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Critical Points of Chi-Fields

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Abstract. We give here a semi-analytic formula for the density of critical values for chi random fields on a general manifold. The result uses Kac-Rice argument and a convenient representation for the Hessian matrix of chi fields, which makes the computation of their expected determinant much more feasible. In the high-threshold limit, the expression for the expected value of critical points becomes very transparent: up to explicit constants, it amounts to Hermite polynomials times a Gaussian density. Our results are also motivated by the analysis of polarization random fields in Cosmology, but they might lead to applications in many different environments.

Contents

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1. Introduction

1.1. *Background.* The investigation of the geometric properties of random fields has represented a major thread of research over the last fifteen years. A major driving force has been given by the publication of very popular research monographs such as Adler and Taylor (2007) and Azaïs and Wschebor (2009); these books have discussed in depth the Kac-Rice approach for the derivation of expected values for critical points of smooth random fields. In broad terms, the Kac-Rice approach leads to an "expectation Metatheorem" (the terminology adopted in Adler and Taylor (2007)), stating that under regularity conditions the expected number of critical points can be expressed in terms of the expectation of the absolute value of the determinant of the Hessian matrix of the field, conditional on the gradient of the field being zero; other conditions can be added to obtain related quantities, for instance on the signature of the Hessian if one is interested in the expected number of minima or maxima. This general approach has led to an impressive amount of results and applications, starting from the celebrated Gaussian Kinematic Formula, which allows the computation of expected value of Lipschitz-Killing Curvatures for excursion sets of Gaussian fields. As noted elsewhere, this area bridges the gap (in a very fascinating way) among different areas of Mathematics, such as Differential Geometry and Random Fields; at the same time, it leads to results which are motivated by fastly growing applied fields, including for instance Cosmology, Neuroimaging, Neural Networks, Optimization, Spin Glasses and many others (see e.g., Auffinger and Ben Arous (2013),Cheng and Schwartzman (2017), Arous et al. (2020), Cheng et al. (2020), Fan et al. (2021), Fyodorov and Tublin (2022), Belius et al. (2022), Azaïs and Delmas (2022), Telschow et al. (2023), just to mention a few recent references).

1.2. *Motivations.* The overwhelming majority of the literature on critical points has so far been confined to the analysis of Gaussian random fields. Indeed, although the Kac-Rice approach is valid in much greater generality than under Gaussianity, it turns out in practice to be extremely hard in non-Gaussian circumstances to derive any analytic expression for critical points: in particular, it is very difficult to compute exactly some extremely cumbersome multiple integrals arising from the absolute values of the Hessian determinants, conditional on the gradient being null. Our purpose in this paper is to move some steps beyond these limitations: more precisely, our goal is to derive some semi-analytic expressions for the density of critical values for chi-square fields defined on the sphere.

The choice of chi-square fields is natural if one has in mind motivations from statistics or machine learning, and it is easy to figure out several applications. Among these, we are motivated by very concrete examples which arise from Cosmological Data Analysis. In particular, it has been shown in Lerario et al. (2025) that chi-square fields may approximate closely the behaviour of the squared norm for *random sections of spin fiber bundles*, i.e. the random fields which model the behaviour of *Cosmic Microwave Background polarisation*, see for instance Marinucci and Peccati (2011), Ch.12, Malyarenko (2013) or LiteBIRD Collaboration (2023). Understanding the distribution of critical points and extrema for polarisation fields is instrumental for the derivation of algorithms allowing point source detection in polarisation data; this would represent an extension of the approach given in the case of scalar random fields (Cosmic Microwave Background temperature data) in Cheng et al. (2020), see also Telschow et al. (2023), Pistolato and Stecconi (2024) and the references therein.

1.3. *Discussion of Main Results.* Our main results can be described as follows. We consider chi random fields fields with k degrees of freedom, defined as the square root of the sum of the squares of k i.i.d., unit variance, *normal* Gaussian fields on a smooth manifold M of dimension m. By this we mean the following.

Definition 1. Let (M, g) be a smooth Riemannian manifold. A *normal field* on (M, g) is a Gaussian field X on M, of class at least $\mathcal{C}^2$, having unit variance: $\mathbb{E}|X(p)|^2 = 1$ and such that $\mathbb{E}|d_pX(v)|^2 = 1$ for any unit tangent vector $v \in T_pM$; here and in the sequel, we are using d_pX to denote the differential of X at p. A *regular chi–field with k degrees of freedom* on (M, g) is a field f_k of the form

$$f_k = \sqrt{X_1^2 + \cdots + X_k^2}, \tag{1.1}$$

where $X_1, \ldots, X_k$ are i.i.d. copies of a normal field X. In this case, we say that f_k is *induced by* X.

For our applications, we have in mind $M = \mathbb{S}^m$ the unit sphere and X isotropic, i.e., invariant under the action of the orthogonal group, but our results do not require this assumption. We will show that the expected number of critical points of these fields can be computed, up to some explicit constants, as the expected value of the determinant of random matrices with Gaussian entries. These random matrices do not fall within any known class (such as the Gaussian Orthogonal Ensemble (GOE) or the Gaussian Unitary Ensemble (GUE)), and because these entries have a complicated dependence structure, these expected values in the general case can only be expressed as rather cumbersome multiple integrals, for which it is difficult to provide explicit analytic expressions. More precisely, let us introduce the following:

Definition 2. Let H be a random symmetric and Gaussian $m \times m$ matrix. We say that H is *Hessian-like* if there exists a Gaussian random variable $\gamma \sim \mathcal{N}(0, 1)$ such that $\mathbb{E}H\gamma = -\mathbb{1}_m$. In this case, we also say that H is *Hessian-like with respect to* γ and, for all $k \in \mathbb{N}$ and $t \geq 0$, we define the real number

$$E_k^t(H) := \mathbb{E}\left\{1_{[t,+\infty]}(\chi_k)\, |\det(A(k-1, m) + \chi_k H + \chi_k(\gamma - \chi_k)\mathbb{1}_m)|\right\} \in \mathbb{R}, \tag{1.2}$$

where χ_k is an auxiliary independent chi random variable of parameter k and $A(k-1, m)$ is an auxiliary independent Wishart random matrix, i.e. it is distributed according to Theorem 11 below.

Note that if H is Hessian-like and γ is as above, then the joint law of H, γ is determined by the law of H. This explains why we write $E_k^t(H)$ and not $E_k^t(H, \gamma)$. We stress the fact that the only dependence relation among the random variables and matrices in the expectation is that $\mathbb{E}[H\gamma] = -\mathbb{1}_m$.

Our first main result is the following; more discussion on the mentioned random fields and their properties is given in the Sections below:

Theorem 3. *Let (M, g) be a smooth m-dimensional Riemannian manifold. Let f_k be a regular chi-field with $k > m$ degrees of freedom on (M, g), induced by the normal field X. We have:*

$$\mathbb{E}[\#C_t] = \frac{\Gamma(\frac{k-m}{2})}{2^{m/2}\Gamma(\frac{k}{2})} \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M E_k^t(H_pX)dM(p), \tag{1.3}$$

where $\#C_t$ denotes the number of critical points of f_k of values equal or larger than t, and H_pX the Hessian matrix of X at $p \in M$.

Clearly in the special case where the random field $X : M \to \mathbb{R}$ is isotropic, in an adequate sense, the previous result simplifies to

$$\mathbb{E}[\#C_t] = \frac{\Gamma(\frac{k-m}{2})}{2^{m/2}\Gamma(\frac{k}{2})} \frac{\text{Vol}(M)}{(2\pi)^{\frac{m}{2}}} E_k^t(HX), \tag{1.4}$$

where the distribution of the Hessian HX does not depend on p.

The previous result is rather general but suffers from two limitations: the result on the expected value is not fully explicit as the computation of $E^t_k(H_pX)$ requires rather cumbersome multiple integrals (or simulations), and the case $k = m$ is not covered, despite for all m, k the total number of critical points of f_k with non zero value is integrable, as we show in Theorem 18. We are able to address at least partially these issues and obtain our second main result, which we describe below.

More precisely, when we focus on maxima, rather than critical points with an arbitrary signature, we are able to transform the problem into the computation of Gaussian extremes on a different domain, and hence to obtain much more explicit results. In particular, as mentioned above the maxima distribution is strongly motivated by statistical applications, such as the implementation of multiple testing with False Discovery Rate control, as in Cheng et al. (2020); to this aim, it is especially important to evaluate the distribution of maxima in the high-threshold tail, and especially in the 2-dimensional case $m = 2$. Indeed in terms of motivations, it is especially relevant the case where $M = \mathbb{S}^2$ and $k = 2$, because as we discussed earlier this corresponds to the modulus of isotropic spin random fields as those emerging from the analysis of Cosmic Microwave Background polarisation data, see also Carones et al. (2024) and the references therein. In this setting, we show that the maxima density takes the form of known polynomials of order 2 times a Gaussian density.

In particular, let us denote by $H_p(t)$ the Hermite polynomials defined as in Adler and Taylor (2007, sec. 11.6), and by $\phi(t) = (2\pi)^{-1/2}\exp(-t^2/2)$ the standard Gaussian density; we prove the following:

Theorem 4. *Let X_1, X_2 be two i.i.d. copies of an isotropic Gaussian field on the two-sphere, of class $\mathcal{C}^2$, with variance $\mathbb{E}|X_i(p)|^2 = 1$ and $\mathbb{E}\|d_pX_i\|^2 = 2r^2$. Let $f_2(p) := \sqrt{X_1(p)^2 + X_2(p)^2}$ and denote by $\mu_t(\mathbb{S}^2, f_2)$ the number of maxima of f_2 where $f_2 \geq t$. Then as $t \to \infty$ we have that, for some $\delta > 0$,*

$$\left(\left(H_2(t)2r^2 + 2\right)(2\pi)^{\frac{1}{2}} \cdot \phi(t)\right)^{-1} \mathbb{E}[\mu_t(\mathbb{S}^2, f_2)] = 1 + O\left(\exp\left(-\delta t^2\right)\right) \ . \tag{1.5}$$

In words, the tail behaviour of the maxima distribution is Gaussian, up to corrections terms which are fully explicit combinations of Hermite polynomials and known constants.

Remark 5. In order to be able to connect more easily with the existing literature on isotropic fields (e.g. Marinucci and Peccati (2011)), we stated the above result for fields that are not normal on the unit sphere, unlike in the rest of the paper. However, the field X_1 in Theorem 4 becomes a normal field if and only if the sphere is endowed with the round metric of radius r, in which case f_2 becomes a regular chi-field with 2 degrees of freedom, induced by X_1.

Theorem 4 can be seen a corollary of a more general result that is valid in full generality and shows that the behavior of the expected number of maxima of a regular chi field resembles in some aspects that of a Gaussian one, which is well documented in the literature. In particular, by the aforementioned passage from f_k to an auxiliary Gaussian field φ, we are able to exploit the results of Gayet (2022) and Adler and Taylor (2007), proving the following.

