Analysis and Mathematical Physics (2025) 15:48
https://doi.org/10.1007/s13324-025-01046-w

Geometry and topology of spin random fields

Antonio Lerario[1] · **Domenico Marinucci**[2] · **Maurizia Rossi**[3] · **Michele Stecconi**[4]

Received: 26 June 2024 / Revised: 10 February 2025 / Accepted: 26 February 2025 /
Published online: 21 March 2025

Abstract

Spin (spherical) random fields are very important in many physical applications, in particular they play a key role in Cosmology, especially in connection with the analysis of the Cosmic Microwave Background radiation. These objects can be viewed as random sections of the s-th complex tensor power of the tangent bundle of the 2-sphere. In this paper, we discuss how to characterize their expected geometry and topology. In particular, we investigate the asymptotic behaviour, under scaling assumptions, of general classes of geometric and topological functionals including Lipschitz–Killing Curvatures and Betti numbers for (properly defined) excursion sets; we cover both the cases of fixed and diverging spin parameters s. In the special case of monochromatic fields (i.e., spin random eigenfunctions) our results are particularly explicit; we show how their asymptotic behaviour is non-universal and we can obtain in particular complex versions of Berry's random waves and of Bargmann–Fock's models as subcases of a new generalized model, depending on the rate of divergence of the spin parameter s.

Keywords Spin random fields · Lipschitz–Killing curvatures · Betti numbers · Spin random eigenfunctions

Antonio Lerario, Domenico Marinucci, Maurizia Rossi and Michele Stecconi have contributed equally to this work.

✉ Michele Stecconi
michele.stecconi@uni.lu

Antonio Lerario
lerario@sissa.it

Domenico Marinucci
marinucc@mat.uniroma2.it

Maurizia Rossi
maurizia.rossi@unimib.it

1 Mathematics Area, SISSA, Trieste, Italy

2 Dipartimento di Matematica, Università di Roma Tor Vergata, Rome, Italy

3 Dipartimento di Matematica, Università di Milano Bicocca, Milan, Italy

4 DMATH, University of Luxembourg, Esch-sur-Alzette, Luxembourg

Contents

1 Introduction and motivations . . . 3
2 An overview of the main results . . . 3
 2.1 Our setting . . . 3
 2.2 Scaling limits and asymptotics . . . 5
 2.3 Non-universal asymptotic geometry of spin eigenfunctions . . . 9
 2.4 Technical novelties . . . 13
 2.5 Plan of the paper . . . 14
3 Spin line bundles . . . 14
 3.1 Intrinsic definition . . . 14
  3.1.1 Tensor representation . . . 15
 3.2 Hermitian metric . . . 17
 3.3 Trivialization via Euler's angles . . . 18
 3.4 Spectral representation of spin sections . . . 20
4 Spin random fields . . . 21
 4.1 Random sections . . . 22
 4.2 Isotropy and gaussianity . . . 22
5 Jet bundles and Type-W singularities . . . 25
 5.1 Examples . . . 27
 5.2 Intrinsic singularity type W . . . 27
 5.3 Semialgebraic type–W singularities . . . 28
 5.4 Transversality . . . 29
 5.5 Whitney stratified subsets of the sphere . . . 30
 5.6 Euler–Poincaré characteristic . . . 30
6 Lipschitz–Killing curvatures . . . 31
 6.1 Normal Morse index . . . 31
 6.2 Explicit formula for stratified subsets of the sphere . . . 32
  6.2.1 Description of β^1 . . . 33
  6.2.2 Description of β^0 . . . 33
 6.3 Stratified Morse theory . . . 34
  6.3.1 Semialgebraic Morse inequalities . . . 35
7 Scaling assumption . . . 37
 7.1 Scaling assumption for the covariance . . . 37
 7.2 The rescaled field . . . 37
 7.3 Smooth convergence of the covariance functions . . . 39
8 Main results . . . 42
 8.1 Setting . . . 42
  8.1.1 Jets and type-W singularities . . . 42
  8.1.2 Supports . . . 43
 8.2 Convergence in distribution . . . 43
 8.3 Convergence of expectations . . . 44
  8.3.1 Outline of the proof of Theorem 3 and Theorem 20 . . . 45
9 Proof of Theorem 3 and Theorem 20 . . . 46
 9.1 Step 1 . . . 46
 9.2 Step 2: Lipschitz–Killing curvatures . . . 48
  9.2.1 The Euler–Poincaré characteristic . . . 49
  9.2.2 The first intrinsic volume . . . 50
  9.2.3 The area . . . 51
 9.3 Step 3: Betti numbers . . . 51
10 Monochromatic waves . . . 52
 10.1 Betti numbers of the excursion set . . . 54
 10.2 Excursion sets – proof of Theorem 5 . . . 55
Appendix A Expected number of zeroes: proof of Theorem 1 . . . 56
 A.1 The covariance function of the rescaled field . . . 56
 A.2 Proof of Theorem 1 . . . 58
Appendix B Proof of technical lemmas . . . 60

B.1 Expected nodal volume of Gaussian fields 60
B.2 Explicit formulas for L–K curvatures for spin equal to zero 61
Appendix C Alternative to Hilb's asymptotic 62
C.1 Limit of the covariance of monochromatic fields 65
Appendix D Berry's complex random Wave model 66
References 68

1 Introduction and motivations

The notion of spin s property ($s \in \mathbb{Z}$) for functions on the sphere was first introduced in the physics literature by Newman and Penrose [45], as follows:
a quantity τ defined on the unit two-dimensional sphere $\mathbb{S}^2$ has spin weight s if, whenever a tangent vector v at any point p on the sphere transforms under coordinate change by $v' = e^{i\psi} v$, then the quantity at this point p transforms by $\tau' = e^{is\psi}\tau$.

In the mathematical literature Newman and Penrose's theory was developed by [24] (see also [42, Chapter 12]), who linked the notion of spin s quantity to that of section of the so-called spin s line bundle on the sphere; later, many other papers such as [14, 33, 38, 39] dealt with these geometric objects, strongly motivated by both theoretical interests and cosmological applications [42, Section 1.2].

Apart from their pure mathematical interest, spin spherical functions have drawn extremely strong attention in the last two decades in the Cosmological literature, in particular, in the context of so-called Cosmic Microwave Background (CMB) polarization data (see e.g. [20], Chapter 5). Such data are modelled as a section of a vector bundle on the sphere, and indeed they are commonly viewed in a probabilistic sense as a single realization of a random section of a Gaussian spin bundle. The analysis of polarization data extends and generalizes the investigation of CMB temperature data, which are viewed as a realization of a Gaussian scalar-valued random field on the sphere; the study of CMB is the major tool to probe Big Bang models and to determine the main cosmological constants, and as such it has been the object of an enormous interest in the last 20 years, leading to two major satellite missions, NASA's WMAP and ESA's Planck, see [7] and the references therein. A similar amount of interest is currently drawn by CMB polarization, which will be the object of the future satellite Mission Lite-Bird and of several ground based observational experiments [6]. For instance, it is expected that polarization data may probe the existence of primordial gravitational waves, thus providing the definite proof for the so-called inflationary scenario in Big Bang dynamics, as discussed for instance in [5]. Spin function emerge also in other very important Cosmological observations, most notably in so-called weak gravitational lensing data, the object of the ESA's satellite Mission Euclid [30].

2 An overview of the main results

2.1 Our setting

The purpose of this paper is to establish a general technique to characterize geometric functionals of spin fiber bundles. These functionals cover, among others, Lipschitz–

Killing curvatures for excursion sets, zeroes, critical points and Betti numbers. The formal statement of our results will require a considerable amount of discussion and definitions which will be given in the following Sections. We believe it is nonetheless useful to provide first a general overlook of the framework we are interested in and our main results.

In particular, we shall be concerned with random sections of spin fiber bundles of possibly varying order $s_n \in \mathbb{Z}$, considering also an asymptotic framework where both the spin order s_n and the variances of the random coefficients are allowed to vary with n. More precisely, we shall be concerned with spin isotropic Gaussian sections, that is Gaussian random sections σ_n of $\mathcal{T}^{\otimes s_n}$, the s_n–th complex tensor power of the tangent bundle $\mathcal{T}^{\otimes 1} = T\mathbb{S}^2$, that is seen as a complex line bundle over the standard two–sphere $\mathbb{S}^2$. We will call $\mathcal{T}^{\otimes s}$ the *spin–s line bundle*. We refer to [3, 22, 25, 26, 32] for some recent results on zero sets of Gaussian random sections, under different settings than ours in this paper.

It is known (see the discussion in Sects. 3 and 4) that spin–s_n random sections can be given as a spectral representation of the form

$$\sigma_n := \sum_{\ell \geq |s_n|}^{\infty} \sum_{m=-\ell}^{\ell} a^{\ell}_{m,s_n}(n) Y^{\ell}_{m,s_n};$$

here, Y^{ℓ}_{m,s_n} denotes the family of *spin–s_n spherical harmonics*, which we view as deterministic sections of the spin line bundle $\mathcal{T}^{\otimes s_n}$, while the *random* array $a^{\ell}_{m,s_n}(n)$ represents the spin spherical harmonic coefficients, that we will take to be jointly complex, circularly–symmetric Gaussian random variables. Spin spherical harmonics were introduced in [45] and their connection with the elements of the Wigner's matrices representations for the group $SO(3)$ is discussed in e.g. [14, 39], [42][Chapter 12]. In particular, σ_n can be represented, via a one–to–one correspondence, as a complex Gaussian random function $X_n : SO(3) \to \mathbb{C}$, called the *pull–back field*. We explain this mechanism in Sect. 4, see Theorem 7 in particular.

Our first important remark is that σ_n is characterized by the *circular covariance* function $k_n : \mathbb{R} \to \mathbb{R}$, where

$$\mathbb{E}\{X_n(\mathbb{1})\overline{X_n(R(\varphi, \theta, \psi))}\} = k_n(\theta) e^{is(\varphi+\psi)};$$

here, $R(\varphi, \theta, \psi) \in SO(3)$ is a rotation characterized by the three Euler angles (φ, θ, ψ), defined as in Sect. 3.3. In the special case where $s = 0$, i.e., for scalar random fields, it is immediately seen that the covariance function depends only on the parameter θ, to be interpreted as an angular distance – but this no longer holds for general $s \neq 0$, see also [52].

The behaviour of scalar-valued Gaussian isotropic random fields is well-known to be fully characterized by their angular power spectra, i.e., the variance of the random spherical harmonic coefficients $a^{\ell}_{m,0}$. This is still the case for a fixed, arbitrary value of the spin parameter; however two random fields with unequal spin parameters $s \neq s'$ will have different geometric and topological properties even if they are endowed the same angular power spectra. In particular, it should be noted that the derivatives of the spin field are not stochastically independent for $s \neq 0$; indeed we can represent σ in local coordinates as a Gaussian field $\xi : \mathbb{D} \to \mathbb{C}$ on the disk. Then, the covariance matrix of the random vector $(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y})$ is the following (see Corollary 1):

$$\mathbb{E}\left\{ \begin{pmatrix} \xi \\ \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \end{pmatrix} \begin{pmatrix} \overline{\xi} & \overline{\frac{\partial \xi}{\partial x}} & \overline{\frac{\partial \xi}{\partial y}} \end{pmatrix} \right\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -k''(0) & -i\frac{s}{2}k(0) \\ 0 & i\frac{s}{2}k(0) & -k''(0) \end{pmatrix}.$$

Remark 1 Note that for $s \neq 0$, the "real and imaginary" components of ξ are not independent as fields for any choice of local coordinates.

A simple consequence of this phenomenon is given by our first result below, where the expected value of the number of zeros for a Gaussian spin bundle is established and shown to depend explicitly on s:

Theorem 1 (Expected number of zeroes) *Let $\sigma : \mathbb{S}^2 \to \mathcal{T}^{\otimes s}$ be a smooth Gaussian isotropic spin s random field. Let $k(\theta)$ be its* circular covariance *function. Then*

$$\mathbb{E}\{\#\{\sigma = 0\}\} = 2\frac{|k''(0)|}{k(0)} + \frac{s^2}{2}\frac{k(0)}{|k''(0)|}.$$

The proof of the above theorem will be given in Appendix A.

Example 1 The previous equation takes an especially simple form in the case of random eigenfunctions of the spin Laplacian (to be discussed below), i.e., when $\sigma = \sum_m a^\ell_{m,s} Y^\ell_{m,s}$ is monochromatic, with spin equal to s; in this case $k(\theta) = d^\ell_{-s,-s}(\theta)$ is a Wigner d-function, see e.g. [42], Section 3.3. Here we have $k(0) = 1$ and $k''(0) = -\frac{1}{2}(\ell(\ell+1) - s^2)$, so that

$$\mathbb{E}\{\#\{\sigma = 0\}\} = \ell(\ell+1) - s^2 + \frac{s^2}{\ell(\ell+1) - s^2}.$$

Note that the corresponding eigenvalue with respect to the spin Laplacian $\eth\overline{\eth}$ (to be discussed below) is $= -(\ell - s)(\ell + s + 1)$.

2.2 Scaling limits and asymptotics

For the results to follow, we shall assume that a scaling condition of the following form holds; for some scaling sequence $\rho_n \to 0$ and such that

$$\lim_{n\to\infty} s_n \rho_n^2 = \beta \in \mathbb{R}, \tag{1}$$

we have that

$$k_n(\rho_n \cdot x) \to k_\infty(x)$$

in addition, the convergence of k_n should hold in the $\mathcal{C}^{\infty}(\mathbb{R})$ sense as a sequence of functions in the variable x. In the examples that we will consider in Sect. 2.3, the value of β is always in the set $\{\frac{1}{2r}|r \in \mathbb{Z}\backslash\{0\}\} \cup \{0\}$; we conjecture that any arbitrary $\beta \in \mathbb{R}$ could be attained, however we are not aware of any specific construction that leads to this condition.

Scaling conditions for sequences of standard (scalar-valued) random fields are known to hold in many circumstances, including random eigenfunctions and needlet fields, see [17, 18, 46, 54, 55]; as noted before, we also admit the possibility that the spin parameter s_n depends on n.

Under these conditions, we will show that the geometry and topology of $\sigma_n|_{B_{\rho_n}}$, where B_{ρ_n} is a spherical ball with radius ρ_n, converge to those of a Gaussian Random Field $\xi_\infty : \mathbb{D} \to \mathbb{C}$ having covariance function

$$\mathbb{E}\{\xi_\infty(z_1)\overline{\xi_\infty(z_2)}\} = k_\infty(|z_1 - z_2|) \exp\left(\beta i \Im(z_1\overline{z_2})\right).$$

With some additional work, we will be able to say something about the *global* geometry and topology of ξ_∞.

Remark 2 $\xi_\infty : \mathbb{C} \to \mathbb{C}$ is circularly symmetric ($\mathbb{E}\{\xi_\infty(z)\xi_\infty(w)\} = 0$, for all $z, w \in \mathbb{C}$, because the same is assumed on X_n, the pullback random field). If $\beta = 0$, it is also stationary and with real covariance function: $\mathbb{E}\{\xi(z)\overline{\xi(w)}\} = k_\infty(|z - w|) \in \mathbb{R}$, by construction. It follows that its real and imaginary parts, as random fields, are independent and identically distributed. This allows to apply directly the formulas from [4] for the Lipschitz–Killing curvatures of the excursion sets of $|\xi_\infty|$.

For the next statement we need to anticipate the notion of type-W singularities of σ_n, denoted $Z^W(\sigma_n)$; a rigorous and detailed definition will be given later in Sect. 5.2, see Sect. 5.1 below for some examples. For the moment, it suffices to say that $Z^W(\sigma_n) \subset \mathbb{S}^2$ is a subset of $\mathbb{S}^2$ identified in terms of conditions, encoded in W, on the modulus of the random section and its higher order derivatives; in practice, the class of random subsets $Z^W(\sigma)$ is general enough to cover for instance excursion sets, level curves, zeroes, critical points and basically all other examples which are usually investigated in stochastic geometry.

Let us define also $Z^W(\xi_\infty) \subset \mathbb{D}$ to be the random subset of the disk defined by the limit field ξ_∞ mentioned above. We are now able to state the two main results of this paper up to some qualifications to be discussed later, the first regarding asymptotic laws, while the second expected values. Let B_{ρ_n} be a spherical ball of radius ρ_n. Heuristically, our objective will be to show a form of convergence in law for the random sets on the shrinking ball to analogous limit random sets for a suitably defined limiting process:

$$Z^W(\sigma_n) \cap B_{\rho_n} \to Z^W(\xi_\infty),$$

These subsets can be characterized as random smooth Whitney stratified subsets (see [4]), although for the time being we do not discuss this issue in full details, see Sect. 5. Such a convergence has to be taken as heuristic, as to make it rigorous one should specify the topology over the space of all stratified subsets of the disk; we will not do that in this paper. Instead, we will focus on the convergence of two types of functionals associated to a Whitney stratified subset $Z \subset \mathbb{D}$: the Lipschitz–Killing curvatures $\mathcal{L}_i(Z)$ (see [4] and Sect. 6 below for more discussions and exact definitions) and the Betti numbers $b_i(Z)$ (the dimension of the Homology groups, see [27]), both indexed by $i \in \{0, 1, 2\}$. Roughly speaking, the former are computed by integrating certain universal functions of the curvature of Z and of its strata, thus they measure the geometric content of Z, while the latter are purely topological.

To give some very simple example our results cover the excursion sets for the norm of random sections; here, the strata are given by the boundary and the interior. With our tools we will also be able to cover much more complicated frameworks, such as the intersections of excursion sets; other examples include the set of critical points for the norm or the set of critical points of the norm of one random section restricted to the excursion set of another, the set of points where the rank of the covariant derivative of the section is one, and many others, see Sect. 5 for a discussion of some of these examples.

We are now ready to give a more precise statement of our next result.

Theorem 2 *Under suitable regularity conditions*

1. *Almost surely, $Z^W(\sigma_n) \subset \mathbb{S}^2$ is regular (i.e. it is a Whitney stratified subset of $\mathbb{S}^2$) for n big enough. The same holds for $Z^W(\xi_\infty) \subset \mathbb{D}$.*
2. *There exists a discrete limiting probability law $p^W_\infty(Z)$ on the set diffeomorphism classes of Whitney stratified subsets of $\mathbb{D}$ such that:*

$$\exists \lim_{n\to+\infty} \mathbb{P}\{Z^W(\sigma_n) \cap B_{\rho_n} \textit{ is diffeomorphic to } Z\} = p^W_\infty(Z). \tag{2}$$

3. *Whenever Z can be realized as a regular type-W singularity of some smooth function $f \in supp(\xi_\infty)$, we have that $p^W_\infty(Z) > 0$.*
4. *There is convergence in law: $\mathcal{L}_i(Z^W(\sigma_n) \cap B_{\rho_n}) \Rightarrow \mathcal{L}_i(Z^W(\xi_\infty))$ and $b_i(Z^W(\sigma_n) \cap B_{\rho_n}) \Rightarrow b_i(Z^W(\xi_\infty))$.*

A more rigorous statement will be given with Theorem 19.

Remark 3 The regularity conditions that we need are going to be discussed below, see Sect. 8.1; in short, these conditions ensure the regularity (transversality) of the equations that define $Z^W(\sigma_n)$. They can be viewed as a generalization to the spin bundle case of the Morse functions requirements that are needed for the application of Kac–Rice arguments for standard scalar valued fields.

Remark 4 Following the heuristic of the convergence in law of $Z^W(\sigma_n) \cap B_{\rho_n}$, let us consider a bounded functional $\mathcal{F}(Z^W(\xi))$ depending continuously on $\xi \in \mathcal{C}^\infty(\mathbb{D})$ with respect to the $\mathcal{C}^\infty$ topology. Then, the same arguments we will use to prove Theorem 2 allow to deduce the convergence in law of $\mathcal{F}(Z^W(\sigma_n) \cap B_{\rho_n})$. The same properties holds for those functionals that have discontinuities contained in the subset of irregular subsets, since these subsets have probability zero of occurring, in virtue of the continuous mapping theorem for random variables taking values in $\mathcal{C}^\infty(\mathbb{D})$. In particular this is the case of the Lipschitz–Killing curvatures $\mathcal{F} = \mathcal{L}_i$ and of the Betti numbers $\mathcal{F} = b_i$.

Remark 5 Theorem 2 implies for instance that for all sequences of spin random fields having covariance functions which satisfy the same scaling limit, then there exists a *universal* discrete law for the limiting topology of their excursions sets intersected with small balls. For instance, as we shall see below this covers the asymptotic topology for the excursion sets of spin eigenfunctions (spin spherical harmonics) for arbitrary, but fixed, values of s. We refer among others to [54] for a recent important universality result on the limiting topology of excursion sets for random eigenfunctions on generic two-dimensional surfaces; however, we stress the fact that the latter result concerns the global behavior of the whole nodal set, while ours is only local.

Remark 6 The result can actually be stated in a stronger form replacing diffeomorphic with diffeotopic.

Our next result refers to the global study of the expected values of Lipschitz–Killing curvatures and Betti numbers.

Theorem 3 *For all* $i = 0, 1, 2$ *we have:*

1. $\mathbb{E}\mathcal{L}_i(Z^W(\sigma_n)) = \rho_n^i \frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})}\left(\mathbb{E}\mathcal{L}_i\left(Z^W(\xi_\infty) \cap \mathrm{int}(\mathbb{D})\right) + o(1)\right)$;
2. *There are constants* $c_i^W \geq 0$, $C_i^W > 0$ *such that*

$$\frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_\ell})} c_i^W \leq \mathbb{E}\left[b_i(Z^W(\sigma_n))\right] \leq \frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})} C_i^W;$$

3. *If there exists a smooth function* $f \in supp(\xi_\infty) \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{C})$ *such that* $Z^W(f)$ *is regular and it has a connected component* $C \subset \mathrm{int}(\mathbb{D})$*, with* $b_i(C) > 0$*, then* $c_i^W > 0$.

As before a more rigorous statement will be given with Theorem 20.

Remark 7 It should be noted here that $\mathcal{L}_0\left(Z^W(\xi_\infty) \cap \mathrm{int}(\mathbb{D})\right)$ is not equal to the Euler–Poincaré characteristic $\chi(Z^W(\xi_\infty))$, because the former does not take into account the intersection with the boundary of the disk, see Sect. 9.2.1. Also, to interpret correctly the above theorem, recall the standard fact that

$$\mathrm{vol}(B_\rho) = 2\pi(1 - \cos(\rho)) = \pi\rho^2 + O(\rho^4).$$

Remark 8 With the same technique we can include many examples of interest, for instance the expected value of the number of critical values and/or extremes in the regions where the modulus of the spin random section exceeds a certain (fixed) threshold value u. As we mentioned earlier these statistics, as well as the Lipschitz–Killing curvatures mentioned before, have many important applications arising in the framework of Cosmic Microwave Background data analysis, see for instance [16].

2.3 Non-universal asymptotic geometry of spin eigenfunctions

The setting considered in the previous theorem can be applied to a number of different circumstances where the asymptotic behavior of spin random fields is of interest. For physical applications, natural examples are spin eigenfunctions and their averages, also known as spin needlet fields.

In this paper for brevity and definiteness we will consider only the former case, i.e. spin eigenfunctions. In the scalar (spin zero) case, the geometry and topology of random eigenfunctions has been the object of very strong interest in the last decade, see among others [31, 47] for the number of nodal domain, [55] for the variance of nodal lines, [43, 44, 46] for their limiting distributions, [54] for universality results on topology and Betti numbers, [19] for Lipschitz–Killing Curvatures, [17] for universality results on two-dimensional manifolds.

We will consider below three different settings. In particular, we shall consider the limiting behavior of the spin eigenfunctions

$$\sigma_\ell = \sum_m a^\ell_{m,s_\ell} Y^\ell_{m,s_\ell} \tag{3}$$

corresponding to eigenvalues $\lambda_{\ell,s} := -(\ell - s)(\ell + s + 1)$, where $n = \ell \to +\infty$ and

$$|s_\ell| = \ell - r_\ell,$$

with $r_\ell \leq \ell$, under three different regimes:

(a.) (The Berry regime) $\liminf_{\ell\to\infty} r_\ell = +\infty$; this covers the cases where $s_\ell = s$ is fixed (and $\ell \to \infty$) or s_ℓ grows with ℓ even linearly, but $\ell - s_\ell$ diverges. In this case the shrinking rate is $\rho_\ell(s_\ell)$, where

$$\rho_\ell(s) = \frac{1}{\sqrt{(r_\ell + 1)(2\ell - r_\ell)}} \sim \frac{1}{\sqrt{(\ell - s_\ell)(\ell + s_\ell + 1)}} = \frac{1}{\sqrt{\lambda_{\ell,s_\ell}}}. \tag{4}$$

and the associated limit field on the disc is the Berry random field [9], indeed the limit of the circular covariance function is $k_\infty(x) = J_0(x)$, the Bessel function of the first kind of order zero, and $\beta = 0$.

(b.) (The middle regime) $r_\ell = r$ for some fixed $r \in \mathbb{N} \setminus \{0\}$. The real part of the covariance k_∞ is an explicit analytic function $M_r(x)$, see Formula (6) below, computed in Appendix, see Eq. C4. In this case the shrinking rate is given by the same formula as in (5):

$$\rho_\ell(s_\ell) = \frac{1}{\sqrt{(r + 1)(2\ell - r)}} \tag{5}$$

and $\beta = \pm\frac{1}{2(r+1)}$, the sign depending on the asymptotic sign of s_ℓ.

(c.) (The Bargmann–Fock regime) when $|s_\ell| = \ell$, i.e. $r_\ell = 0$; this is the only case where the section is holomorphic; the rate of convergence is

$$\rho_\ell = \frac{1}{\sqrt{2\ell}}.$$

Here the associated limit field is the complex Bargmann–Fock field, with $k_\infty(x) = e^{-\frac{x^2}{4}}$ and $\beta = \pm\frac{1}{2}$; remark that the rate of convergence is indeed the same as (5) in the special case where $r = 0$. Note that it can be obtained by specialyzing the formula in (4), but it is slower in general. It is worth stressing that Bargmann–Fock field is receiving a great attention also in percolation theory, see e.g. [50].

The above discussion illustrates the following theorem.

Theorem 4 *For any $r \in \{0, \dots, \infty\}$, there exists a smooth Gaussian random field $b_r : \mathbb{C} \to \mathbb{C}$ having covariance function*

$$\mathbb{E}\left\{b_r(z)\overline{b_r(w)}\right\} = M_r(|z-w|)\exp\left(\frac{i}{2(r+1)}\Im(z\overline{w})\right) \quad \forall z, w \in \mathbb{C},$$

where $M_\infty(x) = J_0(x)$ and, for all $r \in \mathbb{N}$,

$$M_r(x) := \sum_{j=0}^{r} \frac{r!}{(r+1)^j (r-j)!} \frac{(-1)^j}{j!j!} \left(\frac{x}{2}\right)^{2j} e^{-\frac{x^2}{4(r+1)}}. \tag{6}$$

In particular, b_∞ is the complex Berry's random wave model and b_0 is the complex Bargmann–Fock model. In each of the regimes described above: if $r_\ell = r \in \mathbb{N}$ or if $\liminf_{\ell\to\infty} r_\ell = +\infty$, the scaling limit of the monochromatic field σ_ℓ defined as in (3)*, having spin $|s_\ell| = \ell - r_\ell$, is either the field b_r or $\overline{b_r}$, if $\lim_{\ell\to\infty} \mathrm{sign}(s_\ell) = 1$ or -1, respectively. Assumption* 1 *and hence Theorems* 2 *and* 3 *apply to such sequence σ_ℓ with $k_\infty = M_r$, $\xi_\infty = b_r$ and scaling rate*

$$\rho_\ell = \frac{1}{\sqrt{(2\ell - r_\ell)(r_\ell + 1)}}.$$

Remark 9 Note that all rates between $\rho_\ell(s_\ell) = O(\ell^{-1})$ and $\rho_\ell(s_\ell) = O(\ell^{-1/2})$ can be attained for suitable choices of s_ℓ. We can moreover observe that as $r \to +\infty$

$$\lim_{r\to+\infty} M_r(x) = J_0(x).$$

Hence the middle regime can be heuristically viewed as a form of smooth *interpolation* between complex Berry's Random Waves (obtained for $r \to +\infty$) and the complex Bargmann–Fock model (obtained for $r = 0$).

