum_k \frac{g_k}{2}\hat{b}_k$, where $g_0, g_k \in \mathbb{C}$ are coupling amplitudes. To alleviate the notation, we drop the tensor products with identity operators when there is no ambiguity. The total qubit-laser-bath system is described by a density matrix $\hat{\rho}(t)$, which follows a unitary dynamics

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}(t), \hat{\rho}(t)], \tag{19}$$

the solution of which is

$$\hat{\rho}(t) = \hat{U}(t,0)\hat{\rho}(0)\hat{U}^\dagger(t,0), \tag{20}$$

where $\hat{U}(t,0) \equiv e^{-i\hat{H}t}$ is the propagator. The combined qubit-laser system is described by a density matrix $\hat{\rho}_X \equiv \mathrm{Tr}_B[\hat{\rho}]$,

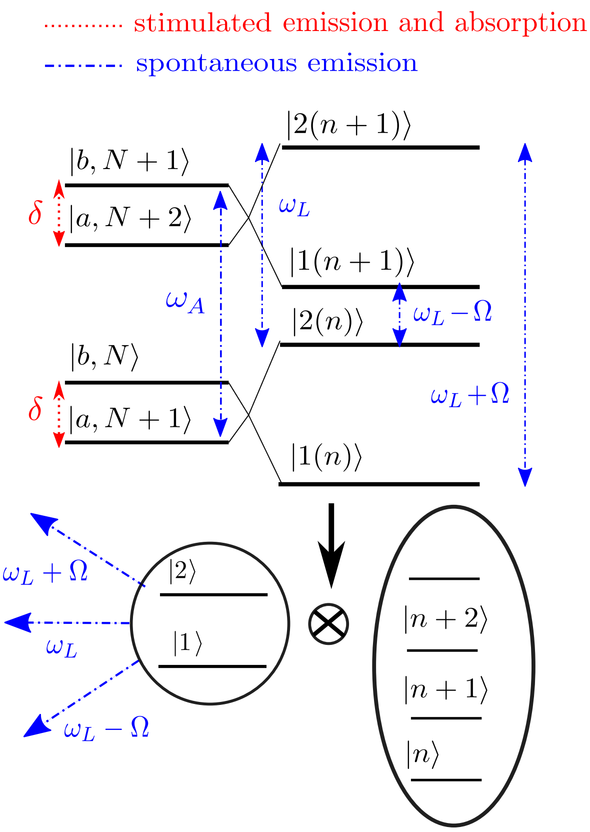


FIG. 2. Schematic representation of the mapping to the dressed qubit space. (top left) The three processes at play represented in the product basis $\{|b,N\rangle, |a,N+1\rangle\}$ of the product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_L$: spontaneous emission and stimulated emission and absorption. (top right) Change of basis to the eigenbasis of $\hat{H}_A + \hat{V}_{AL} + \hat{H}_L$. (bottom) Mapping to $\mathcal{H}_{DA} \otimes \mathcal{H}_{DL}$.

where Tr_B denotes the partial trace over the space $\mathcal{H}_B$. The state of the qubit A is in turn given by the density matrix $\hat{\rho}_A \equiv \mathrm{Tr}_L[\hat{\rho}_X]$, and the state of the laser L is described by the density matrix $\hat{\rho}_L \equiv \mathrm{Tr}_A[\hat{\rho}_X]$. The density matrix is initially factorized,

$$\hat{\rho}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_L^{\mathrm{coh}}(0) \otimes \hat{\rho}_B, \tag{21}$$

where the bath is in a Gibbs state,

$$\hat{\rho}_B = e^{-\beta_B \hat{H}_B}/Z_B, \tag{22}$$

with $Z_B = \mathrm{Tr}[e^{-\beta_B \hat{H}_B}]$ and β_B^{-1} the temperature.

There are two processes involved in the evolution of the qubit: the spontaneous absorption and emission and the stimulated absorption and emission. In the product basis $\{|b,N\rangle, |a,N\rangle\}$ of the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_L$, where $\{|N\rangle\}_{N\in\mathbb{N}}$ is the Fock basis for the photons of the mode ω_L, these processes are represented as follows (see Fig. 2): During a spontaneous emission (absorption), the qubit emits (absorbs) a photon from the bath, while the number of photons in the laser remains constant. Hence, a spontaneous emission (absorption) induces a transition $|b,N\rangle \to |a,N\rangle$ ($|a,N\rangle \to |b,N\rangle$) between states separated by an energy gap ω_A. On the other hand, during a stimulated emission (absorption), the qubit exchanges a photon with the laser, which corresponds to the transitions $|b, N\rangle \rightarrow |a, N+1\rangle$ ($|a, N+1\rangle \rightarrow |b, N\rangle$) between states separated by an energy gap $|\delta|$, where

$$\delta \equiv \omega_A - \omega_L \tag{23}$$

is the detuning between the qubit and laser frequencies.

B. Dressed qubit approach

In this section, we introduce the two assumptions and approximations underlying near-resonant coherent driving, which we make throughout this paper. We then present the dressed qubit approach [3] and show that this approach leads to a change of basis, which allows us to write the Hilbert space $\mathcal{H}_X$ as a tensor product of two new Hilbert spaces.

1. Assumptions

We make the following two assumptions, later referred to as assumptions 1 and 2:

(1) The laser is nearly resonant with the qubit: $\omega_L \gg |\omega_L - \omega_A|$.

(2) The photon statistics of the laser satisfies $\langle N\rangle \gg \sigma(N) \gg 1$,

where $\sigma(N) = (\langle N^2\rangle - \langle N\rangle^2)^{1/2}$ is the standard deviation. Assumption 1 allows us to perform the rotating-wave approximation, i.e., neglect the off-resonant terms $\hat{\sigma}_+\hat{a}^\dagger$ and $\hat{\sigma}_-\hat{a}$ in $\hat{V}_{AL}$ in (17) [2]. Consequently, $\hat{H}_X$ becomes block diagonal in the product basis $\{|a, N+1\rangle, |b, N\rangle\}$, $\hat{H}_X = \sum_{N\in\mathbb{N}} \hat{H}_X^{(N)}$, where $\hat{H}_X^{(N)}$ only acts on the subspace $\mathcal{E}(N)$ spanned by $\{|b, N\rangle, |a, N+1\rangle\}$. Assumption 2 is satisfied in the macroscopic limit $|\alpha| \gg 1$ and implies that the relative variations of $\sqrt{N}$ in the range $\sigma(N)$ around $\langle N\rangle$ are small, which allows us to neglect the fluctuations of $\sqrt{N}$ in the subspace $\mathcal{E}(N)$ and to replace

$$g_0\sqrt{N+1} \approx g_0\sqrt{\langle N\rangle} \equiv g. \tag{24}$$

Consequently,

$$\begin{aligned}\hat{H}_X^{(N)} = &\left[\left(N+1+\frac{1}{2}\right)\omega_L - \frac{\omega_A}{2}\right]|a, N+1\rangle\langle a, N+1| \\ &+ \left[\left(N+\frac{1}{2}\right)\omega_L + \frac{\omega_A}{2}\right]|b, N\rangle\langle b, N| \\ &+ \frac{g}{2}(|a, N+1\rangle\langle b, N| + |b, N\rangle\langle a, N+1|).\end{aligned} \tag{25}$$

2. The dressed qubit and dressed laser Hilbert spaces

Under assumptions 1 and 2, the restrictions $\hat{H}_X^{(N)}$ of $\hat{H}_X$ on $\mathcal{E}(N)$ defined in (25) can be diagonalized by a unitary transformation which is identical in every $\mathcal{E}(N)$,

$$\begin{aligned}\hat{H}_X^{(N)} = &\left[(N+1)\omega_L + \frac{\Omega}{2}\right]|2(n)\rangle\langle 2(n)| \\ &+ \left[(N+1)\omega_L - \frac{\Omega}{2}\right]|1(n)\rangle\langle 1(n)|,\end{aligned} \tag{26}$$

where

$$\begin{aligned}|2(n)\rangle &\equiv \sqrt{\frac{\Omega+\delta}{2\Omega}}|b, N\rangle + \sqrt{\frac{\Omega-\delta}{2\Omega}}|a, N+1\rangle, \\ |1(n)\rangle &\equiv -\sqrt{\frac{\Omega-\delta}{2\Omega}}|b, N\rangle + \sqrt{\frac{\Omega+\delta}{2\Omega}}|a, N+1\rangle,\end{aligned} \tag{27}$$

and where

$$\Omega = \sqrt{\delta^2 + g^2} \tag{28}$$

is the Rabi frequency [3].

Using the eigenbasis $\{|1(n)\rangle, |2(n)\rangle\}_{n\in\mathbb{N}}$, we see that the total Hilbert space $\mathcal{H}_X = \mathcal{H}_A \otimes \mathcal{H}_L$ is equivalent to a tensor product of two new Hilbert spaces, defined by the change of basis (see Fig. 2)

$$\begin{aligned}\mathcal{H}_A \otimes \mathcal{H}_L &\rightarrow \mathcal{H}_{DA} \otimes \mathcal{H}_{DL}, \\ |j(n)\rangle &\mapsto |j\rangle \otimes |n\rangle.\end{aligned} \tag{29}$$

In this new basis, the Hamiltonian (26) becomes

$$\begin{aligned}\hat{H}_X &= \hat{H}_{DA} \otimes \mathbb{I}_{DL} + \mathbb{I}_{DA} \otimes \hat{H}_{DL}, \\ \hat{H}_{DA} &= \frac{\Omega}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|), \\ \hat{H}_{DL} &= \sum_{n\geqslant 0} \omega_L(n+1)|n\rangle\langle n|.\end{aligned} \tag{30}$$

We use different notations N, n in order to distinguish between the Fock basis $\{|N\rangle\}$ of $\hat{H}_L$ and the Fock basis $\{|n\rangle\}$ of $\hat{H}_{DL}$. The subscript DA stands for "dressed qubit" [3], denoting the qubit "dressed" with the photons from the driving field. By symmetry, we introduce the concept of "dressed laser"—the state of the laser slightly modified by the interaction with the qubit, when the laser and the qubit are in a coherent superposition—and denote it with the subscript DL. The identity (30) describes the physical phenomenon at play: the laser and the qubit form a new quantum state, consisting of the dressed qubit and another monochromatic macroscopic field. Moreover, the change of basis (27) and the mapping (29) allow us to rewrite the initial condition (21), up to corrections of the order $1/|\alpha|$, as

$$\hat{\rho}(0) = \hat{\rho}_{DA}(0) \otimes \hat{\rho}_{DL}^{\text{coh}}(0) \otimes \hat{\rho}_B, \tag{31}$$

with $\hat{\rho}_{DL}^{\text{coh}}(0)$ in a coherent state in the basis $\{|n\rangle\}$.

Notice that, by construction, the Hamiltonian $\hat{H}_{DL}$ is equal to

$$\hat{H}_{DL} = \mathbb{I}_A \otimes \hat{H}_L + \frac{\omega_L}{2}\hat{\sigma}_z \otimes \mathbb{I}_L. \tag{32}$$

C. Nonautonomous approach

An alternative approach to study the problem of III A is to trace out the degrees of freedom of the laser and account for its effect using an external, time-periodic field.

Consider the density matrix $\hat{\tilde{\rho}}(t)$ obtained by performing the following unitary transformation on $\hat{\rho}(t)$, called the Mollow transformation [27],

$$\begin{aligned}\hat{\tilde{\rho}}(t) &\equiv \hat{D}^\dagger[\alpha(t)]\hat{\rho}(t)\hat{D}[\alpha(t)] \\ &= \hat{\tilde{U}}(t, 0)\hat{\tilde{\rho}}(0)\hat{\tilde{U}}^\dagger(t, 0),\end{aligned} \tag{33}$$

with $\alpha(t) \equiv \alpha e^{-i\omega_L t}$, $\hat{\tilde{U}}(t,0) \equiv \hat{D}^\dagger[\alpha(t)]\hat{U}(t,0)\hat{D}[\alpha(0)]$, and where we introduced the displacement operator [3] $\hat{D}[\alpha(t)] \equiv e^{\alpha(t)\hat{a}^\dagger - \alpha(t)^*\hat{a}}$, a unitary operator acting on the creation and annihilation operators as $\hat{D}[\alpha(t)]^\dagger \hat{a}\hat{D}[\alpha(t)] = \hat{a} + \alpha(t)$, $\hat{D}[\alpha(t)]\hat{a}\hat{D}^\dagger[\alpha(t)] = \hat{a} - \alpha(t)$, and creating a coherent state from the vacuum, $\hat{D}[\alpha(t)]|0\rangle = |\alpha(t)\rangle$. The density matrix $\hat{\tilde{\rho}}(t)$ is the solution of Ref. [3]

$$\frac{d\hat{\tilde{\rho}}(t)}{dt} = -i[\hat{\tilde{H}}(t), \hat{\tilde{\rho}}(t)], \tag{34}$$

$$\hat{\tilde{\rho}}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}'_B, \tag{35}$$

where $\hat{\rho}'_B \equiv |0\rangle\langle 0| \otimes \hat{\rho}_B$ and

$$\hat{\tilde{H}}(t) = \hat{H}_A(t) + \hat{V}'_{AB} + \hat{H}'_B, \tag{36}$$

with

$$\hat{H}_A(t) = \hat{H}_A + \hat{V}(t), \tag{37}$$

and

$$\hat{V}(t) = \tfrac{1}{2}(g\hat{\sigma}_+ e^{-i\omega_L t} + g^*\hat{\sigma}_- e^{i\omega_L t}), \tag{38}$$

where $g = g_0\alpha$ is the same as defined in (24), and where we regrouped $\hat{V}'_{AB} = \hat{V}_{AL} + \hat{V}_{AB}$ and $\hat{H}'_B = \hat{H}_L + \hat{H}_B$.[1]

1. Floquet basis

Since the Hamiltonian (36) is $2\pi/\omega_L$ periodic, it is convenient to describe the evolution of $\hat{\tilde{\rho}}(t)$ using Floquet states. The Floquet states $\{|u_n(t)\rangle\}$ are by definition $2\pi/\omega_L$ periodic and solutions of the eigenvalue problem

$$(\hat{H}_A + \hat{V}(t) - i\partial_t)|u_n(t)\rangle = \epsilon_n |u_n(t)\rangle. \tag{39}$$

The $\{|u_n(t)\rangle\}$ form an orthonormal basis of $\mathcal{H}_A$. Quite conveniently, in the present case, the $\{|u_n(t)\rangle\}$ are simply related to the states $\{|j\rangle\}$ by (see details in Appendix B)

$$e^{i\omega_L\hat{\sigma}_z t/2}|u_j(t)\rangle = |j\rangle \tag{40}$$

for $j = 1, 2$.

2. Equivalence between dressed qubit and rotating frame

The operation (40), which defines a change of basis from the Floquet basis to the dressed qubit basis, is equivalent to going to the rotating frame, where the qubit-bath system is described by the density matrix

$$\hat{\tilde{\rho}}^{\rm rot}(t) \equiv [e^{i\omega_L\hat{\sigma}_z t/2} \otimes \mathbb{I}_B]\hat{\tilde{\rho}}(t)[e^{-i\omega_L\hat{\sigma}_z t/2} \otimes \mathbb{I}_B], \tag{41}$$

and follows the dynamics

$$\frac{d\hat{\tilde{\rho}}^{\rm rot}(t)}{dt} = -i\big[\hat{H}_A^{\rm rot} + \hat{V}'_{AB}(t) + \hat{H}'_B, \hat{\tilde{\rho}}^{\rm rot}(t)\big], \tag{42}$$

where

$$\begin{aligned}\hat{V}'_{AB}(t) &\equiv e^{i\omega_L\hat{\sigma}_z t/2}\hat{V}'_{AB}e^{-i\omega_L\hat{\sigma}_z t/2} \\ &= e^{i\omega_L t}\hat{\sigma}_+\hat{B} + e^{-i\omega_L t}\hat{\sigma}_-\hat{B}^\dagger.\end{aligned} \tag{43}$$

[1] To obtain the expression for $\hat{\tilde{H}}(t)$, we used $\hat{D}^\dagger(\alpha(t))\hat{H}_L\hat{D}(\alpha(t)) = \hat{H}_L + \omega_L[\alpha(t)^*\hat{a} + \alpha(t)\hat{a}^\dagger + |\alpha|^2]$ and $-i\partial_t[\hat{D}^\dagger(\alpha(t))]\hat{D}(\alpha(t)) = \omega_L[\alpha(t)^*\hat{a} + \alpha(t)\hat{a}^\dagger + |\alpha|^2]$.

As expected from the relation (40), we check that

$$\begin{aligned}\hat{H}_A^{\rm rot} &\equiv \Big[e^{i\omega_L\hat{\sigma}_z t/2}(\hat{H}_A + \hat{V}(t))e^{-i\omega_L\hat{\sigma}_z t/2} - \frac{\omega_L}{2}\hat{\sigma}_z\Big] \\ &= \hat{H}_{DA}.\end{aligned} \tag{44}$$

Moreover, we can show (see proof in Appendix B) that the evolution of the dressed qubit and bath in the autonomous description, obtained by tracing out the dressed laser's degrees of freedom in (19), is equivalent to the evolution of the qubit and bath in the rotating frame in the nonautonomous description. In other words,

$$\hat{\tilde{\rho}}^{\rm rot} = {\rm Tr}_{DL}[\hat{\rho}]. \tag{45}$$

We use this equivalence extensively in the rest of this work.

D. Thermodynamics at the average level

We now derive and compare the first and second laws of thermodynamics in the autonomous and nonautonomous descriptions. Starting with the autonomous description, we show that the laser acts as a work source for the qubit, while the proper work source in the dressed qubit approach is the dressed laser. We then discuss the nonautonomous description, showing the equivalence with the autonomous case: the laws of thermodynamics for the qubit are equivalent in the autonomous and nonautonomous descriptions, and the laws of thermodynamics in the dressed qubit picture (in the autonomous description) are equivalent to the laws of thermodynamics in the rotating frame (in the nonautonomous picture). The results are summarized in Fig. 3.

1. Autonomous description

In the autonomous description, the total qubit-laser-bath system is closed, hence its total energy is conserved, ${\rm Tr}[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}] = 0$ for all $t \geqslant 0$. Since the bath is assumed by be initially at thermal equilibrium, the variation of energy in the bath is identified as minus the heat [28],

$$Q \equiv -{\rm Tr}[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_B]. \tag{46}$$

On the other hand, as discussed in Sec. II, we identify minus the variation of energy in the laser as work,

$$W_L \equiv -{\rm Tr}[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_L]. \tag{47}$$

The conservation of energy leads to define the energy of the qubit as

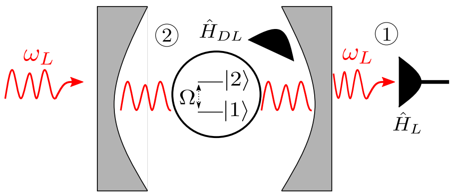
$$E_A(t) \equiv {\rm Tr}[\hat{\rho}(t)\big(\hat{H}_A + \hat{V}_{AL} + \hat{V}_{AB}\big)], \tag{48}$$

which leads to the first law for the qubit,

$$\Delta E_A = Q + W_L. \tag{49}$$

We point out that the rate of the work is proportional to the coherences in the dressed qubit basis: from (19), we obtain the rate $\dot{W}_L = -{\rm Tr}[d_t\hat{\rho}(t)\hat{H}_L]$,

$$\begin{aligned}\dot{W}_L &= i\omega_L g_0 {\rm Tr}[(\hat{a}^\dagger\hat{\sigma}_- - \hat{a}\hat{\sigma}_+)\hat{\rho}(t)] && (50)\\ &= i\omega_L g_0 \sum_{n\geqslant 0}(\langle a, n+1|\hat{\rho}(t)|b, n\rangle - \langle b, n|\hat{\rho}(t)|a, n+1\rangle) \\ &\approx i\omega_L g \sum_{n\geqslant 0}(\langle 2, n|\hat{\rho}(t)|1, n\rangle - \langle 1, n|\hat{\rho}(t)|2, n\rangle) \\ &= -\omega_L g\,{\rm Im}(\langle 2|\hat{\rho}_{DA}(t)|1\rangle), && (51)\end{aligned}$$



(a)

	Thermodynamic quantities	
	① *A-L* splitting	② *DA-DL* splitting
Autonomous	$E_A(t) \equiv Tr[\rho(\hat{t})(\hat{H_A} + V_{AL})]$ $Q \equiv -Tr[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_B]$ $W_L \equiv -Tr[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_L]$	$E_{DA}(t) \equiv Tr[\hat{\rho}(t)\hat{H}_{DA}]$ $Q \equiv -Tr[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_B]$ $W_{DL} \equiv -Tr[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_{DL}]$
Non-autonomous	$\tilde{E}_A(t) \equiv Tr[\hat{\tilde{\rho}}(t)(\hat{H}_A + \hat{V}(t) + \hat{V}'_{AB})]$ $\tilde{Q} \equiv -Tr[(\hat{\tilde{\rho}}(t) - \hat{\tilde{\rho}}(0))\hat{H}'_B]$ $W \equiv \int_0^t ds Tr[d_t\hat{V}(s)\hat{\tilde{\rho}}(s)]$	$\tilde{E}_A^{\rm rot}(t) \equiv Tr[\hat{\tilde{\rho}}^{\rm rot}(t)\hat{H}_{DA}]$ $\tilde{Q} \equiv -Tr[(\hat{\tilde{\rho}}(t) - \hat{\tilde{\rho}}(0))\hat{H}'_B]$ $W^{\rm rot} \equiv Tr[d_t\hat{V}'_{AB}(t)\hat{\tilde{\rho}}^{\rm rot}(t)]$

(b)

	First and second law	
	① *A-L* splitting	② *DA-DL* splitting
Auto.	(1a) $\Delta E_A = Q + W_L$ (1b) $\Delta S_A - \beta_B Q \geq 0$	(2a) $\Delta E_{DA} = Q + W_{DL}$ (2b) $\Delta S_{DA} - \beta_B Q \geq 0$
Non-auto.	(1a') $\Delta\tilde{E}_A = Q + W$ (1b') $\Delta\tilde{S}_A - \beta_B Q \geq 0$	(2a') $\Delta\tilde{E}_A^{\rm rot} = Q + W_{DL}$ (2b') $\Delta\tilde{S}_A^{\rm rot} - \beta_B Q \geq 0$

(c)

Identities	
$\Delta\tilde{E}_A = \Delta E_A$ $W = W_L$ $\tilde{Q} = Q$	$\Delta\tilde{E}_A^{\rm rot} = \Delta E_{DA}$ $W^{\rm rot} = W_{DL}$
$\Delta E_A = \Delta E_{DA} + Tr[\frac{\omega_L}{2}\hat{\sigma}_z(\hat{\rho}(t) - \hat{\rho}(0))]$ $W_L = W_{DL} - Tr[\frac{\omega_L}{2}\hat{\sigma}_z(\hat{\rho}(t) - \hat{\rho}(0))]$	

(d)

FIG. 3. Summary of the laws of thermodynamics and identities at the unitary level. (a) In situation 1, $\hat{H}_L$ is first measured before the interaction is turned on and the last measurement is performed after the interaction is switched off. In this case, it is more convenient to use the nondressed approach. In situation 2, the interaction is always on, but we measure instead $\hat{H}_{DL}$, which implies that the dressed qubit approach is more convenient. (b) Thermodynamic observables in both the dressed and nondressed approaches. (c) Laws of thermodynamics. (d) Identities between the thermodynamic observables of both approaches.

where we used assumption 2 for the third line. We highlight that assumption 2 (macroscopic limit) ensures that these coherences survive the phase averaging. To see this, consider the case where the interaction with the bath is neglected, hence where the evolution of $\hat{\rho}_X(t)$ depends only on $\hat{H}_X$. The generalization to the case where the bath is taken into account is straightforward by repeating the reasoning at the level of quantum maps, which will be introduced in Sec. IV [specifically in (99)]. When the bath is neglected, we have $\hat{\rho}_X(t) = e^{-it\hat{H}_X}\hat{\rho}_X(0)e^{it\hat{H}_X}$. Under assumption 2, $\hat{H}_X$ can be written as a sum of terms (26), which, using (21), yields

$$\langle 2, n|\hat{\rho}(t)|1, n'\rangle \propto e^{-|\alpha|^2}\frac{|\alpha|^{n+n'}e^{i\phi(n-n')}}{\sqrt{n!n'!}}e^{i(n'-n+\Omega)t}, \quad (52)$$

hence when setting $N = N'$ the dependence in the phase ϕ disappears. Numerically, we find that this feature happens as early as $\alpha = 4$, see Fig. 4. This illustrates the general statement of Sec. II B, that the phase of the laser is irrelevant for the work. However, we point out that the coherent term $\langle b|\hat{\rho}_A|a\rangle \equiv \sum_N \langle b, N|\hat{\rho}|a, N\rangle$ does not survive phase averaging, since

$$\langle b, N|\hat{\rho}|a, N\rangle \propto e^{i\phi}, \quad (53)$$

as illustrated in Fig. 5.

Let us now turn to the dressed qubit picture. We define instead

$$E_{DA}(t) \equiv \mathrm{Tr}[\hat{\rho}(t)(\hat{H}_{DA} + \hat{V}_{AB})], \quad (54)$$

and the conservation of energy leads to the first law

$$\Delta E_{DA} = Q + W_{DL}, \quad (55)$$

where,

$$W_{DL} \equiv -\mathrm{Tr}[(\hat{\rho}(t) - \hat{\rho}(0))\hat{H}_{DL}], \quad (56)$$

is identified as work.