Theorem 6. *Let f_k be a regular chi-field with k degrees of freedom (see Theorem 1) on a smooth compact Riemannian manifold M of dimension m. For any Borel subset $A \subset M$, let us denote by $\mu_t(A, f_k)$ the number of maxima of f_k where $f_k \geq t$ that belong to A. If f_k is induced by the normal field X, then we have that*

$$\mathbb{E}\left(\mu_t(A, f_k)\right) = \frac{1}{2^{\frac{m+k-3}{2}}\pi^{\frac{m-1}{2}}\Gamma\left(\frac{k}{2}\right)} \int_A D^t_k\left([H_pX]\right) dM(p), \tag{1.6}$$

where $D^t_k([H_pX])$ depends solely the law of H_pX and is defined in Theorem 33. Moreover, as $t \to +\infty$, the following asymptotic equivalences hold up to an error of order $O\left(\exp\left(-(\frac{1}{2} + \delta)t^2\right)\right)$

for some $\delta > 0$*:*

$$\mathbb{P}\left(\max_M f_k \geq t\right) \sim \mathbb{E}\left(\mu_t(M, f_k)\right) \tag{1.7}$$

$$\sim \mathbb{E}(\#\{df_k = 0, f_k \geq t\}) \sim \mathbb{E}b(f_k \geq t) \sim \mathbb{E}b_0(f_k \geq t) \sim \mathbb{E}b_0(f_k \geq t; \mathbb{B}^m) \tag{1.8}$$

$$\sim \mathbb{E}\chi(f_k \geq t) \sim \sum_{j=0}^{m+k-1} \frac{\mathcal{L}_j(M \times \mathbb{S}^{k-1})}{(2\pi)^{\frac{j}{2}}} H_{j-1}(t)\phi(t) \tag{1.9}$$

$$\overset{(\text{if } M = r\mathbb{S}^2 \text{ and } k=2)}{\sim} (2 + 2r^2 H_2(t))\sqrt{2\pi}\phi(t). \tag{1.10}$$

Here, $b(f_k \geq t)$ *is the sum of all Betti numbers;* $b_0(f_k \geq t)$ *is the number of connected components;* $b_0(f_k \geq t, \mathbb{B}^m)$ *is the number of the connected components that are homeomorphic to the unit ball in* $\mathbb{R}^m$*;* $\chi(f_k \geq t)$ *is the Euler–Poincaré characteristic;* $\mathcal{L}_i$ *is the* i^{th} *Lipschitz–Killing curvature (defined as in Adler and Taylor (2007, sec. 7.6)).*

Note in particular that the asymptotic behavior of the excursion probability of f_k at a high threshold depends only on the geometry of M and not on the inducing normal field X (which is not uniquely determined by the Riemannian metric of M), although the distribution of $\max_M f_k$ might depend on X. Moreover, we can observe that

$$\mathbb{P}\left(\max_{M \times \mathbb{S}^{k-1}} \varphi \geq t\right) \sim \mathbb{E}\chi\left(\varphi \geq t\right), \tag{1.11}$$

for any normal Gaussian field φ defined on $M \times \mathbb{S}^{k-1}$, in virtue of Adler and Taylor (2007, Th. 14.0.2). Indeed, the main idea of our proof will be to show that for a suitable normal field φ, we have that $\mu_t(M, f_k) = \mu_t(M \times \mathbb{S}^{k-1}, \varphi)$, see Section 5 (see also Kuriki and Matsubara (2023) and Bloomfield et al. (2016) for some related results on the geometry of chi-square fields with a view to cosmological applications).

1.4. *Plan of the paper.* The plan of the paper is as follows: in Section 2 we fix our notation and introduce some background material; in Section 3 we give our general result for critical values, which is not fully explicit: for this reason, in Section 4 we study more deeply the structure of the Hessian in two dimension and in Section 5 we exploit these results to give a fully analytic expression for the expected value of the number of maxima, and in Section 6 we prove the high-threshold limits. We first prove Theorem 4 directly, then prove the more general Theorem 6 by relying on more abstract results.

2. Setting and Background

2.1. *Notations.* The following list contains some recurring conventions adopted in the rest of paper.

(i) Unless otherwise specified, every random element is assumed to be defined on an adequate common probability space $(\Omega, \mathfrak{S}, \mathbb{P})$.

(ii) A *random element* (see Billingsley (1999)) of the topological space V (or *with values* in V) is a measurable mapping $X\colon \Omega \to V$, defined on $(\Omega, \mathfrak{S}, \mathbb{P})$. In this case, one writes

$$X \in V \tag{2.1}$$

and denote by $[X] := \mathbb{P}X^{-1}$ the (push-forward) Borel probability measure on V induced by X. We will use the notation

$$\mathbb{P}\{X \in U\} := [X](U) = \mathbb{P}X^{-1}(U) \tag{2.2}$$

to indicate the probability that $X \in U$, for some Borel measurable subset $U \subset V$, and write (as usual)

$$\mathbb{E}\{f(X)\} := \int_V f(v)[X](dv), \tag{2.3}$$

to denote the expectation of the random variable $f(X)$, where $f\colon V \to \mathbb{R}^k$ is a measurable mapping such that the above integral is well-defined. We will sometimes write that X is a *random variable*, a *random vector* or a *random field*, respectively, when V is the real line, a vector space, or a space of functions $\mathcal{C}^r(M, \mathbb{R}^k)$, respectively.

(iii) We will use the special symbol

$$X\colon M \xrightarrow{\Omega} \mathbb{R}^k, \tag{2.4}$$

to indicate that X is a random field (see above), i.e., a random element of $\mathcal{C}^0(M, \mathbb{R}^k)$. The symbol hints at the fact that X is also a measurable function $X\colon M \times \Omega \to \mathbb{R}^k$.

(iv) The sentence: "X has the property $\mathcal{P}$ almost surely" (abbreviated "a.s.") means that the set $S = \{v \in V : v \text{ has the property } \mathcal{P}\}$ contains a Borel set of $[X]$-measure 1. It follows, in particular, that the set S is $[X]$-measurable, i.e. it belongs to the σ-algebra obtained from the completion of the measure space $(V, \mathcal{B}(V), [X])$.

(v) We write $\#(S)$ for the cardinality of the set S.

2.2. *Definition of the main objects.*

2.2.1. *Normal fields.* Let (M, g) be a smooth manifold of dimension m and let $X\colon M \xrightarrow{\Omega} \mathbb{R}$ be a Gaussian random field of class $\mathcal{C}^2$ such that for all $p \in M$ we have $X(p) \sim \mathcal{N}(0, 1)$ and

$$g_p(v, w) = \mathbb{E}\left\{d_pX(v)d_pX(w)\right\}. \tag{2.5}$$

We call $g_p(v, w)$ the *Adler and Taylor metric*, see Adler and Taylor (2007, Section 12.2). Following Mathis and Stecconi (2024, Definition 6.3), in this case we write $X \sim \mathcal{N}(M, g)$ and say that X is a *normal field* on (M, g), as anticipated in Theorem 1. Recall that the Hessian is the random bilinear form $H_pX\colon T_pM \times T_pM \to \mathbb{R}$ such that

$$H_pX(v, w) = \partial_v\partial_w X(p) - d_pX(\nabla_v w), \tag{2.6}$$

where ∇ is the Levi-Civita connection of g. Let us first recall the following standard characterization of the dependence structure for the gradient and Hessian.

Proposition 7. *For every $p \in M$,*

$X(p)$ and d_pX are independent.

(2) d_pX and H_pX are independent.

(3) $\mathbb{E}\{X(p)H_pX\} = -g_p$.

Proof: These results are classical and they are proved, for instance, in Adler and Taylor (2007, Section 12.2). □

2.2.2. *Chi distribution.*

Definition 8. Let $k \in \mathbb{N}$. We say that a random variable $\alpha \in \mathbb{R}$ is a *chi of parameter* k if it has the same law as the random variable

$$\chi_k := \sqrt{\gamma_1^2 + \cdots + \gamma_k^2}, \tag{2.7}$$

where $\gamma_1, \ldots, \gamma_k \sim \mathcal{N}(0, 1)$ are indipendent and identically distributed. In this case, we will write briefly that $\alpha \sim \chi_k$. The following characterization is classical, but we recall it for completeness.

Proposition 9. *Given $a \in \mathbb{R}$, we have that $\chi_k \in L^a$ if and only if $k > -a$.*

Proof: It is sufficient to observe that

$$\mathbb{E}\{\chi_k^a\} = \int_{\mathbb{R}^k} |x|^a \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{k}{2}}} d\mathbb{R}^k(x) \sim \int_0^1 r^{a+k-1} dr. \tag{2.8}$$

□

Before we state our first main result, let us recall a simple property of chi-random variables; by a straightforward computation, for $k > m$ we have that

$$\mathbb{E}\left\{\frac{1}{\chi_k^m}\right\} = \int_0^\infty \frac{1}{x^{m/2}} \frac{x^{k/2-1} exp(-x/2)}{\Gamma(k/2)2^{k/2}} dx \tag{2.9}$$

$$= \frac{\Gamma(\frac{k-m}{2})}{2^{m/2}\Gamma(\frac{k}{2})} \int_0^\infty \frac{x^{(k-m)/2} e^{-x/2}}{\Gamma(k-m)/2)} dx = \frac{\Gamma(\frac{k-m}{2})}{2^{m/2}\Gamma(\frac{k}{2})} . \tag{2.10}$$

2.2.3. *The chi-field.* Now let $Y := (X^1, \dots, X^k) : M \xrightarrow{\Omega} \mathbb{R}^k$ such that all components are i.i.d., $X^i \sim X$. Define $F : M \xrightarrow{\Omega} \mathbb{R}$ as

$$F(p) = \frac{1}{2}|Y(p)|^2 \ ; \tag{2.11}$$

in particular, notice that $F(p) \sim \frac{1}{2}\chi_k^2$ for all $p \in M$. Denote $Z := F^{-1}(0)$ and for $t \geq 0$,

$$\begin{aligned} C_t :&= \mathrm{Crit}(|Y|) \cap \{|Y| \geq t\} \\ &= \mathrm{Crit}(F) \cap \{F \geq \frac{t^2}{2}\} = \left\{p \in M : d_pF = 0, F(p) \geq \frac{t^2}{2}\right\}. \end{aligned} \tag{2.12}$$

Of course, Z denotes the nodal set of Y while C_t counts the number of critical values where the chi-field is larger than some given (positive) value t. Note that $Z \subset C_0 \subset M$ is a random submanifold of dimension $d = m - k$ and $C_t \subset C_0 \smallsetminus Z$ is a random finite set for all $t > 0$. In particular, if $k > m$ then Z is empty with probability one. Our first goal is to compute the expected value $\mathbb{E}[\# C_t]$; indeed, in the language of Section 1, the field $f = |Y|$ is a regular chi-field with k degrees of freedom, on the manifold M.

Remark 10. The Riemannian volume density of (M, g), which we denote as dM, is proportional to the expectation of the Riemannian d-volume of $Z = Y^{-1}(0)$:

$$\frac{1}{s_d}\mathbb{E}\left\{\int_Z \alpha(p) dZ(p)\right\} = \frac{1}{s_m}\int_M \alpha(p) dM(p), \tag{2.13}$$

where s_i is the i-dimensional volume of the unit sphere of dimension i: $\mathbb{S}^i \subset \mathbb{R}^{i+1}$ and α is any Borel function on M. This expression is a consequence of Kac-Rice formula (Azaïs and Wschebor, 2009, Theorem 6.8). The precise constants can be computed by testing the formula on spheres, see Lerario et al. (2025, Proposition 95).