Remark 10 The monochromatic waves in the case $s = \ell$ are holomorphic sections of the line bundle $\mathcal{T}^{\otimes s}$, indeed they are in correspondence (by identifying $\mathbb{S}^2$ with the Riemann sphere $\mathbb{CP}^1$ and $\mathcal{T}^{\otimes s}$ with $O(2s)$) with polynomials of degree $2s$ in one complex variable, see [52]. Moreover, we note that in this case the sequence of monochromatic spin Gaussian fields with $s = \ell$ corresponds to the sequence of complex Kostlan polynomials of degree $2s$, see also [1, 2, 11, 12].

Remark 11 The limit that we obtain in the so called Bargmann–Fock case (i.e. the regime $s = \ell$) can be written explicitely as

$$\xi_\infty(z) = \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{n!}}\gamma_n \left(\frac{z}{\sqrt{2}}\right)^n e^{-\frac{|z|^2}{4}},$$

where the γ_n are i.i.d. standard complex Gaussians. It should be noted that neglecting the factor $e^{-\frac{|z|^2}{4}}$ this model would correspond to the well known Gaussian entire process, considered for instance in [51]. This model is not stationary, indeed the variance grows with $|z|$ as $e^{\frac{|z|^2}{4}}$; heuristically, this can be explained by noting that the stereographic projection, which is holomorphic, over the tangent plane stretches the length of tangent vectors more and more as they get further and further away from the origin of the coordinates. For our model, this would correspond to the variance of the scaling limit getting larger and larger as z grows. Indeed, the factor $e^{-\frac{|z|^2}{4}}$ is a consequence of the fact that we use a trivialization of the bundle and of the sphere that comes from the exponential map instead of the stereographic projection: the metric on the fiber differs by a factor that exactly compensates. Despite the fact that the variance is constant the limit is not stationary, in fact it has covariance

$$K_{\xi_\infty}(z_1, z_2) = \mathbb{E}\left\{\xi_\infty(z_1)\overline{\xi_\infty(z_2)}\right\} = e^{\frac{|z_1-z_2|^2}{4}} \exp\left(\frac{i}{2}\Im(z_1\overline{z_2})\right).$$

On the basis of these results it is possible to give more explicit formulae for the expectations of Lipschitz–Killing curvatures, for instance see Theorem 5 below. In particular, in the Berry regime a., we can provide the following result on the Lipschitz–Killing curvatures for the excursion sets of spin random sections.

Theorem 5 *Assume that* $\mathbb{E}\|\sigma_\ell(p)\|^2 = 1$ *and that* σ_ℓ *is as above in the Berry regime. Then for any* $u > 0$*, we have the following asymptotic identities, with* $\rho_\ell = \frac{1}{\sqrt{(\ell-s_\ell)(\ell+s_\ell+1)}}$*:*

i. $\mathbb{E}\mathrm{vol}_2(\{|\sigma_\ell| \geq u\}) = 4\pi e^{-\frac{u^2}{2}} + o(1)$.
ii. $\mathbb{E}\mathrm{vol}_1(\{|\sigma_\ell| = u\}) = \frac{1}{\rho_\ell} \cdot (2\pi^{\frac{3}{2}} u e^{-\frac{u^2}{2}} + o(1))$.
iii. $\mathbb{E}\chi(\{|\sigma_\ell| \geq u\}) = \frac{1}{\rho_\ell^2} \cdot ((u^2 - 1)e^{-\frac{u^2}{2}} + o(1))$.
iv. $\mathbb{E}\#\{\sigma_\ell = 0\} = \frac{1}{\rho_\ell^2}(1 + o(1))$.

v. *There are positive constants* $c_i^W(u), C_i^W(u) > 0$ *such that for* ℓ *big enough, we have*

$$c_i^W(u)\frac{4}{\rho_\ell^2} \le \mathbb{E}b_i\{|\sigma_\ell| \ge u\} \le C_i^W(u)\frac{4}{\rho_\ell^2}.$$

Remark 12 Notice that $\mathbb{E}\chi(\{|\sigma_0| \ge u\})$ is not continuous at $u = 0$, in that

$$\lim_{u\to 0^+} \chi(\{|\sigma_0| \ge u\}) = 2 - \#\{\sigma_0 = 0\}, \tag{7}$$

while $\chi(\mathbb{S}^2) = 2$. This should not surprise, in that for small values of u, the excursion set $\{|\sigma_0| \ge u\}$ is just the complement of a small neighborhood of the zero set, thus (7) holds almost surely. On the other hand, it is clear that the first two identities $i.$ and $ii.$ are still true for $u = 0$.

Remark 13 Results analogous to Theorem 5 can be established for the two other regimes b. and c. as a consequence of Theorem 3, of course replacing the scaling factors ρ_ℓ appropriately. However, it should be noted that the computation of the multiplicative constants $\mathbb{E}\mathcal{L}_i(Z(\xi_\infty) \cap \text{int}(\mathbb{D}))$ is in these cases more challenging: for instance, in case c. (the complex Bargmann–Fock) the real and imaginary parts of the limit field are not independent and hence the Gaussian kinematic formula [4] does not hold. In any case, we stress that what we are omitting here is just a (tedious) computation concerning only the limit field.

Remark 14 (On the law of large numbers) Using similar techniques as in the proof of Theorem 3, together with the $\mathcal{C}^\infty$ convergence of the covariance kernel of any pair of rescaled fields it is actually possible to prove a law of large numbers result for the Lipschitz–Killing curvatures, i.e.

$$\frac{\mathcal{L}_i(Z^W(\sigma_\ell))}{\mathbb{E}\{\mathcal{L}_i(Z^W(\sigma_\ell))\}} \xrightarrow[\ell\to\infty]{L^2} 1$$

We plan to address these issues and related ones about central limit theorems in a forthcoming paper.

Remark 15 (Non-universal asymptotic topology of spin eigenfunctions) As for the Lipschitz–Killing curvatures the scaling factors appearing in the asymptotic behavior of the Betti numbers are different in each of the three considered regimes.

In the Berry case a. we prove below that $c_0^W > 0$; the proof takes into account the (non)monotonicity property of the Bessel function J_0, see Sect. 10.2, and Alexandrov duality for compact subsets of the sphere. More details are given below. For b_1 one can modify the scaling sequence ρ_n by a constant factor c strictly bigger than the first minimum point of J_0 and then run an analogous argument together with Alexander's duality to prove that $c_1^W > 0$.

The behavior of Betti numbers in cases b. and c. is discussed in Remark 41. The upper bound for the expected values of the Betti numbers b_0 and b_1 takes the same form (the constant C_i^W is finite) as for the Berry case a.; moreover, the lower bound holds with a strictly positive constant c_0^W for b_0 in both cases. On the other hand, it is not possible in those environments to prove that the constant c_1^W appearing in the lower bound for b_1 is strictly positive. In fact condition (3) in Theorem 3 is not satisfied by the complex Bargmann–Fock field, due to the maximum principle for holomorphic functions. However, this does not imply that the lower bound fails, although we conjecture that it does.

It may be further noted that the expected number of connected components for the excursion sets is $O(\ell^2)$ when s is fixed or bounded away from ℓ, it is $O(\ell(\ell - |s_\ell|))$, if $s_\ell < \ell$ can grow as quickly as ℓ, and finally it is $O(\ell)$ in the holomorphic case $s = \ell$; the same asymptotics hold for the first Betti number b_1.

2.4 Technical novelties

The first significant technical novelty of this paper lies in our objects of interest: spin functions, which are not just scalar functions, but smooth random sections of complex line bundles. Similar objects have been considered in a predominantly algebraic context (see for instance [3, 15, 22, 25, 26, 32]) with a few exceptions (see [38, 41, 49]).

The study of type-W singularities of random fields has previously been addressed only in [36] (see also [13, 35, 37]) as an extension of the results of Gayet and Welschinger [25, 26] on the zero sets. Theorem 2 is based on analogous techniques (Thom Isotopy Lemma and the results of [34]). As it was for the former paper, there is a notable novelty compared to past results, even on the zero sets: the existence of the limit (2), rather than just an inequality for the $\liminf$.

From a technical perspective, the most important novelty is presented in Theorem 3, which postulates a convergence of the expectation in addition to the convergence in distribution of Theorem 2. This is the very first instance in the literature that studies the Lipschitz–Killing curvatures of singular sets of random fields, in such vast generality. Our proof hinges on two key ideas: first, reducing the computation of the Lipschitz–Killing curvatures of a stratified set to a counting measure via stratified Morse theory (building upon and improving the ideas introduced in [36]); and second, computing the expectation of such counting measure using the generalized Kac–Rice formula recently developed in [53], and analyzing its asymptotic behavior.

Remark 16 We emphasize that this method would not be feasible with only the standard version of Kac–Rice formula, for which a standard reference is the book [4]. The reason is the following. A type-W singularity—even in the simple case of the excursion set—by definition, can be expressed as the *preimage* of a certain submanifold $W \subset \mathbb{R}^\ell$ via a Gaussian random field $f : \mathbb{D} \to \mathbb{R}^\ell$, and our study requires to compute quantities of the form $\mathbb{E} \int_{f^{-1}(W)} \alpha(f, x) dx$. Now, the standard Kac–Rice formula deals only with the case $W = \{0\}$. The paper [53], on which we rely, gives a formula for a general submanifold W (which entails a non-trivial modification of the integrand), discusses the subtleties involved in such a generalization, and it is taylored for dealing with the case of type–W singularities of Gaussian fields.

Finally, the scaling limit that we obtain depends on the regime of s and ℓ and it is not universal and not always stationary, see Sect. 2.3. This introduces a completely new family of Gaussian fields on the plane, with covariance function M_r defined as in Theorem 4, that includes as limit cases ($r = 0$ and $r = \infty$) two of the most studied Gaussian ensembles, see Remark 9. Of the intermediate regime $r \in \mathbb{N} \setminus \{0\}$ we know very little, for instance we have not worked out an explicit Karhunen-Loève representation for it. We believe that its investigation represents a fruitful, unexplored territory.

2.5 Plan of the paper

Sections 3 and 4 introduce our framework in terms of the formal construction of spin line bundles and the definition of isotropic spin random fields on the sphere; these Sections build upon some previous references, including in particular [14, 24, 39]. Sections 5 and 6 introduce the geometrical tools that we are going to explore, in particular jet bundles, type-W singularities (see [36]) and their description as Whitney stratified subsets of the sphere, Lipschitz–Killing curvatures in their integral form and their alternative expression in terms of critical points of stratified Morse functions. Sections 7 and 8 give our asymptotic framework and main results, Theorems 2 and 3, whose proofs are collected also in Sect. 9. Finally, Sect. 10 specializes our results to the monochromatic case, whereas some technical lemmas are collected in the Appendix.

3 Spin line bundles

In this Section, coherently with Newman and Penrose's theory (see Sect. 1), we introduce the notion of spin line bundles on the sphere giving both the intrinsic definition and the description in terms of an atlas, with great attention to so-called spin sections.

3.1 Intrinsic definition

The 3-dimensional special group of rotations $SO(3)$ acts transitively on the two-dimensional unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with an action that we denote by $g \mapsto gp$, $g \in SO(3)$, $p \in \mathbb{S}^2$. Let us fix once for all a point $p_0 \in \mathbb{S}^2$, and define K to be the isotropy group of p_0, i.e. the subgroup of elements $g \in SO(3)$ such that $gp_0 = p_0$, then $K \cong U(1)$ the circle group, and $\mathbb{S}^2 \cong SO(3)/K$. Let us denote by χ_s, $s \in \mathbb{Z}$, the family of characters of K (χ_0 is the trivial representation); for every $s \in \mathbb{Z}$, the group K acts on $SO(3) \times \mathbb{C}$ as follows: for $(g, z) \in SO(3) \times \mathbb{C}$,

$$k \mapsto k(g, z) := (gk, \chi_s(k^{-1})z).$$

We denote by $SO(3) \times_s \mathbb{C}$ the space of orbits $\{\theta(g, z), (g, z) \in SO(3) \times \mathbb{C}\}$, where $\theta(g, z) = \{k(g, z), k \in K\}$, and consider the (projection) map

$$\begin{aligned} \pi_s : SO(3) \times_s \mathbb{C} &\to \mathbb{S}^2 \\ \theta(g, z) &\mapsto gK. \end{aligned}$$

Let us set $\xi_s := (SO(3) \times_s \mathbb{C}, \pi_s, \mathbb{S}^2)$, then ξ_s is a complex line bundle, indeed the fiber over p is $\pi_s^{-1}(\{p\}) \cong \mathbb{C}$ for every $p \in \mathbb{S}^2$. We call $SO(3) \times_s \mathbb{C}$ (resp. $\mathbb{S}^2$) the total (resp. base) space of ξ_s. Plainly, for $s = 0$ we obtain the trivial bundle, in particular $SO(3) \times_0 \mathbb{C} \cong \mathbb{S}^2 \times \mathbb{C}$.

Definition 1 The *spin s line bundle on the sphere* is the triplet $\xi_s = (SO(3) \times_s \mathbb{C}, \pi_s, \mathbb{S}^2)$.

Let us now recall the notion of section of ξ_s: it is a map $\sigma : \mathbb{S}^2 \to SO(3) \times_s \mathbb{C}$ that associates to each $p \in \mathbb{S}^2$ one element of its fiber $\pi_s^{-1}(\{p\})$, i.e. some $\theta(g, z) \in SO(3) \times_s \mathbb{C}$ such that $gK = p$. We call such a σ a *spin s section*. Plainly, spin 0 sections are identified with complex valued functions on the sphere.

Remark 17 There is a one to one correspondence between spin s sections σ and complex valued functions f on $SO(3)$ such that for every $g \in SO(3)$ and every $k \in K$

$$f(gk) = \chi_s(k^{-1}) f(g). \tag{8}$$

(We call $f : SO(3) \to \mathbb{C}$ satisfying (8) a *function of right spin* $-s$.) Indeed, given f satisfying (8), the corresponding section $\sigma = \sigma^f$ is defined as follows: for $\mathbb{S}^2 \ni p = g_p K$,

$$\sigma(p) := \theta(g_p, f(g_p)).$$

(Note that this definition does not depend on the coset representative.) On the other hand, consider a section σ of ξ_s, then $\sigma(p) = \theta(g_p, z_p)$ where $p = g_p K$. Define the corresponding function $f = f^\sigma$ of right spin $-s$ (called the *pullback function of* σ in [14]) as follows:

$$f(g_p) := z_p, \qquad f(g_p k) := \chi_s(k^{-1}) z_p \ \text{ for } k \in K.$$

Plainly, functions of type 0 are constant on left cosets of K in $SO(3)$, hence they are identified with complex valued functions on $\mathbb{S}^2$.

We will always work with sections that are at least *continuous*: on the total space and the base space of ξ_s we consider the respective Borel σ-fields, in particular this ensures the bundle projection π_s to be continuous itself. Hence a spin s section σ is continuous if and only if its pullback function f^σ is a continuous function of right spin $-s$.

3.1.1 Tensor representation

In this paper, we will extensively use the following tensor representation, an alternative approach to the theory of spin line bundles than the one leading to Definition 1.

Remark 18 There are two choices for the isomorphism $\chi_1 : K \to U(1)$ depending on the orientation of K. To make such choice is equivalent to choose an orientation of the tangent space at $p_0 \in \mathbb{S}^2$ and thus an orientation of $\mathbb{S}^2$: this is due to the fact that K can

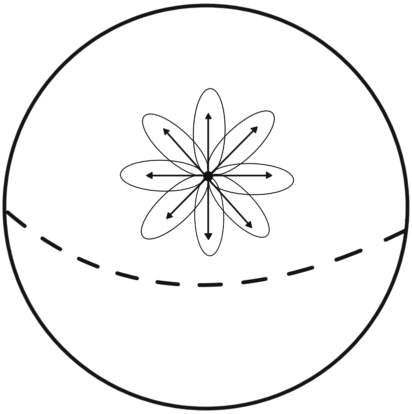

Fig. 1 We can represent an element belonging to $\pi_s^{-1}(\{p\})$ as $\tau = (z_1 v) \otimes \cdots \otimes (z_s v)$, where $z_1, \ldots, z_s$ are the solutions of $z^s = 1$. This can be pictured as flower centered at p with s petals, drawed on the surface of the sphere

be embedded as a small circle around p_0 by drawing the orbit Kp of a point $p \in \mathbb{S}^2$ close to p_0. Then, the tangent bundle on the sphere, denoted by $\mathcal{T} := \left(T\mathbb{S}^2, \pi, \mathbb{S}^2\right)$ and equipped with the rotation of angle $\frac{\pi}{2}$ coherent with the given orientation, is isomorphic to ξ_1 as a complex line bundle. Observing that $\chi_s = (\chi_1)^s$, it follows that for all $s \in \mathbb{N}$, we have

$$\xi_s \cong \mathcal{T}^{\otimes s}, \qquad \xi_{-s} \cong \left(\mathcal{T}^*\right)^{\otimes s},$$

where $\mathcal{T}^*$ is the so-called cotangent bundle on the sphere, equipped with the dual almost complex structure (recall that ξ_0 is the trivial bundle) and $\otimes$ denotes the complex tensor product. In other words, ξ_s is *the* complex line bundle with Euler characteristic $2s$. This holds for whatever choice of orientation of $\mathbb{S}^2$.

For $s \in \mathbb{N}$, bearing in mind (18), let $p \in \mathbb{S}^2$ and $v \in T_p\mathbb{S}^2 \setminus \{0\}$, where $T_p\mathbb{S}^2$ denotes the tangent space at point p, then for the fiber over p we have

$$\pi_s^{-1}(\{p\}) \cong \left\{ \sum_i v_1^i \otimes \cdots \otimes v_s^i : v_j^i \in T_p\mathbb{S}^2 \right\} = \{z \cdot v^{\otimes s} : z \in \mathbb{C}\},$$

where as usual

$$v^{\otimes s} := \underbrace{v \otimes \cdots \otimes v}_{s \text{ times}}; \tag{9}$$

see also Fig. 1 for an alternative representation. When v changes, say $v' = wv$, then the vector $v^{\otimes s}$ changes accordingly to

$$(v')^{\otimes s} = w^s v^{\otimes s}.$$

Remark 19 Note that the coordinates of $\tau = zv^{\otimes s} = z'(v')^{\otimes s}$, identified with an element of the fiber over p via (3.1.1), have spin weight $-s$, i.e.

$$z' = w^{-s} z,$$

indeed "the coordinates of vectors are covectors, hence they belong to the dual".

An analogous representation holds for $s < 0$, it suffices to replace $T_p(\mathbb{S}^2)$ with the cotangent space $T_p^*(\mathbb{S}^2)$ at point p, hence for $v \in T_p^*(\mathbb{S}^2) \setminus \{0\}$

$$\pi_s^{-1}(p) \cong \{z \cdot v^{\otimes -s} : z \in \mathbb{C}\}.$$

It is worth stressing that (18), in light of (3.1.1) and the discussion thereafter, gives the most natural definition of spin s line bundle on the sphere according to Newman and Penrose's theory, see Section 1. From now on, the spin s line bundle ξ_s (Definition 1) will be tacitly identified with $\mathcal{T}^{\otimes s}$ for $s > 0$ (resp. with $(\mathcal{T}^*)^{\otimes -s}$ for $s < 0$), and with the trivial bundle for $s = 0$ – as explained in Remark 18.

Let $s > 0$ and σ be a section of ξ_s. Then from (3.1.1), obviously, for any point $p \in \mathbb{S}^2$ and $v \in T_p\mathbb{S}^2 = p^\perp$, we have

$$\sigma(p) = z v^{\otimes s},$$

for some $z = z_\sigma(p, v) \in \mathbb{C}$. A convenient way to understand this $z_\sigma(p, v)$ is to observe that for any such p, v there exists a unique positive "rotation" $g \in SO(3)$ such that $ge_3 = p$ and $ge_2 = v$. Here e_1, e_2, e_3 are the axes of the coordinate system (we consider the standard, right-handed basis for $\mathbb{R}^3$). It follows that the section $\sigma : \mathbb{S}^2 \to SO(3) \times_s \mathbb{C}$ is uniquely determined by a function $f = f^\sigma : SO(3) \to \mathbb{C}$ such that

$$\sigma(ge_3) = f(g)\,(ge_2)^{\otimes s}\,, \tag{10}$$

where here ge_2 must be intended as an element of $T_{ge_3}\mathbb{S}^2$, cf. Remark 17. Analogous considerations hold for $s < 0$ replacing the tangent space with the cotangent space.

3.2 Hermitian metric

The complex line bundles $\mathcal{T}^{\otimes s}$ are endowed with a natural hermitian metric, defined as follows, via the induced norm, see also [52].

Definition 2 Let $\|\cdot\| : \mathcal{T}^{\otimes s} \to [0, +\infty)$, such that if $v \in T_p\mathbb{S}^2$ has length $\|v\| = 1$, then $\|v^{\otimes s}\| = \frac{1}{s}$, see equation (9).

Remark 20 This is the only choice for which all the maps below are Riemannian coverings, see [52].

$$\begin{aligned} SO(3) \overset{\cong}{\to} S(\mathcal{T}^{\otimes 1}) &\to \frac{1}{s} S(\mathcal{T}^{\otimes s}) \to \frac{1}{ks} S(\mathcal{T}^{\otimes ks}) \\ \begin{pmatrix} u & v & p \end{pmatrix} \mapsto (v, p) &\mapsto (v^{\otimes s}, p) \mapsto (v^{\otimes sk}, p). \end{aligned}$$

Here, $S(\mathcal{T}^{\otimes s}) = \{\|\cdot\| = 1\}$ denotes the unit sphere bundle of $\mathcal{T}^{\otimes s}$ with respect to the chosen metric.

3.3 Trivialization via Euler's angles

Euler's angles are three angles the we denote by φ, θ, ψ describing the orientation of a rigid body with respect to a fixed coordinate system. We use the same convention as in [42, Section 3.2]; let $g \in SO(3)$ be any rotation, [42, Proposition 3.1] ensures that g can be realized as the sequential composition of three elementary rotations, i.e., rotations around the axes e_1, e_2, e_3 of the coordinate system, as follows.

Proposition 6 (Proposition 3.1 in [42]) *Each rotation $g \in SO(3)$ can be realized sequentially as*

$$g = R(\varphi, \theta, \psi) = R_3(\varphi)R_2(\theta)R_3(\psi), \qquad \varphi \in [0, 2\pi),\ \theta \in [0, \pi],\ \psi \in [0, 2\pi),$$

where for $\alpha \in \mathbb{R}$

$$R_3(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2(\alpha) := \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}.$$

Representation (6) is unique whenever $\theta \neq 0, \pi$. If $\theta = 0$, then only the sum $\varphi + \psi$ is uniquely defined. If $\theta = \pi$, then only the difference $\varphi - \psi$ is uniquely defined.

The matrix $g(\varphi, \theta, \psi) \in SO(3)$ can be interpreted as an element of the positive orthonormal frame bundle of $\mathbb{S}^2$ as follows. Let $T_p\mathbb{S}^2 = p^\perp$ be endowed with the standard complex structure: multiplication by i is the anticlockwise rotation by angle $\frac{1}{2}\pi$. Let $p \in \mathbb{S}^2$ have (standard) polar coordinates (θ, φ) and let us define the orthonormal basis of $T_p\mathbb{S}^2$, given by the downward meridian and anticlockwise parallel directions:

$$\hat{\theta}(p) = \frac{\partial}{\partial\theta}(p) = \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}, \qquad \hat{\varphi}(p) = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}(p) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

Then

$$\begin{aligned} g(\varphi, \theta, \psi) &= (e^{i\psi}\hat{\theta}(p), e^{i\psi}\hat{\varphi}(p), p) = (\cos\psi\hat{\theta} + \sin\psi\hat{\varphi}, -\sin\psi\hat{\theta} + \cos\psi\hat{\varphi}, p) \\ &= (\hat{\theta}(p), \hat{\varphi}(p), p)\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{11}$$

Remark 21 Recall Remark 18. In this paper we consider the sphere $\mathbb{S}^2$ to be oriented in the usual way, with respect to the outer normal direction, so that we define, for every $\psi \in \mathbb{R}$,

$$\chi_s \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} := e^{is\psi},$$

where χ_s still denotes the s-th linear character of K, the isotropy group of p_0. Notice that when the sphere $\mathbb{S}^2$ is identified with the Riemann sphere $\mathbb{CP}^1$ by means of the stereographic projection from the north pole, the orientation induced by the complex structure is the opposite.

Remark 22 In [42, p. 287] the transition functions, see e.g. [29, Definition 2.3], for the bundle ξ_s are

$$f_{g_2}(x) = e^{is\psi_{g_2g_1}} f_{g_1}(x), \tag{12}$$

where $\psi_{g_1g_2}$ is the angle between $\frac{\partial}{\partial\varphi_{g_1}}$ and $\frac{\partial}{\partial\varphi_{g_2}}$. In other words,

$$\frac{\partial}{\partial\varphi_{g_1}} = e^{i\psi_{g_2g_1}} \frac{\partial}{\partial\varphi_{g_2}}.$$

Therefore the rule (12) is equivalent to the transition rule for $\mathcal{T}^{\otimes s}$:

$$f_{g_2}(x)\left(\frac{\partial}{\partial\varphi_{g_2}}\right)^{\otimes s} = f_{g_1}(x)\left(\frac{\partial}{\partial\varphi_{g_1}}\right)^{\otimes s}.$$

It is easy to see that a function $f : SO(3) \to \mathbb{C}$ is associated with a section σ of ξ_s if and only if

$$f(gR_3(\psi)) = f(g)e^{-is\psi}, \tag{13}$$

for any $\psi \in \mathbb{R}$. Indeed

$$\begin{aligned} f^\sigma(g)\,(ge_2)^{\otimes s} &= \sigma(ge_3) = \sigma(gR_3(\psi)e_3) = f^\sigma(gR_3(\psi))\,(gR_3(\psi)e_2)^{\otimes s} \\ &= f^\sigma(gR_3(\psi))\,(-\sin\psi\, ge_1 + \cos\psi\, ge_2)^{\otimes s} \\ &= f^\sigma(gR_3(\psi))\left(e^{i\psi} ge_2\right)^{\otimes s} \\ &= f^\sigma(gR_3(\psi))e^{is\psi}\,(ge_2)^{\otimes s}. \end{aligned}$$

Theorem 7 *Sections of ξ_s are in bijections with functions $f : SO(3) \to \mathbb{C}$ that satisfy the rule (13), via the identity (10). We say that $f = f^\sigma$ is the* pullback *of σ (see [14]) and that f has* right spin $-s$.

Remark 23 This change of sign in the spin weight is explained by the fact that $f^\sigma(g)$ is actually a function that expresses the *coordinates* (see Remark 19) of σ in the trivialization of the bundle $\mathcal{T}^{\otimes s}$ determined by g.