Using (32), we can split

$$W_{DL} = W_L - \mathrm{Tr}\left[\frac{\omega_L}{2}\hat{\sigma}_z(\hat{\rho}(t) - \hat{\rho}(0))\right], \quad (57)$$

which leads to the following identity connecting the internal energies of the qubit and dressed qubit:

$$\Delta E_A = \Delta E_{DA} + \mathrm{Tr}\left[\frac{\omega_L}{2}\hat{\sigma}_z(\hat{\rho}(t) - \hat{\rho}(0))\right]. \quad (58)$$

We now examine the second law of thermodynamics. Since the density matrix is initially factorized (21), the von Neumann entropy $S_X \equiv -\mathrm{Tr}[\hat{\rho}_X \ln \hat{\rho}_X]$ can be split into two terms [28], $\Delta S_X = \beta_B Q + D(\hat{\rho}||\hat{\rho}_X(t) \otimes \hat{\rho}_B)$, where $D(\hat{\rho}_1||\hat{\rho}_2) \equiv \mathrm{Tr}[\hat{\rho}_1 \ln \hat{\rho}_1] - \mathrm{Tr}[\hat{\rho}_1 \ln \hat{\rho}_2]$ denotes the relative entropy between two density matrices. Since a relative entropy is always positive, the identity leads to a second law of thermodynamics for X,

$$\Sigma_X \equiv \Delta S_X - \beta_B Q \geqslant 0, \quad (59)$$

where Σ_X is the entropy production for X. Using again (21), the von Neumann entropy variation ΔS_X can then be

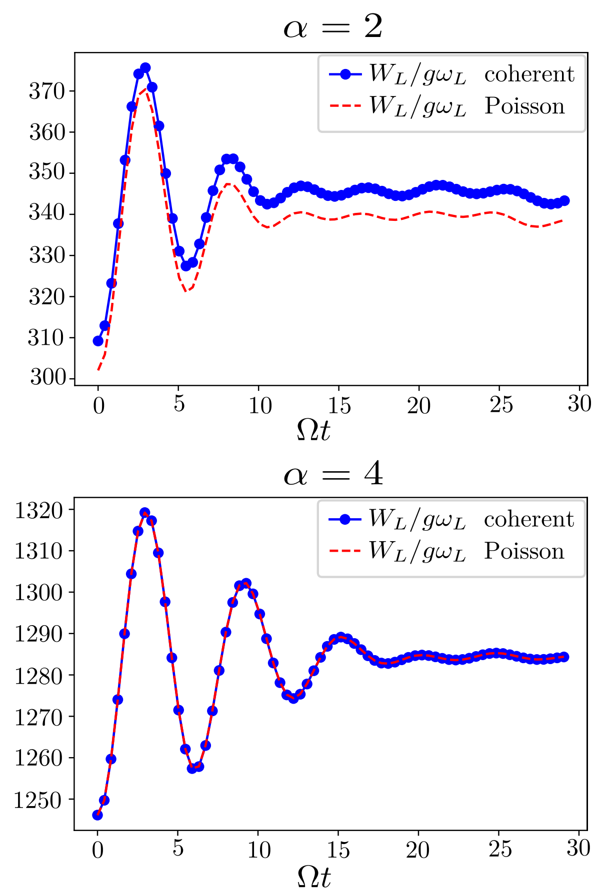


FIG. 4. Work W_L transferred from the laser to the qubit, with the time in units of the inverse Rabi frequency Ω^{-1}. The joint qubit-laser system is coupled to a heat bath. The Hamiltonian is given in (15). The parameters are $\beta = 5/D$, where $D = 20$ is the spectral width of the bath, $g_0 = g_k = 0.1$ for all k, and the bath is modelized with $N_B = 50$ modes. The blue dotted line is the work obtained when the laser is in a coherent state, while the red dashed line corresponds to a Poisson state. (top) $\alpha = 2$. (bottom) $\alpha = 4$. The number of photons is chosen as $2(|\alpha|^2 + |\alpha|)$.

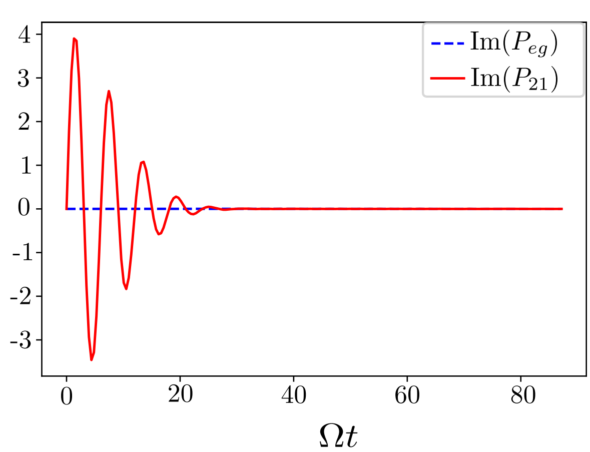


FIG. 5. (dashed line) Coherences in the dressed qubit basis; (full line) coherences in the qubit basis. Parameters: $N_B = 50$ modes in the bath, $2|\alpha|^2 + 2|\alpha|$ photons in the laser, $\beta = 5/D$, where $D = 20$ is the spectral width of the bath, $g_0 = g_k = 0.1$ for all k, and where the time is in units of the Rabi frequency.

written as

$$\Delta S_X = \Delta S_A + \Delta S_L - D(\hat{\rho}_X(t)||\hat{\rho}_A(t) \otimes \hat{\rho}_L(t)), \tag{60}$$

where S_A, S_L are respectively the von Neumann entropies associated with the subsystems A and L. As long as the laser stays close to a coherent state or a Poisson state, ΔS_L is negligible (see Appendix A and Fig. 1), and we obtain the second law for the qubit,

$$\Sigma_A \equiv \Delta S_A - \beta_B Q \geqslant D(\hat{\rho}_X(t)||\hat{\rho}_A(t) \otimes \hat{\rho}_L(t)) \geqslant 0, \tag{61}$$

where Σ_A is the entropy production of the qubit. Likewise, for the dressed qubit, using (31), we find

$$\Sigma_{DA} \equiv \Delta S_{DA} - \beta_B Q \geqslant D(\hat{\rho}(t)||\hat{\rho}_{DA}(t) \otimes \hat{\rho}_{DL}(t)) \geqslant 0. \tag{62}$$

2. *Nonautonomous description*

In the nonautonomous description, the qubit-bath system is isolated, with energy changes due solely to the time-dependence of the Hamiltonian, identified as work,

$$\dot{W} \equiv d_t \mathrm{Tr}[\hat{\tilde{\rho}}(t)\hat{\tilde{H}}(t)] = \mathrm{Tr}[\hat{\tilde{\rho}}(t) d_t \hat{V}(t)]. \tag{63}$$

Interestingly, this definition becomes equivalent to the definition of work in the autonomous description (6) in the macroscopic limit $|\alpha| \gg 1$. Indeed, applying the transformation (33) in (50), we find

$$\dot{W}_L = \dot{W} + i\omega_L g_0 \mathrm{Tr}[(\hat{\sigma}_- \hat{a}^\dagger - \hat{\sigma}_+ \hat{a})\hat{\tilde{\rho}}]. \tag{64}$$

Since the coupling amplitude of $\hat{V}(t)$ is $g = g_0|\alpha|$, the first term on the right-hand side (r.h.s.) is dominant compared with the second one when $|\alpha| \gg 1$.[2]

Defining now

$$\tilde{E}_A(t) \equiv \mathrm{Tr}[(\hat{H}_A + \hat{V}(t) + \hat{V}_{AB'})\hat{\tilde{\rho}}(t)], \tag{65}$$

the conservation of energy leads to

$$d_t \tilde{E}_A(t) = \dot{W} - \mathrm{Tr}[\hat{H}_{B'} d_t \hat{\tilde{\rho}}(t)]. \tag{66}$$

Note that, using (33) which connects $\hat{\tilde{\rho}}(t)$ to $\hat{\rho}(t)$, we obtain the identity $\tilde{E}_A(t) = E_A(t)$. A standard calculation also shows that

$$-\mathrm{Tr}[\hat{H}_{B'} d_t \hat{\tilde{\rho}}(t)] = \dot{Q} + \dot{W}_L - \dot{W}, \tag{67}$$

which simplifies to

$$-\mathrm{Tr}[\hat{H}_{B'} d_t \hat{\tilde{\rho}}(t)] = \dot{Q} \tag{68}$$

in the macroscopic limit, hence the first law for the qubit in the nonautonomous description is consistent with that of the qubit in the autonomous description,

$$d_t \tilde{E}_A(t) = \dot{Q} + \dot{W}_L. \tag{69}$$

Similarly, we find that the first law in the rotating frame coincides with that of the dressed qubit, namely,

$$d_t \tilde{E}_A^{\mathrm{rot}}(t) = \dot{Q} + \dot{W}_{DL}, \tag{70}$$

[2]The correspondence between the autonomous and nonautonomous work rates has also been pointed out in Ref. [29], in the absence of dissipation (no bath).

where (see Appendix C for details)

$$\tilde{E}_A^{\rm rot}(t) \equiv {\rm Tr}\big[\big(\hat{H}_A^{\rm rot} + \hat{V}'_{AB}(t)\big)\hat{\tilde{\rho}}^{\rm rot}(t)\big]. \tag{71}$$

This shows that the equivalence between the dressed qubit in the autonomous picture and the qubit in the rotating frame, in the nonautonomous picture, also holds for the thermodynamics at the average level.

As a final remark, we point out that, on one hand

$$ {\rm Tr}[\hat{\tilde{\rho}}(t) d_t \hat{V}(t)] = -\omega_L g {\rm Im}(\langle b|\hat{\tilde{\rho}}^{\rm rot}|a\rangle), \tag{72}$$

while on the other hand, a standard calculation shows that

$$ {\rm Im}(\langle b|\hat{\tilde{\rho}}^{\rm rot}|a\rangle) = {\rm Im}(\langle 2|\hat{\rho}|1\rangle). \tag{73}$$

This is consistent with the fact that the work in the autonomous picture (51) is equal to the one in the nonautonomous picture (63).

We now turn to the second law. Using the initial condition in (34) and following the same reasoning as in the autonomous description, we obtain

$$\Delta\tilde{S}_A - \beta_B Q \geqslant D(\hat{\rho}_X || \hat{\rho}_A(t) \otimes \hat{\rho}_L(t)) \geqslant 0, \tag{74}$$

and

$$\Delta\tilde{S}_A^{\rm rot} - \beta_B Q \geqslant D\big(\hat{\rho}_X \big|\big|\hat{\tilde{\rho}}_A^{\rm rot}(t) \otimes \hat{\rho}_L(t)\big) \geqslant 0. \tag{75}$$

E. Thermodynamics at the fluctuating level

Here, we focus on the thermodynamics at the fluctuating level. We resort to the two-point measurement technique with counting fields, introduced in Sec. II B 2. This technique is efficient to relate the fluctuations of observables corresponding to operators which commute (e.g., $\hat{H}_A$ and $\hat{H}_L$), when the interactions between these operators are weak, but fails in the presence of strong interactions, which is the case in the macroscopic limit $|\alpha| \gg 1$, for example. In this section, we show that the dressed qubit picture, on the one hand, and the Mollow transformation, on the other hand, allow us to overcome the difficulty raised by the coupling term $\hat{V}_{AL}$, yielding two different work fluctuation theorems, respectively, for W_{DL} and W_L.

1. Joint generating functions

One can measure several observables simultaneously provided that they commute. In the case of simultaneous measurements, we resort to a joint moment-generating function

$$\begin{aligned} \mathcal{G}(t,\boldsymbol{\lambda}) &\equiv {\rm Tr}[\hat{\rho}_{\boldsymbol{\lambda}}(t)], \\ \hat{\rho}_{\boldsymbol{\lambda}}(t) &\equiv \hat{U}_{\boldsymbol{\lambda}}(t,0)\bar{\hat{\rho}}(0)\hat{U}^\dagger_{-\boldsymbol{\lambda}}(t,0), \\ \hat{U}_{\boldsymbol{\lambda}}(t,0) &\equiv e^{i\boldsymbol{\lambda}\cdot\hat{\boldsymbol{H}}/2}\hat{U}(t,0)e^{-i\boldsymbol{\lambda}\cdot\hat{\boldsymbol{H}}/2}, \end{aligned} \tag{76}$$

where $\hat{\boldsymbol{H}}$, $\boldsymbol{\lambda}$ respectively denote a vector of Hamiltonians and a vector of counting fields. By convention, we use the same subscripts for the counting fields and the corresponding Hamiltonians; for instance, when measuring $\hat{\boldsymbol{H}} = (\hat{H}_A(t), \hat{H}_{B'})$ we use $\boldsymbol{\lambda} = (\lambda_A, \lambda_{B'})$, while the counting fields for $\hat{\boldsymbol{H}} = (\hat{H}_{DA}, \hat{H}_{DL}, \hat{H}_B)$ are denoted $\boldsymbol{\lambda} = (\lambda_{DA}, \lambda_{DL}, \lambda_B)$.

Let us now connect the outcomes of the measurements described by (76) with the thermodynamic quantities introduced in Sec. III D. From (46) and (68), it is straightforward to see that the heat Q leaked from the bath is equivalently obtained by measuring $-\hat{H}_B$ or $-\hat{H}_{B'}$, while from (6) and (56) we see that the work terms W_L and W_{DL} are obtained by measuring respectively $-\hat{H}_L$ and $-\hat{H}_{DL}$. Finally, we may measure the energy of the qubit (dressed qubit) defined in (65) [(54)], using a two-point measurement of $\hat{H}_A(t)$ ($\hat{H}_{DA}$) if we require that the coupling $\hat{V}_{AB'}$ ($\hat{V}_{AB}$) is switched on only after the initial measurement and switched off before the final one.

2. Work and entropy fluctuation theorems—autonomous picture

Fluctuation theorems are symmetries relating the energy or entropy fluctuations generated during a given forward process and its time-reversed counterpart. In the context of two-point measurement schemes with counting fields, such theorems can be expressed as symmetries between the moment-generating functions of the forward and backward dynamics.

In this section, we derive a work fluctuation theorem and an entropy fluctuation theorem in the autonomous description.

We first need to introduce the moment generating function of the reversed dynamics, which is defined as [23,30]

$$\begin{aligned} \mathcal{G}^R(\boldsymbol{\lambda},t) &\equiv {\rm Tr}\big[\hat{\rho}^R_{\boldsymbol{\lambda}}(t)\big] \\ &= {\rm Tr}[\hat{U}^\dagger_{\boldsymbol{\lambda}}(t,0)\bar{\hat{\rho}}^R(0)\hat{U}_{-\boldsymbol{\lambda}}(t,0)], \end{aligned} \tag{77}$$

where $\bar{\hat{\rho}}^R(0)$ is the diagonal part of the initial density matrix of the reverse dynamics $\hat{\rho}^R(0)$ in the common eigenbasis of $\hat{H}_{DA}$, $\hat{H}_{DL}$, $\hat{H}_B$ chosen for the measurement. Now, we assume that the initial density matrix of the reverse dynamics is factorized, as for the forward dynamics,

$$\hat{\rho}^R(0) = \hat{\rho}^R_{DA}(0) \otimes \hat{\rho}^R_{DL}(0) \otimes \hat{\rho}_B. \tag{78}$$

Notice that, since $\hat{H}$ is time-independent, (78) implies that the time-reversed density matrix is simply given by $\hat{\rho}^R(t) = \hat{\rho}(-t)$. Let us furthermore assume that

$$\hat{\rho}_{DA}(0) = \hat{\rho}^R_{DA}(0) = e^{-\beta_{DA}\hat{H}_{DA}}/Z_{DA}. \tag{79}$$

As explained in Sec. II B 2, the two-point measurement technique with counting fields allows us to compute rigorously the fluctuations of the work performed by a Poisson state, which correspond to the fluctuations of the work performed by a coherent state during a series of experiments where the initial phase of the coherent state is randomly chosen. We therefore assume that $\hat{\rho}_{DL}(0)$ and $\hat{\rho}^R_{DL}(0)$ are Poisson states (5). As explained in Sec. II B 2, Poisson states can be written as Gibbs states in the macroscopic limit $|\alpha| \gg 1$, specifically (see Appendix D),

$$\hat{\rho}_{DL}(0) = \hat{\rho}^R_{DL}(0) = e^{-\beta_{DL}\hat{H}'_{DL}}/Z_{DL}, \tag{80}$$

with $\beta_{DL} \equiv 1/|\alpha|^2$ and where $\hat{H}'_{DL} = (\hat{a}^\dagger\hat{a} - \langle N\rangle)^2/2$. Under these assumptions, the moment-generating function (76) satisfies the following symmetry (see Appendix D):

$$\mathcal{G}(\boldsymbol{\lambda},t) = \mathcal{G}^R(-\boldsymbol{\lambda} + i\boldsymbol{\nu}, t), \tag{81}$$

with $\boldsymbol{\nu} = (\beta_{DA}, \beta_{DL}, \beta_B)$. Given that, under assumption 2, fluctuations of the order $1/|\alpha|$ can be neglected, we further on replace $\beta_{DL} = 0$.

On its own, the symmetry (81) is formal but, combined with the notion of energy conservation, it yields a work fluctuation theorem. Energy conservation is conveniently expressed using the generating function (76): in the absence of external driving, the total energy of the system should be conserved. Setting $\lambda_{DA} = \lambda_{DL} = \lambda_B \equiv \lambda$, this condition is satisfied at the average level if and only if $\partial_\lambda \mathcal{G}(\boldsymbol{\lambda}, t)|_{\lambda=0} = 0$, which yields the first law (55). Imposing instead energy conservation at the fluctuating level takes the form of a strict energy conservation condition [31],

$$\hat{\rho}_{\boldsymbol{\lambda}}(t) = \hat{\rho}_{\boldsymbol{\lambda}+\chi\mathbf{1}}(t), \tag{82}$$

where $\mathbf{1} = (1, 1, 1)$. Together with the symmetry (81), the strict energy conservation condition (82) implies a work fluctuation theorem: setting $\beta_{DA} = \beta_B \equiv \beta$, we find

$$\begin{aligned}\mathcal{G}(0, \lambda_{DL}, 0, t) &= \mathcal{G}^R(i\beta, -\lambda_{DL}, i\beta, 0)\\ &= \mathcal{G}^R(0, -\lambda_{DL} - i\beta, 0, t).\end{aligned} \tag{83}$$

Applying a reverse Fourier transform allows us to rephrase the above equality in terms of the probabilities $p(W_{DL})$ the [respectively $p^R(W_{DL})$] to observe a variation W_{DL} in the forward (respectively time-reversed) dynamics [see (11)], which yields

$$\frac{p(W_{DL})}{p^R(-W_{DL})} = e^{\beta W_{DL}}. \tag{84}$$

The symmetry (81) and fluctuation theorem (84) constitute important results of this work, and will serve as a criteria of consistency for quantum optical quantum master equations.

A similar relation as (81) can be obtained for the entropy production, Σ, obtained from measuring the variations of the operator

$$\hat{\Sigma}_{DA}(t) \equiv -\ln \hat{\rho}_{DA}(t) - \ln \hat{\rho}_{DL}(t) + \beta_B \hat{H}_B. \tag{85}$$

Assuming the initial conditions (31) and (78), we obtain an entropy fluctuation theorem

$$\mathcal{G}_\Sigma(\lambda_\Sigma, t) = \mathcal{G}^R_\Sigma(-\lambda_\Sigma + i, t), \tag{86}$$

where λ_Σ is the counting field associated with $\hat{\Sigma}$. In particular, $\mathcal{G}_\Sigma(i, t) = 1$, which leads to the integral fluctuation theorem $\langle e^{-\Sigma}\rangle = 1$ and by convexity to $\langle\Sigma\rangle \geqslant 0$, which is the second law (62).

3. Work and entropy fluctuation theorems—nonautonomous picture

We now turn to the nonautonomous description, starting with the work fluctuation theorem.

Performing a two point-measurement of the Hamiltonian $\hat{H}_L$, then applying the Mollow transformation (33) yields the tilted density matrix

$$\begin{aligned}\hat{\tilde{\rho}}_{\lambda_L} &\equiv D^\dagger[\alpha(t)]\hat{\rho}_{\lambda_L} D[\alpha(t)]\\ &= \mathcal{T}\big[e^{-i\int_0^t ds \hat{H}_{\lambda_L}(s)}\big]\hat{\rho}(0)\mathcal{T}\big[e^{i\int_0^t ds \hat{H}_{-\lambda_L}(s)}\big],\end{aligned} \tag{87}$$

where $\hat{\rho}_{\lambda_L}(t)$ is obtained by choosing $\boldsymbol{\lambda} = (0, \lambda_L, 0)$ with $\boldsymbol{H} = (0, \hat{H}_L, 0)$ in (76), where $\mathcal{T}$ denotes time ordering and where $\hat{H}_{\lambda_L}(t) = \hat{H}_A + \hat{H}_{B'} + \hat{V}_{AB} + \hat{V}_{AL}^{\lambda_L} + \hat{V}_{\lambda_L}(t)$ with

$$\hat{V}_{AL}^{\lambda_L} = \frac{g_0}{2}(\hat{\sigma}_+\hat{a}e^{-i\omega_L\lambda_L} + \hat{\sigma}_-\hat{a}^\dagger e^{i\omega_L\lambda_L}), \tag{88}$$

$$\begin{aligned}\hat{V}_{\lambda_L}(t) &= \frac{g_0}{2}(\hat{\sigma}_+\alpha(t)e^{-i\omega_L\lambda_L} + \hat{\sigma}_-\alpha^*(t)e^{i\omega_L\lambda_L})\\ &= \hat{V}(t + \lambda_L).\end{aligned} \tag{89}$$

The fluctuations of $\hat{H}_L$, measured by λ_L, are now carried both by $\hat{V}_{AL}^{\lambda_L}$ and the time-dependent term $\hat{V}_{\lambda_L}(t)$. Performing now additional projective measurements with counting fields on $\hat{H}_A + \hat{V}(t)$ and $\hat{H}_B$ yields the tilted density matrix

$$\hat{\tilde{\rho}}_{\boldsymbol{\lambda}} \equiv \hat{\tilde{U}}_{\boldsymbol{\lambda}}\hat{\tilde{\rho}}(0)\hat{\tilde{U}}^\dagger_{-\boldsymbol{\lambda}}, \tag{90}$$

with $\boldsymbol{\lambda} = (\lambda_A, \lambda_L, \lambda_B)$ and

$$\begin{aligned}\hat{\tilde{U}}_{\boldsymbol{\lambda}} &= e^{i\frac{\lambda_B}{2}\hat{H}_B}e^{i\frac{\lambda_A}{2}(\hat{H}_A+\hat{V}(t))}\mathcal{T}[e^{-i\int_0^t ds\hat{H}_{\lambda_L}(s)}]\\ &\quad\times e^{-i\frac{\lambda_B}{2}\hat{H}_B}e^{-i\frac{\lambda_A}{2}(\hat{H}_A+\hat{V}(0))}.\end{aligned} \tag{91}$$

The tilted density matrix for the reversed dynamics is in turn given by

$$\hat{\tilde{\rho}}^R_{\boldsymbol{\lambda}}(t) \equiv \hat{\tilde{U}}^\dagger_{\boldsymbol{\lambda}}\hat{\tilde{\rho}}(0)\hat{\tilde{U}}_{-\boldsymbol{\lambda}}. \tag{92}$$

Noticing that the Mollow transformation does not change the trace, we may apply the same reasoning as in the autonomous description. Let us introduce the partition function $Z_A(t) \equiv \mathrm{Tr}[e^{-\beta_A(\hat{H}_A+\hat{V}(t))}]$. Since $e^{i\omega_L\hat{\sigma}_z t/2}\hat{V}(t)e^{-i\omega_L\hat{\sigma}_z t/2} = \hat{V}(0)$, the partition function is in fact time-independent, $Z_A(t) = Z_A(0) \equiv Z_A$. We now assume that the initial density matrix of the reverse process is factorized as in (35), and further that

$$\begin{aligned}\hat{\tilde{\rho}}_A(0) &= \frac{e^{-\beta_A(\hat{H}_A+\hat{V}(0))}}{Z_A},\\ \hat{\tilde{\rho}}^R_A(0) &= \frac{e^{-\beta_A(\hat{H}_A+\hat{V}(t))}}{Z_A}.\end{aligned} \tag{93}$$

We point out that, since, in the rotating frame, the evolution of the qubit is equivalent to that of the dressed qubit (as shown in Sec. III C 2), which is governed by a time-independent Hamiltonian, we deduce, as in the autonomous case, the relation $\hat{\tilde{\rho}}^R(t) = \hat{\tilde{\rho}}(-t)$. For an alternative proof using the Floquet states, see Appendix E. We then obtain, in the macroscopic limit $|\alpha| \gg 1$, the following symmetry for the generating function in the nonautonomous description $\tilde{\mathcal{G}}(\boldsymbol{\lambda}, t) \equiv \mathrm{Tr}[\hat{\tilde{\rho}}_{\boldsymbol{\lambda}}]$,

$$\tilde{\mathcal{G}}(\boldsymbol{\lambda}, t) = \tilde{\mathcal{G}}^R(-\boldsymbol{\lambda} + i\boldsymbol{\nu}, t), \tag{94}$$

with $\boldsymbol{\nu} = (\beta_A, 0, \beta_B)$. Similarly as in the autonomous case, the energy conservation condition is here satisfied if $\partial_\lambda\tilde{\mathcal{G}}(\boldsymbol{\lambda}, t)|_{\lambda=0} = 0$ when $\boldsymbol{\lambda} = (\lambda, \lambda, \lambda)$. When satisfied, this condition yields the first law (69). In turn, the strict energy conservation reads

$$\hat{\tilde{\rho}}_{\boldsymbol{\lambda}+\chi\mathbf{1}}(t) = \hat{\tilde{\rho}}_{\boldsymbol{\lambda}}(t). \tag{95}$$

The combination of (95) and (94) then yields the following work fluctuation theorem for the laser, similar to the Crooks relation [32,33]: setting $\beta_A = \beta_B = \beta$, we find

$$\frac{p(W_L)}{p^R(-W_L)} = e^{\beta W_L}. \tag{96}$$

Notice that (96) implies the Jarzynski equality [34].

We now turn to the entropy fluctuation theorem. We use the same reasoning as in the previous section: since the entropy fluctuations of the laser can be neglected in the autonomous picture (see Appendix A), measuring

$$\hat{\Sigma}_A(t) \equiv \ln \hat{\tilde{\rho}}_A(t) + \ln \hat{\tilde{\rho}}_L(t) + \beta_B \hat{H}_B \tag{97}$$

amounts to measuring the entropy production $\Delta\tilde{S}_A - \beta_B Q$. Then, assuming that the density matrices of the forward and time-reversed dynamics are initially factorized with $\hat{\tilde{\rho}}^R(0) = \hat{\tilde{\rho}}_A(t) \otimes \hat{\rho}_{B'}$, we obtain the entropy fluctuation theorem

$$\tilde{\mathcal{G}}_\Sigma(\lambda_{\Sigma_A}, t) = \tilde{\mathcal{G}}^R_\Sigma(-\lambda_{\Sigma_A} + i, t). \tag{98}$$

We mention that recent works [25,26] have examined the full counting statistics of the work performed by a laser using the counting field method, but no symmetry such as (94) had so far been derived. We also highlight that, in those approaches, the statistics of the laser is described solely using the term (89). Our approach shows that the full statistics are in fact given by both the terms (88) and (89). In the macroscopic limit $|\alpha| \gg 1$, the term (89) dominates, and both approaches should become equivalent. It would be interesting to study the low number of photons limit, where (88) and (89) become comparable.