2.2.4. *Random matrices.*

Definition 11. Let $k, m \in \mathbb{N}$. Let $\gamma_1, \dots, \gamma_m \sim N(0, \mathbb{1}_k)$. We define $A(k,m) \in \mathbb{R}^{m \times m}$ to be the random symmetric matrix whose coordinates $A_{a,b}$ have a joint law defined by:

$$A_{a,b} = \langle \gamma_a, \gamma_b \rangle. \tag{2.14}$$

Notice that $A \sim R^T A R$ for any orthogonal matrix $R \in O(m)$. For instance,

$$A(1,2) = \begin{pmatrix} \gamma_1^2 & \gamma_1\gamma_2 \\ \gamma_1\gamma_2 & \gamma_2^2 \end{pmatrix}, \tag{2.15}$$

$$A(2,2) = \begin{pmatrix} \gamma_{11}^2 + \gamma_{12}^2 & \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} \\ \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} & \gamma_{21}^2 + \gamma_{22}^2 \end{pmatrix}, \tag{2.16}$$

$$A(3,2) = \begin{pmatrix} \gamma_{11}^2 + \gamma_{12}^2 + \gamma_{13}^2 & \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} + \gamma_{13}\gamma_{23} \\ \gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22} + \gamma_{13}\gamma_{23} & \gamma_{21}^2 + \gamma_{22}^2 + \gamma_{23}^2 \end{pmatrix}. \tag{2.17}$$

Remark 12. Matrices of the form $A(k,m)$ follow a so-called Wishart distribution $A(k,m) \sim W_m(\mathbb{1}_m, k)$; more precisely, for $k \geq m$ these matrices have densities

$$f_{(k,m)}(A) = \frac{(\det(A))^{(k-m-1)/2} \exp(-\mathrm{tr}(A/2))}{2^{km/2}\pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma((k+1-j)/2)} \mathbb{1}_{\det(A)>0}(A). \tag{2.18}$$

It can be noted that the law of the matrix A depends just on its determinant and its trace - two quantities invariant to rotations, as expected; moreover, these densities are positive only over matrices which are positive definite, and they are zero otherwise. For instance, we have

$$f_{(2,2)}(A) = f_{(2,2)}(a_{11}, a_{12}, a_{21}, a_{22}) = \frac{\exp((-a_{11} - a_{22})/2)}{4\pi(a_{11}a_{22} - a_{12}a_{21})^{1/2}} \mathbb{I}_{\det(A)>0}(A), \tag{2.19}$$

and

$$f_{(3,2)}(A) = f_{(2,2)}(a_{11}, a_{12}, a_{21}, a_{22}) = \frac{\exp((-a_{11} - a_{22})/2)}{4\pi} \mathbb{I}_{\det(A)>0}(A). \tag{2.20}$$

Now recall the notion of Hessian-like matrices in Definition 2, and notice that $E_k^t(H) = E_k^t(R^T HR)$ for any orthogonal matrix $R \in O(m)$. Moreover, the property of being Hessian-like is also invariant under orthogonal changes of coordinates. Therefore, the following definition is well posed.

Definition 13. Let (T, g) be any Euclidean space of dimension m and let $k \in \mathbb{N}$ and $t > 0$. Let $H \colon T \times T \to \mathbb{R}$ be a Hessian-like Gaussian symmetric bilinear form on T. Then we define the deterministic real number

$$E_k^t(H) := E_k^t\left((H(e_a, e_b))_{1 \leq a,b \leq m}\right) \in \mathbb{R}, \tag{2.21}$$

where $e_1, \ldots, e_m$ is any orthonormal basis of T. To keep track the dependence on the metric g, when needed, we will write $E_k^t(g, H)$.

Lemma 14. *$E_k^t(\lambda g, H) = E_k^t(g, \lambda^{-1}H)$ for any $\lambda > 0$.*

Proof: The proof is straightforward and hence omitted. □

Remark 15. $E_k^t(H)$ depends only on covariance matrix of H, that is, it depends on the 4-tensor $\mathbb{E}H_{ab}H_{cd}$. We will not need to consider such object in this paper.

3. The First Main Result: Critical Points

In this Section we give our main result on the expected value of critical points for chi fields. For convenience, we split it into two subsections, when covering the case $k > m$, the other $k = m$ which requires some different argument.

3.1. *The expected value of critical points for $k > m$.* Here is the main result of this subsection.

Theorem 16. *In the setting described above, for all $k > m$, we have:*

$$\mathbb{E}[\#C_t] = \frac{\Gamma(\frac{k-m}{2})}{2^{m/2}\Gamma(\frac{k}{2})} \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M E_k^t(H_p X) dM(p). \tag{3.1}$$

Proof: Notice that when $k > m$, the set $Z = Y^{-1}(0)$ is almost surely empty. Let us observe that

$$d_p F(v) = Y(p)^T d_p Y(v) \in \mathbb{R}; \tag{3.2}$$

$$H_p F(v, w) = d_p Y(v)^T d_p Y(w) + Y(p)^T H_p Y(v, w) \in \mathbb{R}. \tag{3.3}$$

Notice that, by Theorem 7, we have that $d_p Y$ and $Y(p)$ are independent, for every fixed $p \in M$.

We will use the Kac-Rice formula (in particular, we refer to the statement in Mathis and Stecconi (2024, Alpha-formula, Prop. 6.1)). Assuming that the formula is applicable, we have that

$$\begin{aligned}\mathbb{E}\left\{\#C_t\right\} &= \mathbb{E}\left\{\sum_{p\in dF^{-1}(0)} 1_{[\frac{t^2}{2},+\infty]}(F(p))\right\} \\ &= \int_M \mathbb{E}\left\{1_{[\frac{t^2}{2},+\infty]}(F(p))|\det(H_pF)|\Big|d_pF=0\right\}\rho_{d_pF}(0)dM(p),\end{aligned} \tag{3.4}$$

where, for any $p\in M$ fixed, $\rho_{[d_pF]}\colon T_p^*M\to[0,+\infty)$ is the density of the random vector $d_pF\in T_p^*M$, with respect to the volume defined by the (flat) metric g_p. Indeed, $x\mapsto\rho_{[d_pF]}(x)$ exists if and only if $k>m$. In this case, it is continuous with respect to both (p,x). Let us compute it. Let us fix an orthonormal basis of T_p^*M, so that $(T_p^*M,g_p)\cong(\mathbb{R}^m,\mathbb{1}_m)$. Then for all bounded continuous functions $\alpha\colon\mathbb{R}^m\to[0,1]$ we have:

$$\begin{aligned}\int_{\mathbb{R}^m}\alpha(x)\rho_{[d_pF]}(x)d\mathbb{R}^m(x) &= \mathbb{E}\left\{\alpha(d_pF)\right\} \\ &= \int_{\mathbb{R}^k}\mathbb{E}\left\{\alpha(u^Td_pY)\big|Y(p)=u\right\}d[Y(p)](u) \\ &= \int_{\mathbb{R}^k}\mathbb{E}\left\{\alpha(u^Td_pY)\right\}d[Y(p)](u)=\dots\end{aligned} \tag{3.5}$$

where we used the expression $[Y(p)]$ to denote the probability measure induced by $Y(p)\in\mathbb{R}^k$, i.e., the law of $Y(p)$. Now observe that, by construction, the law of the random matrix of d_pY in an orthonormal basis is that of the $k\times m$ matrix:

$$d_pY=\left(\gamma_j^i\right)_{1\le i\le k,1\le j\le m}, \tag{3.6}$$

where $\gamma_j^i\sim\mathcal{N}(0,1)$ are i.i.d. This distribution is invariant under orthogonal transformations, therefore, the integrand above depends only on $|u|$. Observe that the law of $|Y(p)|$ is that of a chi of parameter k, that we have denoted as χ_k. Hence we obtain

$$\begin{aligned}\dots &= \int_0^{+\infty}\mathbb{E}\left\{\alpha(t(e_1)^Td_pY)\right\}d[|Y(p)|](t) \\ &= \mathbb{E}\left\{\alpha(\chi_k\cdot d_pX^1)\right\}=\mathbb{E}\left\{\int_{\mathbb{R}^m}\alpha(\chi_k\cdot x)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{m}{2}}}d\mathbb{R}^m(x)\right\} \\ &= \mathbb{E}\left\{\int_{\mathbb{R}^m}\alpha(y)\frac{e^{-\frac{|y|^2}{2\chi_k^2}}}{(2\pi)^{\frac{m}{2}}\chi_k^m}d\mathbb{R}^m(y)\right\}=\int_{\mathbb{R}^m}\alpha(x)\mathbb{E}\left\{\frac{e^{-\frac{|x|^2}{2\chi_k^2}}}{(2\pi)^{\frac{m}{2}}\chi_k^m}\right\}d\mathbb{R}^m(x)\ .\end{aligned} \tag{3.7}$$

In the 4th identity above, we used the change of variables $y=\chi_k\cdot x$. We conclude that

$$\rho_{[d_pF]}(x)=\mathbb{E}\left\{\frac{e^{-\frac{|x|^2}{2\chi_k^2}}}{(2\pi)^{\frac{m}{2}}\chi_k^m}\right\},\ \text{for almost every } x\in\mathbb{R}^m. \tag{3.8}$$

If $m<k$, this defines a continuous function of (p,x) (if $m\ge k$, it has a pole at $x=0$) and $\rho_{[d_pF]}(0)=\mathbb{E}\left\{(2\pi)^{-\frac{m}{2}}\chi_k^{-m}\right\}$.

Now, let us compute the conditional probability given d_pF. This is interpreted as a family of random vectors parametrized by the possible values of d_pF and we denote it as $[(F(p),H_pF)|d_pF=$

$\xi]$, for $\xi \in T_p^*M$. We are only interested in the case $\xi = 0$. We will do the computation in an orthonormal frame, so that $T_p^*M = \mathbb{R}^m$. Let $\alpha\colon \mathbb{R}^{m\times m} \to [0,1]$ be any continuous function. Then

$$\begin{aligned}\mathbb{E}\left\{\alpha(F(p), H_pF)\,|d_pF = 0\right\} &= \int_{\mathbb{R}^k} \mathbb{E}\left\{\alpha(F(p), H_pF)|d_pF = 0, Y(p) = u\right\} d[Y(p)](u) \\ &= \int_{\mathbb{R}^k} \mathbb{E}\left\{\alpha\left(\frac{|u|^2}{2}, H_pF\right) \middle| \begin{matrix} d_pF = 0, \\ Y(p) = u \end{matrix}\right\} d[Y(p)](u) \\ &= \int_{\mathbb{R}^k} \mathbb{E}\left\{\alpha\left(\frac{|u|^2}{2}, d_pY^T d_pY + u^T H_pY\right) \middle| \begin{matrix} u^T d_pY = 0, \\ Y(p) = u \end{matrix}\right\} d[Y(p)](u) = \dots\end{aligned} \tag{3.9}$$

Recall that the law of d_pY is that given in (3.6) and that it is independent from $(H_pY, Y(p))$. Moreover, by Theorem 7, we have that $[H_pX|X(p) = t] = [H_pX + (X(p) - t)g_p]$. Finally, as before, the integrand depends only on $|u|$ (indeed that $Y \sim RY$ for any $R \in O(k)$), so that we can continue as follows.

$$\begin{aligned}\dots &= \int_0^{+\infty} \mathbb{E}\left\{\alpha\left(\frac{t^2}{2}, \sum_{i=2}^k (d_pX^i)^T d_pX^i + tH_pX^1\right) \middle| \begin{matrix} d_pX^1 = 0, X^1(p) = t, \\ X^i(p) = 0\ \forall i \geq 2 \end{matrix}\right\} d[|Y(p)|](t) \\ &= \mathbb{E}\left\{\alpha\left(\frac{\chi_k^2}{2}, \sum_{i=2}^k (d_pX^i)^T d_pX^i + \chi_k(H_pX + (X(p) - \chi_k)g_p)\right)\right\} = \dots\end{aligned} \tag{3.10}$$

Notice that the random matrix $A := \sum_{i=2}^k (d_pX^i)^T d_pX^i$ has coordinates $A_{a,b} = \sum_{i=2}^k \gamma_a^i \gamma_b^i$, therefore $A \sim A(k-1, m)$, as in Theorem 11.

Moreover, H_pX is obviously a Hessian-like Gaussain matrix, in the sense of Theorem 13 and $\gamma := X(p) \sim \mathcal{N}(0,1)$ is, by Theorem 7, the associated Gaussian random variable such that $\mathbb{E}\{H_pX \cdot X(p)\} = -g_p = -\mathbb{1}_m$. Since the above identities are true for arbitrary α, we can interpret them as identities of probability laws, to conclude that:

$$[(F(p), H_pF)\,|d_pF = 0] = \left[\left(\frac{\chi_k^2}{2}, A(k-1, m) + \chi_k \cdot H_pX + \chi_k(\gamma - \chi_k)\mathbb{1}_m\right)\right], \tag{3.11}$$

where the only dependence relation is $\mathbb{E}H_pX\gamma = -g_p$.