Consider now the Euler angles $\theta, \varphi, \psi \in (0, \pi) \times (0, 2\pi) \times (0, 2\pi)$ on $SO(3) \setminus \{\pm R_3(t) : t \in \mathbb{R}\}$ as coordinates on the frame bundle of $T\mathbb{S}^2$ (which is indeed isomorphic to $SO(3)$). In particular, we see that for any fixed ψ, the angles θ, φ give trivializations of ξ_s over the set $\mathbb{S}^2 \setminus \{\pm e_3\}$ as follows:

$$\begin{aligned}
&(0,\pi)\times \mathbb{R}_{/2\pi\mathbb{Z}} \times \mathbb{C} \cong SO(3)\times_s \mathbb{C}|_{\mathbb{S}^2\setminus\{e_3,-e_3\}} = \bigsqcup_p \{p\}\times \pi_s^{-1}(\{p\}) \\
&\theta,\varphi,z \mapsto \left((R(\varphi,\theta,\psi)e_3)\,, z\cdot (R(\varphi,\theta,\psi)e_2)\otimes\cdots\otimes (R(\varphi,\theta,\psi)e_2)\right) \\
&= \left(\begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}, z\cdot (e^{i\psi}\hat{\varphi})\otimes\cdots\otimes(e^{i\psi}\hat{\varphi})\right) \\
&= \left(p, z\cdot \left(e^{i\psi}\hat{\varphi}\right)^{\otimes s}\right)
\end{aligned} \tag{14}$$

where $g = R(\varphi,\theta,\psi) = (e^{i\psi}\hat{\theta}(p), e^{i\psi}\hat{\varphi}(p), p)$ is interpreted as in (11). Thus, the transformation rule for z (i.e. for local sections of ξ_s), when we pass from $\psi = \psi_1$ to $\psi = \psi_2$ is

$$z_1 e^{is\psi_1} = z_2 e^{is\psi_2}.$$

The local representation of a section $\sigma\colon \mathbb{S}^2 \to SO(3)\times_s \mathbb{C}$ with respect to the trivialization given by ψ is then a function $f_\psi(\theta,\varphi)$ defined by the expression

$$\begin{aligned}
&\sigma(p) = f_\psi(\theta,\varphi)\left(e^{i\psi}\hat{\varphi}\right)^{\otimes s} \\
\text{i.e.}\quad & f_\psi(\theta,\varphi) = F_\sigma\left(R(\varphi,\theta,\psi)\right) = F_\sigma\left(R(\varphi,\theta,0)\right)e^{-is\psi} = f_0(\theta,\varphi)e^{-is\psi}.
\end{aligned}$$

Remark 24 The same reasoning applies for ψ a function $\psi = \psi(\theta,\varphi)$.

Proposition 8 *A section σ of ξ_s can be defined (almost everywhere) by its local expression, i.e. by specifying the function*

$$f_0\colon (0,\pi)\times(0,2\pi)\to\mathbb{C}$$

In this case, the section is continuous if and only if: f_0 is continuous and

$$\sigma(e_3) = \left(\lim_{\theta\to 0^+} f_0(\theta,\varphi)e^{is\varphi}\right)(e_2)^{\otimes s}, \quad \sigma(-e_3) = \left(\lim_{\theta\to\pi^-} f_0(\theta,\varphi)e^{is\varphi}\right)(e_2)^{\otimes s},$$

uniformly with respect to φ.

Proof See [24, Theorem 3.1]. □

3.4 Spectral representation of spin sections

By the Peter-Weyl theorem (see [42, p. 288]), any function $f \in L^2(SO(3))$ with right spin $-s$, i.e. corresponding to a section of ξ_s in the sense of Theorem 7, can be represented by a series (convergent in $L^2(SO(3))$) of the form

$$f(g) = \sum_{\ell \geq |s|} \sum_m b^\ell_{ms} D^\ell_{ms}(g),$$

where $D^\ell_{ms}(g(\varphi, \theta, \psi))$ is the (m, s) entry of the ℓ^{th} Wigner D matrix, see [42], and

$$b^\ell_{ms} := \int_{SO(3)} f(g) \overline{D}^\ell_{ms}(g)\, dg$$

are the Fourier coefficients of f. Therefore the section σ associated to f is determined (on $\mathbb{S}^2 \smallsetminus \{\pm e_3\}$) by the series

$$f_0(\theta, \varphi) = \sum_{\ell \geq |s|} \sum_m b^\ell_{ms} e^{-im\varphi} d^\ell_{ms}(\theta).$$

We recall that the real part of the ℓ^{th} Wigner D-matrix is called Wigner d-matrices whose entries are denoted as $d^\ell_{m,s}$. More precisely, the two are related via the identity $D^\ell_{ms}(g(\varphi, \theta, \psi)) = e^{-im\varphi} d^\ell_{ms}(\theta) e^{-is\psi}$. See [42, Section 3.3.2] for further details.

Definition 3 The m-th *spin s spherical harmonic* of degree ℓ $\sigma_{\ell;ms} : \mathbb{S}^2 \to SO(3) \times_s \mathbb{C}$ (see [42, p. 289]) is the section with

$$f_0(\theta, \varphi) = \sigma_{\ell;ms}(\theta, \varphi) = \sqrt{\frac{2\ell + 1}{4\pi}} \overline{D}^\ell_{m,-s}(R(\varphi, \theta, 0)) = \sqrt{\frac{2\ell + 1}{4\pi}} e^{im\varphi} d^\ell_{m,-s}(\theta).$$

Its pullback function $f_{\ell;ms} : SO(3) \to \mathbb{C}$, with right spin $-s$, is

$$f_{\ell;ms}(g) = \sqrt{\frac{2\ell + 1}{4\pi}} \overline{D}^\ell_{m,-s}(g).$$

Note that $\{\sigma_{\ell;ms}, m = -\ell, \ldots, \ell, \ell \geq |s|\}$, the set of spin s spherical harmonics of degree $\ell \geq |s|$, is an orthonormal basis for the space of square integrable spin s sections.

4 Spin random fields

In this Section we define and study basic properties of so-called spin random fields, which are random sections of the spin line bundles on the sphere introduced in Sect. 3, focusing on their spectral representation. Let us fix once for all a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

4.1 Random sections

Definition 4 A *spin s random field U* is a random section of the spin s line bundle ξ_s, i.e. a measurable map

$$U : \Omega \times \mathbb{S}^2 \to SO(3) \times_s \mathbb{C}$$

such that, for every $\omega \in \Omega$, $U(\omega, \cdot)$ is a section of ξ_s, i.e. $\pi_s(U(\omega, \cdot)) = \mathrm{id}_{\mathbb{S}^2}(\cdot)$ for every $\omega \in \Omega$, where $\mathrm{id}_{\mathbb{S}^2}$ denotes the identity function on the sphere.

To be more precise, in (5) we consider the σ-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{S}^2)$ on $\Omega \times \mathbb{S}^2$. Plainly, in light of Remark 17, there is a one to one correspondence between spin s random fields U and complex-valued random fields X on $SO(3)$ of type s, that is, measurable maps $X : \Omega \times SO(3) \to \mathbb{C}$ whose sample paths are functions of right spin $-s$, i.e. for every $\omega \in \Omega$, every $g \in SO(3)$ and every $k \in K$,

$$X(\omega, gk) = \chi_s(k^{-1})X(\omega, g).$$

For the sake of brevity we omit the dependence on ω from now on. We call X the *pullback random field* of U, as in [14], where this "pullback approach" for spin random fields was first developed.

The Fourier coefficients b^ℓ_{ms} of X as defined in (3.4) are random variables and, if $X \in L^2(SO(3))$ (a.s.), then Peter-Weyl Theorem applies pathwise (up to a negligible set of trajectories), so that (a.s.) in $L^2(SO(3))$ we have the following spectral representation

$$X(g) = \sum_{\ell \geq |s|} \sum_m b^\ell_{ms} D^\ell_{ms}(g).$$

4.2 Isotropy and gaussianity

Assume that U is a.s. square integrable, then the inner product

$$U(h) := \int_{\mathbb{S}^2} \langle U(p), h(p) \rangle_{\pi_s^{-1}(p)} \, dp$$

is well defined for every square integrable spin s section h. We say that U is Gaussian if the vector $(U(h_1), \ldots, U(h_n))$ is Gaussian for any finite number of square integrable spin s sections $h_1, \ldots, h_n$. Hence U is Gaussian if and only if X is Gaussian, seen as a random variable taking values in $L^2(SO(3))$.

Of course, if the spin s random field $U : \mathbb{S}^2 \to SO(3) \times_s \mathbb{C}$ is a.s. continuous (as we shall always assume), then it is Gaussian if and only if its pullback random field $X : SO(3) \to \mathbb{C}$ is Gaussian, namely if and only if the random vector $(X(g_1), \ldots, X(g_n)) \in \mathbb{C}^n$ is complex Gaussian for any finite number of points $g_1, \ldots, g_n \in SO(3)$.

We will restrict to the case of circularly symmetric complex Gaussian random vectors: $\gamma \sim N_{\mathbb{C}}(0, K)$ that is:

$$\mathbb{E}[\gamma] = 0; \quad \mathbb{E}[\gamma\overline{\gamma}^T] = K, \quad \text{and} \quad \mathbb{E}[\gamma\gamma^T] = 0.$$

Remark 25 Given a random field $f: A \to B$, let us denote by $[f]$ its class up to equivalence of fields, namely $[f]$ is the probability measure induced on the space B^A of functions from A to B, endowed with the product $\sigma-$algebra. Notice that the correspondence $[\sigma] \mapsto [X]$ is a bijection, since there is a $\mathbb{C}-$linear isomorphism of vector spaces:

$$\{F: SO(3) \to \mathbb{C} | F \text{ has right spin } = -s\} = \{\sigma: \mathbb{S}^2 \to \mathcal{T}^{\otimes s} | \sigma \text{ is a section}\}, \quad (15)$$

and this induces a bijection on the space of probability measures on those spaces. Moreover, by linearity, this bijection sends Gaussian measures to Gaussian measures. Even more, one can easily see that the bijection (15), when restricted to $\mathcal{C}^r$ functions/sections, is a homeomorphism with respect to the $\mathcal{C}^r$ topologies, for all $r \in \mathbb{N} \cup \{+\infty\}$. In other words, it is completely equivalent to define a ($\mathcal{C}^r$ and/or Gaussian) random section of $\mathcal{T}^{\otimes s}$ or a ($\mathcal{C}^r$ and/or Gaussian) random function $X: SO(3) \to \mathbb{C}$ with right spin$= -s$.

Definition 5 We say that σ is *isotropic* if and only if X is isotropic on the left:

$$X(g\cdot) \sim X(\cdot), \quad \text{for any } g \in \text{ SO } (3).$$

In other words, the random section σ is isotropic if $g_*(\sigma(\cdot)) \sim \sigma(g\cdot)$, for every $g \in SO(3)$, where

$$g_*: \mathcal{T}_p^{\otimes s} \to \mathcal{T}_{gp}^{\otimes s}, \quad g_*(zv^{\otimes s}) = z(gv)^{\otimes s}$$

In terms of the covariance function of X, we have the following characterization.

Remark 26 Recall that $X(gR_3(\psi)) = X(g)e^{-is\psi}$ for all ψ.

Proposition 9 *σ is isotropic if and only if there exists a function $\Gamma: SO(3) \to \mathbb{C}$ such that*

$$Cov\,(X(g), X(h)) = \mathbb{E}\{X(g)\overline{X(h)}\} = \Gamma(g^{-1}h).$$

Moreover, Γ has right spin$= s$ and left spin$= s$:

$$\Gamma\,(R_3(\varphi)hR_3(\psi)) = e^{is\varphi}\Gamma(h)e^{is\psi},$$

and is "hermitian",

$$\Gamma(g^{-1}) = \overline{\Gamma(g)}.$$

Proof Define $\Gamma(h) = \mathbb{E}\{X(\mathbb{1})\overline{X(h)}\}$. The rest is straighfrward. □

It follows that the whole random structure of an isotropic Gaussian section σ of $\mathcal{T}^{\otimes s}$ is determined by the function $k(\theta) = \Gamma(R_2(\theta))$, indeed

$$\Gamma(R(\varphi,\theta,\psi)) = k(\theta)e^{is(\varphi+\psi)}. \tag{16}$$

Proposition 10 *The function* $k(\theta) = \mathbb{E}\left\{X(\mathbb{1})\overline{X(R_2(\theta))}^T\right\}$ *has the following properties*

1. $k\colon \mathbb{R} \to \mathbb{R}$ *is* 2π *periodic and even.*
2. k *is semipositive definite:*

$$\sum_{i,j} k(\theta_i - \theta_j) z_i \overline{z_j} \geq 0 \quad \textit{for any } z_1,\dots,z_n \in \mathbb{C} \textit{ and } \theta_1,\dots,\theta_n \in \mathbb{R}.$$

Proof Of course $k\colon \mathbb{R} \to \mathbb{C}$ is $2\pi-$periodic. Let $\theta \in [0,\pi]$, then the Euler coordinates of $R_2(-\theta)$ are given by:

$$\begin{aligned} R_2(-\theta) &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= R_3(\pi)R_2(\theta)R_3(\pi) = R(\pi,\theta,\pi). \end{aligned}$$

thus k is even

$$k(-\theta) = \Gamma(R_2(-\theta)) = \Gamma(R(\pi,\theta,\pi)) = e^{is\pi}k(\theta)e^{is\pi} = k(\theta),$$

and real

$$\overline{k(\theta)} = \overline{k(-\theta)} = \overline{\Gamma(R_2(\theta)^{-1})} = \Gamma(R_2(\theta)) = k(\theta).$$

Positive definiteness follows from the fact that k is the covariance function of the stationary Gaussian random field $\tau\colon \mathbb{R} \to \mathbb{C}$ defined by $\tau(\theta) = X(R_2(\theta))$. □

Remark 27 We leave as an open issue whether (1) and (2) are enough to classify all functions k coming from an isotropic Gaussian spin s section.

Remark 28 As we can see from equation (16), given $p, q \in \mathbb{S}^2$, the covariance of $\sigma(p)$ and $\sigma(q)$ does not depend only on the angular distance between p, q, i.e. on $\theta = \arccos(\langle p, q\rangle)$. Indeed, if $p = ge_3$ and $q = gR(\varphi,\theta,\psi)e_3$, then

$$\sigma(p) = X(g)\,(ge_2)^{\otimes s}, \quad \sigma(q) = X(gR(\varphi,\theta,\psi))\,(gR(\varphi,\theta,\psi)e_2)^{\otimes s}$$

and

$$\mathbb{E}\{X(g)\overline{X(gR(\varphi,\theta,\psi))}^T\} = \Gamma(R(\varphi,\theta,\psi)) = k(\theta)e^{i(\varphi+\psi)}.$$

Example 2 We will work with sections of the form

$$\sigma = \frac{1}{\sqrt{\frac{2\ell+1}{4\pi}}}\sigma_{\ell;s} := \sum_{m=-\ell}^{\ell} a_{\ell m;s}\sigma_{\ell m;s},$$

where $\sigma_{\ell m;s}$ are the spin spherical harmonics defined in Definition 3 and $a_{\ell m;s} \sim N_{\mathbb{C}}(0,1)$ are iid. The pullback field is

$$X = X_{\ell;s} = \sum_{m=-\ell}^{\ell} a_{\ell m;s}\overline{D}^{\ell}_{m,-s}.$$

Therefore

$$k(\theta) = k_{\ell;s}(\theta) = d^{\ell}_{-s,-s}(\theta).$$

X is isotropic, because $D^{\ell}: SO(3) \to U(2\ell+1)$ and it is a group homomorphism, thus

$$\begin{aligned} X(gR) &= \sum_{m=-\ell}^{\ell} a_{\ell m;s}\overline{D}^{\ell}_{m,-s}(gR) = \\ &= \sum_{m=-\ell}^{\ell} a_{\ell m;s}\sum_{i=-1}^{\ell}\overline{D^{\ell}_{m,i}(g)D^{\ell}_{i,-s}(R)} = \\ &= \sum_{m=-\ell}^{\ell}\left(\sum_{i=-1}^{\ell} a_{\ell i;s}\overline{D^{l}_{i,m}(g)}\right)\overline{D}^{l}_{m,-s}(R) = \\ &= \sum_{m=-\ell}^{\ell} b_{\ell m;s}\overline{D}^{\ell}_{m,-s}(R); \end{aligned}$$

and since $D^{\ell}(g)$ is unitary, it follows that the random variables $b_{\ell m;s} = \left(\sum_{i=-1}^{\ell} a_{\ell i;s}\overline{D^{\ell}_{i,m}(g)}\right)$ are again iid $\sim N_{\mathbb{C}}(0,1)$.

5 Jet bundles and Type-*W* singularities

In this Section we introduce the geometric tools that we are going to exploit to establish our main results. Let us recall that in [36] the authors study the singularities of polynomial maps $\psi: \mathbb{S}^m \to \mathbb{R}$ that arise as preimages via the jet prolongation map $j^r\psi$ of subsets $W \subset J^r(\mathbb{S}^m, \mathbb{R}^k)$ of the jet space. In other words a singularity $Z = (j^r\psi)^{-1}(W)$ is the set of points $p \in \mathbb{S}^m$ where the Taylor polynomial of ψ at p satisfies a given set of conditions, encoded in W, an obvious example being the set of critical points, or extrema. In this paper we will study the same objects, but replacing ψ with a N-tuple of spin functions.

Definition 6 For any $\underline{s} := s_1, \ldots, s_N \in \mathbb{Z}^N$ we define

$$\mathcal{E}^{\underline{s}} := \mathcal{T}^{\otimes s_1} \oplus \cdots \oplus \mathcal{T}^{\otimes s_N}$$

that is the complex hermitian vector bundle whose total space and projections are denoted by:

$$\pi^{\underline{s}} \colon E^{\underline{s}} := (T\mathbb{S}^2)^{\otimes s_1} \oplus \cdots \oplus (T\mathbb{S}^2)^{\otimes s_N} \to \mathbb{S}^2$$

A section $\mathcal{E}^{\underline{s}}$ is a N-tuple of sections of $\mathcal{T}^{\otimes s_i}$, for $i = 1, \ldots, N$. For this reason we will call them *multispin functions* and we will denote them as

$$\underline{\sigma} = (\sigma^1, \ldots, \sigma^N) \colon \mathbb{S}^2 \to E^{\underline{s}}. \tag{17}$$

We denote by $\mathcal{C}^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ the space of all $\mathcal{C}^r$ sections of $\mathcal{E}^{\underline{s}}$ and by $J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ the space of r-jets of sections (we reserve the notation $\mathcal{C}^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ for the larger space of all $\mathcal{C}^r$ functions). The r-jet at p of a $\mathcal{C}^r$ section $\underline{\sigma}$ of $\mathcal{E}^{\underline{s}}$, denoted $j^r_p\underline{\sigma}$, is the equivalence class of all sections that in one (and hence every) trivialization of $\mathcal{E}^{\underline{s}}$ over a neighborhood of p have the same derivatives at p, up to the order r.

The jet is an intrinsic version of the notion of Taylor polynomial, in that it encodes all the properties of the latter which do not depend on the chosen trivialization.

Moreover, the space of all jets

$$J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}}) := \bigsqcup_{p\in\mathbb{S}^2} J^r_p(\mathbb{S}^2|\mathcal{E}^{\underline{s}}) := \left\{ j^r_p\underline{\sigma} | p \in \mathbb{S}^2 \text{ and } \sigma \in \mathcal{C}^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}}) \right\}$$

is a smooth vector bundle over $\mathbb{S}^2$, with the obvious projection map $j^r_p\underline{\sigma} \mapsto p$, called the *source*.

The point of view of jets allows us to put under the same umbrella any set defined by some conditions on the derivatives of a collections of spin functions. Indeed we will view those as the preimage of a given subset $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ via the jet prolongation map, that is the map associated to a $\mathcal{C}^r$ section $\underline{\sigma} \in \mathcal{C}^r(\mathbb{S}^2, \mathcal{E}^{\underline{s}})$ that evaluates the jet at each point:

$$j^r\underline{\sigma} \colon \mathbb{S}^2 \to J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}}), \qquad j^r\underline{\sigma}(p) := j^r_p\underline{\sigma}.$$

We refer to the books [21, 28] for the theory of jet bundles.

Remark 29 If $\underline{\sigma}$ is of class $\mathcal{C}^k$, then $j^r\underline{\sigma}$ is of class $\mathcal{C}^{k-r}$.

Definition 7 Let $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a subset and let $\underline{\sigma} \in \mathcal{C}^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a multispin function. The *type-W singularity* of $\underline{\sigma}$ is the set

$$Z_W(\underline{\sigma}) := \left\{p \in \mathbb{S}^2 : j^r\underline{\sigma} \in W\right\} = (j^r\underline{\sigma})^{-1}(W).$$

We say that $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ is the *singularity type*.

5.1 Examples

Obvious examples of singularities are the excursions sets, the critical points and the extrema for the modulus of a given section $\underline{\sigma} \in \mathcal{C}^r(\mathbb{S}^2, \mathcal{E}^{\underline{s}})$, see below for more explicit computations.

We note first that the excursion sets of the norms of monochromatic spin Gaussian fields give us the possibility to illustrate some very concrete examples of singularity sets. For instance

1. If $r = 0$, then $J^0(\mathbb{S}^2|\mathcal{T}^{\otimes s}) = E(\mathcal{T}^{\otimes s})$ is the total space of the line bundle. Let $B_u(\mathcal{T}^{\otimes s})$ be the total space of the u ball bundle and let W_u be its complement. Then

$$Z_\ell^{W_u} = \{|\sigma_\ell| \geq u\}$$

is the excursion set of the norm of the field. In this case the only meaningful Betti number is b_0, the number of connected components. The number of connected components of the boundary is the $b_0 = b_1$ of

$$Z_\ell^{\mathbb{S}_u(\mathcal{T}^{\otimes s})} = \{|\sigma_\ell| = u\}.$$

2. Given any function $f : \mathbb{S}^2 \to \mathbb{R}$ we can define a singularity $W \subset J^1(\mathbb{S}^m|\mathcal{T}^{\otimes s} \oplus \mathbb{R})$ such that

$$\text{Crit}\left(f|_{\{|\sigma_\ell|=u\}}\right) = \left(j^1(\sigma_\ell, f)\right)^{-1}(W).$$

3. Let us say that a smooth curve $Z \subset \mathbb{R}^3$ has a *flex* at $p \in Z$ if for a (and hence any) regular parametrization $\psi : I \to Z$ such that $\psi(t) = p$, one has that $\dot{\psi}(t) \perp \ddot{\psi}(t)$. The set

$$\{p \in \mathbb{S}^2 : \{|\sigma_\ell| = u\} \text{ has a flex in p}\}$$

is a type-W singularity, with respect to a suitable $W \subset J^2(\mathbb{S}^m|\mathcal{T}^{\otimes s})$.

5.2 Intrinsic singularity type *W*

We need to restrict the class of subsets $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ under consideration, in order to say something meaningful. First of all since we are interested in isotropic spin and multispin random functions, it makes sense to restrict ourselves to the class of W that are isotropic in some sense. In the paper [36] the notion of *intrinsic subset* of a jet space was introduced for the same reason. Let us repeat it here, in a version adapted to our case.

Definition 8 (Intrinsic subset) Let $\underline{s} = (s_1, \dots, s_N) \in \mathbb{Z}^N$. A subset $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ is said to be *intrinsic* if there is a subset $W_0 \subset J^r(\mathbb{D}, \mathbb{C}^N)$, called *model*, such that for any embedding $\varphi \colon \mathbb{D} \hookrightarrow \mathbb{S}^2$ and any metric-preserving trivialization of $\mathcal{E}^{\underline{s}}$ over $\varphi(\mathbb{D})$, namely an isomorphism of vector bundles

$$\tau = (\tau_0, \tau_1) \colon \mathcal{E}^{\underline{s}}|_{\varphi(\mathbb{D})} \to \mathbb{D} \times \mathbb{C}^N,$$

such that $\pi^{\underline{s}} \circ \tau^{-1}(u, z) = \varphi(u)$ and $|\tau^{-1}(u, z)| = |z|$, one has that $(j^r\tau)^*(W) = W_0$, where

$$(j^r\tau)^* \colon J^r\left(\varphi(\mathbb{D})|\mathcal{E}^{\underline{s}}\right) \to J^r\left(\mathbb{D}, \mathbb{C}^N\right), \quad j^r_{\varphi(p)}\underline{\sigma} \mapsto j^r_p(\tau_1 \circ \underline{\sigma} \circ \varphi). \tag{18}$$

The jet space $J^r(\mathbb{D}, \mathbb{C}^N)$ is canonically isomorphic to the product space $\mathbb{D} \times (P_r)^{2N}$, where $P_r \subset \mathbb{R}[x, y]$ denotes the space of real polynomials of degree at most r in two variables (the coordinates on $\mathbb{D}$). Therefore we can make the identification $J^r(\mathbb{D}, \mathbb{C}^N) = \mathbb{D} \times \mathbb{R}^k$, where $k = N(r+1)(r+2)$.

The above definition implies that the model $W_0 \subset J^r(\mathbb{D}, \mathbb{C}^N)$ is itself an intrinsic singularity type of the form

$$W_0 = \mathbb{D} \times \Sigma,$$

for some $\Sigma \subset \mathbb{R}^k$. We will say that a subset $\Sigma \subset \mathbb{R}^k$ is *intrinsic* if the subset $\mathbb{D} \times \Sigma = W_0$ is an intrinsic singularity type.

Remark 30 Note that not all subsets $\Sigma \subset \mathbb{R}^k$ are intrinsic. An obvoius counterexample is $W_0 = \{j^r_p f \subset J^r(\mathbb{S}^2, \mathbb{C}) : f(p) = c\}$ for $c \in \mathbb{R} \setminus \{0\}$, because, of course, the value of f depends from the choice of local coordinates.

Definition 9 (Intrinsic function) Let $\underline{s} = (s_1, \dots, s_N) \in \mathbb{Z}^N$ and let $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be an intrinsic subset with model $W_0 = \mathbb{D} \times \Sigma \subset J^r(\mathbb{D}, \mathbb{C}^N) = \mathbb{D} \times \mathbb{R}^k$. Let $\alpha \colon W \to \mathbb{R}$ be a function. We say that α is *intrinsic* if there is a function $\alpha_0 \colon W_0 \to \mathbb{R}$ such that, under any trivialization of the type described in Equation (18), we have that α corresponds to α_0.

5.3 Semialgebraic type–*W* singularities

We will consider only singularity types $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ that are intrinsic and for which $\Sigma \subset \mathbb{R}^k$ is *semialgebraic* (see [8] or [23]). In particular, this implies that W admits a Whitney stratification (see [4, Sec. 8.1], or [40], or [23, Sec.1.2]), that is a partition $\mathscr{S}$ of W into a locally finite family of disjoint smooth embedded submanifolds $S \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ called the *strata* of the stratification, such that for each $S \in \mathscr{S}$, the set $(\overline{S}-S)\cap W$ is a union of strata (this is known as the *frontier condition*, see [40]) and such that each pair (X,Y) of distinct strata satisfies *Whitney condition B* (cf. [40] or [23, Sec. 1.1]) whenever $\overline{X}\supset Y$ (Whitney condition B implies that in this case $\dim X>\dim Y$, see [40]). A *Whitney stratified subset* of a smooth manifold M is a pair $(W,\mathscr{S})$, such that $\mathscr{S}$ is a Whitney stratification of $W\subset M$. However, we will most frequently just say that W is a Whitney stratified subset, without mentioning the stratification.