IV. THERMODYNAMIC CONSISTENCY OF QUANTUM MASTER EQUATIONS

We now turn to the effective description in terms of quantum master equations and examine their thermodynamic consistency, i.e., under which conditions the laws of thermodynamics derived in Sec. III D, the symmetries (81) and (94), and the fluctuation theorem (84) and (96) hold. We derive the quantum master equations using the theory of quantum maps. To keep track of the energy transfers fluctuations during the derivations, we start from the tilted unitary dynamics defined in (76).

A. For the qubit-laser system X

The coupling to the thermal bath is assumed to be weak. Since the density matrix is initially a tensor product of the matrices of X and B (21), the evolution of $\hat{\rho}^{\boldsymbol{\lambda}}_X(t) \equiv \mathrm{Tr}_B[\hat{\rho}^{\boldsymbol{\lambda}}(t)]$ is described by a quantum map,

$$\begin{aligned}\hat{\rho}^{\boldsymbol{\lambda}}_X(t) &= \sum_{\mu,\nu} \hat{W}^{\boldsymbol{\lambda}}_{\mu,\nu}(t,0)\hat{\rho}_X(0)\hat{W}^{-\boldsymbol{\lambda}\dagger}_{\mu,\nu}(t,0)\\ &\equiv \hat{M}^{\boldsymbol{\lambda}}(t,0)\hat{\rho}_X(0),\end{aligned} \tag{99}$$

where $\hat{W}^{\boldsymbol{\lambda}}_{\mu,\nu}(t,0)$ are Kraus operators [2],

$$\hat{W}^{\boldsymbol{\lambda}}_{\mu,\nu}(t,0) = \sqrt{\eta_\nu}\langle\mu|\hat{U}_{\boldsymbol{\lambda}}(t,0)|\nu\rangle, \tag{100}$$

where $\hat{U}_{\boldsymbol{\lambda}}(t,0)$ was defined in (76) and with $\{|\nu\rangle\}$ the eigenstates of $\hat{H}_B$ of eigenvalues ν and $\eta_\nu = e^{-\beta_B\omega_\nu}/Z_B$.

We then make the Markov approximation, or semigroup hypothesis in the context of quantum maps [2]: $\hat{M}^{\boldsymbol{\lambda}}(t,0) = \hat{M}^{\boldsymbol{\lambda}}(t,s)\hat{M}^{\boldsymbol{\lambda}}(s,0)$ for all $0 \leqslant s \leqslant t$. This leads to a time local equation of motion of the form

$$\begin{aligned}\frac{d\hat{\rho}^{\boldsymbol{\lambda}}_X(t)}{dt} &= \lim_{\delta\to\delta_0}\frac{1}{\delta}(\hat{M}^{\boldsymbol{\lambda}}(t+\delta,t) - \mathbb{I})\hat{\rho}_X(t)\\ &\equiv \mathcal{L}^X_{\boldsymbol{\lambda}}\big(\hat{\rho}^{\boldsymbol{\lambda}}_X(t)\big),\end{aligned} \tag{101}$$

where the coarse graining time δ_0 is chosen larger than the relaxation time of the bath and smaller than the relaxation time of X. We discuss precisely these timescales in Sec. V.

The thermodynamic consistency condition for the master equation (101), where only the thermal bath has been traced out, has been identified in our previous work [31]. It reads

$$\mathcal{L}^{X,R}_{0,0,-\lambda_B}[\ldots] = \mathcal{L}^{X\dagger}_{0,0,-\lambda_B+i\beta_B}[\ldots], \tag{102}$$

where we introduced the adjoint $O^\dagger$ of a superoperator O as the one satisfying $\mathrm{Tr}[(O(X))^\dagger Y] = \mathrm{Tr}[X^\dagger O^\dagger(Y)]$ for all operators X, Y.

B. For the dressed qubit: Autonomous description

We now proceed to trace out the degrees of freedom of the dressed laser and examine the energy exchanges in the dressed qubit picture. We therefore set $\boldsymbol{\lambda} = (\lambda_{DA}, \lambda_{DL}, \lambda_B)$. In the dressed qubit picture, the dressed qubit and dressed laser interact indirectly through the bath. The consistency condition for the master equations for $\hat{\rho}_{DA}$ can then be derived following the same logic as in Ref. [31]. Using the initial condition (31) with (80), tracing out the degrees of freedom of DL in (99) leads to

$$\hat{\rho}^{\boldsymbol{\lambda}}_{DA}(t) = \sum_{\boldsymbol{\kappa},\boldsymbol{\kappa}'} \hat{W}^{\boldsymbol{\lambda}}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}(t,0)\hat{\rho}_{DA}(0)\hat{W}^{-\boldsymbol{\lambda}\dagger}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}(t,0), \tag{103}$$

where the sum runs over the pairs $\boldsymbol{\kappa} = (\mu, n)$, $\boldsymbol{\kappa}' = (\nu, n')$ and where

$$\hat{W}^{\boldsymbol{\lambda}}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'} = \sqrt{\eta_\nu\xi_n}\langle n,\mu|\hat{U}_{\boldsymbol{\lambda}}(t,0)|n',\nu\rangle, \tag{104}$$

with $\xi_n = \langle n|\hat{\rho}_{DL}(0)|n\rangle$. Notice that the Kraus operators (104) satisfy the property

$$\hat{W}^{\boldsymbol{\lambda}}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}(t,0) = e^{\frac{\lambda_{DA}}{2}\hat{H}_{DA}}\hat{W}^{0,\lambda_{DL},\lambda_B}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}(t,0)e^{\frac{-\lambda_{DA}}{2}\hat{H}_{DA}}, \tag{105}$$

which implies that

$$\begin{aligned}\hat{\rho}^{\boldsymbol{\lambda}}_{DA}(t) = {}& e^{i\frac{\lambda_{DA}}{2}\hat{H}_{DA}}e^{t\mathcal{L}_{0,\lambda_{DL},\lambda_B}}[e^{-i\frac{\lambda_{DA}}{2}\hat{H}_{DA}}\hat{\rho}_{DA}(0)e^{-i\frac{\lambda_{DA}}{2}\hat{H}_{DA}}]\\ &\times e^{i\frac{\lambda_{DA}}{2}\hat{H}_{DA}},\end{aligned} \tag{106}$$

where $\mathcal{L}_{\boldsymbol{\lambda}}$ is the superoperator dressed with counting fields describing the evolution of $\hat{\rho}^{\boldsymbol{\lambda}}_{DA}$. The symmetry (81) is then satisfied if

$$\mathcal{L}^R_{0,-\lambda_{DL},-\lambda_B+i\beta_B}[\ldots] = \mathcal{L}^\dagger_{0,-\lambda_{DL},-\lambda_B}[\ldots], \tag{107}$$

as can be seen by computing separately the moment-generating functions $\mathcal{G}(\boldsymbol{\lambda}, t)$ and $\mathcal{G}^R(-\boldsymbol{\lambda} + i\boldsymbol{\nu}, t)$ and using (106) combined with the property of the adjoint $\mathrm{Tr}[(e^{t\mathcal{L}}[\hat{X}])^\dagger\hat{Y}] = \mathrm{Tr}[\hat{X}^\dagger e^{t\mathcal{L}^\dagger}[\hat{Y}]]$. Note that the generalization to many uncoupled heat baths is straightforward by linearity. The condition (107) is another important result of this work: it is a simple criteria of thermodynamic consistency of quantum master equations for quantum systems coupled to heat baths and a coherent light source.

Using the same reasoning as in Sec. III E 2, we deduce that (107) ensures that the entropy fluctuation theorem (86) holds at the level of master equations, hence that the second law is satisfied on average and at the level of the rates,

$$d_t S_{DA} - \beta_B \dot{Q} \geqslant 0. \tag{108}$$

The strict energy conservation condition (82) takes the form

$$\mathcal{L}_{\boldsymbol{\lambda}}[\ldots] = \mathcal{L}_{\boldsymbol{\lambda}+\chi\mathbf{1}}[\ldots] \tag{109}$$

and guarantees that the first law is satisfied at fluctuating level. It also implies energy conservation on average, i.e., that $\partial_\lambda \mathrm{Tr}(\mathcal{L}_{\boldsymbol{\lambda}})|_{\lambda=0} = 0$ when all the counting fields are set equal to λ, which in turn implies that the first law is satisfied on average at the level of the rates,

$$d_t E_{DA} = \dot{Q} + \dot{W}_{DL}. \tag{110}$$

A quantum master equation is said to be fully thermodynamically consistent if and only if it satisfies both (107) and (109). Using the same argument as in Ref. [31], we find that satisfying these two conditions requires us to use the secular approximation in the dressed qubit basis. More precisely, we examine under which condition (103) becomes $\boldsymbol{\lambda}$-independent. Expressing the Kraus operators in the joint eigenbasis of $\hat{H}_{DA}$, $\hat{H}_{DL}$, and $\hat{H}_B$, it appears that the only way to achieve this condition is to perform the secular approximation. We do not provide a detailed proof here since the reasoning and calculations are almost identical as in Ref. [31], where thorough details are provided in the Appendix. We explicitly perform the secular approximation in the Sec. V, when deriving the autonomous Floquet equation.

C. For the qubit: Nonautonomous description

We now turn to the nonautonomous picture. To derive a master equation which preserves the symmetry (94), we begin by noticing that, using the identity $e^{i(\hat{H}_A+\hat{V}(t))} = e^{-i\omega_L\hat{\sigma}_z t/2} e^{i(\hat{H}_A+\hat{V})} e^{i\omega_L\hat{\sigma}_z t/2}$ with $\hat{V} \equiv \frac{g}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$ and the cyclicity of the trace, the generating function in (94) can be rewritten as

$$\tilde{\mathcal{G}}(\boldsymbol{\lambda}, t) = \mathrm{Tr}\big[\hat{U}^{\mathrm{rot}}_{\boldsymbol{\lambda}} \hat{\tilde{\rho}}(0) \hat{U}^{\mathrm{rot}\dagger}_{-\boldsymbol{\lambda}}\big] \tag{111}$$

where

$$\begin{aligned}\hat{U}^{\mathrm{rot}}_{\boldsymbol{\lambda}} &= e^{i\frac{\lambda_B}{2}\hat{H}_B} e^{i\frac{\lambda_A}{2}(\hat{H}_A+\hat{V})} e^{i\omega_L\hat{\sigma}_z t/2}\\ &\quad\times \mathcal{T}[e^{-i\int_0^t ds \hat{H}_{\lambda_L}(s)}] e^{-i\frac{\lambda_B}{2}\hat{H}_B} e^{-i\frac{\lambda_A}{2}(\hat{H}_A+\hat{V})}.\end{aligned} \tag{112}$$

When the counting fields are set to zero, $\hat{U}^{\mathrm{rot}}_{\boldsymbol{\lambda}}$ becomes the propagator of the dynamics in the rotating frame (42). We may now apply the same reasoning as in the autonomous case in Sec. IV B to compute a master equation for $\hat{\tilde{\rho}}^{\mathrm{rot}}_A$. Since the counting fields λ_A are now associated with a time-independent Hamiltonian in (112), the Kraus operators for $\hat{\tilde{\rho}}^{\mathrm{rot}}_A$ will have the same form as (105); applying the same reasoning as in Sec. IV B, we obtain the following condition, which guarantees the symmetry (94),

$$\tilde{\mathcal{L}}^{\mathrm{rot}R}_{0,\lambda_L,\lambda_B+i\beta_B}[\ldots] = \tilde{\mathcal{L}}^{\mathrm{rot}\dagger}_{0,\lambda_L,\lambda_B}[\ldots]. \tag{113}$$

The strict energy conservation condition is written

$$\tilde{\mathcal{L}}_{\boldsymbol{\lambda}+\chi\mathbf{1}}[\ldots] = \tilde{\mathcal{L}}_{\boldsymbol{\lambda}}[\ldots] \tag{114}$$

and is equivalent to (109), since we showed in Sec. III E 3 that the energy conservation conditions are equivalent in the autonomous and nonautonomous pictures. When satisfied, it implies that the first law holds at the level of the rates,

$$d_t E_A = \dot{W}_L + \dot{Q}. \tag{115}$$

V. GENERALIZED BLOCH, OPTICAL BLOCH, AND FLOQUET MASTER EQUATIONS

In this section, we derive three master equations which can be used in practice to study the coherent driving of a qubit and examine whether they satisfy the general conditions of consistency identified in the previous section. Each master equation is derived both in the autonomous and nonautonomous pictures. A schematic representation of the approximations made and of the correspondences between the autonomous and nonautonomous picture is given in Fig. 6.

We first derive a standard Markovian master equation in the joint qubit-laser space, in the autonomous picture. We then trace out the dressed laser to derive master equations in the dressed qubit basis. We begin with a master equation, called the generalized Bloch equation, valid at all qubit-laser coupling strengths. Then, we identify three relevant qubit-laser coupling regimes (or driving regimes): strong, intermediate, and weak. The strong-driving regime leads to the Floquet master equation, while the intermediate- and weak-driving regimes gives rise to the Bloch master equation. We then proceed to show how master equations in the qubit space can be derived using the correspondence between the dressed qubit (in the autonomous picture) and the evolution in the rotating frame (in the nonautonomous picture), showed in Sec. III C 2.

A. Qubit-laser

For convenience, we use the interaction picture [2]. In the interaction picture, the map (99) becomes

$$\hat{\rho}^{\boldsymbol{\lambda}I}_X(t) = \sum_{\mu,\nu} \hat{W}^{\boldsymbol{\lambda}I}_{\mu,\nu}(t,0)\hat{\rho}_X(0)\hat{W}^{-\boldsymbol{\lambda}I\dagger}_{\mu,\nu}(t,0), \tag{116}$$

where we recall that $\boldsymbol{\lambda} = (\lambda_{DA}, \lambda_{DL}, \lambda_B)$ and where the Kraus operators $\hat{W}^{\boldsymbol{\lambda}I}_{\mu,\nu}(t,0)$ are given by

$$\hat{W}^{\boldsymbol{\lambda}I}_{\mu,\nu}(t,0) = \sqrt{\eta_\nu}\langle\mu|\mathcal{T}[e^{-i\int_0^t ds \hat{V}^{\boldsymbol{\lambda}}_{AB}(s)}]|\nu\rangle, \tag{117}$$

and where we recall that $\mathcal{T}$ denotes time ordering. To push the derivation further, we perform a perturbative expansion to second order in $\hat{V}_{AB}$.

The Hamiltonian $\hat{V}^{\boldsymbol{\lambda}}_{AB}(t)$ in (117) is the Hamiltonian $\hat{V}^{\boldsymbol{\lambda}}_{AB}$ in the interaction picture, given by (see Appendix F)

$$\begin{aligned}\hat{V}^{\boldsymbol{\lambda}}_{AB}(t) &\equiv e^{i\hat{H}_0 t}\hat{V}^{\boldsymbol{\lambda}}_{AB}e^{-i\hat{H}_0 t}\\ &= \big(\hat{S}^{\boldsymbol{\lambda}}_z(t) + \hat{S}^{\boldsymbol{\lambda}}_-(t) + \hat{S}^{\boldsymbol{\lambda}}_+(t)\big)\hat{B}^\dagger_{\lambda_B}(t) + \mathrm{H.c.},\end{aligned} \tag{118}$$

with $\hat{B}_{\lambda_B}(t) \equiv \sum_k g_k \hat{b}_k e^{-i\omega_k(t+\lambda_B/2)}$ and

$$\begin{aligned}\hat{S}^{\boldsymbol{\lambda}}_z(t) &= e^{-i\omega_L t}e^{-i\omega_L\lambda_{DL}}\hat{S}_z,\\ \hat{S}^{\boldsymbol{\lambda}}_+(t) &= e^{-i(\omega_L-\Omega)t}e^{-i(\omega_L\lambda_{DL}-\Omega\lambda_{DA})}\hat{S}_+,\\ \hat{S}^{\boldsymbol{\lambda}}_-(t) &= e^{-i(\omega_L+\Omega)t}e^{-i(\omega_L\lambda_{DL}+\Omega\lambda_{DA})}\hat{S}_-,\end{aligned} \tag{119}$$

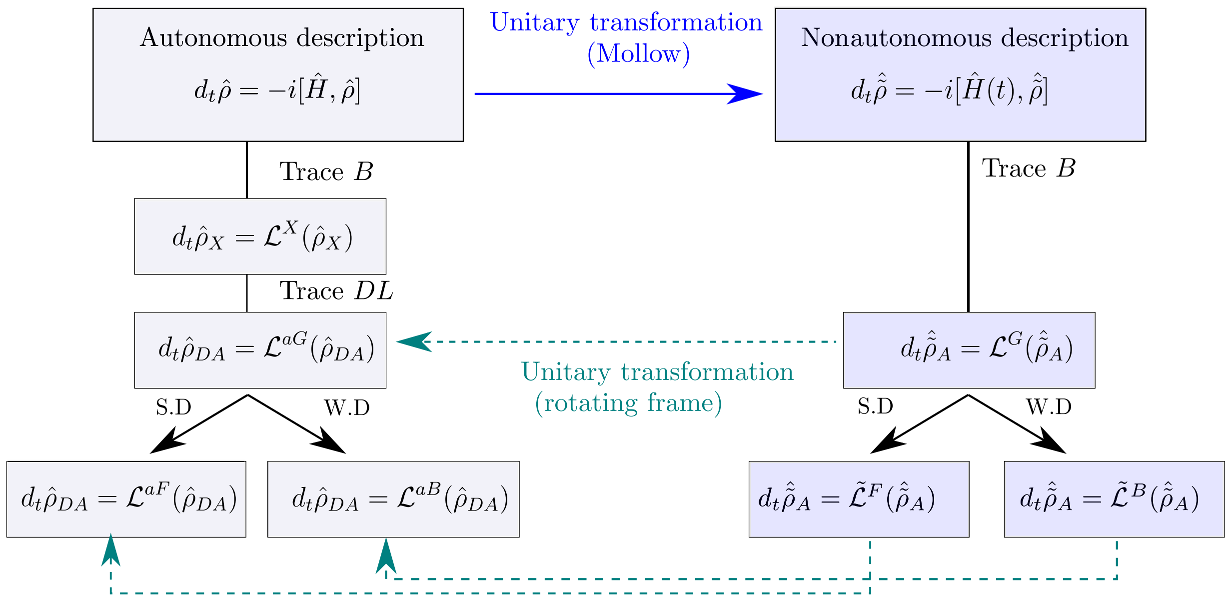


FIG. 6. Schematic representation of the approximations performed in order to derive the generalized Bloch, Bloch and Floquet master equations, and summary of the unitary transformations connecting the autonomous and nonautonomous pictures, both at the unitary level and at the level of the master equations. S.D. stands for strong driving, and the weak- and intermediate-driving regimes are grouped under W.D.

where

$$\begin{aligned}
\hat{S}_z &= \frac{g}{2\Omega}(|2\rangle\langle 2| - |1\rangle\langle 1|) \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n| \\
&\equiv \hat{s}_z \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n|, \\
\hat{S}_+ &\equiv -\frac{\Omega-\delta}{2\Omega}|2\rangle\langle 1| \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n| \\
&\equiv \hat{s}_+ \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n|, \\
\hat{S}_- &\equiv \frac{\Omega+\delta}{2\Omega}|1\rangle\langle 2| \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n| \\
&\equiv \hat{s}_- \otimes \sum_{n\geqslant 0} |n-1\rangle\langle n|,
\end{aligned} \tag{120}$$

where we introduced the reduced operators

$$\begin{aligned}
\hat{s}_z &\equiv \frac{g}{2\Omega}(|2\rangle\langle 2| - |1\rangle\langle 1|) \equiv \frac{g}{2\Omega}\hat{\Sigma}_z, \\
\hat{s}_+ &\equiv -\frac{\Omega-\delta}{2\Omega}|2\rangle\langle 1| \equiv -\frac{\Omega-\delta}{2\Omega}\hat{\Sigma}_+, \\
\hat{s}_- &\equiv \frac{\Omega+\delta}{2\Omega}|1\rangle\langle 2| \equiv \frac{\Omega+\delta}{2\Omega}\hat{\Sigma}_-,
\end{aligned} \tag{121}$$

and where $\hat{\Sigma}_z = |2\rangle\langle 2| - |1\rangle\langle 1|$, $\hat{\Sigma}_+ = |2\rangle\langle 1| = \hat{\Sigma}_-^\dagger$. Later on, we use the reduced operators dressed with counting fields,

$$\begin{aligned}
\hat{s}_z^{\boldsymbol{\lambda}} &\equiv e^{-i\lambda_{DL}\omega_L}\hat{s}_z, \\
\hat{s}_+^{\boldsymbol{\lambda}} &\equiv e^{i\lambda_{DA}\Omega/2}e^{-i\lambda_{DL}\omega_L/2}e^{i\lambda_B(\omega_L-\Omega)/2}\hat{s}_+, \\
\hat{s}_-^{\boldsymbol{\lambda}} &\equiv e^{-i\lambda_{DA}\Omega/2}e^{-i\lambda_{DL}\omega_L/2}e^{i\lambda_B(\omega_L+\Omega)/2}\hat{s}_-.
\end{aligned} \tag{122}$$

It will further be useful to use the identity

$$\hat{S}_z + \hat{S}_+ + \hat{S}_- = \sum_{N_L} |a, N_L\rangle\langle b, N_L|. \tag{123}$$

Let us now introduce the set $\{\hat{\sigma}_{mn}\}$ of jump operators between the eigenstates $\{|j,n\rangle\}$ of $\hat{H}_X$. The set $\{\hat{\sigma}_{mn}\}$ forms a basis of jump operators acting on $\mathcal{H}_X$. To alleviate the notations, we relabel the eigenstates

$$|j,n\rangle \to |E_n\rangle, \tag{124}$$

so that, by definition, $\hat{\sigma}_{mn} = |E_n\rangle\langle E_m|$. We introduce $\omega_{mn} = E_m - E_n$, the corresponding Bohr frequencies. Note that different $\hat{\sigma}_{mn}$ may be associated with the same frequency ω_{mn}. We now express the Kraus operators (117) in the basis $\{\hat{\sigma}_{mn}\}$ and perform the Markov approximation to obtain a time local equation of motion of the form of (101),

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\lambda}}(t)\hat{\rho}_S(t) = \lim_{\delta\to\delta_0}\frac{1}{\delta}\sum_{mn,m'n'} d^{\boldsymbol{\lambda}}_{mn,m'n'}(t,\delta) \\
\times\, \hat{\sigma}_{mn}\hat{\rho}_X^{\boldsymbol{\lambda}I}(t)\hat{\sigma}^\dagger_{m'n'} - \hat{\rho}_X^{\boldsymbol{\lambda}I}(t),
\end{aligned} \tag{125}$$

where

$$\begin{aligned}
d^{\boldsymbol{\lambda}}_{mn,m'n'}(t,\delta) \equiv \sum_{\mu,\nu}\eta_\nu \mathrm{Tr}_S\big[\hat{\sigma}^\dagger_{mn}\hat{W}^{\boldsymbol{\lambda}I}_{\mu,\nu}(t+\delta,t)\big] \\
\times\, \mathrm{Tr}_S\big[\hat{\sigma}_{m'n'}\hat{W}^{-\boldsymbol{\lambda}I\dagger}_{\mu,\nu}(t+\delta,t)\big].
\end{aligned} \tag{126}$$

A perturbative expansion to second order in $\hat{V}_{AB}^{\lambda}$, combined with $\hat{\rho}_B = \sum_\nu \eta_\nu |\eta_\nu\rangle\langle\eta_\nu|$ yields

$$\mathcal{L}_\lambda(t)\hat{\rho}_X^I(t) = \lim_{\delta\to\delta_0}\frac{1}{\delta}\Bigg[\sum_{n,n'}\hat{\sigma}_{nn}\hat{\rho}_X^{\lambda I}(t)\hat{\sigma}_{n'n'}^\dagger - \hat{\rho}_X^{\lambda I}(t) + \sum_{mn,m'n'}\mathrm{Tr}\bigg[\int_t^{t+\delta} ds\,\hat{\sigma}_{mn}^\dagger \hat{V}_{AB}^{\lambda}(s)\hat{\rho}_B \int_t^{t+\delta} ds\,\hat{\sigma}_{m'n'}\hat{V}_{AB}^{-\lambda\dagger}(s)\bigg]\hat{\sigma}_{mn}\hat{\rho}_X^{\lambda I}(t)\hat{\sigma}_{m'n'}^\dagger$$
$$-\frac{1}{2}\sum_{mn,m'n'}\mathrm{Tr}_X[\hat{\sigma}_{m'n'}]\mathrm{Tr}\bigg[\hat{\sigma}_{mn}^\dagger\int_t^{t+\delta}ds\hat{V}_{AB}^{\lambda}(s)\int_t^{s}ds'\hat{V}_{AB}^{\lambda}(s')\hat{\rho}_B\bigg]\hat{\sigma}_{mn}\hat{\rho}_X^{\lambda I}(t)\hat{\sigma}_{m'n'}^\dagger$$
$$-\frac{1}{2}\sum_{mn,m'n'}\mathrm{Tr}_X[\hat{\sigma}_{mn}^\dagger]\mathrm{Tr}\bigg[\hat{\sigma}_{m'n'}\int_t^{t+\delta}ds\hat{V}_{AB}^{-\lambda\dagger}(s)\int_s^{t+\delta}ds'\hat{V}_{AB}^{-\lambda\dagger}(s')\hat{\rho}_B\bigg]\hat{\sigma}_{mn}\hat{\rho}_X^{\lambda I}(t)\hat{\sigma}_{m'n'}^\dagger\Bigg]. \tag{127}$$

Notice that the r.h.s. term of the first line cancels out since the $\hat{\sigma}_{nn}\hat{\rho}_X^{\lambda I}(t)\hat{\sigma}_{n'n'}^\dagger = [\hat{\rho}_X^{\lambda I}(t)]_{nn'}|E_n\rangle\langle E_{n'}|$ and $\{|E_n\rangle\}_n$ is a basis of the system. Writing explicitly $\hat{V}_{AB}^{\lambda}(t)$ in (127), we find that the trace over the bath yields terms of the following form for the coefficients of the master equation (see Appendix F for the full expression):

$$\frac{1}{\delta_0}\int_t^{t+\delta_0}ds\int_t^{t+\delta_0}ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}^\dagger(s)\hat{B}_{-\lambda_B}(s')\hat{\rho}_B\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}$$
$$=\sum_k \mathrm{sinc}\left(\frac{\omega_k-\omega_\alpha}{2}\delta_0\right)\mathrm{sinc}\left(\frac{\omega_k-\omega_{\alpha'}}{2}\delta_0\right)$$
$$\times G_-(\omega_k)e^{i\lambda_B\omega_k}\delta_0 e^{i(t+\delta_0/2)(\omega_{\alpha'}-\omega_\alpha)}, \tag{128}$$

$$\frac{1}{\delta_0}\int_t^{t+\delta_0}ds\int_t^{t+\delta_0}ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}(s)\hat{B}_{-\lambda_B}^\dagger(s')\hat{\rho}_B\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}$$
$$=\sum_k \mathrm{sinc}\left(\frac{\omega_k-\omega_\alpha}{2}\delta_0\right)\mathrm{sinc}\left(\frac{\omega_k-\omega_{\alpha'}}{2}\delta_0\right)$$
$$\times G_+(\omega_k)e^{-i\lambda_B\omega_k}\delta_0 e^{i(t+\delta_0/2)(\omega_{\alpha'}-\omega_\alpha)}, \tag{129}$$

where $G_\pm(\nu)$ is the real part of the half Fourier transform of the bath correlation functions,

$$\int_0^{+\infty}d\tau\mathrm{Tr}[\hat{B}(\tau)\hat{B}^\dagger(0)\hat{\rho}_B]e^{i\nu\tau} \equiv G_+(\nu)+iI_+(\nu),$$
$$\int_0^{+\infty}d\tau\mathrm{Tr}[\hat{B}(\tau)\hat{B}^\dagger(0)\hat{\rho}_B]e^{i\nu\tau} \equiv G_-(\nu)+iI_-(\nu). \tag{130}$$

The product of sinc functions may be approximated by

$$\delta_0\,\mathrm{sinc}\left(\frac{\omega_k-\omega_\alpha}{2}\delta_0\right)\mathrm{sinc}\left(\frac{\omega_k-\omega_{\alpha'}}{2}\delta_0\right) \tag{131}$$
$$\approx \delta_0\,\mathrm{sinc}\left(\frac{2\omega_k-\omega_\alpha-\omega_{\alpha'}}{2}\delta_0\right)$$
$$\approx\begin{cases}\delta[2\omega_k-(\omega_\alpha+\omega_{\alpha'})] & \text{if } |\omega_\alpha-\omega_{\alpha'}|<\delta_0^{-1}\\ 0 & \text{otherwise.}\end{cases}$$

Going back to the Schrödinger picture, (127) takes the form

$$\frac{d\hat{\rho}_X^{\lambda}(t)}{dt} = -i[\hat{H}_X+\hat{H}_{LS}]+\mathcal{D}_\lambda\big(\hat{\rho}_X^{\lambda}(t)\big), \tag{132}$$

where the dissipator dressed with counting fields $\mathcal{D}_\lambda$ is expressed in terms of the operators (120), and where $\hat{H}_{LS}$ is a Lamb shift contribution. The Lamb shift term can be written in terms of the jump operators (120), with amplitudes given by the imaginary part $I_\pm(\nu)$ of the half Fourier transform of the bath correlation functions in (130). We further on neglect the Lamb shift term $\hat{H}_{LS}$, given that it induces negligibly small corrections to the qubit's frequency [5,35,36].