Now that we have all the ingredients, we deduce that when $k > m$, dF satisfies the continuity properties at Mathis and Stecconi (2024, Def. 4.1), hence the Kac-Rice formula (Mathis and Stecconi, 2024, Prop. 6.1) is applicable and we conclude:

$$\begin{aligned}\mathbb{E}\left\{\#C_t\right\} &= \int_M \mathbb{E}\left\{1_{[\frac{t^2}{2}, +\infty]}(F(p))|\det(H_pF)| \middle| d_pF = 0\right\} \rho_{d_pF}(0) dM(p) \\ &= \int_M \mathbb{E}\left\{1_{[t,+\infty]}(\chi_k)|\det\left(A(k-1,m) + \chi_k \cdot H_pX + \chi_k(\gamma - \chi_k)\mathbb{1}_m\right)|\right\} \cdot \\ &\quad \cdot \mathbb{E}\left\{\frac{1}{(2\pi)^{\frac{m}{2}} \chi_k^m}\right\} dM(p) \\ &= \int_M E_k^t(H_pX)\mathbb{E}\left\{\frac{1}{(2\pi)^{\frac{m}{2}} \chi_k^m}\right\} dM(p).\end{aligned} \tag{3.12}$$

□

3.2. *The general case: including $m = k$.* The fact that the theorem holds only for $m < k$ seems to be due to a strange phenomenon of Kac-Rice formula: sometimes the Kac-Rice density, written as "conditional expectation times density", contains some expression of the form: $0 \cdot \infty$. In this case, it might be that $0 \cdot \infty \in \mathbb{R}$. Indeed, in principle, it is possible that there exists another function F_ε

with the same high level critical points as F, for which Kac-Rice formula can be applied. Indeed, when $k=1$, the critical points of F of level $t>0$ are exactly the critical points of the normal field X, of level $\pm t$, and such expectation can be computed with standard computations.

We have failed trying to find a good modification of F. However, we prove below that $\mathbb{E}\{\#C_t\}$ is finite whenever C_t is almost surely a finite set.

There is indeed a generalized version of Kac-Rice formula that can be applied directly to our situation, as we explain below.
Let $\mathbb{R}^k_M = M \times \mathbb{R}^k$ denote the trivial vector bundle over M of rank k. Let us consider the space of one jets of k-valued functions:

$$J^1(M,\mathbb{R}^k) := \mathbb{R}^k \times (T^*M)^{\oplus k} = \left\{(p,y,A) : p \in M, y \in \mathbb{R}^k, A : T_pM \to \mathbb{R}^k \text{ linear}\right\}, \tag{3.13}$$

where a linear map $A : T_pM \to \mathbb{R}^k$ is seen as a k-tuple of covectors $A^1,\dots,A^k \in T_p^*M$, which are its "rows".

Recall (see Hirsch (1994)) that to any smooth function $Y \in \mathcal{C}^1(M,\mathbb{R}^k)$, we can associate a smooth *1-jet prolongation* $j^1Y : M \to J^1(M,\mathbb{R}^k)$ defined as

$$j^1Y(p) := j^1_pY := (p, Y(p), d_pY). \tag{3.14}$$

Clearly, $P : J^1(M,\mathbb{R}^k) \to M$ is a smooth vector bundle and j^1Y is a smooth section. We will use the following standard notation for the fiber of this vector bundle: for any $p \in M$

$$P^{-1}(p) =: J^1_p(M,\mathbb{R}^k). \tag{3.15}$$

For any $t \geq 0$, define the subset $W_t \subset J^1(M,\mathbb{R}^k)$ such that

$$W_t := \left\{(p,y,A) \in J^1(M,\mathbb{R}^k) : |y| > t \text{ and } y^TA = 0\right\}. \tag{3.16}$$

The closure of W_t is just the set $\overline{W_t} = W_t \cup \partial W_t$, where

$$\partial W_t := \left\{(p,y,A) \in J^1(M,\mathbb{R}^k) : |y| = t \text{ and } y^TA = 0\right\}. \tag{3.17}$$

Now, observe that W_t has codimension m and that the set that we are studying is

$$C_t = (j^1Y)^{-1}(\overline{W_t}). \tag{3.18}$$

It is easy to see that $W_t \subset J^1(M,\mathbb{R}^k)$ is an open semialgebraic (locally, because it is defined by polynomial inequalities) submanifold for all $t \geq 0$. Indeed, in a local chart defined on an open subset $O \subset M$ we have that

$$W_t \cap P^{-1}(O) \cong \left\{(p, ry, (A_1,\dots,A_m)) \in O \times \mathbb{R}^k \times \mathbb{R}^{k\times m} : p \in O, r > t, y \in \mathbb{S}^{k-1}, A_i \in y^\perp\right\}, \tag{3.19}$$

where here $A_1,\dots,A_m$ are the columns of $A : \mathbb{R}^m \to \mathbb{R}^k$. Thus, it follows that W_t is locally diffeomorphic to

$$W_t \cap P^{-1}(O) \cong O \times \left((t,+\infty) \times \left(T\mathbb{S}^{k-1}\right)^{\oplus m}\right). \tag{3.20}$$

If $t>0$, the closure $\overline{W_t} = W_t \cup \partial W_t$ is a manifold with boundary. In the case $t=0$, the topological frontier ∂W_t is not a smooth boundary, but rather an additional stratum of codimension k:

$$\overline{W_0} \cap P^{-1}(O) \cong \left(O \times \{0\} \times \mathbb{R}^{k\times m}\right) \cup W_0 \cap P^{-1}(O). \tag{3.21}$$

This stratum is semialgebraic thus, the union $\overline{W_0}$ remains a semialgebraic subset with two strata. Its codimension is, by definition, the minimum of the codimensions of the two strata. Therefore $\overline{W_0}$ has codimension m if and only if $m \leq k$ and $\overline{W_t}$, for $t>0$, has always codimension m.

Remark 17. Observe that when $k > m$, the stratum ∂W_0 is too small and $j^1Y^{-1}(\partial W_0) = \{Y = 0\}$ is almost surely empty. In the general case, $j^1Y^{-1}(\partial W_0) = \{Y = 0\} \subset M$ is almost surely a submanifold of dimension $m - k$. In particular, we will certainly have $\mathbb{E}\#C_0 = \infty$ if $k < m$.

We are in the position to apply Stecconi (2022, Thm. 27) to deduce the following. Notice that the case $k = m$ was not included in Theorem 3.

Theorem 18. *For all $k \in \mathbb{N}$ and $t > 0$, we have $\mathbb{E}\{\#C_t\} \leq \mathbb{E}\{\#C_{0^+}\}$, where*

$$\mathbb{E}\{\#C_{0^+}\} := \mathbb{E}\{\# \cup_{t>0} C_t\} < +\infty. \tag{3.22}$$

Moreover, $\mathbb{E}\{\#Z\} = \mathbb{E}\{\#C_0\} - \mathbb{E}\{\#C_{0^+}\}$ is the expected number of zeroes of Y and satisfies the following: if $k > m$, then $\mathbb{E}\{\#Z\} = 0$; if $k = m$, then $\mathbb{E}\{\#Z\} \in (0, +\infty)$; if $k < m$, then $\mathbb{E}\{\#Z\} = +\infty$.

Proof: Following the discussion above, we have to show that $\mathbb{E}\{\#(j^1Y)^{-1}(W_0)\}$ is finite. Let $W = W_0$ and let $\pi\colon E \to M$ be the trivial vector bundle $E := M \times \mathbb{R}^k$. We will apply Stecconi (2022, Cor. 3.9) to the random field $Y\colon M \xrightarrow{\Omega} \mathbb{R}^k$, that in the language of Stecconi (2022, Cor. 3.9), is a smooth Gaussian random section of E. The fiber over $p \in M$ of its 1-jet extension is $J^1_pE = J^1_p(M, \mathbb{R}^k) = P^{-1}(p)$.

We already observed that $W \subset E$ is a semialgebraic submanifold of codimension m. by Stecconi (2022, Rem. 3.3), this implies that W has sub-Gaussian concentration. The fact that W is transverse to the fibers $P^{-1}(p)$ for all $p \in M$ is obvious from Equation (3.21), in that the local equations of W do not involve p.

The 1-jet of Y at $p \in M$ is

$$j^1_pY = (p, Y(p), d_pY) \in J^1_p(M, \mathbb{R}^k), \tag{3.23}$$

which is non-degenerate by construction since its support is the whole space $\{p\} \times \mathbb{R}^k \times (T^*_pM)^k = J^1_p(M, \mathbb{R}^k)$.

We checked all hypotheses for point 1. of Stecconi (2022, Cor 3.9), applied to the field Y, which implies that $\mathbb{E}\{\#(j^1Y)^{-1}(W)\}$ is finite given that the manifold is compact.

Regarding the set of zeroes $Z = Y^{-1}(0)$, we have that if $k > m$, then Z is almost surely empty while if $k \leq m$, we can use Mathis and Stecconi (2024, Theorem 6.2) using the same argument as in the proof of Mathis and Stecconi (2024, Lemma 6.5) to compute

$$\mathbb{E}\left\{\mathrm{vol}_{m-k}(Z)\right\} = \frac{\mathrm{vol}_{m-k}(\mathbb{S}^{m-k})}{\mathrm{vol}_m(\mathbb{S}^m)} \mathrm{vol}_m(M), \tag{3.24}$$

where vol_j denotes the j^{th} Hausdorff volume measure associated to the Riemannian manifold (M, g). Clearly, (3.24) implies the thesis. □

Remark 19. The condition $k \geq m$ in the above theorem is due to the fact that C_0 includes all critical values v satisfying $v \geq 0$. For instance, when $k = 1$ and $Y = X$, we have that $C_0 = \{X = 0\} \cup \{dX = 0\}$ is clearly infinite, whilst $\mathbb{E}\#C_{0^+} = \mathbb{E}\#\{dX = 0\}$ is finite.

4. A study of the Hessian in dimension 2

Consider the case in which the original Gaussian random function $X : M \xrightarrow{\Omega} \mathbb{R}$ is a random eigenfunction on the sphere $M = \mathbb{S}^m$. Then, X is invariant in law under isometries of $\mathbb{S}^m$. In particular, this implies that for any $R \in O(m+1)$, that fixes a point $p \in \mathbb{S}^m$ we have that

$$R^T H_pXR = H_p(X \circ R) \sim H_pX. \tag{4.1}$$

the same happens for stationary fields on $\mathbb{R}^d$, like Berry or Bargmann-Fock. We may regard such isometry R as an isometry of $p^\perp = T_p\mathbb{S}^m$ and generalize this concept.

Definition 20. Given a Gaussian bilinear form H on a Euclidean space V, we say that H is *rotation invariant* if for every $R\colon V \to V$ linear orthogonal transformation, there is an equivalence in law:

$$H \sim R^T H R. \tag{4.2}$$

Given a Gaussian field $X\colon M \xrightarrow{\Omega} \mathbb{R}$, we say that X has *rotation invariant Hessian* if for every point $p \in M$ the Hessian $H_p X$ is a rotation invariant, as a random bilinear form on $T_p M$.