It follows that a Whitney stratified subset W admits a partition (depending on the stratification) $W=\partial_0W\sqcup\partial_1W\sqcup\partial_2W\sqcup\dots$, where

$$\partial_i W := \bigsqcup_{S\in\mathscr{S},\ \dim S=i} S$$

is a smooth embedded submanifold (because it is a *locally finite* union of smooth embedded submanifolds) of dimension i. Here, we are using the notation of [4].

5.4 Transversality

Let $W\subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a Whitney stratified subset. Let $\underline{\sigma}\in\mathcal{C}^\infty(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a smooth multispin function (see Equation (17) above). Then the jet map $j^r\underline{\sigma}\colon\mathbb{S}^2\to J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ is *transverse* to W if and only if it is transverse to S for each stratum $S\in\mathscr{S}$ of W, see [28]. This is denoted

$$j^r\underline{\sigma}\ \pitchfork\ W. \tag{19}$$

Moreover, in this case we say that the singularity $Z_W(\underline{\sigma})$ is *nondegenerate*. For instance, consider the singularity type

$$W:=\{j^1_p\sigma\in J^1(\mathbb{S}^2|\mathcal{T}^{\otimes s})|\langle\sigma(p),(\nabla\sigma)_p\rangle=0\},$$

then $Z_W(\sigma)=\mathrm{Crit}(|\sigma|^2)$. In this case $j^1\sigma\ \pitchfork\ W$ if and only if the function $|\sigma|^2\colon\mathbb{S}^2\to\mathbb{R}$ is Morse.

Remark 31 There is a little abuse of notation in Equation (19) in that the transversality condition depends also on the chosen stratification.

By classical arguments of differential topology (see [23, 40]), if the singularity $Z_W(\underline{\sigma})$ is nondegenerate, it follows that $Z_W(\underline{\sigma})\subset\mathbb{S}^2$ admits a Whitney stratification $(j^r\underline{\sigma})^{-1}\mathscr{S}$ obtained by taking the preimages of the strata of W having codimension smaller or equal to two.

Proposition 11 *Let $W\subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a semialgebraic subset with a given Whitney stratification $\mathscr{S}$. Let $\underline{\sigma}\in\mathcal{C}^\infty(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ be a smooth multispin function such that $Z_W(\underline{\sigma})$ is nondegenerate. Then the set $(j^r\underline{\sigma})^{-1}\mathscr{S}$ of all subsets $(j^r\underline{\sigma})^{-1}(S)$ with $S\in\mathscr{S}$ is a Whitney stratification of $Z_W(\underline{\sigma})\subset\mathbb{S}^2$. Moreover, if ∂_kW has codimension 2, then*

$$\partial_i Z_W(\underline{\sigma}) = Z_{\partial_{k+i}W}(\underline{\sigma})$$

is a smooth embedded submanifold of dimension $i \in \{0, 1, 2\}$ *in* $\mathbb{S}^2$ *and it is a nondegenerate singularity of* $\underline{\sigma}$ *of type* W_{k+i}*, so that*

$$Z_W(\underline{\sigma}) = Z_{W_k}(\underline{\sigma}) \sqcup Z_{W_{k+1}}(\underline{\sigma}) \sqcup Z_{W_{k+2}}(\underline{\sigma}). \tag{20}$$

Remark 32 The decomposition (20) does not need to be a Whitney stratification. For instance if Z has an isolated point, then it belongs to $\partial_0 Z$, but not to $\overline{\partial_1 Z}$; this would violate the frontier condition.

5.5 Whitney stratified subsets of the sphere

It is worth to spell out explicitely the definition of a closed Whitney stratified subset of $\mathbb{S}^2$, since it is actually quite simple.

Proposition 12 *Let* $Z \subset \mathbb{S}^2$ *be a closed semialgebraic nondegenerate singularity with a partition* $Z = Z_0 \sqcup Z_1 \sqcup Z_2$ *as in* (20)*. Then the following conditions hold.*

i. $Z_0 = \{p_1, \dots, p_{n_0}\}$ *is a finite set.*
ii. $Z_1 = \xi_1(\mathbb{R}) \sqcup \dots \sqcup \xi_{n_1}(\mathbb{R}) \sqcup \gamma_1(\mathbb{S}^1) \sqcup \dots \sqcup \gamma_{n_1'}(\mathbb{S}^1)$*, where* $\xi_i : \mathbb{R} \to \mathbb{S}^2$ *and* $\gamma_i : \mathbb{S}^1 \to \mathbb{S}^2$ *are smooth embeddings with pairwise disjoint image.*
iii. $Z_2 = U_1 \sqcup \dots \sqcup U_{n_2}$ *is a union of open connected components of* $\mathbb{S}^2 \smallsetminus (Z_0 \sqcup Z_1)$*.*
iv. $\exists \lim_{t \to \pm\infty} \xi_i(t) \in Z_0$*.*
v *Let* $\lim_{t \to \pm\infty} \xi_i(t) = y$*. Then the following limits exist and they are equal:*

$$\exists \lim_{t \to \pm\infty} \frac{y - \xi(t)}{|y - \xi(t)|} = \lim_{t \to \pm\infty} \frac{\xi'(t)}{|\xi'(t)|}.$$

Proof The proof follows from standard techniques in semialgebraic geometry and is omitted for brevity's sake. □

Remark 33 If we remove the closedness assumption, none of the above property has to hold. Moreover, property v. is due to the semialgebraicity of Z.

5.6 Euler–Poincaré characteristic

Definition 10 The i^{th} Betti number of a topological space Z is the dimension of the i^{th} real homology group (see [27]):

$$b_i(Z) := \dim H_i(Z, \mathbb{R}).$$

We denote by $b(Z) \in \mathbb{N} \cup \{+\infty\}$ the sum of all Betti numbers. The Euler-Poincaré characteristic of a topological space X is defined whenever $b(Z) < +\infty$ and it is the alternating sum of all Betti numbers:

$$\chi(Z) = b_0(Z) - b_1(Z) + b_2(Z) - \dots$$

By standard arguments, we see that if $Z \subset \mathbb{S}^2$ is a Whitney stratified subset of $\mathbb{S}^2$, the only non-zero Betti numbers are b_0 (which is the number of connected components), b_1 and b_2. Moreover, $b_2(Z)$ is non-zero only if $Z = \mathbb{S}^2$, case in which $b_0 = 1, b_1 = 0$ and $b_2 = 1$.

When Z is closed, one can give it the structure of a CW-complex, using the description given in Proposition 12 (after passing to a finer stratification, possibly), so that the Euler-Poincaré characteristic can also be expressed as follows.

Proposition 13 *Let $Z \subset \mathbb{S}^2$ be a closed Whitney stratified subset and let n_0, n_1, n_2 be as in Proposition 12. Then*

$$\chi(Z) = n_0 - n_1 + n_2 - b_1(Z_2),$$

unless $Z = \mathbb{S}^2$ with the trivial stratification: $Z_0 = \emptyset$, $Z_1 = \emptyset$ and $Z_2 = \mathbb{S}^2$. More generally, let $A \cup B = Z$ such that A, B and $A \cap B$ are closed and are union of connected components of strata of Z. Then

$$\chi(Z) = \chi(A) + \chi(B) - \chi(A \cap B).$$

6 Lipschitz–Killing curvatures

This Section collects some basic definitions and properties of intrinsic volumes/Lipschitz–Killing Curvatures; the presentation is tailored for the main results and proofs to follow in the remaining part of the paper.

6.1 Normal Morse index

Let $Z \subset \mathbb{S}^2$ be a Whitney stratified subset. For any $p \in \partial_0 Z$, we define the set of degenerate covectors (cf. [23]) at p as the set

$$D_p Z := \left\{ \nu \in T_p^* \mathbb{S}^2 : \begin{array}{l} \exists p_n \in \partial_i Z \setminus \partial_0 Z \text{ s.t. } p_n \to p \text{ and} \\ \exists \lim\limits_n T_{p_n} Z_i = Q \subset T_p M \text{ s.t. } \nu(Q) = 0 \end{array} \right\}.$$

In particular if $p \in \partial_0 Z \setminus \overline{Z_1 \cup Z_2}$, then $D_p Z = \emptyset$. Moreover, we define $D_p Z = \emptyset$ for every $p \in Z_2$; $D_p Z = \{0\}$ for every $p \in Z_1 \cap \overline{Z_2}$ and $D_p Z = \emptyset$ for $p \in Z_1 \setminus \overline{Z_2}$. We leave to the reader to check that this definition of *degenerate covectors* corresponds to the general one from [23, Section 1.8] in the special case of stratified subsets of the sphere.

Definition 11 Let us consider $p \in Z_i$, and $v \in T_p^{\perp} Z_i \setminus D_p Z$, we define the (normal) Morse index

$$\alpha(p, v) := 1 - \chi(O_p \cap Z \cap \varphi^{-1}\{x : \langle x, v \rangle \leq -\epsilon\}),$$

where $\varphi : O_p \to \mathbb{D} \subset \mathbb{R}^2$ is a coordinate chart centered at p (i.e. a diffeomorphism such that $\varphi(p) = 0$.).

Due to the cone structure of Whitney stratified subsets, $\varphi^{-1}\{x : \langle x, v\rangle < 0\}$ can be retracted homotopically to a subset of the boundary of O_p which in our case is a finite union of intervals, hence the Euler-Poincaré characteristic is the number of connected components hence

$$\alpha(p, v) = 1 - b_0(O_p \cap Z \cap \varphi^{-1}\{x : \langle x, v\rangle \leq -\epsilon\}).$$

Remark 34 In [4, Equation 8.1.1] the definition of $\chi(T)$ is given with a sign that depends on the dimension of T:

$$\chi_{AT}(T) = (-1)^{\dim T}\chi(T).$$

This is not in agreement with the most standard conventions (in topology), indeed with this defnition χ would not be invariant under homotopy equivalences, because the dimension is not.

6.2 Explicit formula for stratified subsets of the sphere

Let $Z \subset \mathbb{S}^2$ be a closed semialgebraic nondegenerate singularity with a partition $Z = Z_0 \sqcup Z_1 \sqcup Z_2$ as in (20). Now we define Lipschitz–Killing curvature measures as in [4, (10.7.1)]: in our setting the formula becomes, for any $A \subset \mathbb{S}^2$ Borel subset:

$$\begin{aligned}
\mathcal{L}_2(Z, A) &= \mathscr{H}^2(A \cap Z_2)\\
\mathcal{L}_1(Z, A) &= \frac{1}{2}\int_{A\cap Z_1} \beta^1(p)\, dZ_1\\
\mathcal{L}_0(Z, A) &= \frac{1}{2\pi}\sum_{p\in A\cap Z_0} \beta^0(p) + \frac{1}{2\pi}\int_{A\cap Z_1\cap\{\beta^1=1\}} S(p)dZ_1(p) +\\
&\quad + \frac{1}{2\pi}\mathscr{H}^2(A \cap Z_2),
\end{aligned}$$

where for any point $p \in Z_j$, with $j \in \{0, 1\}$, we define $S(T_pZ_j^{\perp}) := \{v \in T_p\mathbb{S}^2 : |v| = 1, T_pZ_j \subset v^{\perp}\}$,

$$\beta^j(p) := \int_{S(T_pZ_j^{\perp})} \alpha(p, v)\mathscr{H}^{1-j}(dv),$$

and $S(p)$ is the geodesic curvature of Z_1 at p, see Equation (21) below.

Remark 35 The only term that is specific to the round sphere is the last summand in the formula for $\mathcal{L}_0(Z)$. To have a formula that is valid on every Riemannian surface, one should replace it with

$$\mathcal{L}_0(Z) = \frac{1}{2\pi} \sum_{p \in Z_0} \beta^0(p) + \frac{1}{2\pi} \int_{Z_1 \cap \{\beta^1 = 1\}} S(p) dZ_1(p) + \frac{1}{2\pi} \int_{Z_2} \kappa(p) dZ_2(p),$$

where $\kappa(p)$ is the Gaussian curvature.

The intrinsic volumes or Lipschitz–Killing curvatures of Z are then defined as $\mathcal{L}_i(Z) := \mathcal{L}_i(Z, Z)$.

Theorem 14 (Chern–Gauss–Bonnet, [4, Theorem 12.6.1]) $\mathcal{L}_0(Z) = \chi(Z)$.

6.2.1 Description of β^1

Concerning the strata of dimension 1, there are three possibilities: let us define for $p \in Z_1$

$$\beta^1(p) = 2 - \#\{\text{strata of dimension 2 adjacent to } Z_1 \text{ at } p\};$$

since we are on the sphere that is two-dimensional, $\beta^1(p) \in \{0, 1, 2\}$.

Let $T_p Z := T_p S$, where S is the stratum of Z, containing $p \in Z$ and let us define the tangent cone of Z at p as the set

$$\hat{T}_p Z := \{v \in T_p \mathbb{S}^2 : \exists \mathcal{C}^1 \text{ curve } \gamma : [0, \varepsilon) \to Z \text{ s.t. } \gamma(0) = p, \dot{\gamma}(0) = v\}.$$

Let $p \in Z_1$ be a boundary point: $\beta^1(p) = 1$. Then we define $S(p)$ as the geodesic curvature of Z_1 at the point p in the inward direction $v \in S(T_p Z_1^{\perp}) \cap \hat{T}_p Z$:

$$S(p) := \langle \nabla_{\dot{\gamma}(0)} \dot{\gamma}, v \rangle,$$

where γ is any $\mathcal{C}^1$ curve parametrizing Z_1 such that $\gamma(0) = p$ and $|\dot{\gamma}| = 1$.

6.2.2 Description of β^0

For $p \in Z_0$, the quantity $\beta^0(p)$ can be understood as follows. We call *tangent link* of Z at p the subset $S(\hat{T}_p Z)$ of unit vectors in the tangent cone. Moreover, define the *link* of Z at p, denoted $\text{link}_p(Z)$ as the topological space $\partial O_p \cap Z$, for a small enough spherical ball O_p around p.

Remark 36 The link and the tangent link are both homeomorphic to a finite union of intervals, but they are not necessarily homeomorphic nor homotopic to each other. The only characterization of them as a pair of spaces is that there is a surjective continuous map $\text{link}_p(Z) \to S(\hat{T}_p Z)$. Indeed the intervals of the link could become points in the tangent link. Moreover, the tangent link may have less connected components than the link. This is due to the existence of semialgebraic cusps.

Proposition 15

$$\beta^0(p) = 2\pi - \mathscr{H}^1\left(S(\hat{T}_p Z)\right) - \chi\left(\text{link}_p(Z)\right)\pi.$$

Proof Let $N := b_0(\text{link}_p(Z)$ be the number of connected components of the link, then $\partial O_p \cap Z$ contracts homotopically to a set of N points, or it is homeomorphic to $\mathbb{S}^1$. Only in this latter case, calling $\varepsilon := b_1(\text{link}_p(Z))$, we have that $\varepsilon = 1$, while $\varepsilon = 0$ otherwise. The formula to prove is:

$$\beta^0(p) = 2\pi - (N - \varepsilon)\pi - \mathscr{H}^1(S(\hat{T}_p Z)). \tag{21}$$

Assume $\varepsilon = 0$. Let $C_1, \dots, C_N$ be the connected components of $(O_p \smallsetminus \{p\}) \cap Z$, for small enough $\varepsilon > 0$ and small enough O_p. For each i, consider the subset $I_i \subset \hat{T}_p Z$ that comes from C_i, that is the subset consisting of those $\dot{\gamma}(0) \in S(\hat{T}_p Z)$, such that $\gamma((0, \varepsilon]) \subset C_i$. Now, let $\theta_i := \text{vol}_1(I_i)$ be the total angle spanned by I_i. Then

$$\begin{aligned}
\beta^0(p) &= \int_{S(T_p\mathbb{S}^2)} 1 - \sum_{i=1}^{N} \chi(C_i \cap \varphi^{-1}\{x \colon \langle x, v\rangle < -\varepsilon\}) \bigoplus H^1(dv) \\
&= 2\pi - 2N\pi + \sum_{i=1}^{N} \int_{S(T_p\mathbb{S}^2)} 1 - \chi(C_i \cap \varphi^{-1}\{x \colon \langle x, v\rangle < -\varepsilon\}) \mathscr{H}^1(dv) \\
&= 2\pi - 2N\pi + \sum_{i=1}^{N} (\pi - \theta_i) \\
&= 2\pi - N\pi - \mathscr{H}^1(S(T_p Z))
\end{aligned}$$

Notice that there might be distinct indices i, j for which $I_i \subset I_j$ or even $I_i = I_j$. However, in any case we have that $\theta_1 + \cdots + \theta_N = \text{vol}^1(S(T_p Z))$. If $\varepsilon = 1$, hence $N = 1$, then both links are homeomorphic to $\mathbb{S}^1$, thus $\mathscr{H}^1(S(T_p Z)) = 2\pi$. In this case $\alpha(p, v) = 0$ for all v, therefore $\beta^0(p) = 0 = 2\pi - (1 - 1)\pi - 2\pi$. □

6.3 Stratified Morse theory

We will make extensive use of the stratified version of Morse theory, for which we refer to the standard textbook by Goresky and Macpherson [23]. The most important result for our purposes are the stratified and probabilistic versions of Morse Theorem and of the Gauss-Bonnet Theorem from the book [4], in which a large portion of stratified Morse theory is reported, including most of the results that we will need here.

The following is the definition of a stratified Morse function specialized to our case.

Definition 12 Given a closed Whitney stratified subset $Z = \partial_0 Z \sqcup \partial_1 Z \sqcup \partial_2 Z$ of $\mathbb{S}^2$, we say that a function $f \colon Z \to \mathbb{R}$ is a *Morse function* if f is the restriction of a smooth function $\tilde{f} \colon \mathbb{S}^2 \to \mathbb{R}$ such that

(a) $f|_{\partial_i Z}$ is a Morse function on $\partial_i Z$, for all $i = 0, 1, 2$. A point $p \in \partial_i Z$ is *critical point* of $f|_Z$ if and only if p is a critical point of $f|_{\partial_i Z}$. All points of $\partial_0 Z$ are critical points, by convention. The set of critical points is denoted by

$$\text{Crit}\,(f|_Z) = \text{Crit}\,\left(f|_{\partial_2 Z}\right) \sqcup \text{Crit}\,\left(f|_{\partial_1 Z}\right) \sqcup \partial_0 Z$$

(b) For every critical point $p \in \text{Crit}$ we have $d_p f \notin D_p Z$, i.e. the covector $d_p f$ is nondegenerate.

If $f|_Z$ is a Morse function and $p \in \text{Crit}(f) \cap \partial_i Z$, we define the *index of* f *at* p, denoted as $\iota_p f \in \mathbb{N}$, as the index of $f|_{\partial Z_i}$, that is the dimension of the negative eigenspace of the second derivative $d_p^2(f|_{\partial_i Z})$.

Theorem 16 (Morse Theorem, see[4, Theorem 9.3.2] or [23]) *Let* $f|_Z$ *be a Morse function, then*

$$\chi(Z) = \sum_{p \in Crit(f)} \alpha(p, d_p f)(-1)^{\iota_p f}.$$

6.3.1 Semialgebraic Morse inequalities

We will need the following specialization of [36, Theorem 8], incorporating also [36, Remark 12], to our setting. The following theorem will be central in the proof Theorem 3 in that it allows to reduce it to the case of zero dimensional singularities, hence to apply effectively a generalized Kac–Rice formula (developed by one of the authors in [53]). See 9.2 and 9.3.

Theorem 17 (See [36, Theorem 8]) *Let* $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ *be an intrinsic semialgebraic subset, with a given semialgebraic Whitney stratification* $W = \sqcup_{S \in \mathscr{S}} S$. *There exists an intrinsic semialgebraic subset* $W' \subset J^{r+1}(\mathbb{S}^2|\mathbb{C} \oplus \mathcal{E}^{\underline{s}})$ *having codimension* ≥ 2, *equipped with a semialgebraic Whitney stratification* $\mathscr{S}'$ *that satisfies the following properties with respect to any couple formed by a smooth section* $\underline{\sigma}\colon \mathbb{S}^2 \to \mathcal{E}^{\underline{s}}$ *and a smooth function* $\sigma_0\colon \mathbb{S}^2 \to \mathbb{C}$. *Let* $\underline{s}' := (0, \underline{s})$ *and* $\underline{\sigma}' := (\sigma_0, \underline{\sigma})\colon \mathbb{S}^2 \to \mathcal{E}^{\underline{s}'} = \mathbb{C} \oplus \mathcal{E}^{\underline{s}}$. *Let* $g := \Re(\sigma_0)\colon \mathbb{S}^2 \to \mathbb{R}$ *be the real part of* σ_0. *Let* $Z_W(\underline{\sigma}) = J^r\underline{\sigma}^{-1}(W)$.

1. *If* $j^r\underline{\sigma} \pitchfork W$ *and* $j^{r+1}\underline{\sigma}' \pitchfork W'$, *then* $g|_{Z_W(\underline{\sigma})}$ *is a Morse function on* $Z_W(\underline{\sigma})$ *with respect to the stratification* $(j^r\underline{\sigma})^{-1}\mathscr{S}$ *and*

$$Crit(g|_{Z_W(\underline{\sigma})}) = Z_{W'}(\underline{\sigma}') = (j^{r+1}\underline{\sigma}')^{-1}(W').$$

More precisely: if $d_p(j^r\underline{\sigma}) \pitchfork TW$, *then* $p \in Crit(g|_{Z_W(\underline{\sigma})})$ *if and only if* $j_p^{r+1}(\sigma_0, \underline{\sigma}) \in W''$; *moreover, if* $j_p^{r+1}(\sigma_0, \underline{\sigma}) \in W'$ *and* $d_p(j^{r+1}(\sigma_0, \underline{\sigma})) \pitchfork T_{j_p^{r+1}(\sigma_0, \underline{\sigma})}W'$ *then* $d_p(j^r\underline{\sigma}) \pitchfork TW$, *thus* $Z_W(\underline{\sigma})$ *is a Whitney stratified subset in a neighborhood* O_p *of* p, *and* p *is a Morse critical point of* $g|_{Z_W(\underline{\sigma}) \cap O_p}$.

2. *If* W *is closed, then* W' *is closed.*

3. *There is a constant $N_W > 0$ depending only on W and $\mathscr{S}$, such that if $j^r\underline{\sigma} \pitchfork W$ and $j^{r+1}\underline{\sigma}' \pitchfork W'$, then*

$$b_i\left(Z_W(\underline{\sigma})\right) \leq N_W \# Z_{W'}(\underline{\sigma}')$$

for all $i = 0, 1, 2$.

4. *There exists a bounded and locally constant and intrinsic function $\alpha' \colon W'' \to \mathbb{Z}$, where*

$$W'' := \left\{ \begin{array}{l} j_p^{r+2}(\sigma_0, \underline{\sigma}) \in J^{r+2}(\mathbb{S}^2|\mathcal{E}^{\underline{s}'}) : \\ j_p^{r+1}(\sigma_0, \underline{\sigma}) \in W', d_p(j^{r+1}(\sigma_0, \underline{\sigma})) \pitchfork T_{j_p^{r+1}(\sigma_0, \underline{\sigma})} W' \end{array} \right\}$$

depending only on W and $\mathscr{S}$, such that if $j^r\underline{\sigma} \pitchfork W$ and $j^{r+1}\underline{\sigma}' \pitchfork W'$, then for every $p \in Z_{W'}(\underline{\sigma}')$ we have

$$\alpha(p, d_p g)(-1)^{\iota_p g} = \alpha'(j_p^{r+2}\underline{\sigma}'),$$

and

$$\chi\left(Z_W(\underline{\sigma})\right) = \sum_{p \in Z_{W'}(\underline{\sigma}')} \alpha'(j_p^{r+2}\underline{\sigma}').$$

5. *The stratification $\mathscr{S}'$ of $\tilde{W}'$ can be taken in such a way that each stratum $S' \in \mathscr{S}'$ is of the form*

$$S' = \left\{ j^{r+1}(\sigma_0, \underline{\sigma}) \in J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}'}) \colon j^r\underline{\sigma} \in S, \; j^1\Re(\sigma_0) \in \mathcal{U}(j_p^{r+1}\underline{\sigma}) \right\},$$

for some stratum $S \in \mathscr{S}$ of W and a family $\{\mathcal{U}(\theta)\}_{\theta \in J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}'})}$ of subsets of $J^1(\mathbb{S}^2, \mathbb{R})$.

Proof This result is a natural generalization (from scalar valued functions to sections of vector bundles) of Theorem 8 in [36]. For this reason the proof is omitted; note that the analogous results of points (4) and (5) were not discussed in [36], however a careful inspection of the proofs reveals easily that these results hold. □

Remark 37 Heuristically, the importance of the previous result can be explained as follows: it shows that it is always possible to define on the singularity set a smooth function such that its critical points form a new singularity involving one more derivative and the auxiliary function, but of dimension zero. The power of this construction is that statistics such as Betti numbers or Euler-Poincaré characteristics can now be equivalently reduced to the study of these random sets with finite cardinality. This trick will be heavily exploited in the sections to follow.

7 Scaling assumption

7.1 Scaling assumption for the covariance

Let $X_n : SO(3) \to \mathbb{C}$ be a sequence of smooth isotropic GRFs with right spin$= -s_n$ (possibly dependent on n), i.e. pullbacks of isotropic Gaussian sections σ_n of $\mathcal{T}^{\otimes s_n}$. Let $\Gamma_n : SO(3) \to \mathbb{C}$ and $k_n : \mathbb{R} \to \mathbb{R}$ be the corresponding *circular covariance functions* (see Sect. 4).

In the following we are going to clarify the assumption of a "scaling limit" for k_n, see 1: Roughly speaking, this happens if the restrictions of σ_n to arbitrary spherical disks of a certain radius $\rho_n > 0$ has a limiting behavior. Indeed the sequence:

$$\sigma_n\Big|_{B_{\rho_n}} : B_{\rho_n} \to \mathcal{T}^{\otimes s_n}\Big|_{B_{\rho_n}}$$

can be interpreted as a sequence of GRFs $\xi_n : \mathbb{D} \to \mathbb{C}$ on a fixed disk and the assumption 1 implies (see Theorem 18 below) that this sequence converges in law to a limit stationary GRF $\xi_\infty : \mathbb{D} \to \mathbb{C}$ with covariance function k_∞. For instance, $k_\infty = J_0$ in the Berry case [9, 10, 46], see Sect. 2.3.

Assumption 1 Let $X_n : SO(3) \to \mathbb{C}$ be a sequence of smooth isotropic GRFs with right spin$= -s_n$, i.e. pullbacks of isotropic Gaussian sections σ_n of $\mathcal{T}^{\otimes s_n}$. Let $\Gamma_n : SO(3) \to \mathbb{C}$ and $k_n : \mathbb{R} \to \mathbb{R}$ be the corresponding *circular covariance functions* (see Sect. 4). Assume that there exists a sequence of positive real numbers $\rho_n \to 0$ such that

$$k_n(\rho_n x) = k_\infty(x) + \varepsilon_n(x),$$

with

$$\nu_n^r := \|\varepsilon_n\|_{\mathcal{C}^r([0,\pi],\mathbb{R})} \xrightarrow[n\to+\infty]{} 0, \quad \forall r \in \mathbb{N},$$

and

$$\lim_{n\to+\infty} s_n \rho_n^2 = \beta \in \mathbb{R}.$$

7.2 The rescaled field

Let $X_n : SO(3) \to \mathbb{C}$ be a sequence of Gaussian random fields with right spin $= -s_n$ that satisfy the Assumption 1 with respect to the sequence $\rho_n \to 0$ and $\beta \in \mathbb{R}$. Then, given any sequence of spherical balls B_n of radius ρ_n, there are trivializations (see 13) of the vector bundle $\mathcal{T}^{\otimes s}|_{B_n} \cong \mathbb{D} \times \mathbb{C}$ for which the local representation of $\sigma_n|_{B_n}$ is given by the following Gaussian smooth function on the standard disk.