We now proceed to deriving master equations for the dressed qubit and the qubit, by tracing out respectively the degrees of freedom of the dressed laser and laser.

B. Dressed qubit

Since, in (131), $\omega_\alpha \in \{\omega_L, \omega_L\pm\Omega\}$, we have, for any $\alpha\neq\alpha'$, $|\omega_\alpha-\omega_{\alpha'}|=\Omega$ or $|\omega_\alpha-\omega_{\alpha'}|=2\Omega$. This allows us to identify three regimes, depending on the value of Ω (equivalently of the coupling g):

$$2\Omega<\delta_0^{-1}\text{: weak driving,}$$
$$\Omega<\delta_0^{-1}<2\Omega\text{: intermediate driving,}$$
$$\delta_0^{-1}<\Omega\text{: strong driving.} \tag{133}$$

However, these definitions are meaningless as long as we do not connect δ_0 with the relevant timescales of the problem. To identify these timescales, we examine the real parts $G_\pm$ of the half Fourier transforms of the bath correlation functions, introduced in (130); we may rewrite them as

$$G_+(\nu)\equiv\frac{1}{2}\int_{-\infty}^{+\infty}d\tau\mathrm{Tr}[\hat{B}(\tau)\hat{B}^\dagger(0)\hat{\rho}_B]e^{i\nu\tau}, \tag{134}$$
$$G_-(\nu)\equiv\frac{1}{2}\int_{-\infty}^{+\infty}d\tau\mathrm{Tr}[\hat{B}^\dagger(\tau)\hat{B}(0)\hat{\rho}_B]e^{i\nu\tau}. \tag{135}$$

The functions $G_\pm(\nu)$ are related to the bath's zero-temperature spectral function, $\Gamma(\nu)\equiv\sum_k|g_k|^2\delta_D(\nu-\omega_k)$, where δ_D is the Dirac δ function, by [5]

$$G_+(\nu)=\Gamma(\nu)[n_B(\nu)+1], \tag{136}$$

$$G_-(\nu)=\Gamma(\nu)n_B(\nu), \tag{137}$$

where $n_B(\nu)\equiv(e^{\beta_B\nu}-1)^{-1}$. Note that $G_+(\nu)=e^{\beta_B\nu}G_-(\nu)$, which is the Kubo–Martin–Schwinger (KMS) condition [2,37]. Let us now define

$$\gamma_{\max}\equiv\max_{\alpha=z,+,-}\{G_\pm(\omega_\alpha)\}, \tag{138}$$

where we recall that ω_α are the frequencies appearing in the Fourier transform of $\hat{V}_{AB}(t)$. These frequencies, together with $\gamma_{\max}$, are the relevant timescales to which δ_0 should be

TABLE I. Summary of the approximations used to derive the Bloch and Floquet master equations, of their regimes of validity and of their thermodynamic consistency. Full consistency means satisfying the symmetries (107), (113) and the strict energy conservation conditions (109), (114); this implies that the laws of thermodynamics are satisfied at the average and fluctuating levels.

<table>
<tr><td>Driving</td><td>Weak
$\Omega < \gamma_{\max}$</td><td>Intermediate
$\Omega \sim \gamma_{\max}$</td><td>(Common regime of validity)
$\omega_L, \omega_A \gg \Omega \gg \gamma_{\max}$</td><td>Strong
$\omega_L, \omega_A, \Omega \gg \gamma_{\max}$</td></tr>
<tr><td>Time scales</td><td>$\omega_L, \omega_A \gg \delta_0^{-1} \gg \Omega, \gamma_{\max}$</td><td>$\omega_L, \omega_A \gg \delta_0^{-1} > \Omega, \gamma_{\max}$</td><td>$\omega_L, \omega_A \gg \Omega \gg \delta_0^{-1} \gg \gamma_{\max}$</td><td>$\omega_L, \omega_A, \Omega \gg \delta_0^{-1} \gg \gamma_{\max}$</td></tr>
<tr><td rowspan="3">QME</td><td colspan="4">Generalized Bloch
•Approximation (141)
•Consistency:
Full consistency in strong coupling
Weak or intermediate: symmetries (107), (113), and 1st and 2nd laws average & rates</td></tr>
<tr><td colspan="3">Bloch
•Approximation: $G_\pm(\nu) \to \overline{G}_\pm$
•Consistency:
Symmetries (107), (113)
1st and 2nd laws: average & rates</td><td></td></tr>
<tr><td colspan="2"></td><td colspan="2">Floquet
•Secular approximation
•Consistency:
Full consistency</td></tr>
</table>

compared in order to push further the derivation of the master equation (132). A necessary condition on δ_0 is that

$$\delta_0^{-1} \gg \gamma_{\max}. \tag{139}$$

Moreover, we require that $\omega_L, \omega_A \gg \delta_0^{-1}$, which is a reasonable assumption in practice. Combined with (131), these conditions allows us to redefine the three driving regimes as

$$\begin{aligned} &\omega_L, \omega_A \gg \delta_0^{-1} \gg \Omega, \gamma_{\max}: \text{weak driving}, \\ &\omega_L, \omega_A \gg \delta_0^{-1} > \Omega, \gamma_{\max}: \text{intermediate driving}, \\ &\omega_L, \omega_A, \Omega \gg \delta_0^{-1} \gg \gamma_{\max}: \text{strong driving}. \end{aligned} \tag{140}$$

We now proceed to deriving master equations. We begin by deriving a new master equation, called generalized Bloch master equation, valid at all coupling strengths. The Floquet and Bloch master equations are then obtained from the generalized Bloch equation by performing additional approximations, respectively, in the strong and weak- or intermediate-driving regimes. A summary of the regimes of validity and approximations made for each master equation is given in the Table I.

1. Generalized Bloch equation

To derive the generalized Bloch equation, we do the following approximation, inspired by the procedure employed in Refs. [31,38,39],

$$\sum_k G_\pm(\omega_k) e^{i\lambda_B \omega_k} \delta_0 \,\mathrm{sinc}\left(\frac{\omega_k - \omega_\alpha}{2}\delta_0\right) \mathrm{sinc}\left(\frac{\omega_k - \omega_{\alpha'}}{2}\delta_0\right) \approx \begin{cases} \sqrt{(G_\pm(\omega_\alpha) G_\pm(\omega_{\alpha'})} e^{i\lambda_B \omega_\alpha} e^{i\lambda_B \omega_{\alpha'}} & \text{if } |\omega_\alpha - \omega_{\alpha'}| < \gamma_{\max} \\ 0 & \text{otherwise.} \end{cases} \tag{141}$$

This procedure makes the superoperator (127) symmetric, although the resulting superoperator takes different forms in the three driving regimes of (133). Since the operators (120) are factorized in the basis $\{|1n\rangle, |2n\rangle\}$, we may readily trace out $\mathcal{H}_{DL}$, and we obtain a master equation for the dressed qubit. In the weak-driving regime ($2\Omega < \gamma_{\max}$), the generalized Bloch equation is

$$\mathcal{L}_{\boldsymbol{\lambda}}^{aG}(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}) = -i[\hat{H}_{DA}, \hat{\rho}_{DA}^{\boldsymbol{\lambda}}] + \mathcal{D}_+^{aG\boldsymbol{\lambda}}(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}) + \mathcal{D}_-^{aG\boldsymbol{\lambda}}(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}), \tag{142}$$

with

$$\begin{aligned} \mathcal{D}_+^{aG\boldsymbol{\lambda}}(\hat{\rho}) &= \hat{T}_+^{\boldsymbol{\lambda}} \hat{\rho} \hat{T}_+^{-\boldsymbol{\lambda}\dagger} - \tfrac{1}{2}(\hat{T}_+^{(\lambda_{DA},\lambda_{DL},0)\dagger} \hat{T}_+^{(\lambda_{DA},\lambda_{DL},0)} \hat{\rho} \\ &\quad + \hat{\rho} \hat{T}_+^{(-\lambda_{DA},-\lambda_{DL},0)\dagger} \hat{T}_+^{(-\lambda_{DA},-\lambda_{DL},0)}), \\ \mathcal{D}_-^{aG\boldsymbol{\lambda}}(\hat{\rho}) &= \hat{T}_-^{\boldsymbol{\lambda}\dagger} \hat{\rho} \hat{T}_-^{-\boldsymbol{\lambda}} - \tfrac{1}{2}(\hat{T}_-^{(\lambda_{DA},\lambda_{DL},0)} \hat{T}_-^{(\lambda_{DA},\lambda_{DL},0)\dagger} \hat{\rho} \\ &\quad + \hat{\rho} \hat{T}_-^{(-\lambda_{DA},-\lambda_{DL},0)} \hat{T}_-^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}), \end{aligned}$$

where the superscript aG stands for autonomous generalized Bloch equation (the nonautonomous counterpart is derived in Sec. V C), $\boldsymbol{\lambda} = (\lambda_{DA}, \lambda_{DL}, \lambda_B)$, and

$$\hat{T}_\pm^{\boldsymbol{\lambda}} = \sum_{\alpha=+,-,z} \sqrt{G_\pm(\omega_\alpha)} \hat{s}_\alpha^{\boldsymbol{\lambda}}. \tag{143}$$

In the strong-driving regime ($\Omega > \gamma_{\max}$), the generalized Bloch equation is equal to the Floquet master equation (in the rotating frame), which we derive in the next section; the expression is given in Eq. (149). The expression of the generalized Bloch equation in the intermediate regime ($\Omega < \gamma_{\max} < 2\Omega$) is given in Appendix G in order to alleviate the text.

It is straightforward to check that the generalized Bloch master equation satisfies the condition (107) in all three

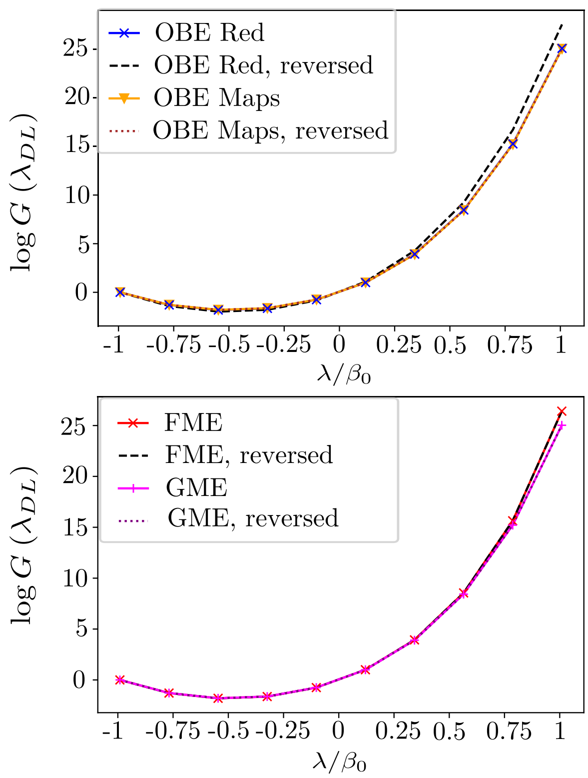


FIG. 7. Work W_{DL} moment generating functions for the master equations discussed. (top) The work fluctuation theorem holds for the Bloch equation derived using quantum maps but breaks down if the Bloch equation is derived with the Redfield equation. (bottom) The work fluctuation theorem is satisfied by the Floquet master equation and the generalized master equation. Parameters: $\alpha = 4$, $\beta = 10/D$ with the spectral width $D = 20$, $\gamma_0 = 0.4\sqrt{D}$.

regimes (see, e.g., Fig. 7 for the case $2\Omega < \gamma_{\max}$). Consequently, the second law of thermodynamics is satisfied on average. The strict energy conservation condition (114), on the other hand, is only valid when $\Omega > \gamma_{\max}$, since, in this case, (141) amounts to performing the secular approximation [2] (the product of sinc functions in (141) is then nonzero only in the case $\alpha = \alpha'$).

We now derive the rates $\dot{W}_{DL}$, $\dot{Q}$ and $d_t E_{DA}$. Let us introduce

$$\begin{aligned} P_1(t) &= \langle 1|\hat{\rho}_{DA}(t)|1\rangle, \\ P_2(t) &= \langle 2|\hat{\rho}_{DA}(t)|2\rangle, \\ P_{21}(t) &= \langle 2|\hat{\rho}_{DA}(t)|1\rangle, \end{aligned} \tag{144}$$

and

$$\begin{aligned} \gamma_{0,\downarrow} &= \frac{g^2}{4\Omega^2} G_+(\omega_L), \\ \gamma_{0,\uparrow} &= \frac{g^2}{4\Omega^2} G_-(\omega_L), \\ \gamma_{1,\downarrow} &= \frac{(\Omega+\delta)^2}{4\Omega^2} G_+(\omega_L+\Omega), \\ \gamma_{1,\uparrow} &= \frac{(\Omega+\delta)^2}{4\Omega^2} G_-(\omega_L+\Omega), \\ \gamma_{2,\downarrow} &= \frac{(\Omega-\delta)^2}{4\Omega^2} G_-(\omega_L-\Omega), \\ \gamma_{2,\uparrow} &= \frac{(\Omega-\delta)^2}{4\Omega^2} G_+(\omega_L-\Omega). \end{aligned} \tag{145}$$

Taking the derivatives in λ_{DL}, λ_B, λ_{DA} in the trace of $\mathcal{L}^{aG}_{\boldsymbol{\lambda}}$, we obtain respectively the rates $\dot{W}_{DL}$, $\dot{Q}$, and $d_t \Delta E_A$. In the weak- and intermediate-driving regimes, the results are

$$\begin{aligned} \dot{W}_{DL} =& \,\omega_L(\gamma_{0\downarrow} - \gamma_{0\uparrow}) \\ &+ \omega_L[(\gamma_{2\uparrow} - \gamma_{1\uparrow})P_1(t) + (\gamma_{1\downarrow} - \gamma_{2\downarrow})P_2(t)] \\ &- 2\omega_L(\sqrt{\gamma_{0\uparrow}\gamma_{2\downarrow}} + \sqrt{\gamma_{0\uparrow}\gamma_{1\uparrow}})\mathrm{Re}[P_{21}(t)] \\ &- 2\omega_L(\sqrt{\gamma_{0\downarrow}\gamma_{2\uparrow}} + \sqrt{\gamma_{0\downarrow}\gamma_{1\downarrow}})\mathrm{Re}[P_{21}(t)], \end{aligned} \tag{146}$$

$$\begin{aligned} \dot{Q} =& \,\omega_L(\gamma_{0\uparrow} - \gamma_{0\downarrow}) \\ &+ [\omega_L(\gamma_{1\uparrow} - \gamma_{2\uparrow}) + \Omega(\gamma_{2\uparrow} + \gamma_{1\uparrow})]P_1(t) \\ &+ [\omega_L(\gamma_{2\downarrow} - \gamma_{1\downarrow}) - \Omega(\gamma_{2\downarrow} + \gamma_{1\downarrow})]P_2(t) \\ &+ 2(\omega_L - \Omega/2)(\sqrt{\gamma_{0\downarrow}\gamma_{2\uparrow}} + \sqrt{\gamma_{0\downarrow}\gamma_{2\downarrow}})\mathrm{Re}[P_{21}(t)] \\ &+ 2(\omega_L + \Omega/2)(\sqrt{\gamma_{0\downarrow}\gamma_{1\uparrow}} + \sqrt{\gamma_{0\downarrow}\gamma_{1\downarrow}})\mathrm{Re}[P_{21}(t)], \end{aligned} \tag{147}$$

$$\begin{aligned} d_t E_{DA} =& \,\Omega(\gamma_{2\uparrow} + \gamma_{1\uparrow})P_1(t) - \Omega(\gamma_{2\downarrow} + \gamma_{1\downarrow})P_2(t) \\ &- \Omega/2(\sqrt{\gamma_{0\downarrow}\gamma_{2\uparrow}} + \sqrt{\gamma_{0\downarrow}\gamma_{2\downarrow}})\mathrm{Re}[P_{21}(t)] \\ &+ \Omega/2(\sqrt{\gamma_{0\downarrow}\gamma_{1\uparrow}} + \sqrt{\gamma_{0\downarrow}\gamma_{1\downarrow}})\mathrm{Re}[P_{21}(t)]. \end{aligned} \tag{148}$$

It is straightforward to check that the first law is satisfied at the level of the rates. The rates in the strong-driving regime are given in the next section, on the Floquet equation.

2. Strong qubit-laser coupling: Floquet master equation

We consider here the strong-driving regime, defined in (140). In this case, the product of sinc functions (141) is nonzero only in the case $\alpha = \alpha'$, which is equivalent to the secular approximation [2]. Performing the secular approximation on (142), we obtain (see Appendix H for the full expression)

$$\begin{aligned} \frac{d\hat{\rho}^{\boldsymbol{\lambda}}_{DA}}{dt} &= \mathcal{L}^{aF}_{\boldsymbol{\lambda}}\left(\hat{\rho}^{\boldsymbol{\lambda}}_{DA}\right) \\ &= -i\left[\hat{H}_{DA}, \hat{\rho}^{\boldsymbol{\lambda}}_{DA}\right] + \mathcal{D}^{aF}_{\boldsymbol{\lambda}}\left(\hat{\rho}^{\boldsymbol{\lambda}}_{DA}\right). \end{aligned} \tag{149}$$

We call this equation the "autonomous Floquet" master equation, denoted by the superscript "aF," since it is equivalent to the Floquet master equation—traditionally used in the nonautonomous picture—which we derive in Sec. V. The dissipator in (149) has three dissipation channels, corresponding to the three frequencies of the Mollow triplet ω_L, $\omega_L \pm \Omega$; we give here its expression the counting fields are set to zero $\boldsymbol{\lambda} = \mathbf{0}$,

$$\begin{aligned} \mathcal{D}^{aF} &\equiv \mathcal{D}^{aF}_+ + \mathcal{D}^{aF}_-, \\ \mathcal{D}^{aF}_+ &= \gamma_{0,\downarrow}\mathcal{D}_{\hat{\Sigma}_z} + \gamma_{1,\downarrow}\mathcal{D}_{\hat{\Sigma}_-} + \gamma_{2,\uparrow}\mathcal{D}_{\hat{\Sigma}_+}, \\ \mathcal{D}^{aF}_- &= \gamma_{0,\uparrow}\mathcal{D}_{\hat{\Sigma}_z^\dagger} + \gamma_{1,\uparrow}\mathcal{D}_{\hat{\Sigma}_+} + \gamma_{2,\downarrow}\mathcal{D}_{\hat{\Sigma}_-}. \end{aligned} \tag{150}$$

Since the autonomous Floquet equation is the restriction of the generalized Bloch equation to the strong-driving regime,

$\mathcal{L}_{\lambda}^{aF}$ satisfies the condition (107) and the strict energy conservation condition (109) (this can also be seen directly from the explicit expression given in Appendix H) and is therefore fully thermodynamically consistent.

Moreover, the dissipator in (149) satisfies the symmetry

$$\mathcal{D}_{0,\lambda_{DL},\lambda_B+i\beta_B}^{aF\dagger}(\dots) = \mathcal{D}_{0,\lambda_{DL},\lambda_B}^{aF}(\dots). \tag{151}$$

Since the steady-state moment-generating function $\mathcal{G}^{ss}(\lambda_{DL},\lambda_B) \equiv \lim_{t\to+\infty}\frac{1}{t}\mathcal{G}(0,\lambda_{DL},\lambda_B,t)$ is given by the dominant eigenvalue of $\mathcal{D}_{0,\lambda_{DL},\lambda_B}$ [12], the identity (151) implies the steady-state work fluctuation theorem

$$\frac{p(W_{DL})}{p(-W_{DL})} \asymp e^{-\beta_B W_{DL}}. \tag{152}$$

We now derive the explicit expressions of the work W_{DL}, dressed qubit energy ΔE_{DA}, and heat by taking the derivatives λ_{DL}, λ_{DA} and λ_B, in the trace of $\mathcal{L}_{\lambda}^{aF}(\hat{\rho}_{DA}^{\lambda})$. We obtain

$$\begin{aligned}\dot{W}_{DL} &= \omega_L(\gamma_{0,\downarrow}-\gamma_{0,\uparrow}) + \omega_L[(\gamma_{1,\downarrow}-\gamma_{2,\downarrow})P_2(t) \\ &\quad - (\gamma_{1,\uparrow}-\gamma_{2,\uparrow})P_1(t)],\end{aligned} \tag{153}$$

$$d_t E_{DA} = \Omega[(\gamma_{1,\uparrow}+\gamma_{2,\uparrow})P_1(t) - (\gamma_{1,\downarrow}+\gamma_{2,\downarrow})P_2(t)], \tag{154}$$

and

$$\begin{aligned}\dot{Q} &= \omega_L(\gamma_{0\uparrow}-\gamma_{0\downarrow}) \\ &\quad + [\omega_L(\gamma_{1\uparrow}-\gamma_{2\uparrow}) + \Omega(\gamma_{2\uparrow}+\gamma_{1\uparrow})]P_1(t) \\ &\quad + [\omega_L(\gamma_{2\downarrow}-\gamma_{1\downarrow}) - \Omega(\gamma_{2\downarrow}+\gamma_{1\downarrow})]P_2(t).\end{aligned} \tag{155}$$

Notice that these expressions are equal to the rates (146), (147), and (148) without the coherent terms $P_{21}(t)$, which is a consequence of the secular approximation.

We also point out that the rate of the heat (155) is consistent with the results obtained in Refs. [7,40], and the rate of the work (153) is consistent with the expression derived in Ref. [5] in the case of a nonautonomous description of the laser. However, it was then assumed that (153) should correspond to the work performed by the laser (and not the dressed laser), which seemed in contradiction with quantum thermodynamics, according to which that work is expected to be of the form (63). Our approach unveils that the work (153) exerted on the dressed qubit is in fact produced by the dressed laser, which partially explains the difference from the anticipated form (153). The expression (153) also suggests that work is mediated by transitions and thus originates from a nonconservative force. This was also pointed out in Refs. [5,12], in the nonautonomous picture. Nonconservative forces typically arise when eliminating an underlying degree of freedom (here the laser), that, over sufficiently long timescales, delivers energy without being affected by it [41]. This is manifest in the form of the rates (145) which satisfy a local detailed balance condition [41,42], which relates the log ratio of the transition rates (145) to the change in entropy in the bath resulting from that transition:

$$\ln\frac{\gamma_{j,\downarrow}}{\gamma_{j,\uparrow}} = \beta_B(\omega_L + l_j\Omega), \tag{156}$$

with $j = 0, 1, 2$ and $l_0 = 0, l_1 = 1, l_2 = -1$. Interestingly, the rates (145) are the same as those appearing in the dissipator of the master equation for $\hat{\rho}_X$, denoted $\mathcal{L}^{aF,X}$ above (149). At the level of the master equation for $\hat{\rho}_X$, the r.h.s. of (156) is equal to (up to the factor β_B) the energy variation of the system X during a transition induced by the dissipator; this implies that the dynamics is detailed balanced and will relax to equilibrium. However, at the level of the master equation for the dressed qubit (149), the r.h.s. of (156) not only contains the difference of energies of the system (i.e., $\pm\Omega$), but also the laser's energy. The fact that the term ω_L remains even after we have traced out the degrees of freedom of the laser results directly from assumption 2. Indeed, this assumption allows us to neglect the variations of the number of photons in the laser during a transition between a state of $|j(n)\rangle \to |i(n\pm 1)\rangle$ $(i, j \in \{1, 2\})$ and justifies the mapping (29), which in turn leads to the simple product structure of the operators (120). This product structure then allows us to trace out the dressed laser DL without changing the rates in the dissipator of the master equation. During this procedure, the variation of the number of photons is treated as an underlying degree of freedom which is traced out and does not appear explicitly in the dynamics, but leaves a fingerprint in the thermodynamics through the term ω_L in (156), which is at the origin of the nonconservative force (153). As a consequence, the steady-state solution of $\mathcal{L}^{aF}$ is a nonequilibrium steady state, as can be checked by noticing that the entropy production rate is strictly positive in the steady state (see Appendix I).