Lemma 21. *Let* $H = \begin{pmatrix} h_1 & h_3 \\ h_3 & h_2 \end{pmatrix}$ *satisfy Equation (4.2). Then there are constants* $\sigma, c \geq 0$ *such that:*

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \sim \mathcal{N}\left(0, \begin{pmatrix} 2\sigma^2 + c & c & 0 \\ c & 2\sigma^2 + c & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}\right). \tag{4.3}$$

Moreover, H *is Hessian-like if and only if* $(\sigma^2 + c) \geq 1$*, with*

$$\gamma = -\operatorname{tr}(H) \frac{1}{2(\sigma^2 + c)} + \gamma_0 \sqrt{1 - \frac{1}{(\sigma^2 + c)}}. \tag{4.4}$$

for some $\gamma_0 \sim \mathcal{N}(0,1)$ *independent from* H*.*

Proof: First, one can easily see that $\mathbb{E}[h_1^2] = \mathbb{E}[h_2^2] =: a^2$, and that $\mathbb{E}[h_1 h_3] = \mathbb{E}[h_2 h_3] =: b$. Let $R(\theta)$ be the matrix of the rotation of angle θ in $\mathbb{R}^2$ and define

$$H(\theta) = \begin{pmatrix} h_1(\theta) & h_3(\theta) \\ h_3(\theta) & h_2(\theta) \end{pmatrix} := R(\theta)^T H R(\theta). \tag{4.5}$$

□

By imposing the condition that $\mathbb{E}[h_i(\theta) h_j(\theta)]$ is constant in θ, one gets, for any choice of i, j, the same condtion:

$$a^2 = 2\sigma^2 + c \quad \text{and} \quad b = 0; \tag{4.6}$$

hence the proposition is proven.

4.0.1. *Stationary plane fields.*

Proposition 22. *Let* $\xi\colon \mathbb{R}^d \xrightarrow{\Omega} \mathbb{R}$ *be a stationary and isotropic Gaussian random field of class* $\mathcal{C}^2$*, with covariance function* $K(|x-y|) = \mathbb{E}\{\xi(x)\xi(y)\}$*. Then the random variables* $\partial_i \partial_j \xi(0)$*, for* $1 \leq i \neq j \leq d$ *have the following covariances:*

$$\mathbb{E}\partial_{i,j}\xi(0)\partial_{i,k}\xi(0) = \mathbb{E}\partial_{i,j}\xi(0)\partial_{h,k}\xi(0) = 0, \tag{4.7}$$

$$\mathbb{E}|\partial_i^2 \xi(0)|^2 = K''''(0), \quad \mathbb{E}|\partial_i \partial_j \xi(0)|^2 = \frac{1}{3} K''''(0), \quad \mathbb{E}\partial_i^2 \xi \partial_j^2 \xi = \frac{1}{3} K''''(0), \tag{4.8}$$

where i, j, h, k *are any 4 distinct indices and* $K''''(0)$ *denotes the fourth derivative of* K *evaluated at the origin. In particular, if* $d = 2$*, then* $H = H_0 \xi$ *satisfies (4.3) with the additional condition that* $c = \sigma^2$*:*

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \sim \mathcal{N}\left(0, K''''(0) \begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}\right). \tag{4.9}$$

and

$$\xi(0) = -\Delta \xi(0) \frac{3}{4K''''(0)} + \gamma_0 \sqrt{1 - \frac{3}{2K''''(0)}}. \tag{4.10}$$

for some $\gamma_0 \sim \mathcal{N}(0,1)$ *independent from* H*,* Δ *denoting as usual the Laplacian operator.*

Remark 23. The above proposition is in accordance with Nicolaescu (2017, Eq. (2.11)). Note that our setting includes the Hessian of Berry's random field, for which Nicolaescu (2017, Prop. B.6) does not hold.

Remark 24. For ξ to be a *normal* field on $\mathbb{R}^d$ with respect to the standard metric (see Section 2), we must have that $K(0) = -K''(0) = 1$.

Proof: $K(t)$ is an even function of class $\mathcal{C}^2$, so its Taylor expansion is a series in t^2. Let us define $K(t) = h(t^2)$, then or all $n \in \mathbb{N}$

$$h^{(n)}(0) = \frac{n!}{(2n)!}K^{(2n)}(0). \tag{4.11}$$

Now, it is sufficient to compute $\partial_i\partial_j\partial_{i'}\partial_{j'}h(|x-y|^2)$ at $x = y = 0$. We report only the computation of $\mathbb{E}\partial_i^2\xi\partial_j^2\xi$.

$$\begin{aligned}
\mathbb{E}\partial_i^2\xi\partial_j^2\xi &= \frac{d^2}{dt^2}\Big|_0\frac{d^2}{ds^2}\Big|_0\mathbb{E}\left[\xi(t,0)\xi(0,s)\right] \\
&= \frac{d^2}{dt^2}\Big|_0\frac{d^2}{ds^2}\Big|_0 h\left(t^2+s^2\right) = \frac{d^2}{dt^2}\Big|_0\frac{d}{ds}\Big|_0 h'\left(t^2+s^2\right)2s \\
&= \frac{d^2}{dt^2}\Big|_0 h'\left(t^2\right)2 = 4h''(0) = 4\frac{2!}{4!}K''''(0).
\end{aligned} \tag{4.12}$$

□

Let $J_0\colon \mathbb{R} \to \mathbb{R}$ be the Bessel function of the first kind, that is,

$$J_0(t) := \sum_{j=0}^{\infty}\frac{(-1)^j}{(j!)^2}\left(\frac{t}{2}\right)^{2j}. \tag{4.13}$$

Proposition 25. *If $\xi : \mathbb{R}^2 \xrightarrow{\Omega} \mathbb{R}$ is the Berry random field, with covariance $\mathbb{E}\{\xi(x)\xi(y)\} = J_0(\sqrt{2}|x-y|)$*[1]*, then ξ is a normal field on $\mathbb{R}^2$, with $K''''(0) = \frac{3}{2}$ and*

$$H_0\xi = \begin{pmatrix}\xi_1''(0) & \xi_{12}''(0)\\ \xi_{12}''(0) & \xi_2''(0)\end{pmatrix} \quad \textit{with} \quad \begin{pmatrix}\xi_1''(0)\\ \xi_2''(0)\\ \xi_{12}''(0)\end{pmatrix} \sim \mathcal{N}\left(0, \frac{3}{2}\begin{pmatrix}1 & \frac{1}{3} & 0\\ \frac{1}{3} & 1 & 0\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\right). \tag{4.14}$$

In particular, $\xi(0) = -\frac{1}{2}\Delta\xi(0)$.

Proposition 26. *If $\xi : \mathbb{R}^2 \xrightarrow{\Omega} \mathbb{R}$ is the Bargmann-Fock random field, with covariance $\mathbb{E}\{\xi(x)\xi(y)\} = e^{-\frac{|x-y|^2}{2}}$, then ξ is a normal field on $\mathbb{R}^2$, with $K''''(0) = 2$ and*

$$H_0\xi = \begin{pmatrix}\xi_1''(0) & \xi_{12}''(0)\\ \xi_{12}''(0) & \xi_2''(0)\end{pmatrix} \quad \textit{with} \quad \begin{pmatrix}\xi_1''(0)\\ \xi_2''(0)\\ \xi_{12}''(0)\end{pmatrix} \sim \mathcal{N}\left(0, 2\begin{pmatrix}1 & \frac{1}{3} & 0\\ \frac{1}{3} & 1 & 0\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\right). \tag{4.15}$$

In particular, $\xi(0) = -\frac{3}{8}\Delta\xi(0) + \frac{1}{2}\gamma_0$.

4.0.2. *Isotropic spherical fields.*

Proposition 27. *Take $\xi : \mathbb{S}^2 \xrightarrow{\Omega} \mathbb{R}$ to be a Gaussian isotropic spherical random field, with covariance $\mathbb{E}\{\xi(p)\xi(q)\} = K(\theta(p,q))$, where $\theta(p,q)$ denotes the spherical distance of p and q; let $K(0) = 1$, $a^2 := K''''(0)$ and $r^2 := -K''(0)$. Then, $\hat{\xi}(p) := \xi(r^{-1}p)$ is a normal field on $r\mathbb{S}^2$. In such case, for*

[1]This normalization, with the factor $\sqrt{2}$, is the only one that ensures that we are in the setting of this paper, namely $K''(0) = -1$. Then, ξ satisfies the almost sure equation $\Delta\xi = -2\xi$.

any fixed $p \in \mathbb{S}^2$ and orthonormal basis u, v of $p^\perp$ we have that the Riemannian Hessian of $\hat{\xi}$ has the following law:

$$H_p\hat{\xi} =: \begin{pmatrix} \hat{\xi}''_u(p) & \hat{\xi}''_{uv}(p) \\ \hat{\xi}''_{uv}(p) & \hat{\xi}''_v(p) \end{pmatrix}, \text{ with } \begin{pmatrix} \hat{\xi}''_u(p) \\ \hat{\xi}''_v(p) \\ \hat{\xi}''_{uv}(p) \end{pmatrix} \sim \mathcal{N}\left(0, \frac{a^2}{r^4}\begin{pmatrix} 1 & \frac{1}{3}+\frac{2r^2}{3a^2} & 0 \\ \frac{1}{3}+\frac{2r^2}{3a^2} & 1 & 0 \\ 0 & 0 & \frac{1}{3}-\frac{r^2}{3a^2} \end{pmatrix}\right). \quad (4.16)$$

In particular, in the notation of Theorem 21, $c - \sigma^2 = \frac{1}{r^2}$ and $H_p\hat{\xi}$ is Hessian-like with respect to

$$\gamma = -\Delta_{\mathbb{S}^2}\xi(0)\frac{3r^4}{(4a^2+2r^2)} + \gamma_0\sqrt{1 - \frac{3r^4}{(2a^2+r^2)}}, \quad (4.17)$$

where $\Delta_{\mathbb{S}^2}$ denotes the spherical Laplacian and $\gamma_0 \sim \mathcal{N}(0,1)$ is independent from $H_p\hat{\xi}$.

Proof: Notice that $K(\theta) = h(\cos\theta)$ for some $\mathcal{C}^1$ function h. If ξ is isotropic, then it has rotation invariant Hessian, thus Theorem 21 holds. For this reason it is enough to compute $\mathbb{E}|\hat{\xi}''_u(p)|^2$ and $\mathbb{E}[\hat{\xi}''_u(p)\hat{\xi}''_v(p)]$. Let $p(\theta,\phi) \in \mathbb{S}^2$ be the point with polar coordinates θ and φ and let us assume that $p = p(0,\varphi)$ is the north pole, so that the curves $t \mapsto p(t,\varphi)$ are geodesics, for any fixed φ.

$$\begin{aligned} \mathbb{E}[\hat{\xi}''_u(p)\hat{\xi}''_v(p)] &= \frac{d^2}{dt^2}\Big|_0 \frac{d^2}{ds^2}\Big|_0 \mathbb{E}[\hat{\xi}(rp(r^{-1}t,0))\hat{\xi}(rp(r^{-1}s,\frac{1}{2}\pi)] \\ &= \frac{d^2}{dt^2}\Big|_0 \frac{d^2}{ds^2}\Big|_0 \mathbb{E}[\xi(p(t,0))\xi(p(s,\frac{1}{2}\pi)]r^{-4} \\ &= \frac{d^2}{dt^2}\Big|_0 \frac{d^2}{ds^2}\Big|_0 h\left(\langle p(t,0), p(s,\frac{1}{2}\pi)\rangle\right) r^{-4} \\ &= \frac{d^2}{dt^2}\Big|_0 \frac{d^2}{ds^2}\Big|_0 h(\cos t\cos s) = h''(1) + h'(1) = \left(\frac{1}{3}K''''(0) - \frac{2}{3}K''(0)\right) r^{-4}. \end{aligned} \quad (4.18)$$

An analogous computation shows that $\mathbb{E}|\hat{\xi}''_u(p)|^2 = K''''(0)r^{-4}$. □