Definition 13 Let us make the identification $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ and let $\mathbb{D} \subset \mathbb{C}$ be the standard disk. For any $g \in SO(3)$, define

$$\phi_\rho^g \colon \mathbb{D} \to B_\rho(ge_3) \subset \mathbb{S}^2$$
$$z = te^{i\varphi} \mapsto g \cdot \begin{pmatrix} \sin(\rho t)\, e^{i\varphi} \\ \cos(\rho t) \end{pmatrix} = \exp_{ge_3}(\rho z \cdot ge_1) \in \mathbb{S}^2$$

Here, ϕ_ρ^g is constructed via the Riemannian exponential map $\exp_{ge_3} \colon T_{ge_3}\mathbb{S}^2 = \{z \cdot ge_1 \colon z \in \mathbb{C}\} \to \mathbb{S}^2$ precomposed with a rescaling of $\mathbb{C}$. In particular, $\phi_\rho^g(u) = g \cdot \phi_1^{\mathbb{1}}(\rho u)$ and the map $\phi_1^{\mathbb{1}}$ corresponds to the one that we called p in (14): $\phi_\rho^{\mathbb{1}}\left(te^{i\varphi}\right) = p(\varphi, \rho t, \psi) = R(\varphi, \rho t, \psi)e_3$, independently from the value of $\psi \in \mathbb{R}$.

Over the ball $B_\rho(ge_3) = \phi_\rho^g(\mathbb{D})$, the line bundle $\mathcal{T}^{\otimes s}$ have a nonvanishing smooth section $(g\hat{\varphi}e^{-i\varphi})^{\otimes s}$ (see proposition 3.3), where $\hat{\varphi}(p(\varphi, \rho t, 0))e^{-i\varphi} = R(\varphi, \rho t, -\varphi)e_2$ over the ball $B_\rho(ge_3)$ (see equation (14)). This defines a trivialization of the vector bundle $\mathcal{T}^{\otimes s}$ over the ball $B = B_\rho(ge_3) \subset \mathbb{S}^2$:

$$\tau_\rho^g \colon \mathbb{D} \times \mathbb{C} \xrightarrow{\cong} \mathcal{T}^{\otimes s}|_B = \{(p; v) |\ p \in B,\ v \in (T_p\mathbb{S}^2)^{\otimes s}\}$$
$$\tau_\rho^g(z, \xi) = \left(\phi_\rho^g(z); \xi \cdot \left(g\hat{\varphi}(\phi_\rho^g(z))e^{-i\varphi}\right)^{\otimes s}\right)$$

It follows that a section $\sigma \in \mathcal{C}^\infty(\mathbb{S}^2|\mathcal{T}^{\otimes s})$ has a local representation over the ball $B \subset \mathbb{S}^2$ as the function $\xi \colon \mathbb{D} \to \mathbb{C}$ such that

$$\begin{aligned}\tau_\rho^g(z, \xi(z)) &= \sigma(\phi_\rho^g(z)) \\ &= \left(R(\varphi, \rho t, -\varphi)e_3; X(gR(\varphi, \rho t, -\varphi)) \cdot (gR(\varphi, \rho t, -\varphi)e_2)^{\otimes s}\right) \\ &= \left(\phi_\rho^g(z); X(gR(\varphi, \rho t, -\varphi)) \cdot \left(g\hat{\varphi}(\phi_\rho^g(z))e^{-i\varphi}\right)^{\otimes s}\right).\end{aligned}$$

where the second equality is the very definition of the pull-back correspondence between σ and X, see Equation (10). The above construction justifies the initial discussion and allows us to reduce the local study of σ_n to the study of the sequence of Gaussian functions so defined:

Definition 14 (The rescaled field) Let $\xi_n \colon \mathbb{D} \to \mathbb{C}$ be the GRF defined, for any $z = te^{i\varphi} \in \mathbb{D}$, as

$$\xi_n(z) = X_n\left(R\left(\varphi, \rho_n t, -\varphi\right)\right).$$

As it is well known, Gaussian random functions $\xi \colon \mathbb{D} \to \mathbb{R}^2$ are characterized by their covariance function: $K_\xi^{\mathbb{R}}(z_1, z_2) = \mathbb{E}\{\xi(z_1)\xi(z_2)^T\}$, with values in $\mathbb{R}^{2\times 2}$. In complex notation ($\mathbb{R}^2 = \mathbb{C}$) it is useful to observe that $K_\xi^{\mathbb{R}}(z_1, z_1)$ is determined by the pair of complex numbers $\mathbb{E}\{\xi(z_1)\overline{\xi(z_2)}\}$ and $\mathbb{E}\{\xi(z_1)\xi(z_2)\}$, and viceversa. In our case the second is always zero, because we are only considering circularly symmetric complex Gaussian fields. Therefore, we will call *covariance function* of a random field $\xi\colon \mathbb{D}\to\mathbb{C}$, the function

$$K_\xi\colon \mathbb{D}\times\mathbb{D}\to\mathbb{C},\quad K_\xi(z_1,z_2)=\mathbb{E}\{\xi(z_1)\overline{\xi(z_2)}\}.$$

If ξ is a smooth Gaussian field, then $K_\xi\in\mathcal{C}^\infty(\mathbb{D}\times\mathbb{D},\mathbb{C})$ and the application $\xi\mapsto K_\xi$ is injective.

Definition 15 Let $\xi_\infty\colon\mathbb{D}\to\mathbb{C}$ be *the* smooth GRF with covariance function

$$K_{\xi_\infty}(z_1,z_2)=k_\infty(|z_1-z_2|)\exp\left(\beta i\Im(z_1\overline{z_2})\right).$$

The well-posedness of the above definition is actually a consequence of Theorem 18 below (see also Remark 38).

7.3 Smooth convergence of the covariance functions

Lemma 1 *Let* $\theta_1,\theta_2\in[0,\pi)$, $\varphi\in\mathbb{R}$, $\tilde\varphi\in\mathbb{R}$, $\tilde\theta\in[0,\pi)$ *and* $\tilde\psi\in[0,2\pi)$ *such that*

$$\cos\left(\frac{\tilde\theta}{2}\right)e^{i\frac{\tilde\varphi+\tilde\psi}{2}}=\cos\left(\frac{\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)e^{i\frac{\varphi}{2}}+\sin\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)e^{-i\frac{\varphi}{2}};$$
$$\sin\left(\frac{\tilde\theta}{2}\right)e^{i\frac{-\tilde\varphi+\tilde\psi}{2}}=\sin\left(\frac{-\theta_1}{2}\right)\cos\left(\frac{\theta_2}{2}\right)e^{i\frac{\varphi}{2}}+\cos\left(\frac{\theta_1}{2}\right)\sin\left(\frac{\theta_2}{2}\right)e^{-i\frac{\varphi}{2}}$$

Then

$$R_2(-\theta_1)R_3(\varphi)R_2(\theta_2)=R_3(\tilde\varphi)R_2(\tilde\theta)R_3(\tilde\psi).\tag{22}$$

Proof Let us lift the equation (22) to $SU(2)$, using the convention in [52, Prop. 17] for the precise definition of the covering $\pi: SU(2)\to SO(3)$. We obtain the equation

$$\begin{pmatrix}\cos(\frac{\theta_1}{2}) & \sin(\frac{\theta_1}{2})\\ -\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2})\end{pmatrix}\begin{pmatrix}e^{i\frac{\varphi}{2}} & 0\\ 0 & e^{-i\frac{\varphi}{2}}\end{pmatrix}\begin{pmatrix}\cos(\frac{\theta_2}{2}) & -\sin(\frac{\theta_2}{2})\\ \sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2})\end{pmatrix}=\varepsilon\begin{pmatrix}\alpha & -\overline{\beta}\\ \beta & \overline{\alpha}\end{pmatrix}$$

where $\alpha=\cos\left(\frac{\tilde\theta}{2}\right)e^{i\frac{\tilde\varphi+\tilde\psi}{2}}$ and $\beta=\sin\left(\frac{\tilde\theta}{2}\right)e^{i\frac{-\tilde\varphi+\tilde\psi}{2}}$ and $\varepsilon\in\{-1,+1\}$. The sign ε is due to the two possible choices of preimages via π. The Euler coordinates of these two preimages differ by a translation $\tilde\psi\mapsto\tilde\psi+2\pi$, therefore we can always restrict to $\tilde\psi\in[0,2\pi)$. □

Theorem 18 *Let* $X_n\colon SO(3)\to\mathbb{C}$ *satisfy the Assumption 1. Let* $\xi_n\colon\mathbb{D}\to\mathbb{C}$ *be the smooth GRF defined in Definition 14. Then* $K_{\xi_n}\to K_{\xi_\infty}$ *in* $\mathcal{C}^\infty(\mathbb{D}\times\mathbb{D},\mathbb{C})$.

Proof Let $z_1 = x_1 e^{i\varphi_1}$ and $z_2 = x_2 e^{i\varphi_2}$, then

$$\begin{aligned} K_{\xi_n}(z_1, z_2) &= \Gamma_n \left(R_2\Big(- \rho_n x_1 \Big) R_3\Big(\varphi_2 - \varphi_1 \Big) R_2\Big(\rho_n x_2 \Big) \right) e^{i s_n (\varphi_1 - \varphi_2)} \\ &= k_n(\tilde{\theta}_n) e^{i s_n (\tilde{\varphi}_n + \tilde{\psi}_n)} e^{i s_n (\varphi_1 - \varphi_2)}. \end{aligned} \tag{23}$$

Where $\tilde{\varphi}_n, \tilde{\theta}_n, \tilde{\psi}_n$ are the Euler angles defined as in Lemma 1, with $\theta_i = \rho_n x_i$ and $\varphi = \varphi_2 - \varphi_1$ so that

$$R\left(\tilde{\varphi}_n, \tilde{\theta}_n, \tilde{\psi}_n\right) = R_2\Big(- \rho_n x_1 \Big) R_3\Big(\varphi_2 - \varphi_1 \Big) R_2\Big(\rho_n x_2 \Big).$$

Then $\tilde{\theta}_n$ is the spherical distance between $\phi_n^{\mathbb{1}}(z_1)$ and $\phi_n^{\mathbb{1}}(z_2)$, hence it is given by the formula

$$\begin{aligned} \cos \tilde{\theta}_n &= \langle \phi_n^{\mathbb{1}}(z_1), \phi_n^{\mathbb{1}}(z_2) \rangle \\ &= \sin(\rho_n x_1) \sin(\rho_n x_2) \cos(\varphi_1 - \varphi_2) + \cos(\rho_n x_1) \cos(\rho_n x_2) \\ &= \left(\rho_n^2 x_1 x_2 + O\left(\rho_n^4\right) \right) \cos(\varphi_1 - \varphi_2) + \\ &\quad + \left(1 - \frac{1}{2} \rho_n^2 x_1^2 + O\left(\rho_n^4\right) \right) \left(1 - \frac{1}{2} \rho_n^2 x_2^2 + O\left(\rho_n^4\right) \right) \\ &= 1 + \rho_n^2 \langle z_1, z_2 \rangle - \rho_n^2 \frac{|z_1|^2}{2} - \rho_n^2 \frac{|z_2|^2}{2} + O\left(\rho_n^4\right) \\ &= 1 - \rho_n^2 \frac{|z_1 - z_2|^2}{2} + O\left(\rho_n^4\right). \end{aligned}$$

This implies that we have the following limit for the radial part of $K_{\xi_n}(z_1, z_2)$:

$$\begin{aligned} \lim_{n \to +\infty} k_n(\tilde{\theta}_n) &= \lim_{n \to +\infty} k_n \left(\arccos \left(1 - \frac{\rho_n^2 |z_1 - z_2|^2 + O(\rho_n^2)}{2} \right) \right) \\ &= \lim_{n \to +\infty} k_n \left(\rho_n |z_1 - z_2| + O(\rho_n^2) \right) \\ &= k_\infty(|z_1 - z_2|). \end{aligned}$$

By definition (see Lemma 1) we have

$$\begin{aligned} \alpha_n := \cos\left(\frac{\tilde{\theta}_n}{2}\right) e^{i \frac{\tilde{\varphi} + \tilde{\psi}}{2}} &= \cos\left(\frac{\rho_n x_1}{2}\right) \cos\left(\frac{\rho_n x_2}{2}\right) e^{i \frac{\varphi}{2}} + \\ &\quad + \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) e^{-i \frac{\varphi}{2}} \\ &= \cos\left(\frac{\rho_n x_1}{2}\right) \cos\left(\frac{\rho_n x_2}{2}\right) e^{i \frac{\varphi}{2}} (1 + t_1 t_2 e^{-i\varphi}), \end{aligned}$$

where $t_i = \tan(\frac{\rho_n x_i}{2}) = \frac{1}{2}\rho_n x_i + O(\rho_n^3)$ and $\varphi = \varphi_2 - \varphi_1$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} e^{is_n(\tilde{\varphi}_n+\tilde{\psi}_n)} e^{is_n(\varphi_1-\varphi_2)} &= \lim_{n\to\infty} \left(\frac{\alpha_n}{|\alpha_n|}\right)^{2s_n} e^{-is_n\varphi} \\
&= \lim_{n\to\infty} \frac{\left(1+t_1t_2e^{-i\varphi}\right)^{2s_n}}{\left|1+2t_1t_2\cos(\varphi)+t_1^2t_2^2\right|^{s_n}} \\
&= \lim_{n\to\infty} \frac{\left(1+\frac{\rho_n^2}{2}x_1x_2e^{-i\varphi}+O(\rho_n^4)\right)^{\frac{2}{\rho_n^2}(s_n\rho_n^2)}}{\left|1+\rho_n^2\frac{x_1x_2}{2}\cos(\varphi)+O(\rho_n^4)\right|^{\frac{1}{\rho_n^2}(s_n\rho_n^2)}} \\
&= \exp\left(\beta x_1x_2e^{-i\varphi}\right)\exp\left(-\beta\frac{x_1x_2}{2}\cos(\varphi)\right) \\
&= \exp\left(-\beta i x_1x_2\sin(\varphi_2-\varphi_1)\right) \\
&= \exp\left(\beta i\Im(z_1\overline{z_2})\right).
\end{aligned}$$

Combining the latter computation, with the first line (23) and with the estimate of $\tilde{\theta}$, we get that

$$\lim_{n\to+\infty} K_{\xi_n}(z_1,z_2) = k_\infty(|z_1-z_2|)\exp\left(\beta i\Im(z_1\overline{z_2})\right).$$

By an analogous argument, the above limit can be shown to hold in the $\mathcal{C}^\infty$ sense, thus we conclude. □

Remark 38 It was proved in [34], that the convergence of the covariance functions in the C^∞ topology is equivalent to the convergence in law $\xi_n \Rightarrow \xi_\infty$ as random elements of $C^\infty(\mathbb{D},\mathbb{C})$, i.e. to the weak-$*$ convergence of the corresponding sequence of probability measures. In particular, it also implies that the limit of the covariance functions, if exists, is the covariance function of a smooth Gaussian field, hence that Definition 15 is well posed. The following are equivalent (in virtue of Portmanteau's theorem) characterizations of such convergence:

1. For any continuous function $\mathcal{F}\colon \mathcal{C}^\infty(\mathbb{D},\mathbb{C}) \to [0,1]$, we have that

$$\lim_{\ell\to+\infty} \mathcal{F}(\xi_\ell) = \mathcal{F}(\xi_\infty).$$

2. For any Borel subset $B \subset \mathcal{C}^\infty(\mathbb{D},\mathbb{C})$, we have that

$$\mathbb{P}\{\xi_\infty \in \mathrm{int}(B)\} \le \liminf_{\ell\to+\infty} \mathbb{P}\{\xi_\ell \in B\} \le \limsup_{\ell\to+\infty} \mathbb{P}\{\xi_\ell \in B\} \le \mathbb{P}\{\xi_\infty \in \overline{B}\}.$$

8 Main results

8.1 Setting

In this section we will consider the following setting. Let $N \in \mathbb{N}$ and let $\underline{s_n} = (s_n^1, \dots, s_n^N) \in \mathbb{Z}^N$ be a sequence of N-tuples of spin weights. Let $\underline{\sigma_n}$ be a sequence of isotropic Gaussian random sections of the complex vector bundle

$$\mathcal{E}^{\underline{s_n}} = \mathcal{T}^{\otimes s_n^1} \oplus \cdots \oplus \mathcal{T}^{\otimes s_n^N},$$

i.e. $\underline{\sigma_n} = (\sigma_n^1, \dots \sigma_n^N)$ is a N-tuple of isotropic spin Gaussian fields and we assume that the whole collection $\{\sigma_n^i\}_{i,n}$ is an independent family. Let $\underline{X_n} = (X_n^1, \dots, X_n^N) \colon SO(3) \to \mathbb{C}^N$ be the corresponding sequence of isotropic Gaussian functions on $SO(3)$ and let $\underline{k_n} = (k_n^1, \dots, k_n^N) \colon \mathbb{R} \to \mathbb{R}^N$ be their circular covariance functions (see Sect. 4). Let $\underline{\beta} = (\beta^1, \dots, \beta^N) \in \mathbb{R}^N$.

Assumption 2 Assume that, for every $i = 1, \dots, N$, the sequence $\{\sigma_n^i\}_n$ satisfies Assumption 1 with $\beta = \beta^i$ with respect to the same infinitesimal sequence of positive real numbers $\rho_n \to 0^+$.

Let $\underline{\xi_n} = (\xi_n^1, \dots, \xi_n^N) \colon \mathbb{D} \to \mathbb{C}^N$ be the sequence of rescaled fields (see Definition 14) and let $\underline{\xi_\infty} = (\xi_\infty^1, \dots, \xi_\infty^N)$ be the N-tuple of limit fields (see Definition 15).

8.1.1 Jets and type-W singularities

As we did in Subsection 5.2, we identify $J^r(\mathbb{D}, \mathbb{C}^N) = \mathbb{D} \times J_0^r(\mathbb{D}, \mathbb{C}^N) = \mathbb{D} \times \mathbb{R}^k$, so that taking the jet at a point $p \in \mathbb{D}$ yields a map $j_p^r \colon \mathcal{C}^\infty(\mathbb{D}, \mathbb{C}^N) \to \mathbb{R}^k$.

Definition 16 For $n \in \mathbb{N} \cup \{\infty\}$, let $Y_n \colon \mathbb{D} \to \mathbb{R}^k$ be the Gaussian field such that $Y(p) = j_p^r \underline{\xi_n}$.

We will consider the random subset of the disk $\mathbb{D}$ given by the type-W singularity

$$Z_W(\underline{\sigma_n}) = j^r \underline{\sigma_n}^{-1}(W) \subset \mathbb{S}^2,$$

defined by a closed intrinsic semialgebraic subset $W \subset J^r(\mathbb{S}^2 | \mathcal{E}^{\underline{s_n}})$ modeled on $W_0 = \mathbb{D} \times \Sigma \subset J^r(\mathbb{C}, \mathbb{C}^N) = \mathbb{C} \times \mathbb{R}^k$ (see Subsection 5.2). Asking for the semialgebraicity of W is equivalent to assume that $\Sigma \subset \mathbb{R}^k$ is semialgebraic. For all $n \in \mathbb{N} \cup \{\infty\}$, let

$$Z_n := Z_{W_0}(\underline{\xi_n}) = Y_n^{-1}(\Sigma) \subset \mathbb{D}.$$

By construction (see the discussion before Definition 14), if $B_n \subset \mathbb{S}^2$ is a sequence of shrinking spherical balls of radius ρ_n, then there is a sequence of diffeomorphisms $\phi_{\rho_n}^g \colon \mathbb{D} \to B_n$ such that

$$\phi_{\rho_n}^g(Z_n) = Z_W(\underline{\sigma_n}) \cap \overline{B_n}. \tag{24}$$

Moroever, let $\mathring{Z}_n := Z_n \cap B_n = Z_n \setminus \partial B_n$.

8.1.2 Supports

Definition 17 For $n \in \mathbb{N} \cup \{\infty\}$, define $F_n := \operatorname{supp}(\underline{\xi_n}) \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{C}^N)$ to be the *topological support* of the law of $\underline{\xi_n}$, i.e.

$$F_n = \left\{ f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{C}^N) \colon \mathbb{P}\left\{ \underline{\xi_n} \in O_f \right\} > 0 \text{ for any } O_f \text{ open neighborhood of } f \right\}.$$

By standard arguments (see [34, 48], for instance), the above definition is well posed and the support F_n is always a closed subspace of $\mathcal{C}^\infty(\mathbb{D}, \mathbb{C}^N)$, indeed it is the smallest closed subset with $[\underline{\xi_n}]$-probability one. By construction we have that

$$\operatorname{supp}(Y_n(p)) = j^r_p F_n = \left\{ j^r_p f \colon f \in F_n \right\} \subset \mathbb{R}^k.$$

The next assumption ensures that the type-W singularities $Z_W(\underline{\sigma_n}) \subset \mathbb{S}^2$ and $Z_n \subset \mathbb{D}$ are nondegenerate, thus they are Whitney stratified subsets in the sense of subsection 5.2. This will be proved in Theorem 2 below.

Assumption 3 For every $n \in \mathbb{N} \cup \{\infty\}$, $j^r_0 F_n \pitchfork \Sigma$.

8.2 Convergence in distribution

In this subsection we provide a rigorous statement and a proof for Theorem 2 in Sect. 2.

Remark 39 We recall that, by construction, we have that the random subsets $Z_W(\underline{\sigma_n}) \cap B_{\rho_n} \cong Z_n$ are diffeomorphic in the sense of (24).

In the next statement, $\Sigma \subset \mathbb{C}^N$ is a Whitney stratified subset, defined as the second factor in the model $W_0 = \mathbb{D} \times \Sigma$ of W, as in Subsection 8.1.1 above.

Theorem 19 *Assume that Σ is closed and that Assumptions 2 and 3 are satisfied. Then the following properties hold.*

(1) Almost surely, $Z_W(\underline{\sigma_n}) \subset \mathbb{S}^2$ is nondegenerate for all $n \in \mathbb{N}$. The same holds for $Z_n \subset \mathbb{D}$ for all $n \in \mathbb{N} \cup \{+\infty\}$.

(2) There exists a discrete limiting probability law $p_W(S)$ on the set of diffeomorphisms classes of Whitney stratified subsets $S \subset \mathbb{D}$:

$$\exists \lim_{n \to +\infty} \mathbb{P}\{Z_n \textit{ is diffeomorphic to } S\} = p_W(S).$$

(3) Whenever S is diffeomorphic to a nondegenerate type-W singularity of some smooth function $f \in F_\infty$, we have that $p_W(Z) > 0$.

(4) There is convergence in law: $\mathcal{L}_i(Z_n) \Rightarrow \mathcal{L}_i(Z_\infty)$ and $b_i(Z_n) \Rightarrow b_i(Z_\infty)$.

Proof (1). We want to apply [34, Theorem 7] to the random section $\underline{\sigma_n}$ and to the finite union of smooth submanifolds $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}_n})$. To see that the hypotheses of the theorem are satisfied, just observe that, if W is intrinsic with model $W_0 = \mathbb{D} \times \Sigma$ and $\underline{\sigma_n}$ is isotropic, then $\text{supp}[j^r_p\underline{\sigma_n}] \pitchfork W$ if and only if $\text{supp}[j^r\underline{\xi_n}] \pitchfork W_0$, if and only if $j^r_0 F \pitchfork \Sigma$. Therefore, by [34, Theorem 7], we have that

$$\mathbb{P}\{j^r\underline{\sigma_n} \pitchfork W\} = 1 \quad \text{and} \quad \mathbb{P}\{Y_n \pitchfork \Sigma, Y_n|_{\partial\mathbb{D}} \pitchfork \Sigma\} = 1,$$

for all $n \in \mathbb{N}$. The second identity holds for $n = \infty$ as well, for the same reason.

(2). Consider the set:

$$U_S := \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) : f^{-1}(W) \text{ is diffeomorphic to } S\}.$$

As it is explained in [34], by Thom's isotopy theorem, if f_t is a homotopy of maps such that $f_t \pitchfork \Sigma$ and $f_t|_{\partial\mathbb{D}} \pitchfork \Sigma$, then the diffeotopy type of the pair $(\mathbb{D}, f_t^{-1}(\Sigma))$ is constant. Moreover, if Σ is closed, then the transversality condition is open in the space of smooth functions, therefore we have that

$$\text{int}(U_S) = U_S \smallsetminus \Delta_\Sigma \quad \text{and} \quad \partial U_S \subset \Delta_\Sigma,$$

where $\Delta_s = \{f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) : f \not\pitchfork \Sigma \text{ or } f|_{\partial\mathbb{D}} \not\pitchfork \Sigma\}$. Notice that by Theorem 18 and Remark 38 we have

$$\mathbb{P}\{Y_\infty \in \text{int}(U_S)\} \leq \liminf_{n\to+\infty} \mathbb{P}\{Y_n \in U_S\} \leq \limsup_{n\to+\infty} \mathbb{P}\{Y_n \in U_S\} \leq \mathbb{P}\{Y_\infty \in \overline{U_S}\}.$$

Therefore, since by point (1) we have that $\mathbb{P}\{Y_n \in \Delta_W\} = 0$, it follows that

$$\exists \lim_{n\to\infty} \mathbb{P}\{Y_n \in U_S\} = \mathbb{P}\{Y_\infty \in U_S\}.$$

(3). Let $f \in U_S \cap F_\infty \smallsetminus \Delta_\Sigma$ and assume that S is diffeomorphic to $f^{-1}(\Sigma)$. Then, by Thom isotopy theorem again, the same holds for all g on a neighborhood $O_f \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k)$ of f. In other words the set $U_S \cap F_\infty \smallsetminus \Delta_\Sigma$ is open. Since Δ_Σ has zero probability, we have that $\mathbb{P}\{Y_\infty \in U_S\} > 0$ if and only if $U_S \cap F_\infty \smallsetminus \Delta_\Sigma \neq \emptyset$. This proves (3).

(4). The convergence in law of Betti numbers follows directly from (2). For the Lipschitz–Killing curvatures $\mathcal{L}_i$ we could essentially repeat the argument used to prove (2). A more direct way is to observe that the functional $L : f \mapsto \mathcal{L}_i(f^{-1}(\Sigma))$ is continuous on $\mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) \smallsetminus \Delta_\Sigma$. Since $\mathbb{P}\{Y_\infty \in \Delta_\Sigma\} = 0$, this implies that the composition $L \circ Y_n$ converges in law in $\mathbb{R}$. □

8.3 Convergence of expectations

In this subsection we provide a rigorous statement and a proof of Theorem 3 in section 2. For the next result we need the following further technical condition.

Assumption 4 We assume that for every $n \in \mathbb{N} \cup \{+\infty\}$, the dimension of $j_0^r F_n$ and that of $j_0^{r+1} F_n$ are constant.

Let $\mathring{Z}_\infty := \mathbb{E}\mathcal{L}_i(Z_{W_0}(\xi_\infty) \cap \mathrm{int}(\mathbb{D}))$.