3. Weak and intermediate coupling: Bloch master equation

We now consider the weak-driving regime defined in (140). Provided that $G_\pm(\nu)$ are smooth on the intervals $[\pm\omega_L - \Omega, \pm\omega_L + \Omega]$, we may replace

$$G_\pm(\omega_L), G_\pm(\omega_L \pm \Omega) \approx G_\pm(\omega_A) \equiv \overline{G}_\pm \tag{157}$$

in (142), which yields the tilted master equation $d_t\hat{\rho}_{DA}^{\lambda}(t) \equiv \mathcal{L}_{\lambda}^{aB}(\hat{\rho}_{DA}(t))$ (see Appendix J for the explicit expression). When the counting fields are set to zero, it is equal to

$$\begin{aligned}\frac{d\hat{\rho}_{DA}(t)}{dt} &= \mathcal{L}^{aB}(\hat{\rho}_{DA}(t)) \\ &= -i[\hat{H}_{DA},\hat{\rho}_{DA}(t)] + \overline{G}_+\mathcal{D}_{\hat{\sigma}_-}(\hat{\rho}_{DA}(t)) \\ &\quad + \overline{G}_-\mathcal{D}_{\hat{\sigma}_+}(\hat{\rho}_{DA}(t)),\end{aligned} \tag{158}$$

where we recall that $\hat{\sigma}_+ = |b\rangle\langle a| = \hat{\sigma}_-^\dagger$. The regime of validity of (158) can be extended to the intermediate-driving regime in (140) since, in this regime, the condition $\delta_0 > \Omega^{-1}$ is still satisfied. The notation aB stands for autonomous Bloch equation, since we will see, in Sec. V, that (158) is equivalent to the optical Bloch master equation [3,5], usually derived in the nonautonomous picture.

It is straightforward to check that $\mathcal{L}_{\lambda}^{aB}$ (see Appendix J) satisfies the condition (107). This implies the second law, at the level of the rates [31],

$$d_t S_{DA} - \beta_B\dot{Q} \geqslant 0. \tag{159}$$

Setting $\lambda_{DA} = \lambda_{DL} = \lambda_B = \lambda$, we find that $\mathcal{L}_{\lambda}^{aB}$ is not λ independent, which means that the strict energy conservation (109) is not satisfied, hence that the first law of thermodynamics does not hold at the fluctuating level.

We obtain $\dot{Q}$, $\dot{W}_{DL}$, and $d_t E_{DA}$ by taking the derivatives in λ_B, λ_{DL}, and λ_{DA} of $\mathrm{Tr}[\mathcal{L}^{aB}_{\boldsymbol{\lambda}}(\hat{\rho}_{DA})]$,

$$\begin{aligned}\dot{Q} =& -\omega_A[\overline{G}_+ P_b(t) - \overline{G}_- P_a(t)]\\ &-\frac{g}{2}(\overline{G}_+ + \overline{G}_-)\mathrm{Re}(P_{ab}(t)),\end{aligned} \tag{160}$$

$$\dot{W}_{DL} = \omega_L[\overline{G}_+ P_b(t) - \overline{G}_- P_a(t)], \tag{161}$$

$$\begin{aligned}d_t E_{DA} =& -\delta(\overline{G}_+ P_b(t) - \overline{G}_- P_a(t))\\ &-\frac{g}{2}(\overline{G}_+ + \overline{G}_-)\mathrm{Re}(P_{ab}(t)),\end{aligned} \tag{162}$$

with

$$\begin{aligned}P_b(t) &\equiv \langle b|\hat{\rho}_{DA}(t)|b\rangle\\ &= \frac{1}{2} + \frac{\delta}{2\Omega}[P_2(t) - P_1(t)] - \frac{g}{\Omega}\mathrm{Re}[P_{21}(t)],\\ P_a(t) &\equiv \langle a|\hat{\rho}_{DA}(t)|a\rangle\\ &= \frac{1}{2} - \frac{\delta}{2\Omega}[P_2(t) - P_1(t)]\frac{g}{\Omega} + \mathrm{Re}[P_{21}(t)],\\ P_{ba}(t) &\equiv \langle b|\hat{\rho}_{DA}(t)|a\rangle\\ &= \frac{g}{2\Omega}[P_2(t) - P_1(t)] + \frac{\delta}{\Omega}\mathrm{Re}[P_{21}(t)] + i\mathrm{Im}[P_{21}(t)],\\ P_{ab}(t) &\equiv \langle a|\hat{\rho}_{DA}(t)|b\rangle.\end{aligned} \tag{163}$$

One can check, using the identity (123), that the above expressions are consistent with (146), (147) and (148). We also check that the first law is satisfied,

$$d_t E_{DA} = \dot{Q} + \dot{W}_{DL}. \tag{164}$$

C. Nonautonomous qubit

As showed in Sec. III C 2, and summarized in Fig. 6, the evolution of the dressed qubit (in the autonomous description) is equivalent to the evolution of the qubit in the rotating frame (in the nonautonomous description). This remains true at the level of master equations. Specifically, the nonautonomous counterparts of the generalized Bloch equation (142) and of the autonomous Bloch (158) and Floquet (149) master equations can be obtained simply by using the correspondence (40). This leads to the following relation between the autonomous master equations and there nonautonomous versions,

$$\begin{aligned}\tilde{\mathcal{L}}^{G,F,B}(\hat{\tilde{\rho}}_A(t)) = e^{-i\omega_L\hat{\sigma}_z t/2}\bigg\{&\mathcal{L}^{aG,aF,aB}[\hat{\rho}_{DA}(t)]\\ &- i\frac{\omega_L}{2}[\hat{\sigma}_z, \hat{\rho}_{DA}(t)]\bigg\}e^{i\omega_L\hat{\sigma}_z t/2},\end{aligned} \tag{165}$$

where we use "G" for generalized Bloch, "F" for Floquet, and "B" for optical Bloch. See Fig. 6 for a summary of the correspondences.

We now examine the thermodynamics in the qubit picture, similarly as in Sec. III where we focused on the dressed qubit picture. We therefore derive tilted master equations with counting fields on the laser and bath, $\boldsymbol{\lambda} = (\lambda_L, \lambda_B)$. We leave out the qubit part, since measuring $\hat{H}_A + \hat{V}(t)$ is difficult in practice and deriving a tilted master equation with a counting field on $\hat{H}_A + \hat{V}(t)$ is technically cumbersome. The nonautonomous master equations with counting fields $\boldsymbol{\lambda} = (\lambda_L, \lambda_B)$ can be obtained using the substitutions $\hat{s}^{\boldsymbol{\lambda}}_{z,\pm} \to \hat{s}^{\boldsymbol{\lambda}}_{z,\pm}(t)$, where

$$\begin{aligned}\hat{s}^{\boldsymbol{\lambda}}_z(t) &\equiv \frac{g}{2\Omega}e^{-i(\lambda_L-\lambda_B)\omega_L/2}\hat{\Sigma}_z\left(t + \frac{\lambda_L}{2}\right),\\ \hat{s}^{\boldsymbol{\lambda}}_+(t) &\equiv -\frac{\Omega-\delta}{2\Omega}e^{-i\lambda_L\omega_L/2}e^{i\lambda_B(\omega_L-\Omega)/2}\hat{\Sigma}_+\left(t + \frac{\lambda_L}{2}\right),\\ \hat{s}^{\boldsymbol{\lambda}}_-(t) &\equiv \frac{\Omega+\delta}{2\Omega}e^{-i\lambda_L\omega_L/2}e^{i\lambda_B(\omega_L+\Omega)/2}\hat{\Sigma}_-\left(t + \frac{\lambda_L}{2}\right),\end{aligned} \tag{166}$$

with

$$\begin{aligned}\hat{\Sigma}_z(t) &\equiv |u_2(t)\rangle\langle u_2(t)| - |u_1(t)\rangle\langle u_1(t)|,\\ \hat{\Sigma}_+(t) &\equiv |u_2(t)\rangle\langle u_1(t)|,\\ \hat{\Sigma}_-(t) &\equiv |u_1(t)\rangle\langle u_2(t)|.\end{aligned} \tag{167}$$

To see this, let us treat $\hat{H}_L$, $\hat{H}_B$ separately. The tilted master equation dressed with the counting field λ_B can be obtained directly from the autonomous equations (142), (149) and (158), by setting $\boldsymbol{\lambda} = (0, 0, \lambda_B)$ in those equations and using the identity (165). For $\hat{H}_L$, we use the identity $\hat{H}_L = \hat{H}_{DL} - \frac{\omega_L}{2}\hat{\sigma}_z$. Since $[\hat{H}_{DL}, \frac{\omega_L}{2}\hat{\sigma}_z] = 0$, we may measure them separately, using counting fields λ_{DL}, λ_σ; then, setting $\lambda_L = \lambda_{DL} = -\lambda_\sigma$ yields the tilted master equation dressed with the counting field λ_L for $\hat{H}_L$. The convenience of this approach is that, since $\frac{\omega_L}{2}\hat{\sigma}_z$ only acts on the Hilbert space of the qubit, the terms $e^{i\lambda_\sigma\omega_L\hat{\sigma}/2}$ can be factorized out of the Kraus operators when tracing out the Hilbert spaces $\mathcal{H}_L$, $\mathcal{H}_B$, similarly as the situation in (105) (but for $\mathcal{H}_{DL}$, $\mathcal{H}_B$). The exponential term $e^{-i\omega_L\lambda_L}$ comes from counting $\hat{H}_{DL}$, while the shifts in the operators $\hat{\Sigma}^F_z(t)$, $\hat{\Sigma}^F_\pm(t)$ are due to counting $\frac{\omega_L}{2}\hat{\sigma}_z$.

1. Generalized Bloch equation

Using (142), (165), and (166), we obtain the generalized Bloch equation with counting fields $\boldsymbol{\lambda} = (\lambda_L, \lambda_B)$,

$$\mathcal{L}^G_{\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A(t)) \tag{168}$$

$$\begin{aligned}&= -i[\hat{H}_A(t + \lambda_L/2)\hat{\tilde{\rho}}_A(t) - \hat{\tilde{\rho}}_A(t)\hat{H}_A(t - \lambda_L/2)]\\ &\quad + \mathcal{D}^{G\boldsymbol{\lambda}}_+(\hat{\tilde{\rho}}_A(t)) + \mathcal{D}^{G\boldsymbol{\lambda}}_-(\hat{\tilde{\rho}}_A(t)),\end{aligned} \tag{169}$$

where we note

$$\hat{H}_A(t) \equiv \hat{H}_A + \hat{V}(t). \tag{170}$$

In the weak-driving regime,

$$\begin{aligned}\mathcal{D}^{G\boldsymbol{\lambda}}_+(\hat{\rho}) &= \hat{T}^{\boldsymbol{\lambda}}_+(t)\hat{\rho}\hat{T}^{-\boldsymbol{\lambda}\dagger}_+(t) - \tfrac{1}{2}(\hat{T}^{(\lambda_L,0)\dagger}_+(t)\hat{T}^{(\lambda_L,0)}_+(t)\hat{\rho} + \hat{\rho}\hat{T}^{(-\lambda_L,0)\dagger}_+(t)\hat{T}^{(-\lambda_L,0)}_+(t)),\\ \mathcal{D}^{G\boldsymbol{\lambda}}_-(\hat{\rho}) &= \hat{T}^{\boldsymbol{\lambda}\dagger}_-(t)\hat{\rho}\hat{T}^{-\boldsymbol{\lambda}}_-(t) - \tfrac{1}{2}(\hat{T}^{(\lambda_L,0)}_-(t)\hat{T}^{(\lambda_L,0)\dagger}_-(t)\hat{\rho} + \hat{\rho}\hat{T}^{(-\lambda_L,0)}_-\hat{T}^{(-\lambda_L,0)\dagger}_-(t)),\end{aligned}$$

with

$$\hat{T}^{\lambda}_{\pm}(t) = \sum_{\alpha=+,-,z} \sqrt{G_{\pm}(\omega_\alpha)}\hat{s}^{\lambda}_{\alpha}(t). \tag{171}$$

The expression of the dissipators in the strong-driving regime corresponds to the Floquet equation, given in the next section in (174) and Appendix L. The expression in the intermediate-driving regime is obtained from the autonomous generalized Bloch equation in that regime, given in Appendix G, by using (165) and (166).

The heat is the same as in the autonomous case, and the rate of the work $\dot{W}_L$ is given in the Appendix K.

2. Floquet master equation

The strong-coupling limit in the nonautonomous description leads to the Floquet quantum master equation [5,7]. We obtain it from (149), using the correspondence (165),

$$\mathcal{L}^F(\hat{\tilde{\rho}}_A(t)) = -i[\hat{H}_A + \hat{V}(t), \hat{\tilde{\rho}}_A(t)] + \mathcal{D}^F(\hat{\tilde{\rho}}_A(t)), \tag{172}$$

where the dissipator is obtained by replacing the operators $\hat{\Sigma}_z$, $\hat{\Sigma}_\pm$ in (150) by $\hat{\Sigma}_z(t)$, $\hat{\Sigma}_\pm(t)$, defined in (167).

The tilted Floquet master equation with counting fields $\boldsymbol{\lambda} = (\lambda_L, \lambda_B)$ is obtained by performing the secular approximation in (168),

$$\mathcal{L}^F_{\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A(t)) = -i[\hat{H}_A(t+\lambda_L/2)\hat{\tilde{\rho}}_A(t) - \hat{\tilde{\rho}}_A(t)\hat{H}_A(t-\lambda_L/2)] + \mathcal{D}^F_{\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A(t)). \tag{173}$$

The dissipator $\mathcal{D}^F_{\boldsymbol{\lambda}}$ is decomposed as

$$\begin{aligned}\mathcal{D}^F_{\boldsymbol{\lambda}} &\equiv \mathcal{D}^F_{\boldsymbol{\lambda},+} + \mathcal{D}^F_{\boldsymbol{\lambda}-},\\ \mathcal{D}^F_{\boldsymbol{\lambda}+} &= \gamma_{0,\downarrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{0,+} + \gamma_{1,\downarrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{1,+} + \gamma_{2,\uparrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{2,+},\\ \mathcal{D}^F_{F\boldsymbol{\lambda}-} &= \gamma_{0,\uparrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{0,-} + \gamma_{1,\uparrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{1,-} + \gamma_{2,\downarrow}\mathcal{D}^{F\boldsymbol{\lambda}}_{2,-}.\end{aligned} \tag{174}$$

The full dissipator is given in Appendix L. It is straightforward to check that $\mathcal{L}^F_{\boldsymbol{\lambda}}$ satisfies the symmetries (113) and (114). The dissipator also satisfies the symmetry

$$\mathcal{D}^{F\dagger}_{0,\lambda_L,\lambda_B+i\beta_B}(\dots) = \mathcal{D}^F_{0,\lambda_L,\lambda_B}(\dots), \tag{175}$$

which, as explained under (152), implies the following steady-state work fluctuation theorem, this time for W_L,

$$\frac{p(W_L)}{p(-W_L)} \asymp e^{\beta_B W_L}. \tag{176}$$

We may now obtain the rate of the work $\dot{W}_L$,

$$\begin{aligned}\dot{W}^F_L &\equiv -\frac{1}{i}\partial_{\lambda_L}\mathrm{Tr}\big[\mathcal{L}^F_{\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A(t))\big]|_{\boldsymbol{\lambda}=\mathbf{0}}\\ &= \dot{W}_{DL} + \mathrm{Tr}\Big[\frac{\omega_L}{2}\hat{\sigma}_z\mathcal{L}^F(\hat{\tilde{\rho}}_A(t))\Big]\\ &= \mathrm{Tr}[d_t\hat{V}(t)\hat{\tilde{\rho}}_A(t))] + \omega_L(\gamma_{0\downarrow} - \gamma_{0\uparrow}) + \omega_L\frac{g}{\Omega}\\ &\quad\times \mathrm{Re}[P_{21(t)}]\bigg(\gamma_{0\downarrow} + \gamma_{0\uparrow} + \frac{\gamma_{1\downarrow}+\gamma_{1\uparrow}+\gamma_{2\downarrow}+\gamma_{2\uparrow}}{2}\bigg)\\ &\quad+ \omega_L P_1(t)\bigg[\gamma_{2\uparrow} - \gamma_{1\uparrow} + \frac{\delta}{\Omega}(\gamma_{2\uparrow}+\gamma_{1\uparrow})\bigg]\\ &\quad- \omega_L P_2(t)\bigg[\gamma_{2\downarrow} - \gamma_{1\downarrow} + \frac{\delta}{\Omega}(\gamma_{2\downarrow}+\gamma_{1\downarrow})\bigg].\end{aligned} \tag{177}$$

where $P_{21}(t) \equiv \langle 2|\hat{\rho}_{DA}(t)|1\rangle$ and where $P_{1,2}(t)$ were defined in (144). Notice that $\mathrm{Tr}[d_t\hat{V}(t)\hat{\tilde{\rho}}_A(t)]$ may be rewritten as

$$\begin{aligned}\mathrm{Tr}[d_t\hat{V}(t)\hat{\tilde{\rho}}_A(t)] &= -g\omega_L\mathrm{Im}(\langle 2|\hat{\rho}_{DA}|1\rangle)\\ &= -g\omega_L\mathrm{Im}(\langle b|\hat{\rho}_{DA}|a\rangle),\end{aligned} \tag{178}$$

similarly to the unitary case (51).

3. Bloch master equation

In the nonautonomous picture, the weak- or intermediate-driving regimes correspond to the regimes of validity of the optical Bloch master equation [3,5]. Using (165) with (158), we obtain the optical Bloch master equation [3]

$$\mathcal{L}^B(\hat{\tilde{\rho}}_A(t)) = -i[\hat{H}_A + \hat{V}(t), \hat{\tilde{\rho}}_A(t)] + \overline{G}_+\mathcal{D}_{\hat{\sigma}_-}(\hat{\tilde{\rho}}_A(t)) + \overline{G}_-\mathcal{D}_{\hat{\sigma}_+}(\hat{\tilde{\rho}}_A(t)). \tag{179}$$

From the discussion in Sec. V B 3, we deduce that the Bloch master equation satisfies the symmetry (113) as well as the first law (55).

The optical Bloch master equation dressed with the counting fields $\boldsymbol{\lambda} = (\lambda_L, \lambda_B)$ is obtained by performing the approximation $G_\pm(\nu) \to \overline{G}_\pm$ in (168),

$$\frac{d\hat{\tilde{\rho}}^{\boldsymbol{\lambda}}_A}{dt} \equiv \mathcal{L}^B_{\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) \tag{180}$$

$$= -i[\tilde{H}_A + \hat{V}_{\lambda_L}(t+\lambda_L/2), \hat{\tilde{\rho}}_A] + \overline{G}_+\mathcal{D}^{\lambda_B}_{\hat{\Sigma}}(\hat{\tilde{\rho}}_A) + \overline{G}_-\mathcal{D}^{\lambda_B}_{\hat{\Sigma}^\dagger}(\hat{\tilde{\rho}}_A), \tag{181}$$

where $\mathcal{D}^{\lambda_B}$ is in fact equal to by setting to the dissipator of $\mathcal{L}^{aB}$ with $\boldsymbol{\lambda} = (0, 0, \lambda_B)$. We can then obtain the rate of the work, $\dot{W}_L = \frac{1}{i}\partial_{\lambda_L}\mathrm{Tr}[\hat{\rho}^{\boldsymbol{\lambda}}_A]$, and we find

$$\dot{W}_L = \mathrm{Tr}[d_t\hat{V}(t)\hat{\tilde{\rho}}_A(t)] = -g\omega_L\mathrm{Im}(\langle b|\hat{\rho}_{DA}|a\rangle). \tag{182}$$

The heat is the same as in (160); we may decompose it as

$$\begin{aligned}\dot{Q} &= \omega_A[(\overline{n}+1)P_a(t) - \overline{n}P_b(t)] + \gamma g\mathrm{Re}[P_{ab}(t)]\\ &= \mathrm{Tr}[\mathcal{L}^B(\hat{\tilde{\rho}}_A)\hat{H}_A] + \mathrm{Tr}[\mathcal{L}^B(\hat{\tilde{\rho}}_A)\hat{V}(t)],\end{aligned} \tag{183}$$

which then gives the first law

$$\begin{aligned}d_t\tilde{E}_A &= \mathrm{Tr}[(\hat{H}_A + \hat{V}(t))\mathcal{L}^B(\hat{\rho}_A(t))] + \mathrm{Tr}[d_t\hat{V})\hat{\rho}_A(t)]\\ &= \dot{Q} + \dot{W}_L.\end{aligned} \tag{184}$$

VI. QUANTUM MAPS VS REDFIELD EQUATION

In the Sec. V, we derived the Bloch and Floquet equation using the formalism of quantum maps and the Kraus operators (117). We saw that this procedure preserves the fluctuation theorems. However, the Bloch equation is usually derived using the Redfield equation [2,5,43]. We show in this short section that, although both procedures result in the same equation in the absence of counting fields, the tilted master equations differ, and that the Bloch equation derived from the Redfield method breaks the symmetry (107).

To see this, we repeat the derivation of the Bloch equation via the Redfield equation, which can be found in Ref. [2,5], but adding here the counting fields. The first step is to write the Liouville equation in the interaction picture, with counting fields, which is done by taking the time derivative of

$\hat{\rho}_{\boldsymbol{\lambda}}(t)$ in (76), and going to the interaction picture with respect to $\hat{H}_X + \hat{H}_B$,

$$\frac{d\hat{\rho}^{\boldsymbol{\lambda}I}(t)}{dt} = -i\big[\hat{V}_{AB}^{\boldsymbol{\lambda}}(t)\hat{\rho}^{\boldsymbol{\lambda}I}(t) - \hat{\rho}^{\boldsymbol{\lambda}I}(t)\hat{V}_{AB}^{-\boldsymbol{\lambda}}(t)\big]. \tag{185}$$

The system X evolves over a timescale $\approx\gamma_{\max}^{-1}$. Assuming that the relaxation time τ_B of the heat bath B satisfies $\tau_B \ll \gamma_{\max}^{-1}$, we can coarse-grain the dynamics over a timescale δ_0 such that $\tau_B \ll \delta_0 \ll \gamma_{\max}^{-1}$. Integrating (185) over δ_0 and re-injecting the solution in (185) yields, assuming $[\hat{V}_{AB}(0), \hat{\rho}_{AL}^{I}(0)] = 0$, an equation similar to the Redfield master equation,

$$\begin{aligned}\frac{d\hat{\rho}_X^{\boldsymbol{\lambda}}}{dt} = &-\mathrm{Tr}_B\bigg[\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\int_t^{t'}du\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t')\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(u)\hat{\rho}_X^{\boldsymbol{\lambda}}(u)\otimes\hat{\rho}_B + \hat{\rho}_X^{\boldsymbol{\lambda}}(u)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(u)\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t')\\ &- \hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t')\hat{\rho}_X^{\boldsymbol{\lambda}}(u)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(u) - \hat{V}_{AB}^{\boldsymbol{\lambda}/2}(u)\hat{\rho}_X^{\boldsymbol{\lambda}}(u)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t')\bigg].\end{aligned} \tag{186}$$

In the weak-coupling limit, we can perform the Born-Markov approximation [2] and replace $\hat{\rho}(u) \equiv \hat{\rho}_X(t)\otimes\hat{\rho}_B$. We then do the change of variable $\tau = t' - u$ in the second integral; since $\tau_B \ll \delta_0$, we replace the upper bound of the second integral by $+\infty$, and we obtain, finally,

$$\begin{aligned}\frac{d\hat{\rho}_X^{\boldsymbol{\lambda}}}{dt} = &-\mathrm{Tr}_B\bigg[\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\int_0^{+\infty}d\tau\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t')\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t'-\tau)\hat{\rho}_X^{\boldsymbol{\lambda}}(t)\otimes\hat{\rho}_B + \hat{\rho}_X^{\boldsymbol{\lambda}}(t)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t'-\tau)\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t')\\ &- \hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t')\hat{\rho}_X^{\boldsymbol{\lambda}}(t)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t'-\tau) - \hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t'-\tau)\hat{\rho}_X^{\boldsymbol{\lambda}}(t)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t')\bigg].\end{aligned} \tag{187}$$

The key difference between (187) and the master equation (127), obtained from a perturbative expansion of (101), is the final approximation made for the Redfield: replacing the upper bound of the second integral by $+\infty$. We did not make this approximation in Sec. IV, see Eqs. (127) to (131). This approximation is known to break the positivity of the Redfield equation [2]. We also showed in a previous work that it breaks a fluctuation theorem in the case of a quantum system connected to heat baths [31]. The same happens here, in the presence of the laser: to see this, suffices to finish the calculation of the Bloch equation, using the approximation $G_\pm(\nu) \approx \overline{G}_\pm$; details are provided in the Appendix M. The expression of the Bloch equation with counting fields is then

$$\begin{aligned}\frac{d\hat{\rho}_{DA}^{\boldsymbol{\lambda}}}{dt} &\equiv \mathcal{L}_{\boldsymbol{\lambda}}^{aB,\mathrm{red}}\\ &= -i\big[\hat{H}_{DA}, \hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big] + \overline{G}_+\mathcal{D}_{\hat{s}}^{\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big) + \overline{G}_-\mathcal{D}_{\hat{s}^\dagger}^{\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big),\end{aligned} \tag{188}$$

with

$$\begin{aligned}\mathcal{D}_{\hat{s}}^{\boldsymbol{\lambda}}(\hat{\rho}) = &-\frac{1}{2}\big(\hat{s}_{\lambda_{DA},0,0}^\dagger\hat{s}_{\lambda_{DA},0,0}\hat{\rho} + \hat{\rho}\hat{s}_{-\lambda_{DA},0,0}^\dagger\hat{s}_{-\lambda_{DA},0,0}\big)\\ &+ \frac{1}{2}e^{i\omega_L\lambda_B}\hat{s}_{\lambda_{DA},\lambda_{DL},0}\hat{\rho}\hat{s}_{-\boldsymbol{\lambda}}^\dagger\\ &+ \frac{1}{2}e^{i\omega_L\lambda_B}\hat{s}_{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_{-\lambda_{DA},-\lambda_{DL},0}^\dagger,\end{aligned} \tag{189}$$

$$\begin{aligned}\mathcal{D}_{\hat{s}^\dagger}^{\boldsymbol{\lambda}}(\hat{\rho}) = &-\frac{1}{2}\big(\hat{s}_{\lambda_{DA},0,0}\hat{s}_{\lambda_{DA},0,0}^\dagger\hat{\rho} + \hat{\rho}\hat{s}_{-\lambda_{DA},0,0}\hat{s}_{-\lambda_{DA},0,0}^\dagger\big)\\ &+ \frac{1}{2}e^{-i\omega_L\lambda_B}\hat{s}_{\lambda_{DA},\lambda_{DL},0}^\dagger\hat{\rho}\hat{s}_{-\boldsymbol{\lambda}}\\ &+ \frac{1}{2}e^{-i\omega_L\lambda_B}\hat{s}_{\boldsymbol{\lambda}}^\dagger\hat{\rho}\hat{s}_{-\lambda_{DA},-\lambda_{DL},0}.\end{aligned} \tag{190}$$

The explicit expressions in (189) allow us to see directly that the symmetry (107) is not satisfied. See also Fig. 7 for a numerical check.