Remark 28. It is well-known that under isotropy the covariance function can be expressed as

$$\mathbb{E}\{\xi(p)\xi(q)\} = K(\theta(p,q)) = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell P_\ell(\cos\theta(p,q)),$$

the non-negative sequence $(C_\ell)_{\ell=0,1,2,\dots}$ denoting the angular power spectrum of the field and $P_\ell(.)$ representing Legendre polynomials (see e.g. Marinucci and Peccati (2011)). Then standard computations yield (see e.g. Cammarota et al. (2016))

$$K(0) = \sum_\ell \frac{2\ell+1}{4\pi}C_\ell,$$

$$K''(0) = -\sum_\ell \frac{2\ell+1}{4\pi}\frac{\lambda_\ell}{2}C_\ell,$$

where we wrote $\lambda_\ell = \ell(\ell+1)$, and

$$K''''(0) = \sum_\ell \frac{2\ell+1}{4\pi}\left(3\frac{\lambda_\ell(\lambda_\ell-2)}{8} + \frac{\lambda_\ell}{2}\right)C_\ell.$$

4.1. *Behavior of the Hessian under scaling limit.*

Proposition 29. *Assume that $X_\lambda \colon M \xrightarrow{\Omega} \mathbb{R}$ is a sequence of $\mathcal{C}^2$ GRFs of unit variance with Adler-Taylor metric g^λ (so that $X_\lambda \sim \mathcal{N}(M, g^\lambda)$) and such that the following limit holds:*

$$X_\lambda \left(\exp_p^{g_\lambda} \left(\frac{u}{\sqrt{\lambda}} \right) \right) \xrightarrow[\lambda \to +\infty]{\mathcal{C}^2(T_pM)\text{-law}} \xi(u), \tag{4.19}$$

in distribution in the space of $\mathcal{C}^2$ functions of $u \in T_pM$, where $\xi \colon T_pM \xrightarrow{\Omega} \mathbb{R}$ is some GRFs on $T_pM \cong \mathbb{R}^m$. Let H^{g^λ} denote the Hessian operator with respect to the metric g^λ. Then, we have that for every $p \in M$,

$$\frac{1}{\lambda} H_p^{g^\lambda} X_\lambda(u, v) \xrightarrow[\lambda \to +\infty]{\mathbb{R}-law} H_0 \xi(u, v). \tag{4.20}$$

Proof: It is enough to check the limit for $u = v$, since the symmetric form $H_p X_\lambda(u, v)$ can be recovered from the quadratic form $H_p X_\lambda(u, v)$ by means of the polarization formula. Moreover, by Skorohod's theorem, we can assume that the convergence in Equation (4.19) holds almost surely. We have that

$$\frac{1}{\lambda} H_p^{g^\lambda} X_\lambda(u, u) = \frac{d^2}{dt^2}\Big|_{t=0} X_\lambda \left(\exp_p^{g_\lambda} \left(\frac{tu}{\sqrt{\lambda}} \right) \right) \xrightarrow[\lambda \to +\infty]{\mathbb{R}} \frac{d^2}{dt^2}\Big|_{t=0} \xi(tu). \tag{4.21}$$

□

The hypotheses of the above proposition are, in particular, satisfied for Gaussian Laplace eigenfunctions: $\Delta_{\mathbb{S}^m} X_\lambda = -\lambda X_\lambda$ on the sphere $M = \mathbb{S}^m$, with $\lambda \in \{\ell(\ell + m - 1) \colon \ell \in \mathbb{N}\}$ tending to $+\infty$. In this case, we have that $g^\lambda = \frac{\lambda}{m} g$, where g is the standard round metric on $\mathbb{S}^m$ and therefore $\exp_x^g = \exp_x^{g^\lambda}$, so that Equation (4.19) is the usual scaling limit, with ξ being Berry's random field on $T_p\mathbb{S}^m \cong \mathbb{R}^m$.

In this situation, as $\lambda \to +\infty$, we can approximate:

$$E_k^t \left(g^\lambda, H_p^{g^\lambda} X_\lambda \right) = E_k^t \left(g, \frac{m}{\lambda} H_p^{g^\lambda} X_\lambda \right) \sim E_k^t(g, m H_0 \xi), \tag{4.22}$$

which by Theorem 3 gives as $\lambda \to +\infty$

$$\mathbb{E}\# C_t^\lambda \sim \mathbb{E} \left\{ \frac{1}{\chi_k^m} \right\} \frac{1}{(2\pi)^{\frac{m}{2}}} \text{vol}(M) \cdot E_k^t(g, m H_0 \xi) \cdot \left(\frac{\lambda}{m} \right)^{\frac{m}{2}}. \tag{4.23}$$

5. An exact formula for local maxima

For applications in Statistics, Mathematical Physics and Machine Learning it is of course very common to focus on local maxima, especially at high threshold. These are the random quantities that must be considered, for instance, when probing for galactic point sources among Cosmic Microwave background (CMB) polarisation data, or when investigating the convergence properties of statistics and machine learning optimization algorithms. In this Section, we show how much more explicit results can be obtained, in the limit of high thresholds u.

Let us first introduce the following auxiliary Gaussian random function $\varphi \in \mathcal{C}^\infty(M \times \mathbb{S}^{k-1})$

$$\varphi : M \times \mathbb{S}^{k-1} \to \mathbb{R}, \quad \varphi(p, u) := Y(p)^T u. \tag{5.1}$$

Indeed, we have that if $F = \frac{|Y|^2}{2}$ has non-degenerate maxima (true a.s.), then there is a bijection:

$$\begin{gathered} \{p \in M : \text{ local maxima of } F\} \xrightarrow{\cong} \left\{ (p, v) \in M \times \mathbb{S}^{k-1} : \text{ local maxima of } \varphi \right\} \\ p \to \left(p, \frac{Y(p)}{|Y(p)|} \right). \end{gathered} \tag{5.2}$$

Notice also that $F(p) = \frac{1}{2}\varphi\left(p, \frac{Y(p)}{|Y(p)|}\right)^2$. It follows that almost surely we have that, for all $t \geq 0$

$$\begin{aligned} C_t \cap \{p \in M : H_pF < 0\} &= \left\{p \in M : \text{ local maxima of } F, \text{ with value} \geq \frac{t^2}{2}\right\} \\ &\cong \left\{(p,v) \in M \times \mathbb{S}^{k-1} : \text{ local maxima of } \varphi \text{ of value} \geq t\right\} \\ &= \left\{(p,v) \in M \times \mathbb{S}^{k-1} : d_{(p,v)}\varphi = 0, d^2_{(p,v)}\varphi < 0, \varphi(p,v) \geq t\right\} \\ &=: C_t^{\mathrm{Max}} \end{aligned} \tag{5.3}$$

Recall that $Y = (X^1, \ldots, X^k)$, where $X^i \sim X$ are i.i.d. copies of $X \sim \mathcal{N}(M, g)$.

Lemma 30. *Let $(p,u) \in M \times \mathbb{S}^{k-1}$ and let us choose orthonormal bases to identify $T_pM \cong \mathbb{R}^m$ and $T_u\mathbb{S}^{k-1} \cong \mathbb{R}^{k-1}$. Then d_pX is identified with a standard Gaussian* row *in $\mathbb{R}^m$ and H_pX is identified with a symmetric Gaussian $m \times m$ matrix. The 2-jet of φ has the following joint distribution:*

$$\begin{aligned} \varphi(p,v) &= X^1(p) \in \mathbb{R} \\ d_{(p,v)}\varphi &= \left(d_pX^1, X^2(p), \ldots, X^k(p)\right) \in \mathbb{R}^m \times \mathbb{R}^{k-1} \\ H_{(p,v)}\varphi &= \begin{pmatrix} H_pX^1 & (d_pX^2)^T & \ldots & (d_pX^k)^T \\ d_pX^2 & & & \\ \ldots & & -X^1(p)\mathbb{1}_{k-1} & \\ d_pX^k & & & \end{pmatrix} \in \left(\mathbb{R}^m \times \mathbb{R}^{k-1}\right) \otimes \left(\mathbb{R}^m \times \mathbb{R}^{k-1}\right). \end{aligned} \tag{5.4}$$

In particular, we have that $d_{(p,v)}\varphi$ and $H_{(p,v)}\varphi$ are independent and the dependence between $\varphi(p,v)$ and $H_{(p,v)}\varphi$ is that $\mathbb{E}\{H_{(p,v)}\varphi \cdot X^1(p)\} = -\mathbb{1}_{m+k-1}$.

Proof: The result is the same (in law) for all $v \in \mathbb{S}^{k-1}$, so that we can choose $v = e_1$. Then $T_v\mathbb{S}^{k-1}$ is identified with $e_1^\perp = \mathbb{R}^{k-1}$. In a neighborhood of e_1 in $\mathbb{S}^{k-1}$, we take affine coordinates $u = u^2, \ldots u^k \in \mathbb{R}^{k-1}$, to parametrize the point $v(u) = (\sqrt{1-|u|^2}, u) \in \mathbb{S}^{k-1}$, so that for any curve $t \mapsto u(t)$, the velocity $\frac{d}{dt}(v(u)) \in T_{e_1}\mathbb{S}^{k-1}$ is isometrically identified with $\dot{u} \in \mathbb{R}^{k-1}$. Then, we have $\varphi(p,v) = Y(p)^Tv(u)$. For every $\dot{p} \in T_pM$, and $\dot{u} \in \mathbb{R}^{k-1}$, we compute the Hessian as follows. Let $p(t)$ be a geodesic in M such that $p(0) = p$ and $\dot{p}(0) = \dot{p}$ and let $u(t)$ parametrize a geodesic $v(u(t))$ on $\mathbb{S}^{k-1}$, with $u(0) = 0$ and $\dot{u}(0) = \dot{u}$, that is, $u(t) = \left(\sin(t|\dot{u}|)\frac{\dot{u}}{|\dot{u}|}\right)$. Then,

$$\begin{aligned} H_{(p,e_1)}\varphi\left((\dot{p},\dot{u}),(\dot{p},\dot{u})\right) &= \frac{d^2}{dt^2}\left[X^1(p)\sqrt{1-|u|^2} + \sum_{i=2}^k X^i(p)u^i\right] \\ &= H_pX^1(\dot{p},\dot{p}) + X^1(p)\frac{d^2}{dt^2}\sqrt{1-|u|^2} + 2\sum_{i=2}^k d_pX^i(\dot{p})\dot{u}^i \\ &= H_pX^1(\dot{p},\dot{p}) - X^1(p)|\dot{u}|^2 + 2\sum_{i=2}^k d_pX^i(\dot{p})\dot{u}^i. \end{aligned} \tag{5.5}$$

□

Observe that the law above depends uniquely on the metric g at p, that is essentially the covariance of d_pX, and on the law of H_pX.

Lemma 31. *Let* $(p, u) \in M \times \mathbb{S}^{k-1}$ *and let us choose orthonormal bases to identify* $T_pM \cong \mathbb{R}^m$ *and* $T_u\mathbb{S}^{k-1} \cong \mathbb{R}^{k-1}$. *The* 2*-jet of* φ *has the following joint distribution:*

$$\varphi(p,v) = \gamma_1 \in \mathbb{R}$$

$$d_{(p,v)}\varphi = (\gamma_{1,1}, \dots, \gamma_{m,1}, \gamma_2, \dots, \gamma_k) \in \mathbb{R}^m \times \mathbb{R}^{k-1}$$

$$H_{(p,v)}\varphi = \begin{pmatrix} & & & \gamma_{1,2} & \cdots & \gamma_{1,k} \\ & H_pX & & & \cdots & \\ & & & \gamma_{m,2} & \cdots & \gamma_{m,k} \\ \gamma_{1,2} & \cdots & \gamma_{m,2} & & & \\ & \cdots & & & -\gamma_1 \mathbb{1}_{k-1} & \\ \gamma_{1,k} & \cdots & \gamma_{m,k} & & & \end{pmatrix} \in \left(\mathbb{R}^m \times \mathbb{R}^{k-1}\right) \otimes \left(\mathbb{R}^m \times \mathbb{R}^{k-1}\right). \tag{5.6}$$

where $\gamma_{i,j} \sim \mathcal{N}(0,1)$ *are i.i.d. and independent from* (γ_1, H_pX) *and* $\mathbb{E}\{\gamma_1 H_pX\} = -\mathbb{1}_m$. *In particular, the above law is invariant under orthonormal changes of basis in* T_pM.