Theorem 20 *Assume that $\Sigma \subset \mathbb{C}^N$ is closed and that Assumption 2, Assumptions 3 and 4 are satisfied. Assume that $j_0^r F_n \pitchfork \Sigma$ for all $n \in \mathbb{N} \cup \{\infty\}$. For all $i = 0, 1, 2$ we have:*

1. $\mathbb{E}\mathcal{L}_i(Z_W(\sigma_n)) = \rho_n^i \frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})} \left(\mathring{Z}_\infty + O(1)\right).$
2. *There are constants $c_i^W \geq 0$, $C_i^W > 0$ such that*

$$\frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_\ell})} c_i^W \leq \mathbb{E}b_i(Z_W(\sigma_n)) \leq \frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})} C_i^W;$$

3. *If there exists a smooth function $f \in F_\infty$ such that $Z_{W_0}(f)$ is regular and it has a connected component $C \subset \mathrm{int}(\mathbb{D})$, with $b_i(C) > 0$, then $c_i^W > 0$.*

Remark 40 We will see below, see Lemma 5, that the assumptions of Theorem 20 above are satisfied in the case of the excursion set of a monochromatic spin field; from this we will deduce Theorem 5.

8.3.1 Outline of the proof of Theorem 3 and Theorem 20

1. As it will be clear from points (2) and (3), it is enough to prove the theorem in the case when $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}} \oplus \mathbb{C})$ has codimension 2, i.e. when the only nontrivial quantity is the cardinality $\# = b_0$ of the random finite set $Z^W(\underline{\sigma})$. However, we need to prove this case in a slightly more general form to include weighted count of such set of points: Let $\alpha : \mathcal{C}^\infty(\mathbb{S}^2|\mathcal{E}^{\underline{s}}) \times \mathbb{S}^2 \to \mathbb{R}$ be a measurable function and define, for any $A \subset \mathbb{S}^2$

$$\#^\alpha_{j^r \underline{\sigma} \in W}(A) := \sum_{p \in Z^W(\underline{\sigma}) \cap A} \alpha(\underline{\sigma}, p).$$

In [53, Theorem 4.1] it is shown that these kinds of counting measures admit an integral formula (Kac–Rice-style).

$$\mathbb{E}\left\{\#^\alpha_{j^r \underline{\sigma} \in W}(A)\right\} = \int_A \delta^\alpha_{j^r \underline{\sigma} \in W}$$

Arguing as in the proof of [36, Theorem 27] and [53, Corollary 3.9] we will be able to understand their asymptotic behavior. This, together with items 2 and 3 below, will also prove the general case automatically.

2. We then exploit Theorem 17 to show that for any $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$, and every $i = 0, 1, 2$, there exists another singularity type $W_i \subset J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}} \oplus \mathbb{C})$, having

codimension 2 and $\alpha = \alpha_{W_i} : \mathcal{C}^\infty(\mathbb{S}^2|\mathcal{E}^{\underline{s}} \otimes \mathbb{C}) \times \mathbb{S}^2 \to \mathbb{R}$ measurable and bounded such that

$$\mathcal{L}_i(Z^W(\underline{\sigma})) = \sum_{p \in Z^{W_i}(\underline{\sigma}, \sigma_0)} \alpha((\underline{\sigma}, \sigma_0), p) := \#^\alpha_{j^{r+1}(\underline{\sigma}, \sigma_0) \in W_i},$$

where $\sigma_0 : \mathbb{S}^2 \to \mathbb{C}$ is a random function with spin 0.

3. Finally, we establish a similar statement for the Betti number, although here we are only able produce an inequality (derived from Morse inequalities):

$$b_i(Z^W(\underline{\sigma})) \leq C_W \sum_{p \in Z^{\hat{W}}(\underline{\sigma}, \sigma_0)} \alpha((\underline{\sigma}, \sigma_0), p) := \#^\alpha_{j^{r+1}(\underline{\sigma}, \sigma_0) \in W'},$$

for some constant $C_W > 0$ depending only on W and some higher singularity type $\hat{W} \subset J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}} \oplus \mathbb{C})$ of codimension 2. This follows again from Theorem 17.

9 Proof of Theorem 3 and Theorem 20

In this section we give a full proof of the convergence of the expectation; the proof is split into three steps, as described above; the proof is based upon the generalized Kac–Rice formula proved in [53].

9.1 Step 1

The following theorem is the main technical result of this section.

Theorem 21 *Let $\underline{\sigma_n}$ be the sequence of isotropic Gaussian multi-spin functions that falls in the setting described in Sect. 8.1, in addition to assumptions 2, 3 and 4, assume that $W \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}})$ has codimension 2. Let $\alpha : W' \to \mathbb{R}$ be a bounded continuous and intrinsic function (see Definitions 8 and 9), where*

$$W' := \left\{ j_p^{r+h}\underline{\sigma} \in J^{r+h}(\mathbb{S}^2|\mathcal{E}^{\underline{s}}) : j_p^r\underline{\sigma} \in W, d_p\left(j^r\underline{\sigma}\right) \pitchfork T_{j_p^r\underline{\sigma}}W \right\}$$

and define, for $A \subset \mathbb{S}^2$,

$$\#^\alpha_{j^r\underline{\sigma} \in W}(A) := \sum_{p \in Z_W(\underline{\sigma}) \cap A} \alpha(j_p^{r+h}\underline{\sigma}).$$

Then

$$\mathbb{E}\#^\alpha_{j^r\underline{\sigma} \in W}(\mathbb{S}^2) = \frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})} \left(\mathbb{E}\#^{\alpha_0}_{j^r\xi_\infty \in W_0}(\mathbb{D}) + o(1) \right).$$

Proof Since α and W are assumed to be intrinsic, then W' is also intrinsic because of Theorem 17 and it follows that the signed measure $A \mapsto \mathbb{E}\#^{\alpha}_{j^r \underline{\sigma} \in W}(A)$ is a multiple of the volume measure on $\mathbb{S}^2$. This reduces the problem to its local counterpart, which is Lemma 2, applied to the sequence of fields Y_n defined as in Definition 16. □

Let us take up the notations introduced in Subsection 8.1.1 above. Define the subset $\Sigma' \subset J^h(\mathbb{D}, \mathbb{R}^k)$ such that

$$\Sigma' := \left\{ j^h_p f \in J^h(\mathbb{D}, \mathbb{R}^k) : f(p) \in \Sigma \text{ and } d_p f \pitchfork T_{f(p)}\Sigma \right\}.$$

Here, recall that $T_y\Sigma$ is the tangent space to the stratum of Σ containing y.

Definition 18 Let $T_\Sigma \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k)$ be the set

$$T_\Sigma := \{ f \in \mathcal{C}^\infty(\mathbb{D}, \mathbb{R}^k) : f \pitchfork \Sigma \}.$$

Let $\alpha : T_\Sigma \to \mathbb{R}$, let $f : \mathbb{D} \to \mathbb{R}^k$ and $A \subset \mathbb{D}$. We define

$$\#^{\alpha}_{f \in \Sigma}(A) := \sum_{p \in f^{-1}(\Sigma) \cap A} \alpha(p, f).$$

If $\alpha : \Sigma' \to \mathbb{R}$, we use the same letter to denote $\alpha : T_\Sigma \to \mathbb{R}$, such that $(p, f) \mapsto \alpha(j^h_p f)$.

Notice that $f \in T_\Sigma$ if and only if $j^h_p f \in \Sigma'$ for every $p \in f^{-1}(\Sigma)$.

Lemma 2 *Let Y_n be as in Definition 16 under the assumptions 2, 3 and 4, let Σ be closed, semialgebraic and have codimension 2. Let $\alpha : T_\Sigma \to \mathbb{R}$ be continuous and bounded.*

$$\lim_{n \to \infty} \mathbb{E}\#^{\alpha}_{Y_n \in \Sigma}(\mathbb{D}) = \mathbb{E}\#^{\alpha}_{Y_\infty \in \Sigma}(\mathbb{D}).$$

In the following, we will take up the notations of [53], in which a Kac–Rice formula for the expectation $\mathbb{E}\#^{\alpha}_{Y_n \in \Sigma}$ is proved: the formula (26) below is given by [53, Theorem 4.1]. In particular, given two subspaces $V, W \subset \mathbb{R}^k$, the quantity $\sigma_{\mathbb{R}^k}(V, W)$ is the product of the sines of the principal angles in $\mathbb{R}^k$ between the vector subspaces V and W, See [53, Appendix B]. We will omit the subscript and write just $\sigma(V, W)$, when the ambient space is clear. Moreover, if S is a Riemannian manifold we denote its Riemannian volume density at $y \in S$ as $dS(y)$, so that the integral of a function $f : S \to \mathbb{R}$ with respect to the Riemannian volume density will be written as $\int_S f dS = \int_S f(y) dS(y)$, see [53, Appendix A].

Proof A consequence of Assumption 3 is that $Y_n \pitchfork \Sigma$ with probability one (see 19), therefore $Z_n = Y_n^{-1}(\Sigma)$ is a random discrete subset. Observe that for any fixed n, the support of $Y_n(z)$ is of the form

$$\text{supp}[Y_n(z)] = j^r_z F_n = \text{supp}\left[j^r_z(\underline{\sigma}_n \circ \phi^g_{\rho_n}) \right], \tag{25}$$

hence it is canonically determined from the supports of $j^s_{\phi^g_{\rho_n}(z)}(\underline{\sigma}_n) = j^s_0(\underline{\sigma}_n)$, with $s \leq r$ through a general formula (that of the Talylor polynomial of a composition) involving the derivatives of $\phi^g_{\rho_n}$ at z. This entails that dim supp$[Y_n(z)]$ is constant in z and that we can see Y_n as a non-degenerate $\mathcal{C}^\infty$ Gaussian random section of the vector bundle $\pi : E^n \subset \mathbb{D} \times \mathbb{R}^k \to \mathbb{D}$, with fiber $E^n_z := \{z\} \times j^r_z F_n$. Then, we shall consider the set

$$W^{(n,\Sigma)} := \sqcup_{z\in\mathbb{D}}\{z\} \times (j^r_z F_n \cap \Sigma) = (\mathbb{D} \times \Sigma) \cap E^n$$

noting that, Assumption 3 entails that we have $W^{(n,\Sigma)} \pitchfork E^n_z$ (meaning that the top strata are transverse) for all $z \in \mathbb{D}$. This construction and the observation at Equation (25) also ensures that $W^{(n,\Sigma)}$ so constructed has *sub-Gaussian concentration* in the sense of [53, Definition 3.7]; to see this, one can argue as for [53, Remark 3.3 and Lemma 9.2], by observing that the derivatives of $\phi^g_{\rho_n}$ are trigonometric polynomials. Using [53, Theorem 3.8] (in the form of Theorem [53, Theorem 4.1], for general α), we deduce that the following formula is finite for any $A \subset \mathbb{D}$ Borel subset:

$$\begin{aligned} \mathbb{E}\#^\alpha_{Y_n\in\Sigma}(A) &= \int_A \delta^\alpha_n(z)dz \\ &= \int_A \int_{\Sigma\cap j^r_p F_n} \mathbb{E}\left\{\alpha(j^h_z Y_n) J_z Y_n \frac{\sigma_y(d_z Y_n, \Sigma)}{\sigma_y(j^r_p F_n, \Sigma)} \Big| Y_n(z) = y\right\} \\ &\quad \times \rho_{Y_n(z)}(y) d\left(\Sigma \cap j^r_p F_n\right)(y) dz, \end{aligned} \tag{26}$$

where, for $y \in$ supp$[Y_n(z)]$, we define $\rho_{[Y_n(z)]}(y)$ to be the density of the Gaussian random vector $Y_n(z)$ evaluated at $y \in$ supp$[Y_n(z)] = j^r_z F_n$. The quantity $\sigma_y(d_z Y_n, \Sigma)$ is the product of the sines of the principal in $\mathbb{R}^k$ between the vector subspaces $d_z Y_n(\mathbb{R}^2)$ and $T_y\Sigma$ and it is defined for all $y \in \Sigma$ as $\sigma_y(d_z Y_n, \Sigma) = \sigma_{\mathbb{R}^k}(d_z Y_n(\mathbb{R}^2), T_y\Sigma)$, see [53, Appendix B] for the precise definition and more details.

Finally, the Assumption 4 ensures that all vector bundles E^n have the same rank for all $n \in \mathbb{N} \cup \{+\infty\}$ and Theorem 2 implies that $E^n \to_{n\to+\infty} E^\infty$ in a smooth sense (i.e., as smooth maps to the Grassmannian). Therefore, by composing Y_n with a sequence of bundle diffeomorphisms $\psi_n : E^n \to E^\infty$, we can argue as if $E^n = E$ was a fixed vector bundle and apply the second assertion of [53, Theorem 3.8], which states that

$$\mathbb{E}\#^\alpha_{Y_n\in\Sigma}(A) \to \mathbb{E}\#^\alpha_{Y_\infty\in\Sigma}(A)$$

as wanted. □

9.2 Step 2: Lipschitz–Killing curvatures

Let $\sigma^n_0 : \mathbb{S}^2 \to \mathbb{C}$ be a reindexing of the sequence of isotropic smooth Gaussian random function defined in Example 2 (see also Eq. (3)), with spin equal to zero.

$$\sigma_0^n = \sum_{m=-\ell(n)}^{\ell(n)} a_{m,0}^{\ell(n)} D_{m,0}^{\ell(n)}.$$

The circular covariance function of σ_0^n is $k_{\sigma_0^n}(\theta) = d_{0,0}^{\ell(n)}$, which satisfies the scaling Assumption 1 with rate $\ell(n)^{-1}$ and $k_\infty = J_0$. Clearly we can choose $\ell(n)$ so that $\ell(n)^{-1} \sim \rho_n$, by repeating or skipping some ℓs. Define $h_n := \Re(\sigma_0^n)$.

Let $\underline{s}_n' := (0, \underline{s}_n)$, $\mathcal{E}^{\underline{s}_n}{}' = \mathbb{C} \oplus \mathcal{E}^{\underline{s}_n}$ and define $\underline{\sigma_n'} := (\sigma_0^n, \underline{\sigma_n}) \in \mathcal{C}^\infty(\mathbb{S}^2|\mathcal{E}^{\underline{s}_n}{}')$. Now, observe that this new sequence of Gaussian isotropic multi-spin sections $\underline{\sigma_n'}$ satisfies Assumption 2, with the same shrinking rate $\rho_n \to 0$.

We can now exploit Theorem 21 to prove each case (Lipschitz–Killing curvatures and Betti numbers) of Theorem 20.

9.2.1 The Euler–Poincaré characteristic

By Assumption 3 and Theorem 2.(1), we know that $j^r \underline{\sigma_n} \pitchfork W$. Consider the semialgebraic intrinsic subset $W' \subset J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}_n}{}')$ defined in Theorem 17. We claim that $\underline{\sigma_n'}$ satisfies Assumption 2 with respect to W', as well. The reasons why this is true are two: first, the support of $j_p^1 h_n$ is the whole fiber of the jet space: $J_0^1(\mathbb{S}^2, \mathbb{R})$ and second, the structure of W', established by Theorem 17.(4), implies that the normal bundle $N_w S'$ of any stratum S' of W' at a point $w \in S'$ projects onto the space $J^1(\mathbb{S}^2, \mathbb{R}) \times N_{\pi_1(w)} S$ via the natural map

$$\begin{gathered} J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}_n}{}')/T_w S' = N_w S' \xrightarrow{\pi=(\pi_1,\pi_2)} N_{\pi_1(w)} S \times J^1(\mathbb{S}^2, \mathbb{R}), \\ j_p^{r+1}(\sigma_0, \underline{\sigma}) + T_w S' \mapsto (j^r \underline{\sigma} + T_{\pi_1(w)} S, \, j^1 \Re(\sigma_0)) \end{gathered}$$

where $N_{\pi_1(w)} S$ is the normal bundle of the stratum S of W (meant as a quotient of the ambient space modulo $T\mathbb{S}$), containing $\pi_1(w)$ (by definition $w \in W'$ only if $\pi_1(w) \in W$). Therefore, if $\text{supp}(j_p^r \underline{\sigma_n}) \pitchfork W$ and $\text{supp}(j^1 \Re(\sigma_0^n)) = J_p^1(S^2, \mathbb{R})$, then $\text{supp}(j_p^{r+1} \underline{\sigma_n'}) \pitchfork W'$. The latter condition is equivalent to Assumption 3. By Theorem 17, we have that, almost surely,

$$\mathcal{L}_0\left(Z_W(\underline{\sigma_n})\right) = \chi\left(Z_W(\underline{\sigma_n})\right) = \sum_{p \in Z_{W'}(\underline{\sigma_n'})} \alpha'(j_p^{r+2} \underline{\sigma}') = \#^{\alpha'}_{j^{r+1}\underline{\sigma_n'} \in W''}(A).$$

By the previous discussion, we see that we are now in position to apply Theorem 21 to the sequence $\underline{\sigma_n'}$, the semialgebraic intrinsic submanifold W'' and the intrinsic function α', therefore

$$\mathbb{E}\mathcal{L}_0\left(Z_W(\underline{\sigma_n})\right) = \mathbb{E}\#^{\alpha'}_{j^{r+1}\underline{\sigma_n'} \in W''}(\mathbb{S}^2) = \frac{\text{vol}(\mathbb{S}^2)}{\text{vol}(B_{\rho_n})} \left(\mathbb{E}\#^{\alpha_0'}_{j^{r+1}\xi_\infty \in W_0''}(\mathbb{D}) + o(1)\right). \tag{27}$$

Finally, we conclude by observing that

$$\mathbb{E}\#^{\alpha_0'}_{j^{r+1}\xi_\infty \in W_0''}(\mathbb{D}) = \mathcal{L}_0\left(Z_\infty, \mathring{Z}_\infty\right) = \mathcal{L}_0\left(\mathring{Z}_\infty\right).$$

This follows from the fact that both $\mathbb{E}\#^{\alpha'}_{j^{r+1}\underline{\sigma_n'}\in W''}(\cdot)$ and $\mathbb{E}\mathcal{L}_0\left(Z_W(\underline{\sigma_n}), \cdot\right)$ are invariant measures on $\mathbb{S}^2$, thus equation (27) implies that they are equal. Evaluating on the (open) ball B_n gives

$$\mathbb{E}\#^{\alpha_0'}_{j^{r+1}\xi_n \in W_0''}(\mathbb{D}) = \mathbb{E}\#^{\alpha'}_{j^{r+1}\underline{\sigma_n'}\in W''}(B_n) = \mathbb{E}\mathcal{L}_0\left(Z_W(\underline{\sigma_n}), B_n\right) = \mathbb{E}\mathcal{L}_0\left(Z_n, \mathring{Z}_n\right).$$

9.2.2 The first intrinsic volume

Notice that $h_n \in \ker(\Delta - \lambda(n))$, with $\lambda(n) = \ell(n)(\ell(n)+1) \sim \rho_n^{-2}$. Indeed, $k''_{\sigma_0^n}(0) = (d^\ell_{0,0})''(0) = \frac{\ell(\ell+1)}{2}$, thus, by Proposition 23, we see that the conformal factor of the Adler-Taylor metric g^h of h is $\frac{\lambda(n)}{2}$. Therefore, expressing $\mathcal{L}_1$ as in Sect. 6.2 and using the formula 22, we have the identity:

$$\begin{aligned}\mathcal{L}_1(Z_W(\underline{\sigma_n})) &= \frac{1}{2}\int_{\partial_1 Z_W(\underline{\sigma_n})} \beta^1(j^r_p\underline{\sigma_n})d\mathcal{H}^1(p)\\ &= \left(\frac{\lambda(n)}{2}\right)^{-\frac{1}{2}} \frac{\pi}{2}\mathbb{E}\left(\sum_{p\in\partial_1 Z_W(\underline{\sigma_n})\cap\{h_n=0\}} \beta^1(j^r_p\underline{\sigma_n})\right),\end{aligned}$$

where $\beta_1(j^r_p\underline{\sigma_n})$ is defined as 2 minus the number of 2−dimensional strata of $Z_W(\underline{\sigma_n}) = j^r\underline{\sigma_n}^{-1}(W)$ that are adjacent to p (see Sect. 6.2). Therefore, reasoning as for the Euler-Poincaré characteristic, we can easily define $W' \subset J^r(\mathbb{S}^2|\mathcal{E}^{\underline{s}'})$ and α intrinsic such that $\partial_1 Z_W(\underline{\sigma_n}) \cap \{h_n = 0\} = Z_{W'}(\underline{\sigma_n'})$ and $\beta^1(j^r_p\underline{\sigma_n}) = \alpha(j^r_p\underline{\sigma_n'})$, so that Theorem 21 yields:

$$\begin{aligned}\mathbb{E}\mathcal{L}_1(Z_W(\underline{\sigma_n})) &= \sqrt{\lambda(n)}\frac{\pi}{2\sqrt{2}}\mathbb{E}\#^{\alpha}_{\underline{\sigma_n'}\in W'}(\mathbb{S}^2)\\ &= \rho_n\frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})}\left(\frac{\pi}{2\sqrt{2}}\mathbb{E}\#^{\alpha_0}_{\xi_\infty'\in W_0'}(\mathbb{D}) + o(1)\right)\\ &= \rho_n\frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})}\left(\mathbb{E}\mathcal{L}_1(Z_\infty) + o(1)\right).\end{aligned}$$

The last identity is due to another application of Proposition 22. Indeed, by construction, we have that $(\xi'_\infty)^{-1}(W_0') = \xi_\infty^{-1}(W_0) \cap \{h_\infty = 0\}$ and $\xi_\infty^{-1}(W) = Z_\infty$, where h_∞ is the scaling limit of h_n, that is the real Berry field h_∞, with covariance J_0. Here, $\lambda = 1$ (indeed h_∞ is a solution of the Helmoltz equation, see Appendix C thus the conformal factor is $\frac{1}{2}$.

9.2.3 The area

The case of $\mathcal{L}_2$ is the easiest and it can be proven directly by changing the order of integration in

$$\mathcal{L}_2(Z_W(\underline{\sigma_n})) = \int_{\mathbb{S}^2} 1_W(j^r\underline{\sigma_n}),$$

or by reasoning analogously to the previous case, taking an additional random function h'_n as an independent copy of h_n and using Proposition 22 to obtain the identity:

$$\mathcal{L}_2(Z_W(\underline{\sigma_n})) = \lambda(n)\pi\mathbb{E}\left(\#\partial_2 Z_W(\underline{\sigma_n}) \cap \{h_n = 0\} \cap \{h'_n = 0\}\right).$$

9.3 Step 3: Betti numbers

Let us define $\underline{\sigma'_n}$ as above and let us consider the same $W' \subset J^{r+1}(\mathbb{S}^2|\mathcal{E}^{\underline{s}_n}{}')$ as in the case of the Euler-Poincaré characteristic (see 9.2.1), i.e. the one coming from Theorem 17. Let $i \in \{0, 1, 2\}$. By point (2) of Theorem 17, we have, for some $N_W > 0$, the inequality

$$b_i\left(Z_W(\underline{\sigma})\right) \le N_W \# Z_{W'}(\underline{\sigma}').$$

Taking the expectation on both sides and using Theorem 21, we deduce the upper bound:

$$\mathbb{E}b_i\left(Z_W(\underline{\sigma})\right) \le \frac{\text{vol}(\mathbb{S}^2)}{\text{vol}(B_{\rho_n})}\left(\mathbb{E}\#Z_{W'_0}(\xi_\infty) + o(1)\right).$$

It remains to show the lower bound, with the additional hypothesis that there exists $f \in F_\infty$ such that $Z_{W_0}(f)$ is regular and contains a closed connected component $C \subset Z_{W_0}(f) \cap \text{int}(\mathbb{D})$ such that $b_i(C) > 0$. Recall that, by definition, the regularity of $Z_{W_0}(f)$ is equivalent to the transversality of the map $j^r f : \mathbb{D} \to J^r(\mathbb{D}, \mathbb{C}^N)$ to each stratum of W. Since W is assumed to be closed, such condition is open, i.e. there is a whole open subset $\mathcal{U} \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{C}^N)$ such that for all $g \in U$, we have that $j^r g \pitchfork W$. It follows that $\mathcal{U} \cap F_\infty$ is a non-empty open subset of F_∞, because it contains f. Define $U \subset F_\infty$ to be the path connected component of $\mathcal{U} \cap F_\infty$, so that for any $g \in U$ there is a homotopy $g_t \in F_\infty$ of smooth maps, such that $g_0 = f$ and $g_1 = g$ and such that $j^r g_t \pitchfork W$ for every t. By Thom Isotopy Theorem, it follows that the isotopy type of $j^r g_t^{-1}(W)$ is constant, hence that there is a connected component C_t with $b_i(C_t) \ge 1$. As a consequence we get that

$$\mathbb{E}b_i(Z_\infty^{(\text{int})}) \ge \mathbb{P}(\xi_\infty \in U) > 0,$$

where $Z_\infty^{(\text{int})}$ is the union of all the connected components of Z_∞ that are contained in the interior of $\mathbb{D}$. because U is a non-empty open subset of F_∞, the topological support of ξ_∞.

After this consideration we can repeat the argument used in [36] to prove the lower bound on the expectation of Betti numbers of Kostlan singularities. Consider a sequence of subsets $\mathcal{B}_n \subset \mathbb{S}^2$ such that for every n, $\mathcal{B}_n$ is a disjoint union of L_n balls of radius ρ_n. Here, we can assume that

$$L_n \geq c\frac{\mathrm{vol}(\mathbb{S}^2)}{\mathrm{vol}(B_{\rho_n})}.$$

for some fixed constant $c > 0$. Then we have that

$$\mathbb{E}b_i(Z_W(\underline{\sigma_n})) \geq \mathbb{E}\left(b_i\left((Z_W(\underline{\sigma_n}) \cap \mathcal{B}_n)^{(\mathrm{int})}\right)\right) = L_n\mathbb{E}\left(b_i\left(Z_n^{(\mathrm{int})}\right)\right),$$

where $X^{(\mathrm{int})}$ is the union of the closed connected components of X. The last identity is due to the isotropy of $\underline{\sigma_n}$ and to the fact that W is intrinsic. To conclude, it is sufficient to show that

$$\liminf_{d\to+\infty} \mathbb{E}\left(b_i\left(Z_n^{(\mathrm{int})}\right)\right) \geq \mathbb{E}b_i(Z_\infty^{(\mathrm{int})}).$$

The latter inequality follows from Fatou's Lemma applied to the sequence of random variables $b_i(Z_n^{(\mathrm{int})})$, which converges in law because of Theorem 2.

Proof (This concludes the proof of Theorem 20 and of Theorem 3) □

10 Monochromatic waves

The previous sections established a general framework to investigate the geometry of spin fields. In the present section, we specify those results to the more definite cases where the singular set $Z_W(\sigma_\ell)$ are the excursion sets of a sequence of monochromatic fields σ_ℓ with spin $s(\ell)$. To this aim, our first tool is to establish the local scaling behavior of the circular covariance function in this particular case.