The positivity can however be restored by applying the secular approximation [2]. The secular approximation also restores the fluctuation theorems [31]. Note that the Floquet master equation with counting fields is identical with both methods, as a consequence of the secular approximation.

VII. STEADY-STATE SOLUTIONS

Here, we briefly discuss and compare the steady-state solutions for thermodynamics quantities predicted by the Floquet and Bloch master equations.

First, we point out that the rates $\dot{W}_L$ and $\dot{W}_{DL}$ become equal in the steady-state: In both the Floquet and Bloch master equations, we have

$$\dot{W}_L^{ss} = \dot{W}_{DL}^{ss}. \tag{191}$$

This result is obtained by replacing the steady-state solutions of the Floquet and Bloch master equations, given in Appendixes I and N, in (153) and (177) (Floquet) and (161) and (182) (Bloch). This result was expected, since, at the microscopic level, W_L and W_{DL} only differ by the expectation value of $\frac{\omega_L}{2}\hat{\sigma}_z$, which vanishes in the steady-state since it is a state variable.

We make a second observation, that, in the common regime of validity of the Floquet and Bloch master equations, characterized by $\omega_L, \omega_A \gg \Omega \gg \gamma_{\max}$, with the assumption that the spectral density $\Gamma(\nu)$ is smooth, the steady-state expectation values of heat $\dot{Q}^{ss}$ and work $\dot{W}_L^{ss}$ can equivalently be obtained from either equation. More precisely, using the steady-state

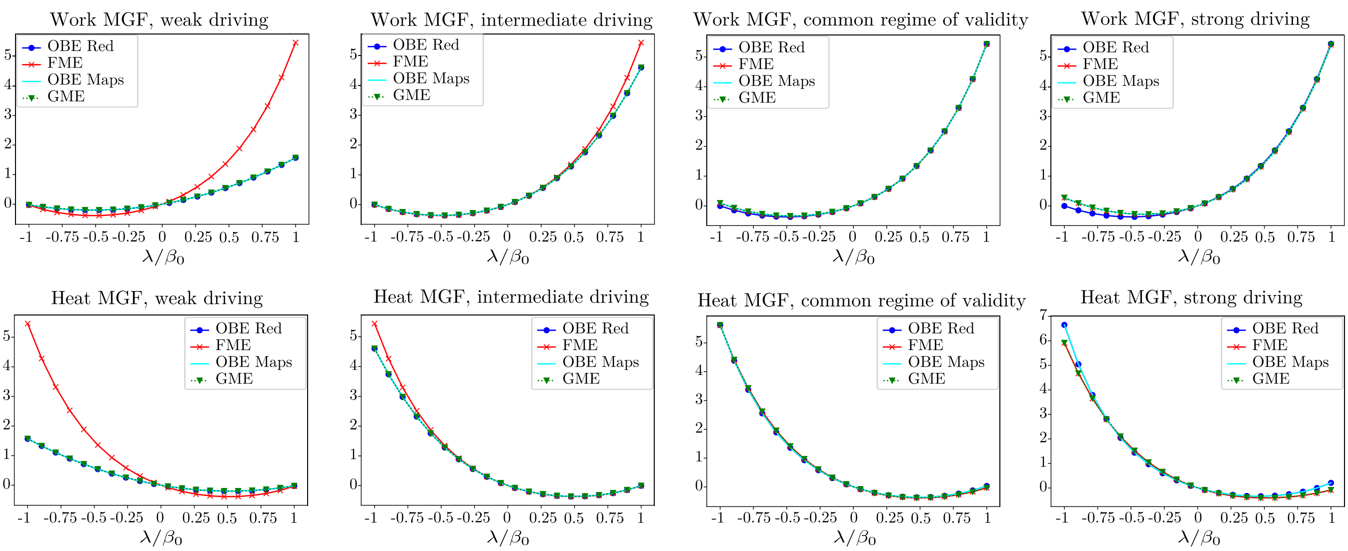


FIG. 8. Steady-state moment-generating functions for the work W_{DL} (top) and heat Q (bottom), for increasing driving values (from left to right). The parameters are $D = 1000$, $\beta = 100/D$ and $\gamma_0 = 0.1\sqrt{D}$, $\delta = 10^{-8}D$, and $\omega_A = 0.02D$. The values of the laser-qubit coupling are such that $g/\gamma_{\max} = 0.8$ for the weak driving, $g/\gamma_{\max} = 8$ for the intermediate regime, $g/\gamma_{\max} = 800$ for the common regime of validity, and $g/\gamma_{\max} = 2000$ for the strong-driving regime.

solutions given in Appendixes I and N in (153), (155), (161), and (160), we find

$$\frac{\dot{Q}^{ss,F} - \dot{Q}^{ss,B}}{\overline{\Gamma}\omega_L} = -\frac{1/2}{\frac{\delta^2+\Omega^2}{g^2}\left(1 + 2\frac{\delta^2+\Omega^2}{\gamma^2(2\overline{n}+1)^2}\right)}, \tag{192}$$

$$\frac{\dot{W}_L^{ss,F} - \dot{W}_L^{ss,B}}{\overline{\Gamma}\omega_L} = \frac{\delta^2/g^2 + \overline{\Gamma}^2(2\overline{n}+1)^2/4g^2}{1 + 2\frac{\delta^2}{g^2} + \overline{\Gamma}^2\frac{(2\overline{n}+1)^2}{2g^2}}, \tag{193}$$

where the superscripts F and B stand respectively for Floquet and Bloch, where $\overline{\Gamma} \equiv \overline{G}_+/(\overline{n}+1)$ with $\overline{n} \equiv n_B(\omega_L)$, and where we approximated $\gamma_{0\downarrow}, \gamma_{1\downarrow}, \gamma_{2\uparrow} \sim \overline{G}_+/4$ and $\gamma_{0\uparrow}, \gamma_{1\uparrow}, \gamma_{2\downarrow} \sim \overline{G}_-/4$. Since $\omega_L, \omega_A \gg \Omega \gg \gamma_{\max}$ and $\gamma_{\max} = \overline{G}_+$, and since $\Omega \sim g$, we find that

$$\frac{\dot{Q}^{ss,F} - \dot{Q}^{ss,B}}{\overline{\Gamma}\omega_L} = O\left(\frac{\gamma_{\max}^2}{g^2}\right), \tag{194}$$

$$\frac{\dot{W}_L^{ss,F} - \dot{W}_L^{ss,B}}{\overline{\Gamma}\omega_L} = O\left(\frac{\gamma_{\max}^2}{g^2}\right). \tag{195}$$

Let us now show that variations of the order $\gamma_{\max}^2/g^2$ are too small to be captured by the Floquet and Bloch master equations in the common regime of validity. Given that the master equations were obtained from a perturbative expansion, to second order, of $\hat{W}_{\mu,\nu}^{\lambda I}(t+\delta_0, t)$ defined in (117), the accuracy of the master equations is of the order $\delta_0^2\gamma_{\max}^2$. In the common regime of validity, δ_0 needs to satisfy $\Omega \gg \delta_0^{-1} \gg \gamma_{\max}$. Choosing for example $\delta_0^{-1} = \sqrt{\gamma_{\max}\Omega}$, it follows that

$$\frac{\dot{Q}^{ss,F} - \dot{Q}^{ss,OB}}{\overline{\Gamma}\omega_L} = \frac{\dot{W}_L^{ss,F} - \dot{W}_L^{ss,OB}}{\overline{\Gamma}\omega_L} = o\left(\delta_0^2\gamma_{\max}^2\right), \tag{196}$$

hence $\dot{Q}^{ss,F} = \dot{Q}^{ss,OB}$ and $\dot{W}_L^{ss,F} = \dot{W}_L^{ss,OB}$ up to negligible corrections.

We highlight that the equivalences (196) assume that the spectral density $\Gamma(\nu)$ is "smooth enough" on the interval $[\omega_A - \Omega, \omega_A + \Omega]$. If this was not the case, we expect the Floquet and Bloch master equation to predict different rates for $\dot{W}_L$.

We conclude with a few plots of the heat and work moment generating functions, from times $t = 0$ to the steady state, for different values of the driving strength, see Fig. 8. The initial density matrix is $|b\rangle\langle b|$. We observe that, in the weak- and intermediate-driving regimes, the generalized Bloch equation coincides with the Bloch equations (derived whether from the Redfield equation or from the Kraus operators), but not with the Floquet master equation. In the common regime of validity, all the master equations give the same result, except at large λ_{DL} where the Redfield Bloch equation slightly diverges. In the strong-driving regime, the generalized Bloch equation matches instead with the Floquet master equation.

VIII. SUMMARY

In Secs. IV and V, we developed a toolbox for deriving quantum master equations for coherently driven systems. In this section, we briefly sum up the results of most practical use.

(1) The generalized Bloch equation is valid at all driving regimes, thermodynamically consistent [satisfies the symmetries (107), (113) and the laws of thermodynamics on average], and fully consistent in the strong drive regime. The Floquet master equation is valid at strong driving ($\Omega \gg \gamma_{\max}$) and is fully consistent. The Bloch equation, derived from the maps, is valid at weak ($\Omega < \gamma_{\max}$) and intermediate ($\Omega \sim \gamma_{\max}$) driving and in the common regime of validity ($\omega_A, \omega_L \gg \Omega \gg \gamma_{\max}$) and satisfies the symmetries (107), (113) and the laws of thermodynamics on average, but not the strict energy conservation. See Table I for a summary.

(2) At the unitary level and at the level of quantum master equations for the qubit, the evolution of the dressed qubit (in the autonomous description) is equivalent to the evolution in the rotating frame (in the nonautonomous description). See

Fig. 6 for a summary of the unitary operations connecting the autonomous and nonautonomous pictures.

(3) The work source for the dressed qubit is the dressed laser; the work source for the qubit is the laser. The laws of thermodynamics in both approaches are summarized in Fig. 3.

(4) In their common regimes of validity, the Bloch and Floquet master equations predict similar steady-state thermodynamics, on average (194), (195), and at the level of moment generating functions, see Fig. 8.

IX. CONCLUSION

In this work, we analyzed the thermodynamics of a qubit interacting with a coherent, macroscopic, electromagnetic field, in the weak-, intermediate-, and strong-driving (140) regimes. We point out that our method can be readily extended to d-level qubits and collective sets on qubits. A summary of the result is presented in Fig. 3 and Table I. We derived two symmetries (81) and (94), which serve as criteria of thermodynamic consistency for quantum master equations, and translate them into the symmetries (107) and (113) at the level of the master equations. We derived a master equation, the generalized Bloch equation, valid in all drive regimes and satisfying (107) and (113). The generalized Bloch equation also satisfies the strict energy conservation condition at strong drives, making it fully consistent in this limit. The Floquet master equation corresponds to the restriction of the generalized Bloch equation in the strong drive regime, while the Bloch master equation is obtained by performing an additional approximation in the weak- or intermediate-driving regimes, which preserve the symmetries (107) and (113). We also pointed out the importance of using quantum maps rather than the Redfield equation when deriving master equations, since the Redfield equation breaks the symmetries (107) and (113).

The present work could be useful for assessing the energy cost of qubit manipulation using coherent light sources, in the spirit of Ref. [44]. Furthermore, our findings are relevant in the context designing and optimizing autonomous heat engines [45–47] using far from equilibrium states of radiation as work sources, in the spirit of Ref. [48]. Our findings could also be relevant for studying energy fluctuations in hybrid optomechanical systems [40,49] where the development of the precision of such systems [50] might allow one to measure work fluctuations directly. More generally, our framework could be easily adapted to models in low-temperature solid-state physics to study the interaction between phonons and defects of the material (often modeled as two-level systems), which reproduces Jaynes-Cummings-like physics [51]. In the context of work measurement schemes in cold atoms systems, our findings complements the method developed by Refs. [52,53] using coherent light as a probe to reconstruct the work statistics from homodyne detection: in these works, the laser is solely seen as a probe, and the work transferred by the laser is not taken into account. Applying our results to these schemes could yield a complete thermodynamic description of work measurement in cold atom setups. Finally, our results should motivate further investigation of the thermodynamics of nonequilibrium steady states generated when coherent light drives a system out of equilibrium. Such states have been studied in mesoscopic physics, in setups where coherent light propagates through random scattering media [54–56], and have been shown to yield fluctuation-induced forces, but a thorough thermodynamic description of these features is still lacking.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: LASER AS A WORK SOURCE

Assume that $\hat{\rho}_L(0)$ is either a coherent state or a Poisson state, and that the final state is not far from $\hat{\rho}_L(0)$; we write it in the form

$$\hat{\rho}_L(t) = \hat{\rho}_L(0) + \epsilon \hat{r}, \tag{A1}$$

with $\mathrm{Tr}[\hat{r}] = 0$ and ϵ small. Consider now the relative entropy

$$D[\hat{\rho}_L(t)||\hat{\rho}_L(0)] = \mathrm{Tr}[\hat{\rho}_L(t)\ln\hat{\rho}_L(t)] - \mathrm{Tr}[\hat{\rho}_L(t)\ln\hat{\rho}_L(0)]. \tag{A2}$$

Substituting (A1) in (A2) and expanding the logarithm to second order in ϵ, we find that

$$D[\hat{\rho}_L(t)||\hat{\rho}_L(0)] = O(\epsilon^2). \tag{A3}$$

On the other hand, we have the identity

$$\begin{aligned} D[\hat{\rho}_L(t)||\hat{\rho}_L(0)] &= -\Delta S_L + \mathrm{Tr}[(\hat{\rho}_L(0) - \hat{\rho}_L(t))\ln\hat{\rho}_L(0)] \\ &= -\Delta S_L - \epsilon \mathrm{Tr}[\hat{r}\ln\hat{\rho}_L(0)]. \end{aligned} \tag{A4}$$

We now show that $\mathrm{Tr}[\hat{r}\ln\hat{\rho}_L(0)]$ is negligible both for a coherent state and a Poisson state. We begin with the case of a Poisson state. As discussed in Sec. II, a Poisson state is equivalent to a Gibbs state at infinite temperature $\beta_L^{-1} \to +\infty$, which implies that $\mathrm{Tr}[\hat{r}\ln\hat{\rho}_L(0)] \propto -\beta_L$, hence $\mathrm{Tr}[\hat{r}\ln\hat{\rho}_L(0)] \approx 0$.

In the case of a coherent state, the logarithm of $\hat{\rho}_L(0)$ is in fact ill defined; to fix this issue, let us introduce

$$\hat{\rho}_\eta \equiv \hat{D}(\alpha)[|0\rangle_L\langle 0|_L + e^{-\eta\hat{A}}]\hat{D}^\dagger(\alpha), \tag{A5}$$

where

$$\hat{A} \equiv e^{-\eta}\sum_{N\geqslant 1}|N\rangle_L\langle N|_L, \tag{A6}$$

such that $\hat{\rho}_L(0) = \lim_{\eta\to+\infty}\hat{\rho}_\eta$ [recall that $\hat{D}(\alpha)|0\rangle_L\langle 0|_L\hat{D}^\dagger(\alpha) = |\alpha\rangle_L\langle\alpha|_L$]. Then,

$$\mathrm{Tr}[\hat{r}\ln\hat{\rho}_\eta] = \eta e^{-\eta}\langle 0|\hat{D}^\dagger(\alpha)\hat{r}\hat{D}(\alpha)|0\rangle = O(\eta e^{-\eta}), \tag{A7}$$

where we used $\mathrm{Tr}[\hat{r}] = 0$ and the fact that $|\langle 0|\hat{D}^\dagger(\alpha)\hat{r}\hat{D}(\alpha)|0\rangle|$ is bounded. Hence, taking the limit $\eta \to +\infty$ and using (A3) and (A4), we obtain finally that

$$\Delta S_L = O(\epsilon^2). \tag{A8}$$

To complete the proof, it is sufficient to notice that

$$\Delta E_L = \mathrm{Tr}[\hat{H}_L(\hat{\rho}_L(t) - \hat{\rho}_L(0))] = O(\epsilon). \tag{A9}$$

APPENDIX B: CORRESPONDENCE BETWEEN THE FLOQUET AND THE DRESSED QUBIT BASES

1. Proof of (40)

To prove the relation (40), it is sufficient to show that the states $e^{-i\omega_L\hat{\sigma}_z t/2}|j\rangle$, $j = 1, 2$ are solutions of the eigenvalue problem (39). This is straightforward using the facts that

$$e^{i\omega_L\hat{\sigma}_z t/2}(\hat{H}_A + \hat{V}(t))e^{-i\omega_L\hat{\sigma}_z t/2} = \hat{H}_{DA} + \frac{\omega_L}{2}\hat{\sigma}_z, \tag{B1}$$

and

$$-i\partial_t e^{-i\omega_L\hat{\sigma}_z t/2}|j\rangle = -\frac{\omega_L}{2}\hat{\sigma}_z e^{-i\omega_L\hat{\sigma}_z t/2}|j\rangle. \tag{B2}$$

Indeed, replacing $|u_j(t)\rangle = e^{-i\omega_L\hat{\sigma}_z t/2}|j\rangle$ in (39), and applying $e^{i\omega_L\hat{\sigma}_z t/2}$ on both sides of the equality, we obtain

$$\left(\hat{H}_{DA} + \frac{\omega_L}{2}\hat{\sigma}_z - \frac{\omega_L}{2}\hat{\sigma}_z\right)|j\rangle = \epsilon_j|j\rangle, \tag{B3}$$

which is true when choosing $\epsilon_1 = -\frac{\Omega}{2}$, $\epsilon_2 = \frac{\Omega}{2}$. Note that this feature can be generalized for any system with SU(2) symmetry described by Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$.

2. Proof of (45)

We now show that the relation (40) implies that the evolution of the system in the dressed basis, in the autonomous picture, is equivalent to the evolution in the rotating frame, in the nonautonomous picture.

We start with the simple case where the bath is not taken into account. In the autonomous picture, the total Hamiltonian is then given by $\hat{H}_X = \hat{H}_{DA} + \hat{H}_{DL}$, and we assume that the initial density matrix is factorized:

$$\hat{\rho}(0) = \hat{\rho}_{DA}(0) \otimes \hat{\rho}_{DL}(0), \tag{B4}$$

hence

$$\hat{\rho}(t) = e^{-it\hat{H}_X}\hat{\rho}_{DA}(0) \otimes \hat{\rho}_{DL}(0)e^{it\hat{H}_X}. \tag{B5}$$

Is then straightforward to obtain that

$$\begin{aligned}\hat{\rho}_{DA}(t) &= \mathrm{Tr}_{DL}[\hat{\rho}(t)] \\ &= \sum_{j,j'=1,2} e^{-it(\epsilon_j - \epsilon_{j'})}\hat{\rho}_{DA}^{jj'}(0)|j\rangle\langle j'|,\end{aligned} \tag{B6}$$

where $\hat{\rho}_{DA}^{jj'}(0) \equiv \langle j|\hat{\rho}_{DA}(0)|j'\rangle$. On the other hand, in the nonautonomous picture, we have

$$\hat{\tilde{\rho}}_A(t) = \mathcal{T}_{\leftarrow}[e^{-i\int_0^t ds\hat{H}_A + \hat{V}(s)}]\hat{\tilde{\rho}}_A(0)\mathcal{T}_{\leftarrow}[e^{i\int_0^t ds\hat{H}_A + \hat{V}(s)}]. \tag{B7}$$

Using Floquet theory [7], we can write the propagator in the Floquet basis

$$\mathcal{T}_{\leftarrow}[e^{-i\int_0^t ds\hat{H}_A + \hat{V}(s)}] = \sum_{j=1,2} e^{-it\epsilon_j}|u_j(t)\rangle\langle u_j(0)|. \tag{B8}$$

Using (40), we may replace $|u_j(t)\rangle = e^{-i\omega_j\hat{\sigma}_z t/2}|j\rangle$ in (B8), and, using the definition (41), we obtain

$$\hat{\tilde{\rho}}^{\mathrm{rot}}(t) = \sum_{j,j'=1,2} e^{-it(\epsilon_j - \epsilon_{j'})}\hat{\tilde{\rho}}_A^{\mathrm{rot}\,jj'}(0)|j\rangle\langle j'|. \tag{B9}$$

Comparing with (B5), it is then sufficient to assume that $\hat{\rho}_{DA}(0) = \hat{\tilde{\rho}}^{\mathrm{rot}}(0)$ to conclude that the two density matrices coincide at all times.

Let us now turn to the general proof of (45), when the coupling with the bath is taken into account. It is convenient to go in the interaction picture. In the autonomous case, the density matrix of the total system in the interaction picture with respect to $\hat{H}_X + \hat{H}_B$ is given by

$$\hat{\rho}^I(t) = \hat{U}_0^\dagger(t,0)\hat{\rho}(t)\hat{U}_0(t,0) = \hat{U}^I(t,0)\hat{\rho}(0)\hat{U}^{I\dagger}(t,0), \tag{B10}$$

with $\hat{U}_0(t,0) = e^{-it(\hat{H}_X + \hat{H}_B)}$ and

$$\hat{U}^I(t,0) \equiv \mathcal{T}[e^{-i\int_0^t ds\hat{V}_{AB}(s)}], \tag{B11}$$

where $\hat{V}_{AB}(t)$ is the Hamiltonian $\hat{V}_{AB}$ in the interaction picture, given by

$$\hat{V}_{AB}(t) \equiv e^{i\hat{H}_0 t}\hat{V}_{AB}e^{-i\hat{H}_0 t}. \tag{B12}$$

To compute $\hat{V}_{AB}(t)$, we express the operators $\hat{\sigma}_- \otimes \mathbb{I}_L$, $\hat{\sigma}_+ \otimes \mathbb{I}_L$ in the eigenbasis of $\hat{H}_X$ (27). In this basis, the operator $\hat{\sigma}_- \otimes \mathbb{I}_L$ writes

$$\begin{aligned}\hat{\sigma}_- \otimes \mathbb{I}_L &= |a\rangle\langle b| \otimes \sum_{N_L}|N_L\rangle\langle N_L| \\ &= \sum_n\left(\sqrt{\frac{\Omega-\delta}{2\Omega}}|1(n-1)\rangle + \sqrt{\frac{\Omega+\delta}{2\Omega}}|2(n-1)\rangle\right) \\ &\quad\times\left(\sqrt{\frac{\Omega+\delta}{2\Omega}}\langle 1(n)| - \sqrt{\frac{\Omega-\delta}{2\Omega}}\langle 2(n)|\right) \\ &= \hat{S}_z + \hat{S}_+ + \hat{S}_-,\end{aligned} \tag{B13}$$

where

$$\begin{aligned}\hat{S}_z &= \frac{g}{2\Omega}(|2\rangle\langle 2| - |1\rangle\langle 1|) \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n| \\ &\equiv \hat{s}_z \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n|, \\ \hat{S}_+ &\equiv -\frac{\Omega-\delta}{2\Omega}|2\rangle\langle 1| \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n| \\ &\equiv \hat{s}_+ \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n|, \\ \hat{S}_- &\equiv \frac{\Omega+\delta}{2\Omega}|1\rangle\langle 2| \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n| \\ &\equiv \hat{s}_- \otimes \sum_{n\geqslant 0}|n-1\rangle\langle n|,\end{aligned} \tag{B14}$$

with

$$\begin{aligned}\hat{s}_z &\equiv \frac{g}{2\Omega}(|2\rangle\langle 2| - |1\rangle\langle 1|) \equiv \frac{g}{2\Omega}\hat{\Sigma}_z, \\ \hat{s}_+ &\equiv -\frac{\Omega-\delta}{2\Omega}|2\rangle\langle 1| \equiv -\frac{\Omega-\delta}{2\Omega}\hat{\Sigma}_+, \\ \hat{s}_- &\equiv \frac{\Omega+\delta}{2\Omega}|1\rangle\langle 2| \equiv \frac{\Omega+\delta}{2\Omega}\hat{\Sigma}_-,\end{aligned} \tag{B15}$$

and where $\hat{\Sigma}_z = |2\rangle\langle 2| - |1\rangle\langle 1|$, $\hat{\Sigma}_+ = |2\rangle\langle 1| = \hat{\Sigma}_-^\dagger$, as defined in the main text in (120) and (121). The term $\hat{\sigma}_+ \otimes \mathbb{I}_L$ is obtained by taking the Hermitian conjugate of (B13). We therefore obtain

$$\hat{V}_{AB}(t) = (\hat{S}_z(t) + \hat{S}_-(t) + \hat{S}_+(t))\hat{B}^\dagger(t) + \text{H.c.}, \tag{B16}$$

with

$$\begin{aligned} \hat{S}_z(t) &= e^{-i\omega_L t}\hat{S}_z, \\ \hat{S}_+(t) &= e^{-i(\omega_L-\Omega)t}\hat{S}_+, \\ \hat{S}_-(t) &= e^{-i(\omega_L+\Omega)t}\hat{S}_-. \end{aligned} \tag{B17}$$

Let us assume that the dressed laser is initially in a Poisson state (5). The generalization to a coherent state is straightforward using the fact that the distribution $e^{-|\alpha|^2/2}\frac{\alpha^N}{\sqrt{N!}}$ is peaked around $N = |\alpha|^2$ in the macroscopic limit $|\alpha| \gg 1$. Since the density matrix is initially factorized (31), the evolution of the density matrix of the dressed qubit, obtained after tracing out the degrees of freedom of the dressed laser DL and of the bath, is given by the quantum map [2],

$$\begin{aligned} \hat{\rho}_{DA}^I(t) &\equiv \text{Tr}_{DL,B}[\hat{\rho}^I(t)] \\ &= \sum_{\boldsymbol{\kappa},\boldsymbol{\kappa}'} \hat{W}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}^I(t,0)\hat{\rho}_{DA}(0)\hat{W}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}^I(t,0), \end{aligned} \tag{B18}$$

where the sum runs over the pairs $\boldsymbol{\kappa} = (\mu, n)$, $\boldsymbol{\kappa}' = (\nu, n')$, with

$$\hat{W}_{\boldsymbol{\kappa},\boldsymbol{\kappa}'}^I = \sqrt{\eta_\nu \xi_n}\langle n, \mu|\hat{U}^I(t,0)|n', \nu\rangle, \tag{B19}$$

where $\xi_n = e^{-|\alpha|^2}|\alpha|^{2n}/n!$, and with $\{|\nu\rangle\}$ the eigenstates of $\hat{H}_B$ of eigenvalues ν and $\eta_\nu = e^{-\beta_B\omega_\nu}/Z_B$. At this stage, we do not want to carry out the trace over the bath (this is the object of Secs. IV and V). However, we now use the fact that the expectation value of the product of operators $\hat{B}^\dagger$, $\hat{B}$ over the Gibbs state $\hat{\rho}_B$ is nonzero only when there is the same number of repetition of $\hat{B}^\dagger$ and $\hat{B}$. This allows us, after writing the exponential in (B11) in a series, to deduce that the only relevant terms are such that the partial trace over DL is equal to one. Indeed, the relevant terms are those with an equal number of $\hat{B}^\dagger$ and $\hat{B}$, which, given the form of $\hat{V}_{AB}(t)$ and (B14), implies that the operator acting on DL is the identity (since a product of the same number of $\sum_n |n-1\rangle\langle n|$ and $\sum_n |n\rangle\langle n-1|$ is the identity). Therefore, we obtain

$$\hat{\rho}_{DA}^I(t) = \text{Tr}_B[\hat{U}^{I,\text{red}}(t,0)\hat{\rho}_{DA}(0) \otimes \hat{\rho}_B \hat{U}^{I,\text{red}\dagger}(t,0)], \tag{B20}$$

where the reduced propagator $\hat{U}^{I,\text{red}}(t,0)$ is

$$\hat{U}^{I,\text{red}}(t,0) = \mathcal{T}[e^{-i\int_0^t ds \hat{V}_{AB}^{I,\text{red}}(s)}], \tag{B21}$$

with

$$\begin{aligned} \hat{V}_{AB}^{I,\text{red}}(t) = (&e^{-i\omega_L t}\hat{s}_z + e^{-i(\omega_L-\Omega)t}\hat{s}_+ + e^{-i(\omega_L+\Omega)t}\hat{s}_-)\hat{B}^\dagger(t) \\ &+ \text{H.c.} \end{aligned} \tag{B22}$$

Let us now turn to the nonautonomous picture. The goal is to show that the evolution of the qubit in the rotating frame is equivalent to that of the dressed qubit (in the autonomous picture), given by (B20). We therefore go to the interaction picture, here with respect to $\hat{H}_{DA} + \hat{H}_B$ (since the degrees of the freedom of the laser have already been traced out),

$$\hat{\tilde{\rho}}^{\text{rot},I} \equiv \hat{\tilde{U}}^I(t,0)\hat{\tilde{\rho}}^{\text{rot}}(0)\hat{\tilde{U}}^{I\dagger}(t,0), \tag{B23}$$

where

$$\hat{\tilde{U}}^I(t,0) \equiv \mathcal{T}[e^{-i\int_0^t ds \hat{V}_{AB}'^I(s)}], \tag{B24}$$

with

$$\hat{V}_{AB}'^I(t) = e^{it(\hat{H}_{DA}+\hat{H}_B)}\hat{V}_{AB}'(t)e^{-it(\hat{H}_{DA}+\hat{H}_B)}, \tag{B25}$$

where $\hat{V}_{AB}'(t)$ is given in (43). Using (B13) and comparing with (B22), it is straightforward to check that

$$\hat{V}_{AB}^{I,\text{red}}(t) = \hat{V}_{AB}'^I(t), \tag{B26}$$

which concludes the proof.