Remark 32. The above lemma shows that φ is a normal field on $M \times \mathbb{S}^{k-1}$.

Definition 33. Let (H, γ_1) be as in Theorem 13: H is an $m \times m$ Hessian-like Gaussian matrix and $\mathbb{E}\{\gamma_1 H\} = -\not{\Bbbk}_m$. Let $\tilde{H}$ be distributed as $H_{(p,v)}\varphi$ in Theorem 31, that is:

$$\tilde{H} = \begin{pmatrix} & & & \gamma_{1,2} & \cdots & \gamma_{1,k} \\ & H & & & \cdots & \\ & & & \gamma_{m,2} & \cdots & \gamma_{m,k} \\ \gamma_{1,2} & \cdots & \gamma_{m,2} & & & \\ & \cdots & & & -\gamma_1 \mathbb{1}_{k-1} & \\ \gamma_{1,k} & \cdots & \gamma_{m,k} & & & \end{pmatrix}, \tag{5.7}$$

where $\gamma_{i,j} \sim \mathcal{N}(0,1)$ are i.i.d. and independent from (γ_1, H). Let us use the notation $\mathfrak{G}(m) \subset \mathbb{R}^{m\times m}$ to denote the subset of positive definite symmetric matrices. [2] Define

$$D_k^t(H) := \mathbb{E}\left[|\det\left(\tilde{H}\right)| \cdot 1_{\mathfrak{G}(m+k-1)}(-\tilde{H}) 1_{[t,+\infty)}(\gamma_1)\right]. \tag{5.8}$$

We can now exploit the previous expressions to derive an explicit formula for the critical values of chi fields.

Theorem 34. *For any* $A \subset M$, *we have that*

$$\mathbb{E}\left\{\#\left(C_t^{\text{Max}} \cap A\right)\right\} = \frac{\text{vol}(\mathbb{S}^{k-1})}{(2\pi)^{\frac{m+k-1}{2}}} \int_A D_k^t\left([H_pX]\right) dM(p). \tag{5.9}$$

Proof: We apply the Alpha-Kac-Rice formula (see Mathis and Stecconi (2024)) to $\varphi : M \times \mathbb{S}^{k-1} \xrightarrow{\Omega} \mathbb{R}$, with

$$\alpha(\varphi, p, v) = 1_{\mathfrak{G}(m+k-1)}(-H_{(p,v)}\varphi) \cdot 1_{[t,+\infty)}(\varphi(p,v)). \tag{5.10}$$

Mathis and Stecconi (2024, Prop. 4.10) shows that we can, because $d\varphi$ is Gaussian and $d_{(p,v)}\varphi$ is non-degenerate for all $(p,v) \in M \times \mathbb{S}^{k-1}$. Since $d_{(p,v)}\varphi$ and $\alpha(\varphi, p, v)$ are independent, the formula says that

$$\begin{aligned} \mathbb{E}\left\{\#\left(C_t^{\text{Max}} \cap A\right)\right\} &= \mathbb{E}\left\{\sum_{(p,v)\in A\times\mathbb{S}^{k-1} \text{ s.t. } d_{(p,v)}\varphi=0} \alpha(\varphi,p,v)\right\} \\ &= \int_{\mathbb{S}^{k-1}} \int_A \mathbb{E}\left\{|\det\left(H_{(p,v)}\varphi\right)| \cdot \alpha(\varphi,p,v)\right\} \rho_{[d_{(p,v)}\varphi]}(0) dM(p) d\mathbb{S}^{k-1}(v) \end{aligned} \tag{5.11}$$

[2] In general, the space of positive definite symmetric matrices is the space of scalar products, so we prefer to work with that instead than with the set of negative definite matrices. Of course, the two are identical.

Observe that, by Theorem 31, the density of $d_{(p,v)}\varphi$ at zero is equal to

$$\rho_{[d_{(p,v)}\varphi]}(0) = \frac{1}{(2\pi)^{\frac{m+k-1}{2}}} \tag{5.12}$$

and that the expectation term is exactly $D^t_k(H_pX)$, which is constant in v, hence we conclude. $\square$

6. High-threshold asymptotics

Let us try to get more explicit formulae in the case $m = k = 2$, and X is an isotropic Gaussian field on M, and M is the round sphere (of some radius r) or the plane. In this case, we can write

$$\tilde{H} = \begin{pmatrix} h_1 & h_2 & \gamma_1 \\ h_2 & h_3 & \gamma_2 \\ \gamma_1 & \gamma_2 & -\gamma \end{pmatrix}, \tag{6.1}$$

where (γ_1, γ_2) is independent from (h_1, h_2, h_3, γ) with

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{6.2}$$

and, since this field have rotation invariant Hessian in the sense of Theorem 20, we have from Theorem 21 that there are constants $\sigma, c \geq 0$ such that

$$\begin{pmatrix} h_1 \\ h_3 \\ h_2 \\ \gamma \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\sigma^2 + c & c & 0 & -1 \\ c & 2\sigma^2 + c & 0 & -1 \\ 0 & 0 & \sigma^2 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}\right). \tag{6.3}$$

In fact, one can compute c, σ^2 by a universal formula depending only on the fourth derivative of the covariance of the field and on the model chosen: Theorem 22 for the plane and Theorem 27 for the sphere. It follows from Theorem 7 that the vector $(\tilde{h}_1, \tilde{h}_3, h_2)$ is zero mean and independent from γ, where $\tilde{h}_i = h_i + \gamma$. With this notation, we have

$$\det(\tilde{H}) = \det \begin{pmatrix} \tilde{h}_1 - \gamma & h_2 & \gamma_1 \\ h_2 & \tilde{h}_3 - \gamma & \gamma_2 \\ \gamma_1 & \gamma_2 & -\gamma \end{pmatrix} \tag{6.4}$$

$$= -\tilde{h}_1\tilde{h}_3\gamma + h_2^2\gamma - \gamma^2(\tilde{h}_1 + \tilde{h}_3) - \gamma^3 - \gamma_1^2\tilde{h}_3 + \gamma\gamma_1^2 - \gamma_2^2\tilde{h}_1 + \gamma_2^2\gamma + 2h_2\gamma_2\gamma_1 . \tag{6.5}$$

Remark 35. It is easy to see that the expected value of this determinant in the region where $(\tilde{h}_1, h_2, \tilde{h}_3)$ is in $\mathbb{R}^3$, (γ_1, γ_2) is in $\mathbb{R}^2$ and $\gamma \geq t$, is equal to

$$A_1 := \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\gamma^2}{2}\right)(\gamma^3 - (3 - c + \sigma^2)\gamma)d\gamma = (H_2(t) + (c - \sigma^2))\varphi(t) .$$

The idea that we will follow in this section is to show that the difference between this term and the one with the absolute value is of smaller order in t.

Remark 36. It should be noted that, for $m = k = 2$,

$$D^t_k(H) := \mathbb{E}\left[|\det\left(\tilde{H}\right)| \cdot 1_{\mathfrak{G}(3)}(-\tilde{H})1_{[t,+\infty)}(\gamma)\right] = \mathbb{E}\left[\det\left(-\tilde{H}\right) \cdot 1_{\mathfrak{G}(3)}(-\tilde{H})1_{[t,+\infty)}(\gamma)\right]$$

(because the determinant of $\tilde{H}$ is necessarily negative)

$$= \mathbb{E}\left[\det\left(-\tilde{H}\right) \cdot 1_{[t,+\infty)}(\gamma)\right] - \mathbb{E}\left[\det\left(-\tilde{H}\right) \cdot \left(1 - 1_{\mathfrak{G}(3)}(-\tilde{H})\right) 1_{[t,+\infty)}(\gamma)\right]$$

We are therefore able to establish the following result.

Theorem 37. *Let $m = k = 2$ and let X have rotation invariant Hessian, in the sense of Theorem 20. As $t \to +\infty$, we have that*

$$\frac{D_2^t(H)}{(H_2(t)+(c-\sigma^2))\phi(t)} = \frac{\mathbb{E}\left[|\det\left(\tilde{H}\right)| \cdot 1_{\mathfrak{G}(3)}(-\tilde{H})1_{[t,+\infty)}(\gamma)\right]}{(H_2(t)+(c-\sigma^2))\phi(t)} = 1 + O\left(\exp(-\delta t^2)\right) \ .$$

Proof: Note first that

$$\begin{aligned} A_1 &= \mathbb{E}\left[\det\left(-\tilde{H}\right)\cdot 1_{[t,+\infty)}(\gamma)\right] \\ &= \mathbb{E}\left[(\tilde{h}_1\tilde{h}_3\gamma - h_2^2\gamma + \gamma^2(\tilde{h}_1+\tilde{h}_3) + \gamma^3 + \gamma_1^2\tilde{h}_3 - \gamma\gamma_1^2 + \gamma_2^2\tilde{h}_1 - \gamma_2^2\gamma - 2h_2\gamma_2\gamma_1)\cdot 1_{[t,+\infty)}(\gamma)\right] \quad (6.6) \\ &= \mathbb{E}\left[(\gamma^3 - (3-c+\sigma^2)\gamma)\cdot 1_{[t,+\infty)}(\gamma)\right] = \int_t^\infty \left(H_3(x)+(c-\sigma^2)H_1(x)\right)\phi(x)dx \\ &= \left(H_2(t)+(c-\sigma^2)\right)\phi(t) \ , \end{aligned}$$

because

$$\mathbb{E}[h_2^2] = \sigma^2,\ \mathbb{E}[\tilde{h}_1\tilde{h}_3] = c-1, \quad \mathbb{E}[\gamma_1^2] = \mathbb{E}[\gamma_2^2] = 1\ ,\ \mathbb{E}[\tilde{h}_1] = \mathbb{E}[\tilde{h}_3] = \mathbb{E}[h_2] = 0\ ,$$

and using one of the defining property of Hermite polynomials, saying that $\int_t^\infty H_{n+1}(\gamma)\phi(\gamma)d\gamma = H_n(t)\phi(t)$. Now let us focus on

$$A_2 := -\mathbb{E}\left[\det\left(-\tilde{H}\right)\cdot\left(1 - 1_{\mathfrak{G}(3)}(-\tilde{H})\right)1_{[t,+\infty)}(\gamma)\right] \ ;$$

in the above integral, the Hessian must be negative definite, which implies that some of the mixed products involving h's and γ must be larger than γ^3; we shall show that this probability is exponentially small in the regime where $\gamma > t$ and t grows to infinity. Precisely, observe that $\tilde{H}$ is not negative definite if and only if there exists a vector $\lambda \in \mathbb{R}^3$, such that $\lambda^T\tilde{H}\lambda \geq 0$. Let $\mu := \max\{|\tilde{h}_1|, |h_2|, |\tilde{h}_3|, |\gamma_1|, |\gamma_2|\}$ and observe that if $\mu < \frac{3}{8}t$ and $\gamma \geq t$, then we for all $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\|\lambda\| \leq 1$ we have that

$$\begin{aligned} \lambda^T\tilde{H}\lambda &= (\tilde{h}_1-\gamma)\lambda_1^2 + (\tilde{h}_3-\gamma)\lambda_2^2 + (-\gamma)\lambda_3^2 + 2h_2\lambda_1\lambda_2 + 2\gamma_1\lambda_1\lambda_2 + 2\gamma_2\lambda_1\lambda_2 \\ &\leq 8\mu - 3t < 0. \end{aligned} \quad (6.7)$$

Therefore, we can deduce the following:

$$\left(1 - 1_{\mathfrak{G}(3)}(-\tilde{H})\right)1_{[t,+\infty)}(\gamma) \leq 1_{[\frac{3t}{8},+\infty)}(\mu)1_{[t,+\infty)}(\gamma). \quad (6.8)$$