We recall from the introduction that a spin monochromatic Gaussian random wave takes the form:

$$\sigma_\ell = \sum_{m=-\ell}^{\ell} a_{m,s(\ell)}^{\ell} Y_{m,s(\ell)}^{\ell} \in \mathcal{C}^\infty(\mathbb{S}^2|\mathcal{T}^{\otimes s(\ell)}),$$

where $Y_{m,s(\ell)}^{\ell}$ denote spin spherical harmonics and $a_{m,s(\ell)}^{\ell}$ are i.i.d. complex Gaussian variables. The field is normalized to have unit variance $\mathbb{E}\{\|\sigma_\ell(p)\|^2\} = 1$ for every $p \in \mathbb{S}^2$. As mentioned in the introduction, we will allow the spin value to depend on ℓ:

$$|s_\ell| = \ell - r_\ell$$

and we will focus on three different cases:

a. (The Berry regime) $\liminf_{\ell\to\infty} r_\ell = +\infty$, thus the Assumption (1) is satisfied with $\beta = 0$; the shrinking rate is $\rho_\ell = \rho_\ell(s_\ell)$ defined in Eq. (4), hence in particular, $\rho_\ell = \frac{1}{\ell}$ when s is fixed. For the covariance function $k_\ell = d^\ell_{-s,-s}$, it can be checked by Hilb's asymptotics (see Appendix D)

$$d^\ell_{-s-s}\left(\frac{t}{\ell+\frac{1}{2}}\right) = J_0(t)\left(\sqrt{\frac{\frac{t}{(\ell+\frac{1}{2})}}{\sin\left(\frac{t}{(\ell+\frac{1}{2})}\right)}}\right) + \delta_\ell\left(\frac{t}{(\ell+\frac{1}{2})}\right),$$

and hence we have

$$k_\ell\left(\rho_\ell\ \cdot\right) = d^\ell_{-s_\ell,-s_\ell}\left(\frac{\cdot}{2\sqrt{(r_\ell+1)(2\ell-r_\ell)}}\right) \xrightarrow[\ell\to+\infty]{\mathcal{C}^\infty} J_0(\cdot).$$

B. (Middle regime) In this case $s_\ell = \ell - r$ for some fixed $r \in \mathbb{N}$; it is possible to establish the asymptotic convergence of the covariance function to an explicit analytic function M_r, see Eq. (C4). The shrinking rate is:

$$\rho_\ell = \frac{1}{\sqrt{2(r_\ell+1)\ell}}.$$

C. (Complex Bargmann–Fock/Gaussian entire process) In the particular case $\ell = s_\ell$, that is $r_\ell = 0$, we see that scaling Hypothesis 1 is again satisfied, with the shrinking rate of $\rho_\ell = \sqrt{\frac{1}{2\ell}}$ and $k_\infty(x) = e^{-\frac{x^2}{4}}$, so that $\beta = \frac{1}{2}$ and

$$k_\ell\left(\rho_\ell\ \cdot\right) = d^\ell_{-\ell,-\ell}\left(\frac{\cdot}{\sqrt{2\ell}}\right) \xrightarrow[\ell\to+\infty]{\mathcal{C}^\infty} k_\infty(\cdot).$$

This confirms the fact that, in the case $\ell = s$, the spin field σ_ℓ is an holomorphic section of $\mathcal{T}^{\otimes s} = O(2s)$, in that the limit field ξ_∞ is a deterministic multiple of the complex Bargmann–Fock GRF, which is almost surely holomorphic:

$$\xi_\infty(z) = \left(\sum_{n=0}^{\infty}\gamma_k\left(\frac{1}{n!}\right)^{\frac{1}{2}}\left(\frac{z}{\sqrt{2}}\right)^n\right)e^{-\frac{|z|^2}{4}},$$

where $\gamma_k \sim N_{\mathbb{C}}(0,1)$ are i.i.d. This can be seen by computing the covariance function:

$$\begin{aligned} K_{\xi_\infty}(z_1,z_2) &= \mathbb{E}\{\xi_\infty(z_1)\overline{\xi_\infty(z_2)}\} = \exp\left(\frac{z_1\overline{z_2}}{2}\right)e^{-\frac{|z_1|^2}{4}}e^{-\frac{|z_2|^2}{4}} \\ &= \exp\left(-\frac{|z_1-z_2|^2}{4}\right)\exp\left(\frac{i}{2}\Im(z_1\overline{z_2})\right) = \\ &= k_\infty\left(|z_1-z_2|\right)\exp\left(\frac{i}{2}\Im(z_1\overline{z_2})\right). \end{aligned}$$

10.1 Betti numbers of the excursion set

A particular case of Theorems 2 and 3 (i.e., Theorems 19 and 20) is when the singular set is the excursion set of the norm, that is, when $W \subset J^0(\mathbb{S}^2|\mathcal{T}^{\otimes s}) = \mathcal{T}^{\otimes s}$ is the complement of the radius u ball bundle:

$$Z^W(\sigma_n) = \{p \in \mathbb{S}^2 \colon |\sigma(p)| \geq u\}; \\ W = B_u^c(\mathcal{T}^{\otimes s}) = \{(v^{\otimes s}, p) \colon p \in \mathbb{S}^2, v \in T_p\mathbb{S}^2, |v| \geq 1\}. \tag{28}$$

Let $\xi \colon \mathbb{D} \to \mathbb{C}$ be the Gaussian random field arising as the local scaling limit of a sequence of isotropic spin Gaussian fields σ_n. Thus, its covariance function is of the form:

$$K_\xi(z, w) = \mathbb{E}\{\xi(z)\overline{\xi(w)}\} = k_\infty(|z-w|)\exp\left(i\beta\Im(z\overline{w})\right).$$

In this section we give two simple sufficient conditions to apply point (3) of Theorem 3. They are both based on the observation that the support $F := \text{supp}(\xi) \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{C})$ of the limit field must contain the function $f : x \mapsto k_\infty(|x|)$ and all of its real multiples. Indeed, by [34, Theorem 6] the support $F := \text{supp}(\xi_r) \subset \mathcal{C}^\infty(\mathbb{D}, \mathbb{C})$ is the closed vector subspace generated by functions of the form $K_{\xi_r}(z, \cdot)$, for all points $z \in \mathbb{D}$. Therefore, $f \in F$ because

$$f(z) := K_{\xi_r}(0, z) = k_\infty(|z|).$$

Notice that k_∞ has always a local maximum at 0, due to Cauchy–Schwartz inequality: $k_\infty(|z|) \leq k_\infty(0)$.

Lemma 3 *If k_∞ is not constant, then for all $u > 0$ there exists $f_u \in \text{supp}(\xi)$ such that $\{|f| \geq u\}$ is non-degenerate and has a connected component entirely contained in* $\text{int}(\mathbb{D})$.

Proof By the Cauchy–Schwartz inequality, if k_∞ is not constant then there exists radiuses $t_1, \in (-1, 1)$ and $\varepsilon > 0$ such that $k_\infty(0) > k_\infty(t_1) + \varepsilon > k_\infty(t_1) > 0$. We see that choosing the function $f_u := \frac{u}{k_\infty(t_1)+\varepsilon} f \in F$, we have that the excursion set

$$\{z \in \mathbb{D} \colon |f_u(z)| \geq u\} = \{z \in \mathbb{D} \colon k_\infty(|z|) \geq k_\infty(t_1) + \varepsilon\}$$

must have a (non-empty) connected component $C \subset \{|z| \leq t_1\}$ thus contained in the interior of $\mathbb{D}$. Observe that under these hypotheses we also have that $k_\infty(0) > 0$, which implies that the excursion set of ξ is non-degenerate with probability one, by Theorem 2; therefore, the non-degeneracy of the equation $f = 0$ can be achieved by a small perturbation of f in the $\mathcal{C}^0$ topology and within the support, since the property established above is stable under $\mathcal{C}^0$ perturbations. □

For what concerns the first Betti number b_1, an analogous lemma could be stated with the hypotheses that k_∞ has a strict local maximum in $(-1, 1)$. However, we can do something a little bit better by exploiting a topological property of the sphere: namely, *Alexander duality*, which tells us that, almost surely,

$$b_1(\{p \in \mathbb{S}^2 \colon |\sigma(p)| \geq u\}) = b_0(\{p \in \mathbb{S}^2 \colon |\sigma(p)| \leq u\}) - 1.$$

Therefore, to prove that the lower bound in point (2) of Theorem 3 is non-trivial (i.e., $c_i^W > 0$), it is enough to show the validity of point (3), for the complement of the excursion set, which requires only that $|k_\infty|$ is not monotone on [0, 1]. This strategy is strictly better because, due to the shape of the covariance function k_∞ it is easier to have minima than maxima. Indeed, there may be cases in which point (3) of Theorem 3 does not hold, but the Lemma below does.

Lemma 4 *If there are* $0 < t_1 < t_2 < 1$ *such that* $|k_\infty(t_1)| < |k_\infty(t_2)|$, *then for all* $u > 0$ *there exists* $f_u \in \operatorname{supp}(\xi)$ *such that* $\{|f| \leq u\}$ *is non-degenerate and has a connected component entirely contained in* $\operatorname{int}(\mathbb{D})$.

Proof The proof follows the same lines as that of the previous lemma. We choose again the same function $f_u := \frac{u}{|k_\infty(t_1)|+\varepsilon} f \in F$, but this time, the hypothesis implies that the set $\{|f_u| \leq u\}$ has a connected component C contained in $\{|z| \leq t_2\}$. □

10.2 Excursion sets – proof of Theorem 5

To establish Theorem 5, it is sufficient to notice that the conditions for the validity of Theorem 20 are met with $W \subset J^0(\mathbb{S}^2|\mathcal{T}^{\otimes s})$ being as in Equation (28), as shown in the next lemma.

Lemma 5 *The sequence of random fields* σ_ℓ *satisfies Assumption* 2, *Assumptions* 3 *and* 4, *with* $r = 0$.

Proof Assumption 2 follows from Theorem 18. Both Assumptions 3 and 4 follows from the simple observation that the first jet $j^1_0\sigma_\ell$ of the monochromatic field (see Eq. 28) has full rank. □

As a consequence, to derive the expected values of Lipschitz–Killing curvatures it is sufficient to investigate the case with spin zero; this is done in Proposition 23 in Appendix B, exploiting the general form of the Gaussian kinematic formula, see [4]. For the number of connected components, see Lemma 3. For b_1 one can modify the scaling sequence ρ_n by a constant factor c strictly bigger than the second zero of J_0 and then run the argument discussed in Sect. 10.1 to prove that $c_1^W > 0$ using Lemma 4 and Alexander's duality.

Remark 41 (Excursion sets in the middle and Bargmann–Fock regimes) Of course, Theorem 3 can be applied to the case $|s_\ell| = \ell - r$, with $r \in \mathbb{N}$. Moreover, by Lemma 3, the lower bound for the number of connected components is non-trivial. However, it should be noted that in this framework we are not able to give a lower bound for the first Betti number b_1. In the case of Bargmann–Fock Limiting Behaviour, the reason for this failure is easy to get: as well known, because of the maximum principle the excursion set of the norm of a holomorphic function must be convex. This property continues to hold when the function is multiplied by the concave function $\exp(-\frac{|x|^2}{4})$.

Appendix A Expected number of zeroes: proof of Theorem 1

A.1 The covariance function of the rescaled field

Let $X\colon SO(3) \to \mathbb{C}$ be the pull-back of a Gaussian isotropic spin-s function σ, with circular covariance function $k\colon \mathbb{R} \to \mathbb{R}$, defined by:

$$\Gamma\{R(\varphi, \theta, \psi)\} = k(\theta)e^{is(\varphi+\psi)},$$

where $\Gamma(g) := \mathbb{E}\{X(\mathbb{1})\overline{X(g)}\}$, see Sect. 4. Define $\xi\colon \mathbb{C} \to \mathbb{C}$ as the field:

$$\xi(\theta e^{i\varphi}) := X(R(\varphi, \theta, -\varphi)).$$

Lemma 6 *Let $K_\xi(z, w) = \mathbb{E}\{\xi(z)\overline{\xi(w)}\}$ be the covariance function of ξ. Then*

(i) $K_\xi(0, 0) = k(0)$;
(ii) $\frac{\partial}{\partial x} K_\xi(x, 0)|_{x=0} = \frac{\partial}{\partial y} K_\xi(iy, 0)|_{y=0} = 0$;
(iii) $\frac{\partial^2}{\partial x^2} K_\xi(x, 0)|_{x=0} = \frac{\partial^2}{\partial y^2} K_\xi(iy, 0)|_{y=0} = k''(0)$;
(iv) $\frac{\partial^2}{\partial x \partial y} K_\xi(x, iy)|_{x=y=0} = -i\frac{s}{2}k(0)$.

Proof We start by observing that:

$$\begin{aligned} K_\xi(xe^{i\varphi}, ye^{i\psi}) &= \mathbb{E}\left\{X(R(\varphi, x, -\varphi))\overline{X(R(\psi, y, -\psi))}\right\} \\ &= \Gamma\left\{R(\varphi, x, -\varphi)^{-1}R(\psi, y, -\psi)\right\} \\ &= \Gamma\{R_2(-x)R_3(\psi - \varphi)R_2(y)\}\, e^{-is(\psi-\varphi)} \end{aligned}$$

Thus (i) is obvious, while (ii) and (iii) follow from

$$k(t) = K_\xi(0, te^{i\psi}) = K_\xi(-te^{i\varphi}, 0)$$

and the fact that k is even. The difficult case is:

$$\frac{\partial^2}{\partial x \partial y} K_\xi(x, iy)|_{x=y=0} = \frac{\partial^2}{\partial x \partial y}|_{x=y=0}\Gamma\left\{R_2(-x)R_3\left(\frac{\pi}{2}\right)R_2(y)\right\}(-i)^s = \dots$$

Let us write $R_2(-x)R_3\left(\frac{\pi}{2}\right)R_2(y) = R(\varphi, \theta, \psi)$, with φ, θ, ψ determined from Lemma 1:

$$\begin{aligned} \alpha &:= \cos\left(\frac{\theta}{2}\right)e^{i\frac{\varphi+\psi}{2}} = \cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)e^{i\frac{\pi}{4}} + \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)e^{-i\frac{\pi}{4}}; \\ \beta &:= \sin\left(\frac{\theta}{2}\right)e^{i\frac{-\varphi+\psi}{2}} = \sin\left(\frac{-x}{2}\right)\cos\left(\frac{y}{2}\right)e^{i\frac{\pi}{4}} + \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)e^{-i\frac{\pi}{4}}. \end{aligned}$$

Then we have:

$$\begin{aligned}
\dots &= (-i)^s \frac{\partial^2}{\partial x \partial y}|_{x=y=0} k(\theta) e^{is(\varphi+\psi)} \\
&= (-i)^s \frac{\partial}{\partial x}\Big|_{x=0} \left(k(x) \frac{\partial}{\partial y}\Big|_{y=0} e^{is(\varphi+\psi)} + e^{is\frac{\pi}{2}} \frac{\partial}{\partial y}\Big|_{y=0} k(\theta) \right) \\
&= (-i)^s \frac{\partial}{\partial x}\Big|_{x=0} \left(k(x) \frac{\partial}{\partial y}\Big|_{y=0} \left(\frac{\alpha^2}{|\alpha|^2} \right)^s + e^{is\frac{\pi}{2}} k'(x) \frac{\partial \theta}{\partial y}\Big|_{y=0} \right) \\
&= (-i)^s \frac{\partial}{\partial x}\Big|_{x=0} \left(k(x) s \left(\frac{\alpha}{|\alpha|} \right)^{2s-2} \frac{\partial}{\partial y} \left(\frac{\alpha^2}{|\alpha|^2} \right)\Big|_{y=0} + e^{is\frac{\pi}{2}} k'(x) \frac{\partial \theta}{\partial y}\Big|_{y=0} \right) = \dots
\end{aligned}$$

Let us examine each derivative separately. Using that $\frac{\partial \alpha}{\partial y}|_{y=0} = \frac{1}{2}\sin(\frac{x}{2}) e^{-i\frac{\pi}{4}}$ and that $\alpha|_{y=0} = \cos(\frac{x}{2}) e^{i\frac{\pi}{4}}$ we get:

$$\begin{aligned}
\frac{\partial}{\partial y}\Big|_{y=0} \left(\frac{\alpha^2}{|\alpha|^2} \right) &= 2 \frac{\alpha}{|\alpha|^2} \frac{\partial \alpha}{\partial y}\Big|_{y=0} - \frac{\alpha^2}{|\alpha|^4} \langle 2\alpha, \frac{\partial \alpha}{\partial y}\Big|_{y=0} \rangle \\
&= \tan\left(\frac{x}{2}\right) - (\dots) \langle e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}} \rangle = \tan\left(\frac{x}{2}\right).
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial y}\Big|_{y=0} \left(\frac{\alpha}{|\alpha|} \right) = \left(\frac{\alpha}{|\alpha|} \right)^{-1} \frac{\partial}{\partial y}\Big|_{y=0} \left(\frac{\alpha^2}{|\alpha|^2} \right) = \frac{1}{2} \tan\left(\frac{x}{2}\right) e^{-i\frac{\pi}{4}},$$

from which we deduce that

$$\begin{aligned}
\frac{1}{2}\sin\left(\frac{x}{2}\right) e^{-i\frac{\pi}{4}} &= \frac{\partial \alpha}{\partial y}\Big|_{y=0} \\
&= -\frac{1}{2}\sin\left(\frac{x}{2}\right) \left(\frac{\alpha}{|\alpha|} \right) \frac{\partial \theta}{\partial y}\Big|_{y=0} + \cos\left(\frac{x}{2}\right) \frac{\partial}{\partial y}\Big|_{y=0} \left(\frac{\alpha}{|\alpha|} \right) \\
&= -\frac{e^{i\frac{\pi}{4}}}{2} \sin\left(\frac{x}{2}\right) \frac{\partial \theta}{\partial y}\Big|_{y=0} + \cos\left(\frac{x}{2}\right) \frac{1}{2} \tan\left(\frac{x}{2}\right) e^{-i\frac{\pi}{4}}.
\end{aligned}$$

Thus $\frac{\partial \theta}{\partial y}\Big|_{y=0} = 0$. Now, we can take up the main line of computations to conclude the proof:

$$\begin{aligned}
\dots &= (-i)^s \frac{\partial}{\partial x}\Big|_{x=0} \left(k(x) s e^{i(2s-2)\frac{\pi}{4}} \tan\left(\frac{x}{2}\right) \right) \\
&= -ik(0)s\frac{1}{2}.
\end{aligned}$$

□

Corollary 1 *The first jet of the rescaled field* $(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y})$ *has the following covariance matrix:*

$$\mathbb{E}\left\{\begin{pmatrix}\xi \\ \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y}\end{pmatrix}\begin{pmatrix}\overline{\xi} & \overline{\frac{\partial \xi}{\partial x}} & \overline{\frac{\partial \xi}{\partial y}}\end{pmatrix}\right\} = \begin{pmatrix}1 & 0 & 0 \\ 0 & -k''(0) & -i\frac{s}{2}k(0) \\ 0 & i\frac{s}{2}k(0) & -k''(0)\end{pmatrix}.$$

Remark 42 Note that for $s \neq 0$, the first order derivatives and hence the "real and complex" components of the spin bundle are not independent for any choice of local coordinates.

A.2 Proof of Theorem 1

Since σ is isotropic, there exists a constant $c \geq 0$ such that

$$\mathbb{E}\#\{p \in A \colon \sigma(p) = 0\} = c \cdot \mathrm{vol}(A)$$

for every (Borel) subset $A \subset \mathbb{S}^2$. Let us consider the field $\xi \colon \mathbb{D} \to \mathbb{C}$ given by

$$\xi(\theta e^i t) = X(R(\varphi, \theta, -\varphi)),$$

where $X \colon SO(3) \to \mathbb{C}$ is the pull-back random field of σ. Notice that ξ corresponds to the rescaled field of Definition 14 in the case $\rho_\ell = 1$, thus it represents the section σ with respect to a trivialization of the bundle $\mathcal{T}^{\otimes s}$ over the local chart given by the exponential map at the north pole. In particular, the number of zeroes of σ on a spherical disk of radius ε around the north pole equals the number of zeroes of ξ in $\varepsilon\mathbb{D}$, for all $\varepsilon > 0$. Moreover, by Kac–Rice formula, applied to ξ we have

$$\begin{aligned}&\mathbb{E}\#\{z = x + iy \in A \colon \xi(z) = 0\} \\ &\quad = \int_A \mathbb{E}\left\{\left|\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix}\right| \,\middle|\, \xi(z) = 0\right\} \rho_{\xi(z)}(0) dx dy.\end{aligned}$$

Where $\rho_{\xi(z)} \colon \mathbb{C} \to \mathbb{R}_+$ is the density of the random variable $\xi(z)$ and $A \subset \mathbb{D}$ is a Borel subset. Combining these two formulas we deduce that

$$c = \mathbb{E}\left\{\left|\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix}\right| \,\middle|\, \xi(z) = 0\right\} \rho_{\xi(z)}(0) \frac{1}{\sqrt{g(z)}} \tag{A1}$$

for any $z \in \mathbb{D}$, where $\sqrt{g(z)} dx dy$ is the area form of $\mathbb{S}^2$ written in the coordinates x, y. Evaluating (A1) at the point $z = 0$ yields

$$\begin{aligned}\mathbb{E}\#\{\xi = 0\} &= \mathbb{E}\left\{\left|\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix}\right| \,\middle|\, \xi(0) = 0\right\} \rho_{\xi(z)}(0)\mathrm{vol}(\mathbb{S}^2) \\ &= \mathbb{E}\left\{\left|\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix}\right|\right\} \frac{4}{k(0)}.\end{aligned}$$

Where in the last equality we used the fact that $\xi(0)$ is independent from $d_0\xi$, as it can be seen from Corollary 1.

To end the computation it is convenient to express the differential of ξ in terms of the Wirtinger derivatives:

$$\frac{\partial \xi}{\partial z} := \frac{1}{2}\left(\frac{\partial \xi}{\partial x} - i\frac{\partial \xi}{\partial y}\right), \qquad \frac{\partial \xi}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial \xi}{\partial x} + i\frac{\partial \xi}{\partial y}\right).$$

As a consequence of the shape of the covariance matrix established by Corollary 1, we have that the two complex random variables $\frac{\partial \xi}{\partial z}$ and $\frac{\partial \xi}{\partial \bar{z}}$ are independent with variances:

$$\mathbb{E}\left\{\left|\frac{\partial \xi}{\partial z}\right|^2\right\} = -\frac{1}{2}k''(0) + \frac{s}{4}k(0), \qquad \mathbb{E}\left\{\left|\frac{\partial \xi}{\partial \bar{z}}\right|^2\right\} = -\frac{1}{2}k''(0) - \frac{s}{4}k(0).$$

Moreover,

$$\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix} = \left|\frac{\partial \xi}{\partial z}\right|^2 - \left|\frac{\partial \xi}{\partial \bar{z}}\right|^2.$$

Lemma 7 *Let γ_1, γ_2 be two independent complex normal variables with variances $a, b > 0$, we have*

$$\mathbb{E}\left\{\left||\gamma_1|^2 - |\gamma_2|^2\right|\right\} = \frac{a^2 + b^2}{a + b}$$

Proof The proof is a straightforward computation of an integral and is omitted. □

A direct computation concludes the proof of Theorem 1.

$$\begin{aligned}
\mathbb{E}\#\{\xi = 0\} &= \frac{4}{k(0)}\mathbb{E}\left\{\left|\det\begin{pmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\end{pmatrix}\right|\right\} \\
&= \frac{4}{k(0)}\mathbb{E}\left\{\left|\left|\frac{\partial \xi}{\partial z}\right|^2 - \left|\frac{\partial \xi}{\partial \bar{z}}\right|^2\right|\right\} \\
&= \frac{4}{k(0)}\frac{(-\frac{1}{2}k''(0) + \frac{s}{4}k(0))^2 + (-\frac{1}{2}k''(0) - \frac{s}{4}k(0))^2}{-k''(0)} \\
&= \frac{2k''(0)^2 + \frac{1}{2}s^2k(0)^2}{-k(0)k''(0)}.
\end{aligned}$$

Appendix B Proof of technical lemmas

B.1 Expected nodal volume of Gaussian fields

A smooth Gaussian random field $X\colon M \to \mathbb{R}$, defines a semipositive definite scalar product on T_pM via the following formula (see [4])

$$g_p^X(v, w) := \mathbb{E}\left\{d_pX(v)d_pX(w)\right\}.$$

Such tensor is a Riemannian metric (i.e. it is positive definite) if and only if d_pX is a non-degenerate Gaussian vector for every $p \in M$. In this case, we call it the *Adler–Taylor metric* of X. Many probabilistic features of X are related to the Riemannian geometry of (M, g^X), starting from the expected nodal volume, i.e. the Hausdorff measure of $\mathcal{H}^{m-1}(X^{-1}(0))$. The formula that we use, in particular in the proof of Theorem 3 is the following. Let $s_k := \mathcal{H}^k(\mathbb{S}^k)$ be the volume of the k-dimensional sphere.

Proposition 22 *Let (M, g) be a compact Riemannian manifold and let Let $X_i \sim X\colon M \to \mathbb{R}$ be i.i.d. copies of a smooth Gaussian random field X such that $g^X = \frac{\lambda}{2}g$ and with constant variance $\mathbb{E}\{|X(p)|^2\} = \sigma^2$. Let $C \subset M$ be a smooth immersed submanifold of dimension d and let $f \in \mathcal{C}^\infty(M)$. Then the integral of $f|_C$ with respect to the d-dimensional Hausdorff measure $\mathcal{H}_g^d$ (i.e. the Riemannian volume measure of C) is*

$$\int_C f(p)d\mathcal{H}_g^d(p) = \left(\frac{\lambda}{2\sigma^2}\right)^{-\frac{d}{2}} \frac{s_d}{2}\mathbb{E}\left\{\sum_{p\in C\cap\{X_1=\cdots=X_d=0\}} f(p)\right\}.$$

Proof It is sufficient to show the formula for $f = 1$ and then extend it by dominated convergence. Moreover, observe that $Y = X|_C$ is a smooth Gaussian field on C and $g^Y = \frac{\lambda}{2}g|_C$, therefore we can assume that $C = M$. Thus, we only have to prove that

$$\mathcal{H}_g^m(M) = \left(\frac{\lambda}{2\sigma^2}\right)^{-\frac{m}{2}} \frac{s_m}{2}\mathbb{E}(\#\left(\{X_1 = \cdots = X_m = 0\}\right)). \tag{B2}$$

We can further reduce to the case $\sigma = \frac{\lambda}{2} = 1$, by replacing X with $Y = \frac{1}{\sigma}X$ and g with g^Y. Indeed observe that the right hand side doesn't change, while the left hand side changes as:

$$g^Y = \frac{\lambda}{2\sigma^2}g \quad \text{and} \quad \mathcal{H}_g^m(M)\varepsilon^{\frac{m}{2}} = \mathcal{H}_{\varepsilon g}^m(M).$$

Kac–Rice formula tells us in particular that the two quantities in (B2) are proportional. The correct constant can be thus deduced from a simple case without making computations. The simplest case is that of the standard sphere: $M = \mathbb{S}^m$ with $X(p) := \gamma^T p$

for $\gamma \sim N(0, \mathbb{1}_k)$, so that g^X is the standard round metric. Since the random set $C \cap \{X_1 = \cdots = X_m = 0\}$ consists almost surely of 2 points, we conclude. □

Remark 43 The intuition behind the identity $g^X = \frac{\lambda}{2} g$ is that the conformal factor is $\frac{\lambda}{2}$, when X is a Gaussian eigenfunction in $\ker(\Delta - \lambda)$ on $\mathbb{S}^2$ (see 23).

B.2 Explicit formulas for L–K curvatures for spin equal to zero

The next result follows quite directly from the general Gaussian kinematic formula of [4].