APPENDIX C: FIRST LAW IN THE ROTATING FRAME

We provide here details on the derivation of (70). From the definition of $\tilde{E}_A^{\text{rot}}(t)$, conservation of energy yields

$$d_t\tilde{E}_A^{\text{rot}}(t) = \dot{Q} + \text{Tr}[d_t\hat{V}_{AB}'(t)\hat{\tilde{\rho}}^{\text{rot}}(t)]. \tag{C1}$$

Then, using the definition $\hat{\tilde{\rho}}^{\text{rot}}(t) = e^{i\omega_L\hat{\sigma}_z t/2}\hat{D}^\dagger(\alpha e^{-i\omega_L t})\hat{\rho}(t)\hat{D}^\dagger(\alpha e^{-i\omega_L t})e^{-i\omega_L\hat{\sigma}_z t/2}$, we obtain

$$\begin{aligned} \text{Tr}(d_t\hat{V}_{AB}'(t)\hat{\tilde{\rho}}^{\text{rot}}(t)) &= \text{Tr}(i\omega_L[\hat{\sigma}_z, \hat{V}_{AB}']\hat{\tilde{\rho}}(t)) = \text{Tr}(i\omega_L[\hat{\sigma}_z, \hat{V}_{AB} + \hat{V}_{AL} + g(\hat{\sigma}_+\alpha(t) + \hat{\sigma}_-\alpha(t)^*)]\hat{\rho}(t)) \\ &= i\omega_L\text{Tr}(\hat{\sigma}_z[\hat{V}_{AB} + \hat{V}_{AL}, \hat{\rho}(t)]) + \text{Tr}(d_t\hat{V}(t)\hat{\rho}(t)) \\ &= -\omega_L\text{Tr}(\hat{\sigma}_z d_t\hat{\rho}(t)) - i\omega_L \underbrace{\text{Tr}([\hat{\sigma}_z, \hat{H}_A + \hat{H}_L + \hat{H}_B]\hat{\rho}(t))}_{=0} + \text{Tr}(d_t\hat{V}(t)\hat{\rho}(t)) = \dot{W}_{DL}, \end{aligned} \tag{C2}$$

where in the last equality we used (57) and (64). Replacing in (C1), we obtain the first law (70).

APPENDIX D: PROOF OF SYMMETRY (81)

We begin by justifying (80). In the macroscopic limit $|\alpha| \gg 1$, the Poisson distribution effectively becomes equivalent to a Gaussian distribution,

$$e^{-|\alpha|^2}\frac{|\alpha|^{2N}}{N!} \sim \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(N-\mu)^2}{2\sigma^2}}, \tag{D1}$$

where $\sigma \equiv |\alpha|$ and $\mu \equiv |\alpha|$. In turn, a Gaussian state can be understood as a Gibbs state, by rewriting

$$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(N-\mu)^2}{2\sigma^2}} = \frac{1}{Z(\beta_{DL})}e^{-\beta_{DL}E(N)}, \tag{D2}$$

where $\beta_{DL} \equiv 1/\sigma^2 = 1/|\alpha|^2$, $E(N) \equiv (N-\mu)^2/2$, and $Z(\beta_{DL}) \equiv \sum_N -\beta_{DL}E(N)$, which leads to (80).

Let us now prove (81). Using (79) and (80), we can write the generating function of the forward (76) and time-reversed (77) processes explicitly,

$$G(\boldsymbol{\lambda}, t) = \frac{1}{Z}\mathrm{Tr}[\hat{U}(t,0)e^{-i\lambda_{DA}\hat{H}_{DA}-i\lambda_{DL}\hat{H}_{DL}-i\lambda_B\hat{H}_B}e^{-\beta_{DA}\hat{H}_{DA}-\beta_{DL}\hat{H}'_{DL}-\beta_B\hat{H}_B}\hat{U}^\dagger(t,0)e^{i\lambda_{DA}\hat{H}_{DA}+i\lambda_{DL}\hat{H}_{DL}+i\lambda_B\hat{H}_B}], \tag{D3}$$

$$G^R(-\boldsymbol{\lambda}+i\boldsymbol{\beta}, t) = \frac{1}{Z}\mathrm{Tr}[\hat{U}^\dagger(t,0)e^{i\lambda_{DA}\hat{H}_{DA}+i\lambda_{DL}\hat{H}_{DL}+i\lambda_B\hat{H}_B}e^{\beta_{DL}\hat{H}_{DL}}e^{-\beta_{DL}\hat{H}'_{DL}}\hat{U}(t,0)e^{-i\lambda_{DA}\hat{H}_{DA}-i\lambda_{DL}\hat{H}_{DL}-i\lambda_B\hat{H}_B}e^{-\beta_{DA}\hat{H}_{DA}-\beta_{DL}\hat{H}_{DL}-\beta_B\hat{H}_B}], \tag{D4}$$

where $Z = Z_{DA}Z(\beta_{DL})Z_B$. Since, in the limit $\beta_{DL} \to 0$,

$$e^{\beta_{DL}\hat{H}_{DL}}e^{-\beta_{DL}\hat{H}'_{DL}} \to \mathbb{I}, \tag{D5}$$

we may replace $e^{-\beta_{DL}\hat{H}_{DL}}$ by $e^{-\beta_{DL}\hat{H}'_{DL}}$. Using finally the cyclic property of the trace, it is then straightforward to check that

$$G^R(-\boldsymbol{\lambda}+i\boldsymbol{\nu}, t) = G(\boldsymbol{\lambda}, t), \tag{D6}$$

where $\boldsymbol{\nu} = (\beta_{DA}, \beta_{DL}, \beta_B)$, which proves the fluctuation theorem (81).

Note also that, by linearity, the theorem (81) can readily be extended to the case where the system is weakly coupled to many heat baths like B, as long as the baths do not interact with each other.

APPENDIX E: WORK FLUCTUATION THEOREM IN THE NONAUTONOMOUS PICTURE

We prove here the relation $\hat{\tilde{\rho}}^R(t) = \hat{\tilde{\rho}}(-t)$. Let us consider first, for simplicity, the case where the bath is not taken into account, such that the total Hamiltonian is $\hat{H}_A + \hat{V}(t)$. Using (B8), the density matrices of the forward and backward processes write

$$\hat{\tilde{\rho}}_A(t) = \sum_{j,j'=1,2} e^{-it(\epsilon_j-\epsilon_{j'})}|u_j(t)\rangle\langle u_j(0)|\hat{\tilde{\rho}}_A(0)|u_{j'}(0)\rangle\langle u_{j'}(t)|,$$
$$\hat{\tilde{\rho}}^R_A(t) = \sum_{j,j'=1,2} e^{it(\epsilon_j-\epsilon_{j'})}|u_j(0)\rangle\langle u_j(t)|\hat{\tilde{\rho}}^R_A(0)|u_{j'}(t)\rangle\langle u_{j'}(0)|. \tag{E1}$$

Using now (40), the fact that $e^{i\omega_L\hat{\sigma}_z t/2}\hat{V}(t)e^{-i\omega_L\hat{\sigma}_z t/2} = \hat{V}(0)$, we notice that, given the initial conditions (93), we have

$$\begin{aligned}\langle u_j(0)|\hat{\tilde{\rho}}^R_A(t)|u_{j'}(0)\rangle &= e^{it(\epsilon_j-\epsilon_{j'})}\langle u_j(t)|\hat{\tilde{\rho}}^R_A(0)|u_{j'}(t)\rangle\\ &= e^{it(\epsilon_j-\epsilon_{j'})}\langle j|\hat{\tilde{\rho}}_A(0)|j'\rangle\\ &= \langle u_j(-t)|\hat{\tilde{\rho}}_A(-t)|u_{j'}(-t)\rangle,\end{aligned} \tag{E2}$$

from which we deduce that

$$\hat{\tilde{\rho}}^R_A(t) = \hat{\tilde{\rho}}_A(-t). \tag{E3}$$

The generalization to the case where the bath is taken into account is obtained by repeating the reasoning, tracing out first the degrees of freedom of the bath and introducing Kraus operators as in (B18).

APPENDIX F: DERIVATION OF THE MASTER EQUATION (132)

We provide details on the derivation of the master equation (132).

The tilted operator $\hat{V}^{\boldsymbol{\lambda}}_{AB}(t)$ in (118) is simply obtained using the expression (B16) and the identities $e^{i(\lambda_{DA}\hat{H}_{DA}+\lambda_{DL}\hat{H}_{DL})}|1,n\rangle = e^{i(-\lambda_{DA}\Omega+\lambda_{DL}n\omega_L}|1,n\rangle$, $e^{i(\lambda_{DA}\hat{H}_{DA}+\lambda_{DL}\hat{H}_{DL})}|2,n\rangle = e^{i(\lambda_{DA}\Omega+\lambda_{DL}n\omega_L}|2,n\rangle$, and $e^{i\lambda_B\hat{H}_B}\hat{b}_k e^{-i\lambda_B\hat{H}_B} = e^{-i\lambda_B\omega_k}\hat{b}_k$.

Let us now introduce

$$\begin{aligned}d^{\boldsymbol{\lambda}}_{mn,m'n'}(t) &= \sum_{\mu,\nu}\eta_\nu \mathrm{Tr}_S\big[\hat{\sigma}^\dagger_{mn}\hat{W}^{\boldsymbol{\lambda}I}_{\mu,\nu}(t+\delta,t)\big]\mathrm{Tr}_S\big[\hat{\sigma}_{m'n'}\hat{W}^{-\boldsymbol{\lambda}I\dagger}_{\mu,\nu}(t+\delta,t)\big]\\ &= \sum_{\mu,\nu}\eta_\nu\langle E_n,\mu|\mathcal{T}_\to\{e^{-i\int_t^{t+\delta}ds\hat{V}^{\boldsymbol{\lambda}}_{AB}(s)}\}|E_m,\nu\rangle\langle E_{m'},\nu|\mathcal{T}_\to\{e^{i\int_t^{t+\delta}ds\hat{V}^{-\boldsymbol{\lambda}\dagger}_{AB}(s)}\}|E_{n'},\mu\rangle,\end{aligned} \tag{F1}$$

which leads to (125). A perturbative expansion to second order in $\hat{V}_{AB}^{\boldsymbol{\lambda}}$ then yields (127). Since $\{\hat{\sigma}_{mn}\}$ form an orthogonal basis, the only terms $\hat{\sigma}_{mn}$ which remain in (127), and $\sum_{mn}\mathrm{Tr}_X[\hat{\sigma}_{mn}]\hat{\sigma}_{mn} = \mathbb{I}$, we can rewrite (127) as

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\lambda}}\big(\hat{\rho}_X^I(t)\big) = & \frac{1}{\delta_0}\sum_{\alpha,\alpha'=z,+,-}\int_t^{t+\delta_0} ds\int_t^{t+\delta_0} ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}^\dagger(s)\hat{\rho}_B\hat{B}_{-\lambda_B}(s')\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{S}_\alpha^{\boldsymbol{\lambda}}\hat{\rho}_X(t)\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}\dagger}\\
&+\frac{1}{\delta_0}\sum_{\alpha,\alpha'}\int_t^{t+\delta_0} ds\int_t^{t+\delta_0} ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}(s)\hat{\rho}_B\hat{B}_{-\lambda_B}^\dagger(s')\big]e^{i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{S}_\alpha^{\boldsymbol{\lambda}\dagger}\hat{\rho}_X(t)\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}}\\
&-\frac{1}{2}\frac{1}{\delta_0}\sum_{\alpha,\alpha'}\int_t^{t+\delta} ds\int_t^{s} ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}^\dagger(s)\hat{B}_{\lambda_B}(s')\hat{\rho}_B\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{S}_\alpha^{\boldsymbol{\lambda}}\hat{S}_{\alpha'}^{\boldsymbol{\lambda}\dagger}\hat{\rho}_X(t)\\
&-\frac{1}{2}\frac{1}{\delta_0}\sum_{\alpha,\alpha'}\int_t^{t+\delta_0} ds\int_t^{s} ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}(s)\hat{B}_{\lambda_B}^\dagger(s')\hat{\rho}_B\big]e^{i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{S}_\alpha^{\boldsymbol{\lambda}\dagger}\hat{S}_{\alpha'}^{\boldsymbol{\lambda}}\hat{\rho}_X(t)\\
&-\frac{1}{2}\frac{1}{\delta_0}\sum_{\alpha,\alpha'}\int_t^{t+\delta_0} ds\int_X^{t+\delta_0} ds'\mathrm{Tr}_B\big[\hat{B}_{-\lambda_B}^\dagger(s)\hat{B}_{-\lambda_B}(s')\hat{\rho}_B\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{\rho}_X(t)\hat{S}_\alpha^{-\boldsymbol{\lambda}}\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}\dagger}\\
&-\frac{1}{2}\frac{1}{\delta_0}\sum_{\alpha,\alpha'}\int_t^{t+\delta_0} ds\int_s^{t+\delta_0} ds'\mathrm{Tr}_B\big[\hat{B}_{-\lambda_B}(s)\hat{B}_{-\lambda_B}^\dagger(s')\hat{\rho}_B\big]e^{i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{\rho}_X(t)\hat{S}_\alpha^{-\boldsymbol{\lambda}\dagger}\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}},
\end{aligned}
\tag{F2}
$$

where $\hat{S}_z^{\boldsymbol{\lambda}}$, $\hat{S}_+^{\boldsymbol{\lambda}}$, $\hat{S}_-^{\boldsymbol{\lambda}}$ are the operators (120) in the Heisenberg picture,

$$
\hat{S}_z^{\boldsymbol{\lambda}} \equiv e^{-i\omega_L\lambda_{DL}/2}\hat{S}_z,\quad \hat{S}_+^{\boldsymbol{\lambda}} \equiv e^{-i(\omega_L\lambda_{DL}-\Omega\lambda_{DA})/2}\hat{S}_+,\quad \hat{S}_-^{\boldsymbol{\lambda}} \equiv e^{-i(\omega_L\lambda_{DL}+\Omega\lambda_{DA})/2}\hat{S}_-. \tag{F3}
$$

Performing the double integrals then leads to (132). We write explicitly the double integral in first line of the r.h.s. of (F2) (the other terms have similar forms),

$$
\begin{aligned}
&\frac{1}{\delta_0}\int_t^{t+\delta_0} ds\int_t^{t+\delta_0} ds'\mathrm{Tr}_B\big[\hat{B}_{\lambda_B}^\dagger(s)\hat{\rho}_B\hat{B}_{-\lambda_B}(s')\big]e^{-i(\omega_\alpha s-\omega_{\alpha'}s')}\hat{S}_\alpha^{\boldsymbol{\lambda}}\hat{\rho}_X(t)\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}\dagger}\\
&=\sum_k|g_k|^2(n_B(\omega_k)+1)e^{i\lambda_B\omega_k}\delta_0\,\mathrm{sinc}\bigg(\frac{\omega_k-\omega_\alpha}{2}\delta_0\bigg)\,\mathrm{sinc}\bigg(\frac{\omega_k-\omega_{\alpha'}}{2}\delta_0\bigg)\hat{S}_\alpha^{\boldsymbol{\lambda}}\hat{\rho}_X(t)\hat{S}_{\alpha'}^{-\boldsymbol{\lambda}\dagger}e^{i(t+\delta_0/2)(\omega_{\alpha'}-\omega_\alpha)}.
\end{aligned}
\tag{F4}
$$

APPENDIX G: EXPRESSION OF THE GENERALIZED BLOCH EQUATION WHEN $\Omega < \gamma_{\max} < 2\Omega$

When $\Omega < \gamma_{\max} < 2\Omega$, the terms of (127) involving the jump operators $\hat{S}_+$ and $\hat{S}_-$ are removed. We obtain

$$
\mathcal{L}_{\boldsymbol{\lambda}}^G\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big) = -i\big[\hat{H}_{DA},\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big]+\mathcal{D}_+^{G\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big)+\mathcal{D}_-^{G\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big),
$$

with

$$
\begin{aligned}
\mathcal{D}_+^{G\boldsymbol{\lambda}}(\hat{\rho}) = &\sum_{\alpha=+,-,z}G_+(\omega_\alpha)\hat{s}_\alpha^{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_\alpha^{-\boldsymbol{\lambda}\dagger}+\sqrt{G_+(\omega_z)G_+(\omega_+)}\big(\hat{s}_+^{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_z^{-\boldsymbol{\lambda}\dagger}+\hat{s}_z^{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_+^{-\boldsymbol{\lambda}\dagger}\big)+\sqrt{G_+(\omega_z)G_+(\omega_+)}\big(\hat{s}_+^{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_z^{-\boldsymbol{\lambda}\dagger}+\hat{s}_z^{\boldsymbol{\lambda}}\hat{\rho}\hat{s}_+^{-\boldsymbol{\lambda}\dagger}\big)\\
&-\frac{1}{2}\Bigg[\sum_{\alpha=+,-,z}G_+(\omega_\alpha)\hat{s}_\alpha^\dagger\hat{s}_\alpha+\sqrt{G_+(\omega_z)G_+(\omega_+)}\big(\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)\dagger}\hat{s}_+^{(\lambda_{DA},\lambda_{DL},0)}+\hat{s}_+^{(\lambda_{DA},\lambda_{DL},0)\dagger}\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)}\big)\\
&+\sqrt{G_+(\omega_z)G_+(\omega_-)}\big(\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)\dagger}\hat{s}_-^{(\lambda_{DA},\lambda_{DL},0)}+\hat{s}_-^{(\lambda_{DA},\lambda_{DL},0)\dagger}\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)}\big)\Bigg]\hat{\rho}\\
&-\frac{1}{2}\hat{\rho}\Bigg[\sum_{\alpha=+,-,z}G_+(\omega_\alpha)\hat{s}_\alpha^\dagger\hat{s}_\alpha+\sqrt{G_+(\omega_z)G_+(\omega_+)}\big(\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\hat{s}_+^{(-\lambda_{DA},-\lambda_{DL},0)}+\hat{s}_+^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)}\big)\\
&+\sqrt{G_+(\omega_z)G_+(\omega_-)}\big(\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\hat{s}_-^{(-\lambda_{DA},-\lambda_{DL},0)}+\hat{s}_-^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)}\big)\Bigg],\\
\mathcal{D}_-^{G\boldsymbol{\lambda}}(\hat{\rho}) = &\sum_{\alpha=+,-,z}G_-(\omega_\alpha)\hat{s}_\alpha^{\boldsymbol{\lambda}\dagger}\hat{\rho}\hat{s}_\alpha^{-\boldsymbol{\lambda}}+\sqrt{G_-(\omega_z)G_-(\omega_+)}\big(\hat{s}_+^{\boldsymbol{\lambda}\dagger}\hat{\rho}\hat{s}_z^{-\boldsymbol{\lambda}}+\hat{s}_z^{\boldsymbol{\lambda}\dagger}\hat{\rho}\hat{s}_+^{-\boldsymbol{\lambda}}\big)+\sqrt{G_-(\omega_z)G_-(\omega_+)}\big(\hat{s}_+^{\boldsymbol{\lambda}\dagger}\hat{\rho}\hat{s}_z^{-\boldsymbol{\lambda}}+\hat{s}_z^{\boldsymbol{\lambda}\dagger}\hat{\rho}\hat{s}_+^{-\boldsymbol{\lambda}}\big)\\
&-\frac{1}{2}\Bigg[\sum_{\alpha=+,-,z}G_-(\omega_\alpha)\hat{s}_\alpha\hat{s}_\alpha^\dagger+\sqrt{G_-(\omega_z)G_-(\omega_+)}\big(\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)}\hat{s}_+^{(\lambda_{DA},\lambda_{DL},0)\dagger}+\hat{s}_+^{(\lambda_{DA},\lambda_{DL},0)}\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)\dagger}\big)
\end{aligned}
$$

$$+\sqrt{G_-(\omega_z)G_-(\omega_-)}\big(\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)}\hat{s}_-^{(\lambda_{DA},\lambda_{DL},0)\dagger}+\hat{s}_-^{(\lambda_{DA},\lambda_{DL},0)}\hat{s}_z^{(\lambda_{DA},\lambda_{DL},0)\dagger}\big)\Bigg]\hat{\rho}$$
$$-\frac{1}{2}\hat{\rho}\Bigg[\sum_{\alpha=+,-,z}G_-(\omega_\alpha)\hat{s}_\alpha\hat{s}_\alpha^\dagger+\sqrt{G_-(\omega_z)G_-(\omega_+)}\big(\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)}\hat{s}_+^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}+\hat{s}_+^{(-\lambda_{DA},-\lambda_{DL},0)}\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\big)$$
$$+\sqrt{G_-(\omega_z)G_-(\omega_-)}\big(\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)}\hat{s}_-^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}+\hat{s}_-^{(-\lambda_{DA},-\lambda_{DL},0)}\hat{s}_z^{(-\lambda_{DA},-\lambda_{DL},0)\dagger}\big)\Bigg],$$

where $\boldsymbol{\lambda}=(\lambda_{DA},\lambda_{DL},\lambda_B)$.

APPENDIX H: EXPRESSION OF $\mathcal{L}^{aF}$ WITH COUNTING FIELDS

In the strong qubit-laser-coupling limit defined in (140), the product of sinc functions (131) is nonzero only in the case $\alpha=\alpha'$, which is equivalent to the secular approximation in (142). The resulting master equation is

$$\mathcal{L}_{\boldsymbol{\lambda}}^{aF,X}\big(\hat{\rho}_X^{\boldsymbol{\lambda}}\big)=-i\big[\hat{H}_X+\hat{H}_{LS},\hat{\rho}_X^{\boldsymbol{\lambda}}\big]-\tfrac{1}{2}\big\{G_+(\omega_L)\hat{S}_z^\dagger\hat{S}_z+G_+(\omega_L-\Omega)\hat{s}_+^\dagger\hat{S}_++G_+(\omega_L+\Omega)\hat{s}_-^\dagger\hat{s}_-,\hat{\rho}_X^{\boldsymbol{\lambda}}\big\}$$
$$-\frac{1}{2}\big\{G_-(\omega_L)\hat{s}_z\hat{s}_z^\dagger+G_-(\omega_L-\Omega)\hat{s}_+\hat{s}_+^\dagger+G_+(\omega_L+\Omega)\hat{s}_-\hat{s}_-^\dagger,\hat{\rho}_X^{\boldsymbol{\lambda}}\big\}+G_+(\omega_L)e^{i\omega_L\lambda_B}\hat{s}_z^{\boldsymbol{\lambda}}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_z^{-\boldsymbol{\lambda}\dagger}$$
$$+G_+(\omega_L-\Omega)e^{i(\omega_L-\Omega)\lambda_B}\hat{s}_+^{\boldsymbol{\lambda}}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_+^{-\boldsymbol{\lambda}\dagger}+G_+(\omega_L+\Omega)e^{i(\omega_L+\Omega)\lambda_B}\hat{s}_-^{\boldsymbol{\lambda}}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_-^{-\boldsymbol{\lambda}\dagger}$$
$$+G_-(\omega_L)e^{-i\omega_L\lambda_B}\hat{s}_z^{\boldsymbol{\lambda}\dagger}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_z^{-\boldsymbol{\lambda}}+G_-(\omega_L-\Omega)e^{-i(\omega_L-\Omega)\lambda_B}\hat{s}_+^{\boldsymbol{\lambda}\dagger}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_+^{-\boldsymbol{\lambda}}+G_-(\omega_L+\Omega)e^{-i(\omega_L+\Omega)\lambda_B}\hat{s}_-^{\boldsymbol{\lambda}\dagger}\hat{\rho}_X^{\boldsymbol{\lambda}}\hat{s}_-^{-\boldsymbol{\lambda}},\qquad\text{(H1)}$$

where the Lamb shift contribution is

$$\hat{H}_{LS}=I_+(\omega_L)\hat{S}_z^\dagger\hat{S}_z+I_+(\omega_L-\Omega)\hat{S}_+^\dagger\hat{S}_++I_+(\omega_L+\Omega)\hat{S}_-^\dagger\hat{S}_-+I_-(\omega_L)\hat{S}_z\hat{S}_z^\dagger+I_-(\omega_L-\Omega)\hat{S}_+\hat{S}_+^\dagger+I_-(\omega_L+\Omega)\hat{S}_-\hat{S}_-^\dagger,$$

with

$$I_+(\omega)\equiv\mathrm{Im}\bigg(\frac{1}{2}\int_0^{+\infty}d\tau\,\mathrm{Tr}[\hat{B}(\tau)\hat{B}^\dagger(0)\hat{\rho}_B]e^{i\nu\tau}\bigg),\qquad\text{(H2)}$$

$$I_-(\omega)\equiv\mathrm{Im}\bigg(\frac{1}{2}\int_0^{+\infty}d\tau\,\mathrm{Tr}[\hat{B}^\dagger(\tau)\hat{B}(0)\hat{\rho}_B]e^{i\nu\tau}\bigg).\qquad\text{(H3)}$$

As explained in the main text, the Lamb shift contribution may be neglected.

It is straightforward to trace out the degrees of freedom of DL, which gives the master equation (149),

$$\mathcal{L}_{\boldsymbol{\lambda}}^{aF}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big)=-i\big[\hat{H}_{DA},\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big]-\frac{1}{2}\big\{\gamma_{0,\downarrow}\hat{\Sigma}_z^\dagger\hat{\Sigma}_z+\gamma_{2,\uparrow}\hat{\Sigma}_+^\dagger\hat{\Sigma}_++\gamma_{1,\downarrow}\hat{\Sigma}_-^\dagger\hat{\Sigma}_-,\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big\}$$
$$-\frac{1}{2}\big\{\gamma_{0,\uparrow}\hat{\Sigma}_z\hat{\Sigma}_z^\dagger+\gamma_{2,\downarrow}\hat{\Sigma}_+\hat{\Sigma}_+^\dagger+\gamma_{1,\uparrow}\hat{\Sigma}_-\hat{\Sigma}_-^\dagger,\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big\}+\gamma_{0,\downarrow}e^{i\omega_L(\lambda_B-\lambda_{DL})}\hat{\Sigma}_z\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_z^\dagger$$
$$+\gamma_{2,\uparrow}e^{i\Omega\lambda_{DA}-i\omega_L\lambda_{DL}+i(\omega_L-\Omega)\lambda_B}\hat{\Sigma}_+\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_+^\dagger+\gamma_{1,\downarrow}e^{-i\Omega\lambda_{DA}-i\omega_L\lambda_{DL}+i(\omega_L+\Omega)\lambda_B}\hat{\Sigma}_-\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_-^\dagger+\gamma_{0,\uparrow}e^{-i\omega_L(\lambda_B-\lambda_{DL})}\hat{\Sigma}_z^\dagger\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_z$$
$$+\gamma_{2,\downarrow}e^{-i\Omega\lambda_{DA}+Wi\omega_L\lambda_{DL}-i(\omega_L-\Omega)\lambda_B}\hat{\Sigma}_+^\dagger\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_++\gamma_{1,\uparrow}e^{i\Omega\lambda_{DA}+i\omega_L\lambda_{DL}-i(\omega_L+\Omega)\lambda_B}\hat{\Sigma}_-^\dagger\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\hat{\Sigma}_-.\qquad\text{(H4)}$$

From (H4), it is clear that $\mathcal{L}_{\boldsymbol{\lambda}}^{aF}$ satisfies the strict energy conservation condition (109).

APPENDIX I: STEADY STATE OF $\mathcal{L}^{aF}$

As explained in the main text, the fixed point of $\mathcal{L}^{aF}$ is defined by $\mathcal{L}^{aF}(\hat{\rho}_{DA}^{ss})=0$, which implies $d_tE_{DA}=0$. Replacing in (154) leads to

$$P_1^{ss}=\frac{\gamma_{1,\downarrow}+\gamma_{2,\downarrow}}{\gamma_{1,\uparrow}+\gamma_{2,\uparrow}}P_2^{ss},\qquad\text{(I1)}$$

where P_1^{ss}, P_2^{ss} denote the values of $P_1(t)$, $P_2(t)$ in the steady state. Combined with the normalization condition, $P_1^{ss}+P_2^{ss}=1$ we obtain

$$P_1^{ss}=\frac{\gamma_{1,\downarrow}+\gamma_{2,\downarrow}}{\gamma_{1,\downarrow}+\gamma_{2,\downarrow}+\gamma_{1,\uparrow}+\gamma_{2,\uparrow}},\quad P_2^{ss}=\frac{\gamma_{1,\uparrow}+\gamma_{2,\uparrow}}{\gamma_{1,\downarrow}+\gamma_{2,\downarrow}+\gamma_{1,\uparrow}+\gamma_{2,\uparrow}}. \tag{I2}$$

The resonances vanish in the steady state, as can be seen directly from the form of $\mathcal{L}^{aF}$.

Substituting now in (155) yields

$$\dot{Q}^{ss}=-\omega_L(\gamma_{0,\downarrow}-\gamma_{0,\uparrow})-2\omega_L\frac{\gamma_{1,\downarrow}\gamma_{2,\uparrow}-\gamma_{1,\uparrow}\gamma_{2,\downarrow}}{\gamma_{1,\downarrow}+\gamma_{2,\downarrow}+\gamma_{1,\uparrow}+\gamma_{2,\uparrow}}. \tag{I3}$$

Using (145) and recalling that $G_+(\omega)>G_-(\omega)$ for all ω, we deduce that $\dot{Q}^{ss}<0$. Replacing now in (108) and using the fact that, in the steady state, $d_tS_{DA}=0$, we obtain that the entropy production rate of the dressed qubit, defined in (62), is strictly positive in the steady state,

$$d_tS_{DA}^{ss}-\beta_B\dot{Q}^{ss}>0. \tag{I4}$$

APPENDIX J: EXPRESSION OF $\mathcal{L}^{aB}$ WITH COUNTING FIELDS

Assuming that $G_\pm$ are smooth enough in the range $[\omega_L-\Omega,\omega_L+\Omega]$, we replace $G_\pm(\omega_L)$, $G_\pm(\omega_L\pm\Omega)\approx G_\pm(\omega_A)\equiv\overline{G}_\pm$ in (142). Tracing out DL, which is straightforward using (120), we obtain the tilted master equation $\mathcal{L}_\lambda^{aB}$,

$$\mathcal{L}_\lambda^{aB}(\hat{\rho}_{DA}^\lambda)=-i[\hat{H}_{DA},\hat{\rho}_{DA}^\lambda]+\overline{G}_+\mathcal{D}_+^\lambda(\hat{\rho}_{DA}^\lambda)+\overline{G}_-\mathcal{D}_-^\lambda(\hat{\rho}_{DA}^\lambda), \tag{J1}$$

with

$$\begin{aligned}\mathcal{D}_+^\lambda(\hat{\rho})=&\left(\sum_{\alpha=+,-,z}e^{i\lambda_B\omega_\alpha/2}\hat{s}_\alpha^\lambda\right)\hat{\rho}\left(\sum_{\alpha=+,-,z}e^{-i\lambda_B\omega_\alpha/2}\hat{s}_\alpha^{-\lambda}\right)^\dagger\\&-\frac{1}{2}\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^\lambda\right)^\dagger\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^\lambda\right)\hat{\rho}-\frac{1}{2}\hat{\rho}\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^{-\lambda}\right)^\dagger\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^{-\lambda}\right),\end{aligned} \tag{J2}$$

$$\begin{aligned}\mathcal{D}_-^\lambda(\hat{\rho})=&\left(\sum_{\alpha=+,-,z}e^{i\lambda_B\omega_\alpha/2}\hat{s}_\alpha^\lambda\right)^\dagger\hat{\rho}\left(\sum_{\alpha=+,-,z}e^{-i\lambda_B\omega_\alpha/2}\hat{s}_\alpha^{-\lambda}\right)\\&-\frac{1}{2}\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^\lambda\right)\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^\lambda\right)^\dagger\hat{\rho}-\frac{1}{2}\hat{\rho}\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^{-\lambda}\right)\left(\sum_{\alpha=+,-,z}\hat{s}_\alpha^{-\lambda}\right)^\dagger,\end{aligned} \tag{J3}$$

where the operators $\hat{s}_{+,-,z}^\lambda$ were defined in (122).

APPENDIX K: $\dot{W}_L$ FOR THE GENERALIZED BLOCH EQUATION

The rate $\dot{W}_L$ for the generalized Bloch equation is, in the weak and intermediate regimes,

$$\dot{W}_L=\dot{W}_{DL}+\mathrm{Tr}\left[\frac{\omega_L}{2}\hat{\sigma}_z\mathcal{L}^G(\hat{\rho})\right], \tag{K1}$$

with

$$\mathrm{Tr}\left[\frac{\omega_L}{2}\hat{\sigma}_z\mathcal{L}^{aG}(\hat{\rho})\right]=-g\omega_L\mathrm{Im}[P_{21}(t)]+\frac{\omega_L}{2}\mathrm{Tr}[\hat{\sigma}_z\mathcal{D}_+^G(\hat{\rho})]+\frac{\omega_L}{2}\mathrm{Tr}[\hat{\sigma}_z\mathcal{D}_-^G(\hat{\rho})], \tag{K2}$$

where

$$\begin{aligned}\mathrm{Tr}[\hat{\sigma}_z\mathcal{D}_+^G(\hat{\rho})]=&-\frac{g^2}{4\Omega^2}\left(\sqrt{G_+(\omega_L)G_+(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega}+\sqrt{G_+(\omega_L)G_+(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega}\right)\\&+P_1(t)\left[-\frac{g^2}{2\Omega^2}\sqrt{G_+(\omega_L)G_+(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega}+\frac{\delta}{\Omega}G_+(\omega_L-\Omega)\left(\frac{\Omega-\delta}{2\Omega}\right)^2\right]\\&+P_2(t)\left[-\frac{g^2}{2\Omega^2}\sqrt{G_+(\omega_L)G_+(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega}-\frac{\delta}{\Omega}G_+(\omega_L+\Omega)\left(\frac{\Omega+\delta}{2\Omega}\right)^2\right]\end{aligned}$$

$$+ [P_{21}(t) + P_{12}(t)]\left(\frac{g}{2\Omega}\right)^3 [4G_+(\omega_L) + 2\sqrt{G_+(\omega_L+\Omega)G_+(\omega_L-\Omega)}]$$
$$+ P_{21}(t)\left[\frac{g}{\Omega}G_+(\omega_L-\Omega)\left(\frac{\Omega-\delta}{2\Omega}\right)^2 + \frac{\delta g}{2\Omega^2}\sqrt{G_+(\omega_L)G_+(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega}\right]$$
$$+ P_{12}(t)\left[\frac{g}{\Omega}G_+(\omega_L+\Omega)\left(\frac{\Omega+\delta}{2\Omega}\right)^2 - \frac{\delta g}{2\Omega^2}\sqrt{G_+(\omega_L)G_+(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega}\right],$$

and

$$\begin{aligned}\mathrm{Tr}[\hat{\sigma}_z\mathcal{D}_-^G(\hat{\rho})] = &-\frac{g^2}{4\Omega^2}\left(\sqrt{G_-(\omega_L)G_-(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega} + \sqrt{G_-(\omega_L)G_-(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega}\right)\\
&+ P_1(t)\left[-\frac{g^2}{2\Omega^2}\sqrt{G_-(\omega_L)G_-(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega} - \frac{\delta}{\Omega}G_-(\omega_L+\Omega)\left(\frac{\Omega+\delta}{2\Omega}\right)^2\right]\\
&+ P_2(t)\left[-\frac{g^2}{2\Omega^2}\sqrt{G_-(\omega_L)G_-(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega} + \frac{\delta}{\Omega}G_-(\omega_L-\Omega)\left(\frac{\Omega-\delta}{2\Omega}\right)^2\right]\\
&- [P_{21}(t) + P_{12}(t)]\left(\frac{g}{2\Omega}\right)^3 [4G_-(\omega_L) + 2\sqrt{G_-(\omega_L+\Omega)G_-(\omega_L-\Omega)}]\\
&- P_{21}(t)\left[\frac{g}{\Omega}G_-(\omega_L+\Omega)\left(\frac{\Omega+\delta}{2\Omega}\right)^2 - \frac{\delta g}{2\Omega^2}\sqrt{G_-(\omega_L)G_+(\omega_L-\Omega)}\frac{\Omega-\delta}{\Omega}\right]\\
&- P_{12}(t)\left[\frac{g}{\Omega}G_-(\omega_L-\Omega)\left(\frac{\Omega-\delta}{2\Omega}\right)^2 + \frac{\delta g}{2\Omega^2}\sqrt{G_-(\omega_L)G_-(\omega_L+\Omega)}\frac{\Omega+\delta}{\Omega}\right].\end{aligned} \tag{K3}$$

In the strong-driving regimes, it is equal to the rate obtained with the Floquet master equation, in (177).

APPENDIX L: DERIVATION OF THE FLOQUET MASTER EQUATION DRESSED WITH COUNTING FIELDS

We give here the full expression of (173). Following the method explained in the main text, we obtain

$$\begin{aligned}\mathcal{D}_{0,+}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{-i\omega_L\lambda_L}e^{i\omega_L\lambda_B}\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_z^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_z^F(t-\lambda_L/2)\hat{\Sigma}_z^F(t-\lambda_L/2),\end{aligned} \tag{L1}$$

$$\begin{aligned}\mathcal{D}_{0,-}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{i\omega_L\lambda_L}e^{-i\omega_L\lambda_B}\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_z^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\Sigma}_z^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_z^F(t-\lambda_L/2)\hat{\Sigma}_z^F(t-\lambda_L/2),\end{aligned} \tag{L2}$$

$$\begin{aligned}\mathcal{D}_{1,+}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{-i\omega_L\lambda_L}e^{i(\omega_L+\Omega)\lambda_B}\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_+^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_+^F(t-\lambda_L/2)\hat{\Sigma}_-^F(t-\lambda_L/2),\end{aligned} \tag{L3}$$

$$\begin{aligned}\mathcal{D}_{1,-}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{i\omega_L\lambda_L}e^{-i(\omega_L+\Omega)\lambda_B}\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_-^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_+^F(t-\lambda_L/2)\hat{\Sigma}_-^F(t-\lambda_L/2),\end{aligned} \tag{L4}$$

$$\begin{aligned}\mathcal{D}_{2,+}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{-i\omega_L\lambda_L}e^{i(\omega_L-\Omega)\lambda_B}\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_-^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_-^F(t-\lambda_L/2)\hat{\Sigma}_+^F(t-\lambda_L/2),\end{aligned} \tag{L5}$$

$$\begin{aligned}\mathcal{D}_{2,-}^{F\boldsymbol{\lambda}}(\hat{\tilde{\rho}}_A) = &e^{i\omega_L\lambda_L}e^{-i(\omega_L-\Omega)\lambda_B}\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\hat{\Sigma}_+^F(t-\lambda_L/2) - \tfrac{1}{2}\hat{\Sigma}_+^F(t+\lambda_L/2)\hat{\Sigma}_-^F(t+\lambda_L/2)\hat{\tilde{\rho}}_A\\
&- \tfrac{1}{2}\hat{\tilde{\rho}}_A\hat{\Sigma}_+^F(t-\lambda_L/2)\hat{\Sigma}_-^F(t-\lambda_L/2).\end{aligned} \tag{L6}$$

APPENDIX M: REDFIELD EQUATION WITH COUNTING FIELDS

We give here the first and last terms of the tilted master equation (187),

$$
\begin{aligned}
&-\mathrm{Tr}_B\left[\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\int_0^{+\infty}d\tau\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t')\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t'-\tau)\hat{\rho}(t)\otimes\hat{\rho}_B\right]\\
&=-\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\big[G_+(\omega_L)\hat{S}_z\hat{S}_z^\dagger+G_+(\omega_L-\Omega)\hat{S}_-\hat{S}_-^\dagger+G_+(\omega_L+\Omega)\hat{S}_+\hat{S}_+^\dagger\\
&+G_+(\omega_L-\Omega)e^{i\Omega t'}e^{i\Omega\lambda_{DA}/2}\hat{S}_z\hat{S}_-^\dagger+G_+(\omega_L+\Omega)e^{-i\Omega t'}e^{-i\Omega\lambda_{DA}/2}\hat{S}_z\hat{S}_+^\dagger\\
&+G_+(\omega_L)e^{-i\Omega\lambda_{DA}/2}e^{-i\Omega t'}\hat{S}_-\hat{S}_z^\dagger+G_+(\omega_L)e^{i\Omega\lambda_{DA}/2}e^{i\Omega t'}\hat{S}_+\hat{S}_z^\dagger\\
&+G_-(\omega_L)\hat{S}_z^\dagger\hat{S}_z+G_+(\omega_L-\Omega)\hat{S}_-^\dagger\hat{S}_-+G_+(\omega_L+\Omega)\hat{S}_+^\dagger\hat{S}_+\\
&+G_-(\omega_L-\Omega)e^{-i\Omega t'}e^{-i\Omega\lambda_{DA}/2}\hat{S}_z^\dagger\hat{S}_-+G_-(\omega_L+\Omega)e^{i\Omega t'}e^{i\Omega\lambda_{DA}/2}\hat{S}_z^\dagger\hat{S}_+\\
&+G_-(\omega_L)e^{i\Omega\lambda_{DA}/2}e^{i\Omega t'}\hat{S}_-^\dagger\hat{S}_z+G_-(\omega_L)e^{-i\Omega\lambda_{DA}/2}e^{-i\Omega t'}\hat{S}_+^\dagger\hat{S}_z\big]\hat{\rho}(t),
\end{aligned}
\tag{M1}
$$

and

$$
\begin{aligned}
&\mathrm{Tr}_B\left[\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\int_0^{+\infty}d\tau\hat{V}_{AB}^{\boldsymbol{\lambda}/2}(t'-\tau)\hat{\rho}(t)\otimes\hat{\rho}_B\hat{V}_{AB}^{-\boldsymbol{\lambda}/2}(t')\right]\\
&=\frac{1}{\delta_0}\int_t^{t+\delta_0}dt'\big[e^{i\omega_L\lambda_{DL}}\big(G_-(\omega_L)e^{-i\omega_L\lambda_B}\hat{S}_z\hat{\rho}(t)\hat{S}_z^\dagger\\
&+G_-(\omega_L-\Omega)e^{-i(\omega_L-\Omega)\lambda_B}e^{-i\Omega\lambda_{DA}}\hat{S}_-\hat{\rho}(t)\hat{S}_-^\dagger+G_-(\omega_L+\Omega)e^{-i(\omega_L+\Omega)\lambda_B}e^{i\Omega\lambda_{DA}}\hat{S}_+\hat{\rho}(t)\hat{S}_+^\dagger\\
&+G_-(\omega_L)e^{-i\omega_L\lambda_B}e^{-i\Omega\lambda_{DA}/2}e^{i\Omega t}\hat{S}_z\hat{\rho}(t)\hat{S}_-^\dagger+G_-(\omega_L)e^{-i\omega_L\lambda_B}e^{i\Omega\lambda_{DA}/2}e^{-i\Omega t}\hat{S}_z\hat{\rho}(t)\hat{S}_+^\dagger\\
&+G_-(\omega_L-\Omega)e^{-i\lambda_B(\omega_L-\Omega)}e^{-i\Omega\lambda_{DA}/2}e^{-i\Omega t'}\hat{S}_-\hat{\rho}(t)\hat{S}_z^\dagger+G_-(\omega_L+\Omega)e^{-i\lambda_B(\omega_L+\Omega)}e^{i\Omega\lambda_{DA}/2}e^{i\Omega t'}\hat{S}_+\hat{\rho}(t)\hat{S}_z^\dagger\\
&+G_-(\omega_L-\Omega)e^{-i\lambda_B(\omega_L-\Omega)}e^{-2i\Omega t'}\hat{S}_-\hat{\rho}(t)\hat{S}_+^\dagger+G_-(\omega_L+\Omega)e^{-i\lambda_B(\omega_L+\Omega)}e^{2i\Omega t'}\hat{S}_+\hat{\rho}(t)\hat{S}_-^\dagger\big)\\
&+e^{-i\omega_L\lambda_{DL}}\big(G_+(\omega_L)e^{i\omega_L\lambda_B}\hat{S}_z^\dagger\hat{\rho}(t)\hat{S}_z\\
&+G_+(\omega_L-\Omega)e^{i(\omega_L-\Omega)\lambda_B}e^{i\Omega\lambda_{DA}}\hat{S}_-^\dagger\hat{\rho}(t)\hat{S}_-+G_+(\omega_L+\Omega)e^{i(\omega_L+\Omega)\lambda_B}e^{-i\Omega\lambda_{DA}}\hat{S}_+^\dagger\hat{\rho}(t)\hat{S}_+\\
&+G_+(\omega_L)e^{i\omega_L\lambda_B}e^{i\Omega\lambda_{DA}/2}e^{-i\Omega t}\hat{S}_z^\dagger\hat{\rho}(t)\hat{S}_-+G_+(\omega_L)e^{i\omega_L\lambda_B}e^{-i\Omega\lambda_{DA}/2}e^{i\Omega t}\hat{S}_z^\dagger\hat{\rho}(t)\hat{S}_+\\
&+G_+(\omega_L-\Omega)e^{i\lambda_B(\omega_L-\Omega)}e^{i\Omega\lambda_{DA}/2}e^{i\Omega t'}\hat{S}_-^\dagger\hat{\rho}(t)\hat{S}_z+G_+(\omega_L+\Omega)e^{i\lambda_B(\omega_L+\Omega)}e^{-i\Omega\lambda_{DA}/2}e^{-i\Omega t'}\hat{S}_+^\dagger\hat{\rho}(t)\hat{S}_z\\
&+G_+(\omega_L-\Omega)e^{i\lambda_B(\omega_L-\Omega)}e^{2i\Omega t'}\hat{S}_-^\dagger\hat{\rho}(t)\hat{S}_++G_+(\omega_L+\Omega)e^{i\lambda_B(\omega_L+\Omega)}e^{-2i\Omega t'}\hat{S}_+^\dagger\hat{\rho}(t)\hat{S}_-\big)\big].
\end{aligned}
\tag{M2}
$$

Performing the approximation $G_\pm(\nu)\approx\overline{G}_\pm$, we obtain

$$
\begin{aligned}
\frac{d\hat{\rho}_{DA}^{\boldsymbol{\lambda}}}{dt}&\equiv\mathcal{L}_{\boldsymbol{\lambda}}^{aB,\mathrm{red}}\\
&=-i\big[\hat{H}_{DA},\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big]+\overline{G}_+\mathcal{D}_{\hat{s}}^{\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big)+\overline{G}_-\mathcal{D}_{\hat{s}^\dagger}^{\boldsymbol{\lambda}}\big(\hat{\rho}_{DA}^{\boldsymbol{\lambda}}\big),
\end{aligned}
\tag{M3}
$$

with

$$
\begin{aligned}
\mathcal{D}_{\hat{s}}^{\boldsymbol{\lambda}}(\hat{\rho})&=-\tfrac{1}{2}\big(\hat{s}_{\lambda_{DA},0}^\dagger\hat{s}_{\lambda_{DA},0}\hat{\rho}+\hat{\rho}\hat{s}_{-\lambda_{DA},0}^\dagger\hat{s}_{-\lambda_{DA},0}\big)+\tfrac{1}{2}e^{i\omega_L(\lambda_B-\lambda_{DL})}\big(\hat{s}_{\lambda_{DA},0}\hat{\rho}\hat{s}_{-\lambda_{DA},-\lambda_B}^\dagger+\hat{s}_{\lambda_{DA},\lambda_B}\hat{\rho}\hat{s}_{-\lambda_{DA},0}^\dagger\big),\\
\mathcal{D}_{\hat{s}^\dagger}^{\boldsymbol{\lambda}}(\hat{\rho})&=-\tfrac{1}{2}\big(\hat{s}_{\lambda_{DA},0}\hat{s}_{\lambda_{DA},0}^\dagger\hat{\rho}+\hat{\rho}\hat{s}_{-\lambda_{DA},0}\hat{s}_{-\lambda_{DA},0}^\dagger\big)+\tfrac{1}{2}e^{-i\omega_L(\lambda_B-\lambda_{DL})}\big(\hat{s}_{\lambda_{DA},0}^\dagger\hat{\rho}\hat{s}_{-\lambda_{DA},-\lambda_B}+\hat{s}_{\lambda_{DA},\lambda_B}^\dagger\hat{\rho}\hat{s}_{-\lambda_{DA},0}\big),
\end{aligned}
\tag{M4}
$$

where

$$
\hat{s}_{\lambda_{DA},\lambda_B}=\hat{s}_z+e^{i\Omega(\lambda_{DA}/2-\lambda_B)}\hat{s}_++e^{-i\Omega(\lambda_{DA}/2-\lambda_B)}\hat{s}_-.
\tag{M5}
$$

The explicit expressions (189) allow us to see directly that the symmetry (107) is not satisfied. It is also clear that the strict energy conservation condition (109) is not satisfied.

APPENDIX N: STEADY STATE OF $\mathcal{L}^{aB}$

The fixed point of $\mathcal{L}^{aB}$, defined by $\mathcal{L}^{aB}(\hat{\rho}_{DA}^{ss}) = 0$, is

$$P_b^{ss} = \frac{1}{\overline{G}_+ + \overline{G}_-}\left(\overline{G}_- + \frac{1}{2}\frac{\overline{G}_+ - \overline{G}_-}{1 + 2\frac{\delta^2}{g^2} + \frac{(\overline{G}_+ + \overline{G}_-)^2}{2g^2}}\right), \quad P_{ba}^{ss} = -\frac{\frac{\delta(\overline{G}_+ - \overline{G}_-)}{g(\overline{G}_+ + \overline{G}_-)} + i\frac{(\overline{G}_+ - \overline{G}_-)}{2g}}{1 + 2\frac{\delta^2}{g^2} + \frac{(\overline{G}_+ + \overline{G}_-)^2}{2g^2}}. \tag{N1}$$

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