Proposition 23 (L–K curvatures for spin = 0) *Let $\sigma_0 : \mathbb{S}^2 \to \mathbb{C}$ be a complex isotropic smooth Gaussian random field having independent real and imaginary parts and with $\sigma(p) \sim N_{\mathbb{C}}(0, 1)$. Let $k(\theta)$ be its circular covariance function. Then for any $u > 0$, we have the following identities.*

i. $k(\theta) = \mathbb{E}\{\sigma_0(p)\sigma_0(q)\}$ *for every* $p, q \in \mathbb{S}^2$ *such that* $\operatorname{dist}_{\mathbb{S}^2}(p, q) = \theta$.
ii. $\frac{\lambda}{2} := |k''(0)| = \mathbb{E}\{|\partial_v \sigma_0|^2\}$ *for any unit tangent vector* $v \in T\mathbb{S}^2$. *This is the conformal factor of the Adler–Taylor metric* g^{σ_0} *induced by* σ_0, *see* [4], *meaning that* $g^{\sigma_0} = \frac{\lambda}{2} g_{\mathbb{S}^2}$.
iii. $\mathbb{E}\#\{\sigma_0 = 0\} = \lambda$.
iv. $\mathbb{E}\mathrm{vol}_2(\{|\sigma_0| \geq u\}) = 4\pi e^{-\frac{u^2}{2}}$.
v. $\mathbb{E}\mathrm{vol}_1(\{|\sigma_0| = u\}) = \lambda^{\frac{1}{2}} \cdot 2\pi^{\frac{3}{2}} u e^{-\frac{u^2}{2}}$.
vi. $\mathbb{E}\chi(\{|\sigma_0| \geq u\}) = (\lambda \cdot (u^2 - 1) + 2)e^{-\frac{u^2}{2}}$.

Proof i. and the first part of ii. are a straightforward consequence of isotropy. Moreover, notice that iii. follows from Theorem 1. The fact that $\frac{\lambda}{2}$ is indeed the conformal factor can be deduced combining iii. with Proposition 22.

$$\mathbb{E}\#\{\sigma_0 = 0\} = \frac{2}{4\pi} \mathrm{vol}_{g^{\sigma_0}}(\mathbb{S}^2).$$

This ends the proof of ii. To show iv, v and vi, we apply [4, Theorem 15.9.4], which states that for every $j = 0, 1, 2$, we have the formula:

$$\mathbb{E}\mathcal{L}_i\left(\{|\sigma_0| \geq u\}\right) = \sum_{j=0}^{2-i} \binom{i+j}{j} \frac{\omega_{i+j}}{\omega_i \omega_j} \mathcal{L}_{i+j}(\mathbb{S}^2) \rho_j(u^2), \tag{B3}$$

where $\omega_n = \pi^{\frac{n}{2}} \Gamma(\frac{n}{2} + 1)^{-1}$ is the volume of the standard unit ball of dimension n; $\mathcal{L}_j$ are the Lipschitz–Killing curvatures computed with respect to the metric g^{σ_0}; the coefficients $\rho_j(u)$ are universal functions of u, that can be computed via the formula in [4, Theorem 15.10.1] (the formula in the book is for the Lipschitz–Killing curvatures of the set $\{|\sigma_0|^2 \geq u\}$, thus we have to evaluate the formula in u^2 instead than u):

$$\rho_2(u^2) = \frac{e^{-\frac{u^2}{2}}}{2\pi}(u^2 - 1), \qquad \rho_1(u^2) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} u, \qquad \rho_0(u^2) = e^{-\frac{u^2}{2}}.$$

For $A \subset \mathbb{S}^2$ smooth submanifold with boundary, we have

$$\mathcal{L}_2(A) = \frac{\lambda}{2}\mathrm{vol}_2(A), \qquad \mathcal{L}_1(A) = \frac{1}{2}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}} \mathrm{vol}_1(\partial A), \qquad \mathcal{L}_0(A) = \chi(A).$$

Here, vol_1 and vol_2 are now meant with respect to the standard metric of $\mathbb{S}^2$. Therefore formula (B3) gives:

$$\mathbb{E}\chi\{|\sigma| \geq u\} = e^{-\frac{u^2}{2}}\left(\chi(\mathbb{S}^2) + \frac{(u^2-1)}{2\pi}\frac{\lambda}{2}\mathrm{vol}_2(\mathbb{S}^2)\right) = e^{-\frac{u^2}{2}}\left(2 + (u^2-1)\lambda\right);$$
$$\frac{1}{2}\left(\frac{\lambda}{2}\right)^{\frac{1}{2}}\mathbb{E}\mathrm{vol}_1\{|\sigma| = u\} = e^{-\frac{u^2}{2}}\binom{2}{1}\frac{\omega_2}{\omega_1^2}\frac{u}{\sqrt{2\pi}}\frac{\lambda}{2}\mathrm{vol}_2(\mathbb{S}^2) = e^{-\frac{u^2}{2}}\frac{\pi^{\frac{3}{2}}}{\sqrt{2}}u\lambda;$$
$$\frac{\lambda}{2}\mathbb{E}\mathrm{vol}_2\{|\sigma| \geq u\} = e^{-\frac{u^2}{2}}\frac{\lambda}{2}\mathrm{vol}_2(\mathbb{S}^2) = e^{-\frac{u^2}{2}}\frac{\lambda}{2}4\pi.$$

□

Appendix C Alternative to Hilb's asymptotic

This appendix collects some explicit computations which are instrumental for the derivation of the limiting behavior of the covariances of monochromatic spin fields.

Definition 19 (Bessel functions of the first kind) Let $n \in \mathbb{Z}$. The *Bessel function of the first kind* of order n, denoted by $J_n(x)$, is a (regular at 0) solution of the *Bessel equation* $x^2 J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n = 0$. It is an anaytic function $J_n : \mathbb{R} \to \mathbb{R}$ described by the following power series: if $n \geq 0$,

$$J_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+n)!}\left(\frac{x}{2}\right)^{2j+n};$$
$$J_{-n}(x) = \sum_{j \geq n}^{\infty} \frac{(-1)^j}{j!(j-n)!}\left(\frac{x}{2}\right)^{2j-n} = (-1)^n J_n(x).$$

Lemma 8 *Let* $\alpha, \beta : SO(3) \to \mathbb{C}$ *be the two functions defined (see in* [42, Sec 3.2.1]*) by*

$$\alpha(R(\varphi, \theta, \psi)) = \cos\left(\frac{\theta}{2}\right)e^{i(\frac{\varphi+\psi}{2})}; \quad \beta(R(\varphi, \theta, \psi)) = \sin\left(\frac{\theta}{2}\right)e^{i(\frac{\varphi-\psi}{2})}$$

Then (with a little abuse of notation), for all $\ell \geq |s|$, we have

$$
\begin{aligned}
&D^{\ell}_{m,-s}(\alpha,\beta) := D^{\ell}_{m,-s}(g)\\
&= \sum_{j\geq \max\{0,-(m+s)\}}^{\min\{\ell-s,\ell-m\}} \frac{(-1)^{s+m}\sqrt{(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!}}{(\ell-s-j)!(\ell-m-j)!} \frac{(-1)^{j}}{j!(s+m+j)!}\\
&\quad\times \alpha^{\ell-m-j}\overline{\alpha}^{\ell-s-j}\beta^{j}\overline{\beta}^{j+m+s}.
\end{aligned}
$$

Proof (It is Proposition 3.7 in the book [42]). By Definition, we have

$$
\binom{2\ell}{\ell+s}^{\frac{1}{2}}(\alpha z_1-\overline{\beta}z_2)^{\ell+s}(\beta z_1+\overline{\alpha}z_2)^{\ell-s} = \sum_{m=-\ell}^{\ell}\binom{2\ell}{\ell-m}D^{\ell}_{m,-s}(\alpha,\beta)z_1^{\ell-m}z_2^{\ell+m}.
$$

Expanding the first term we get

$$
\begin{aligned}
&\binom{2\ell}{\ell+s}^{\frac{1}{2}}(\alpha z_1-\overline{\beta}z_2)^{\ell+s}(\beta z_1+\overline{\alpha}z_2)^{\ell-s}\\
&=\binom{2\ell}{\ell+s}^{\frac{1}{2}}\sum_{i=0}^{\ell+s}\sum_{j=0}^{\ell-s}\binom{\ell+s}{I}\binom{\ell-s}{j}\alpha^{i}\overline{\alpha}^{\ell-s-j}\beta^{j}\overline{\beta}^{\ell+s-i}(-1)^{\ell+s-i}z_1^{i+j}z_2^{2\ell-(i+j)}\\
&=\binom{2\ell}{\ell+s}^{\frac{1}{2}}\sum_{m=-\ell}^{\ell}\sum_{j=0}^{\ell-s}\binom{\ell-s}{j}\binom{\ell+s}{\ell-m-j}\alpha^{\ell-m-j}\overline{\alpha}^{\ell-s-j}\beta^{j}\overline{\beta}^{j+m+s}\\
&\quad\times(-1)^{j+m+s}z_1^{\ell-m}z_2^{\ell+m},
\end{aligned}
$$

where m and j must satisfy the additional constraints $\ell-m-j=i\geq 0$ and $m+s+j=(\ell+s)-i\geq 0$. Therefore we can conclude by observing that

$$
\frac{\binom{2\ell}{\ell+s}^{\frac{1}{2}}\binom{\ell-s}{j}\binom{\ell+s}{\ell-m-j}\alpha^{\ell-m-j}}{\binom{2\ell}{\ell-m}^{\frac{1}{2}}} = \frac{\sqrt{(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!}}{j!(\ell-s-j)!(\ell-m-j)!(s+m+j)!}.
$$

□

Corollary 2 *For $\ell \geq |s|$, we have*

$$
\begin{aligned}
d^{\ell}_{m,-s}(\theta) = &\sum_{j\geq \max\{0,-(m+s)\}}^{\min\{\ell-s,\ell-m\}} \frac{(-1)^{s+m}\sqrt{(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!}}{(\ell-s-j)!(\ell-m-j)!}\\
&\times\left(\cos\frac{\theta}{2}\right)^{2(\ell-j)-(m+s)}\frac{(-1)^{j}}{j!(s+m+j)!}\left(\sin\frac{\theta}{2}\right)^{2j+(m+s)}.
\end{aligned}
$$

Theorem 24 *For $\ell \to +\infty$, we have that*

$$d^{\ell}_{m,-s}\left(\frac{x}{\ell}\right) \xrightarrow[\ell\to+\infty]{} (-1)^{m+s} J_{m+s}(x).$$

The convergence holds in the $\mathcal{C}^{\infty}$ topology.

Proof Since for $\ell \to +\infty$ we have that

$$\alpha = \left(\cos \frac{x}{2\ell}\right)^{\ell} \sim_{\mathcal{C}^{\infty}} 1; \quad \text{and} \quad \beta = \sin \frac{x}{2\ell} \sim_{\mathcal{C}^{\infty}} \frac{x}{2\ell},$$

we can restrict our study to the function

$$\begin{aligned} &D^{\ell}_{m,-s}\left(1, \frac{x}{\ell}\right) \\ &= \sum_{j\geq \max\{0,-(m+s)\}}^{\min\{\ell-s,\ell-m\}} \frac{(-1)^{s+m}\sqrt{(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!}}{(\ell-s-j)!(\ell-m-j)!} \\ &\times \frac{(-1)^j}{j!(s+m+j)!}\left(\frac{x}{2\ell}\right)^{2j+(m+s)}. \end{aligned}$$

Since the above function is a power series with convergence radius $= +\infty$, its convergence in $\mathcal{C}^{\infty}$ can be checked one coefficient at a time. Now, observe that (assume, for simplicity, that $m \geq 0$ and $s \geq 0$. In the other cases, the argument is essentially the same)

$$\frac{\sqrt{(\ell+s)!(\ell-s)!(\ell+m)!(\ell-m)!}}{(\ell-s-j)!(\ell-m-j)!} \frac{1}{\ell^{2j+m+s}} = 1 + o(1).$$

It follows that

$$\begin{aligned} D^{\ell}_{m,-s}\left(1, \frac{x}{\ell}\right) &\xrightarrow[\ell\to+\infty]{} \sum_{j=\max\{0,-(m+s)\}}^{\infty} (-1)^{s+m} \frac{(-1)^j}{j!(s+m+j)!}\left(\frac{x}{2}\right)^{2j+(m+s)} \\ &= (-1)^{s+m} J_{s+m}(x). \end{aligned}$$

(both if $s+m \geq 0$ and if $s+m < 0$). □

Remark 44 As $s \to +\infty$,

$$d^{s}_{-s,-s}\left(\frac{x}{\sqrt{s}}\right) = \left(\cos\left(\frac{x}{2\sqrt{s}}\right)\right)^{2s} \sim e^{-\frac{x^2}{4}}.$$

C.1 Limit of the covariance of monochromatic fields

Let us take, as before, $|s_\ell| = \ell - r_\ell$.

Now note that

$$d^\ell_{-s_\ell,-s_\ell}\left(\frac{x}{\sqrt{(r+1)(2\ell-r)}}\right) = \sum_{j=0}^{r} \frac{(2\ell-r)!r!}{(r-j)!(2\ell-r-j)!}\frac{(-1)^j}{j!j!}$$
$$\times \sin\left(\frac{x}{2\sqrt{(r+1)(2\ell-r)}}\right)^{2j} \cos\left(\frac{x}{2\sqrt{(r+1)(2\ell-r)}}\right)^{2(\ell-j)}$$
$$= \sum_{j=0}^{r} \frac{(2\ell-r)!}{(2\ell-r-j)!(2\ell-r)^j}\frac{r!}{(r-j)!(r+1)^j}\frac{(-1)^j}{j!j!}$$
$$\times \left(\frac{x}{2}\right)^{2j}\left(1-\frac{x^2}{4((r+1)(2\ell-r))}\right)^{2\ell} + o_{\ell\to+\infty}(1) = \dots$$

Note also that we have always $2\ell - r_\ell \to +\infty$.

We must consider two cases. In the first $r_\ell \to r \in \mathbb{N}$, which is equivalent to r_ℓ being fixed; in this case the shrinking rate is $\rho_\ell = O(\frac{1}{\sqrt{2(r+1)\ell}})$ and thus $\beta = \frac{1}{2(r+1)}$, and, moreover, we obtain easily

$$\dots \xrightarrow[s\to+\infty]{} \sum_{j=0}^{r} \frac{r!}{(r+1)^j(r-j)!}\frac{(-1)^j}{j!j!}\left(\frac{x}{2}\right)^{2j} e^{-\frac{x^2}{4(r+1)}} =: M_r(x). \tag{C4}$$

On the other hand, in the second case $r_\ell \to +\infty$ and we obtain

$$\dots \xrightarrow[s\to+\infty]{} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!j!}\left(\frac{x}{2}\right)^{2j} = J_0(x).$$

In this second case, we have

$$\beta = \lim_{\ell\to+\infty} \frac{\ell-r}{(r+1)(2\ell-r)} = \lim_{\ell\to+\infty} \frac{\ell-r}{(\ell+(\ell-r))(r+1)} = 0.$$

Note that this second scenario covers the Berry regime. Notice also that $M_r(x) = 1 - \frac{x^2}{4} + O(x^4) = J_0(x) + O(x^4)$ and $M_0(x) = e^{-\frac{x^2}{4}}$ is the real part of the covariance function of the complex Bargmann–Fock field.

Appendix D Berry's complex random Wave model

Berry's Complex Random Wave Model is a complex Gaussian random field $\xi := \{\xi(x), x \in \mathbb{R}^2\}$ on $\mathbb{R}^2$ represented as

$$\xi(x) = \sum_{n\in\mathbb{Z}} a_n J_{|n|}(r)e^{in\theta}, \qquad x = (r, \theta),$$

where J_α denotes the Bessel function of the first kind of order α and $(a_n)_n$ is a sequence of i.i.d. standard complex Gaussian random variables. The sample paths are a.s. $\mathcal{C}^\infty$ functions.

It is straightforward to check that ξ a.s. solves the Helmholtz equation on the Euclidean plane, i.e.

$$\Delta_{\mathbb{R}^2}\xi = -\xi \qquad a.s.$$

Indeed, writing $\Delta_{\mathbb{R}^2}$ in polar coordinates

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\xi(x) = \sum_{n\in\mathbb{Z}} \frac{a_n}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - n^2\right) J_{|n|}(r)e^{in\theta}$$
$$= \sum_{n\in\mathbb{Z}} \frac{a_n}{r^2}\underbrace{\left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - n^2 + r^2\right) J_{|n|}(r)\, e^{in\theta}}_{=0} - \sum_{n\in\mathbb{Z}} a_n J_{|n|}(r)e^{in\theta} = -\xi(x).$$

Lemma 9 *Every (smooth) solution f of the Helmholtz equation is of the form*

$$f(x) = \sum_{n\in\mathbb{Z}} \gamma_n J_{|n|}(r)e^{in\theta}. \tag{D5}$$

Proof Let $\mathbb{D}$ denote the unit disc, then f restricted to $\mathbb{D}$ can be written as

$$f(x) = \sum_{n\in\mathbb{Z}} R_n(r)e^{in\theta},$$

for some C^∞ functions R_n. Since f solves the Helmholtz equation, the following holds true for every r, θ

$$\sum_{n\in\mathbb{Z}} \frac{1}{r^2}\left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - n^2 + r^2\right) R_n(r)e^{in\theta} = 0$$

which implies for every n

$$\left(r^2\frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} - n^2 + r^2\right) R_n = 0.$$

The two fundamental solutions of this PDE are J_n and Y_n, the Bessel function of order n of the first and second type respectively. Hence there are coefficients γ_n, η_n such that

$$R_n = \gamma_n J_n + \eta_n Y_n$$

and f is of the form

$$f(x) = \sum_{n\in\mathbb{Z}} (\gamma_n J_n(r) + \eta_n Y_n(r))\, e^{in\theta}.$$

The function f is $\mathcal{C}^\infty$, and so is

$$\int_{[0,2\pi]} f(r,\theta)e^{im\theta}\, d\theta = R_m(r)$$

for every $m \in \mathbb{Z}$, hence $\eta_m = 0$ for every m and f is of the form

$$f(x) = \sum_{n\in\mathbb{Z}} \gamma_n J_{|n|}(r)e^{in\theta}$$

at least on the disc (note that J_n and J_{-n} solve the same PDE, and $J_{-n} = (-1)^n J_n$), thus on the whole plane. □

Let f be of the form (D5), then for every $r \geq 0$

$$\int_{[0,2\pi]} f(r,\theta)\, d\theta = \gamma_0 J_0(r), \tag{D6}$$

in particular if $\bar{r}$ is any positive zero of J_0, then the mean of f over the circle of radius $\bar{r}$ is zero. Moreover, $f(0) = \gamma_0$ hence Eq. (D6) can be rewritten as

$$\int_{[0,2\pi]} f(r,\theta)\, d\theta = f(0)J_0(r).$$

This identity can be thought as a modified mean value theorem valid for solutions of the Helmholtz equation.

Acknowledgements The authors would like to thank an anonymous referee for insightful remarks and useful suggestions. DM acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome "Tor Vergata", CUP E83C18000100006 and INdAM. The research of MR has been supported by the ANR-17-CE40-0008 Project UNIRANDOM and INdAM. MS is supported by the grant TROPICOUNT of Région Pays de la Loire, the ANR Project ENUMGEOM NR-18-CE40-0009-02 and by the Luxembourg National Research Fund (Grant 021/16236290/HDSA).

Author Contributions All authors contributed equally to this work

Data Availibility Statement No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

1. Armentano, D., Azaïs, J.-M., Dalmao, F., León, J.R.: Central limit theorem for the number of real roots of Kostlan Shub Smale random polynomial systems. Am. J. Math. **143**(4), 1011–1042 (2021)
2. Ancona, M., Letendre, T.: Roots of Kostlan polynomials: moments, strong Law of Large Numbers and Central Limit Theorem. Ann. Henri Lebesgue **4**, 1659–1703 (2021)
3. Ancona, M.: Random sections of line bundles over real Riemann surfaces. Int. Math. Res. Not. IMRN **9**, 7004–7059 (2021)
4. Adler, R.J., Taylor, J.E.: Random Fields and Geometry. Springer Monographs in Mathematics. Springer, New York (2007)
5. Ade, P.A.R., the Bicep/Keck Collaboration: Improved constraints on primordial gravitational waves using planck, WMAP, and BICEP/Keck observations through the 2018 observing season. Phys. Rev. Lett. **127**(15), 151301 (2021)
6. Allys, E., the LiteBIRD Collaboration: Probing cosmic inflation with the LiteBIRD cosmic microwave background polarization survey. arXiv e-prints arXiv:2202.02773, February (2022)
7. Aghanim, N., the Planck Collaboration: Planck 2018 results. I. Overview and the cosmological legacy of Planck. Astron. Astrophys. **641**, A1 (2020)
8. Bochnak, J., Coste, M., Roy, M.-F.: Real Algebraic Geometry. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 36. Springer, Berlin (1998)
9. Berry, M.V.: Regular and irregular semiclassical wavefunctions. J. Phys. A Math. Gen. **10**(12), 2083–2091 (1977)
10. Berry, M.V.: Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. J. Phys. A Math. Gen. **35**(13), 3025–3038 (2002)
11. Beliaev, D., Muirhead, S., Rivera, A.: A covariance formula for topological events of smooth Gaussian fields. Ann. Probab. **48**(6), 2845–2893 (2020)
12. Beliaev, D., Muirhead, S., Wigman, I.: Russo–Seymour–Welsh estimates for the Kostlan ensemble of random polynomials. Ann. Inst. Henri Poincaré Probab. Stat. **57**(4), 2189–2218 (2021)
13. Breiding, P., Kenehlou, H., Lerario, A.: Quantitative singularity theory for random polynomials. Int. Math. Res. Not. **22**(8), 5685–5719 (2022)
14. Baldi, P., Rossi, M.: Representation of Gaussian isotropic spin random fields. Stoch. Process. Appl. **124**(5), 1910–1941 (2014)
15. Bleher, P., Shiffman, B., Zelditch, S.: Universality and scaling of correlations between zeros on complex manifolds. Invent. Math. **142**, 351–395 (2000)
16. Cheng, D., Cammarota, V., Fantaye, Y., Marinucci, D., Schwartzman, A.: Multiple testing of local maxima for detection of peaks on the (celestial) sphere. Bernoulli **26**(1), 31–60 (2020)
17. Canzani, Y., Hanin, B.: Local universality for zeros and critical points of monochromatic random waves. Commun. Math. Phys. **378**(3), 1677–1712 (2020)
18. Cammarota, V., Marinucci, D.: On the limiting behaviour of needlets polyspectra. Ann. Inst. Henri Poincaré Probab. Stat. **51**(3), 1159–1189 (2015)
19. Cammarota, V., Marinucci, D.: A quantitative central limit theorem for the Euler–Poincaré characteristic of random spherical eigenfunctions. Ann. Probab. **46**(6), 3188–3228 (2018)
20. Durrer, R.: The Cosmic Microwave Background. Cambridge University Press, London (2008)
21. Eliashberg, Y., Mishachev, N.M., Ariki, S.: Introduction to the h-Principle. Graduate studies in mathematics. American Mathematical Society, New York (2002)
22. Feng, R., Zelditch, S.: Critical values of random analytic functions on complex manifolds. Indiana Univ. Math. J. **63**(3), 651–686 (2014)
23. Goresky, M., MacPherson, R.: Stratified Morse Theory, volume 36 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag (1988)
24. Geller, D., Marinucci, D.: Spin wavelets on the sphere. J. Fourier Anal. Appl. **16**(6), 840–884 (2010)
25. Gayet, D., Welschinger, J.-Y.: Betti numbers of random real hypersurfaces and determinants of random symmetric matrices. J. Eur. Math. Soc. (JEMS) **18**(4), 733–772 (2016)

26. Gayet, D., Welschinger, J.-Y.: Universal components of random nodal sets. Commun. Math. Phys. **347**(3), 777–797 (2016)
27. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2000)
28. Hirsch. M.W.: Differential Topology, Volume 33 of Graduate Texts in Mathematics. Springer, New York (1994) (**Corrected reprint of the 1976 original**)
29. Husemoller, D.: Fibre Bundles, Volume 20 of Graduate Texts in Mathematics, 3rd edn. Springer, New York (1994)
30. Ilić, S., the Euclid Collaboration: Euclid preparation. XV. Forecasting cosmological constraints for the Euclid and CMB joint analysis. Astron. Astrophys. **657**, A91 (2022)
31. Jung, J., Zelditch, S.: Topology of the nodal set of random equivariant spherical harmonics on $\mathbb{S}^3$. Int. Math. Res. Not. IMRN **11**, 8521–8549 (2021)
32. Letendre, T.: Expected volume and Euler characteristic of random submanifolds. J. Funct. Anal. **270**(8), 3047–3110 (2016)
33. Leonenko, N., Sakhno, L.: On spectral representations of tensor random fields on the sphere. Stoch. Anal. Appl. **30**(1), 44–66 (2012)
34. Lerario, A., Stecconi, M.: Differential topology of Gaussian random fields. Preprint ArXiv:1902.03805 (2019)
35. Lerario, A., Stecconi, M.: What is the degree of a smooth hypersurface? J. Singul. **23**, 205–235 (2021)
36. Lerario, A., Stecconi, M.: Maximal and typical topology of real polynomial singularities. Annales de l'Institut Fourier **74**(2), 589–626 (2024)
37. Lerario, A., Stecconi, M.: The probabilistic method in real singularity theory. Arnold Math J. **10**, 355–370 (2024)
38. Malyarenko, A.: Invariant random fields in vector bundles and application to cosmology. Ann. Inst. Henri Poincaré Probab. Stat. **47**(4), 1068–1095 (2011)
39. Malyarenko, A.: Invariant Random Fields on Spaces with a Group Action. Probability and its Applications (New York). Springer, Heidelberg (2013)
40. Mather, J.: Notes on topological stability (1970)
41. Mathis, L., Stecconi, M.: Expectation of a random submanifold: the zonoid section. Ann. Henri Lebesgue **7**, 903–967 (2024)
42. Marinucci, D., Peccati, G.: Random Fields on the Sphere: Representation, Limit Theorems and Cosmological Applications. London Mathematical Society Lecture Note Series. Cambridge University Press, London (2011)
43. Marinucci, D., Peccati, G., Rossi, M., Wigman, I.: Non-universality of nodal length distribution for arithmetic random waves. Geom. Funct. Anal. **26**(3), 926–960 (2016)
44. Marinucci, D., Rossi, M., Wigman, I.: The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. Ann. Inst. Henri Poincaré Probab. Stat. **56**(1), 374–390 (2020)
45. Newman, E.T., Penrose, R.: Note on the Bondi–Metzner–Sachs group. J. Math. Phys. **7**, 863–870 (1966)
46. Nourdin, I., Peccati, G., Rossi, M.: Nodal statistics of planar random waves. Commun. Math. Phys. **369**(1), 99–151 (2019)
47. Nazarov, F., Sodin, M.: On the number of nodal domains of random spherical harmonics. Am. J. Math. **131**(5), 1337–1357 (2009)
48. Nazarov, F., Sodin, M.: Correlation functions for random complex zeroes: strong clustering and local universality. Commun. Math. Phys. **310**(1), 75–98 (2012)
49. Nicolaescu, L.: A stochastic Gauss–Bonnet–Chern formula. Probab. Theory Relat. Fields **165**, 235–265 (2016)
50. Rivera, A., Vanneuville, H.: The critical threshold for Bargmann–Fock percolation. Ann. Henri Lebesgue **169–215**, 3 (2020)
51. Sodin, M., Tsirelson, B.: Random complex zeroes. I. Asymptotic normality. Israel J. Math. **144**, 125–149 (2004)
52. Stecconi, M.: Isotropic random spin weighted functions on S^2 vs isotropic random fields on S^3. Theor. Probab. Math. Stat. **107**, 77–109 (2022)
53. Stecconi, M.: Kac–Rice formula for transverse intersections. Anal. Math. Phys. **12**(2), 44 (2022)
54. Sarnak, P., Wigman, I.: Topologies of nodal sets of random band-limited functions. Commun. Pure Appl. Math. **72**(2), 275–342 (2019)

55. Wigman, I.: Fluctuations of the nodal length of random spherical harmonics. Commun. Math. Phys. **298**(3), 787–831 (2010)