On the other hand, it is easy to see that for any $\gamma > 0$,

$$|\det\left(-\tilde{H}\right)| \leq 11(\gamma^3 + \mu^3). \quad (6.9)$$

Now, μ^3 is bounded by the multiple of a chi-distributed random variable, meaning that $\mu \leq C\chi_5$, for a constant $C > 0$ (which should be comparable with $\sqrt{\sigma^2 + c}$) and a chi-distributed random variable χ_5 of parameter 5, independent from γ. It follows from Theorem 38 below that, by combining Equation (6.8) and Equation (6.9), that the corresponding expected value A_2 is bounded above as follows

$$\begin{aligned} A_2 &\leq 11\int_{\frac{3t}{8C}}^\infty \int_t^\infty (\chi^3+\gamma^3)\chi^4\exp(-\chi^2/2)\exp(-\gamma^2/2)d\gamma d\chi \\ &= O\left(\left(\int_t^\infty (t^6 + t^3\gamma^3)\exp(-\gamma^2/2)d\gamma\right)\exp\left(-2\delta t^2\right)\right) \\ &= O\left((t^6 + t^3\cdot t^2)\exp\left(-\frac{t^2}{2}\right)\exp\left(-2\delta t^2\right)\right) \\ &= O\left(\exp\left(-t^2\left(\frac{1}{2}+\delta\right)\right)\right), \end{aligned} \quad (6.10)$$

for some small constant $\delta > 0$, depending on σ and c. This integral is exponentially smaller than the leading term. □

Lemma 38. *As $t \to +\infty$, we have*

$$\int_{\frac{t}{C}}^{+\infty} x^n \exp\left(-\frac{x^2}{2}\right) dx = O\left(t^{n-1} \exp\left(-\frac{t^2}{2C^2}\right)\right). \tag{6.11}$$

Proof: The proof is straightforward and hence omitted. □

By combining the latter with Theorem 34, we obtain the following, which proves also our main result, Theorem 4.

Corollary 39. *In the same setting as above, with $m = k = 2$ when X is normal and isotropic, or whenever the value of $D_k^t([H_p X])$ is constant in p, we have that as $t \to +\infty$*

$$\mathbb{E}\left\{\#\left(C_t^{\text{Max}}\right)\right\} \cdot \left(\frac{1}{(2\pi)^{\frac{1}{2}}} \text{vol}(M)\left(H_2(t) + (c - \sigma^2)\right)\phi(t)\right)^{-1} = 1 + O\left(\exp(-\delta t^2)\right). \tag{6.12}$$

Recalling Theorem 27, for X a normal isotropic field on $M = r\mathbb{S}^2$, we have $c - \sigma^2 = \frac{1}{r^2}$, where σ, c, r are defined as in Theorem 27. Therefore, the above result implies Theorem 4. We stress that the field $\xi := X_1$ of Theorem 4 is not necessarily a normal field (Theorem 1) on the standard unit sphere $\mathbb{S}^2$ (unless $r = 1$), as one can easily see that $\mathbb{E}|d_p\xi(v)|^2 = -K''(0) = r^2$ (i.e., $\mathbb{E}\|d_p\xi\|^2 = 2r^2$), see Theorem 27. However, Theorem 4 is deduced from Theorem 39 above, by applying it to the field $X = \hat{\xi}$ from Theorem 27, which is a normal isotropic field on the sphere $r\mathbb{S}^2$ of radius r.

6.1. *Proof of Theorem 6.* The first statement of Theorem 6 is Theorem 34, so it remains to show the validity of the asymptotic equivalences. We will address and justify each one of them in the following. The idea is to exploit the fact that φ is a normal field (in the sense of Theorem 1 on $M \times \mathbb{S}^{k-1}$, by Theorem 32.

6.1.1. *The connection with the Euler-Poincaré Characteristic and excursion probabilities.* We note also that, by Adler and Taylor (2007, Eq. (14.0.2)) and for small enough $\delta > 0$, we have

$$\begin{aligned}\mathbb{E}\chi(\varphi \geq t) &= \mathbb{P}(\max_{M \times \mathbb{S}^1} \varphi \geq t) + O\left(\exp\left(-\left(\frac{1}{2} + \delta\right)t^2\right)\right) \\ &= \mathbb{P}(\max_M f_2 \geq t) + O\left(\exp\left(-\left(\frac{1}{2} + \delta\right)t^2\right)\right),\end{aligned} \tag{6.13}$$

since, by construction, $\max_{M\times\mathbb{S}^1} \varphi = \max_M f_2$.

6.1.2. *The connection with Betti numbers.* The Euler characteristic $\chi(E)$ of a manifold with boundary E of dimension m is defined as the alternating sum of its Betti numbers (see Milnor et al. (1969)), and by Equation (6.18) (a classical identity in Morse theory, see Milnor et al. (1969)) it coincides with the alternating sum of the number of critical points of a Morse function. Specifically, in the setting of Theorem 34, Milnor et al. (1969, Th. 5.2)[3] yields

$$\chi(\varphi \geq t) \overset{\text{def}}{=} \sum_{i=0}^{m} (-1)^i b_i(\varphi \geq t) \overset{\text{Milnor et al. (1969, Th. 5.2)}}{=} \sum_{i=0}^{m} (-1)^i C_i(\varphi \geq t) \tag{6.14}$$

[3]The theorem is stated for a compact manifold, but its proof and Milnor et al. (1969, Th. 3.5) implies that it can be applied also for the excursion set of the Morse function

where we denote $C_i(\varphi \geq t) := \#\{d\varphi = 0, \mathrm{index}(H\varphi) = i, \varphi \geq t\}$ and moreover, the weak Morse inequality holds:

$$b_i(\varphi \geq t) \overset{\text{Milnor et al. (1969, Th. 5.2)}}{\leq} C_i(\varphi \geq t). \tag{6.15}$$

In particular, $b_0(\varphi \geq t)$ denotes the number of connected components of the excursion set and $C_0(\varphi \geq t)$ is the number of local maxima of φ with value exceeding t.

By studying the asymptotic behavior the Kac-Rice formulas for $\mathbb{E}C_i(\varphi \geq t)$, Gayet (2022) showed (in the more general context of stratified manifolds) that for $i \geq 1$, we have

$$\mathbb{E}C_i(\varphi \geq t) \overset{\text{Gayet (2022, Th. 3.6)}}{=} O\left(\exp\left(-(\frac{1}{2}+\delta)t^2\right)\right). \tag{6.16}$$

Therefore, up to an exponentially small error in expectation, as $t \to +\infty$, all the critical points of φ in the excursion set $\{\varphi \geq t\}$ are the local maxima.

An additional observation of Gayet (2022) is that Morse theory implies that

$$b_0(\varphi \geq t) \overset{\text{Gayet (2022, Cor. 2.5)}}{\leq} b_0(\varphi \geq t, \mathbb{B}) + \sum_{i=1}^{m} C_i(\varphi \geq t), \tag{6.17}$$

where $b_0(\varphi \geq t, \mathbb{B})$ denotes the number of connected components that are homeomorphic to a unit ball $\mathbb{B}$ of dimension m. As a consequence, one deduces that the only Betti number of the excursion set that is asymptotically relevant is $b_0(\varphi \geq t, \mathbb{B})$, see Gayet (2022, Thm 5.19). The proof of Equation (6.17) is the following: if $E \subset \{\varphi \geq t\}$ is a connected component that contains k critical points that are all of index 0, then by Milnor et al. (1969, Th. 3.5) it follows that the Betti numbers of E are $b_0 = k, b_1 = 0, \ldots, b_m = 0$, which means that $k = 1$ and thus that the flow of $-\nabla\varphi$ deforms E into a small ball around the only maximum, so that E must be homeomorphic to $\mathbb{B}$.

6.1.3. *The case of the two-sphere.* Let $M = r\mathbb{S}^2$ be the round sphere of radius $r > 0$. By showing that in the high-threshold limit the dominant term corresponds to the expected value of the determinant without the modulus, we are actually proving that the number of maxima taking values larger than t is asymptotically equivalent to the Euler-Poincaré characteristic for the excursion set of $\varphi\colon r\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$ which we introduced in Section 5, that is

$$\chi(\varphi \geq t) \overset{\text{Adler and Taylor (2007, Cor. 9.3.5)}}{=} \sum_{\{(p,v) \in r\mathbb{S}^2 \times \mathbb{S}^1\colon\ d_{(p,v)}\varphi = 0\}} \mathrm{sgn}\left(\det\left(-H_{(p,v)}\varphi\right)\right) \cdot 1_{[t,+\infty]}(\varphi(p,v)), \tag{6.18}$$

since $(-1)^{\mathrm{index}(-H)} = \mathrm{sgn}(\det(-H))$. Indeed the expectation of the determinant without the absolute value, i.e., the term A_1 in the proof of Theorem 37, is indeed the Kac-Rice density of the right-hand side. Hence our result Theorem 4 is equivalent to the following limit:

$$\left(\ \mathbb{E}\chi(\varphi \geq t)\ \right)^{-1} \mathbb{E}[\mu_t(r\mathbb{S}^2, f_2)] = 1 + O\left(\exp\left(-\delta t^2\right)\right)\ , \tag{6.19}$$

Indeed, the Adler-Taylor formula for $\mathbb{E}\chi(\varphi \geq t)$, allowing to express the above in terms of the Lipschitz-Killing curvatures $\mathcal{L}_i$ of the space $r\mathbb{S}^2 \times \mathbb{S}^1$ (see Adler and Taylor (2007, Thm. 12.4.2)), yields

$$\begin{aligned}
\mathbb{E}\chi(\varphi \geq t) &\overset{\text{Adler and Taylor (2007, Th. 12.4.1)}}{=} \mathcal{L}_0(r\mathbb{S}^2 \times \mathbb{S}^1)\rho_0(t) + \mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1)\rho_1(t) \\
&\quad + \mathcal{L}_2(r\mathbb{S}^2 \times \mathbb{S}^1)\rho_2(t) + \mathcal{L}_3(r\mathbb{S}^2 \times \mathbb{S}^1)\rho_3(t) \\
&= 0 + \mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1) \cdot \frac{1}{\sqrt{2\pi}}\phi(t) + 0 + r^2 \mathrm{vol}(\mathbb{S}^2)\mathrm{vol}(\mathbb{S}^1) \cdot \frac{H_2(t)}{2\pi}\frac{1}{\sqrt{2\pi}}\phi(t) \\
&= \left(\mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1) + r^2 \mathrm{vol}(\mathbb{S}^2)\mathrm{vol}(\mathbb{S}^1)\frac{H_2(t)}{2\pi}\right) \cdot \frac{1}{\sqrt{2\pi}}\phi(t) \\
&= \left(2H_2(t)r^2 + \frac{\mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1)}{2\pi}\right) \cdot (2\pi)^{\frac{1}{2}}\phi(t) \\
&= \left(2H_2(t)r^2 + 2\right) \cdot (2\pi)^{\frac{1}{2}}\phi(t).
\end{aligned} \tag{6.20}$$

To compute $\mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1)$, for any r, we can use Pistolato and Stecconi (2024, Prop. 3.0.1), implying that for a 3-dimensional closed manifold M with constant scalar curvature $\mathrm{scal}(M)$, one has $4\pi\mathcal{L}_1(M) = \mathrm{scal}(M)\mathrm{vol}(M)$; combining such formula with: $\mathrm{scal}(M \times N) = \mathrm{scal}(M) + \mathrm{scal}(N)$, $\mathrm{scal}(r\mathbb{S}^2) = \frac{2}{r^2}$ and $\mathrm{scal}(\mathbb{S}^1) = 0$, we get that $\mathcal{L}_1(r\mathbb{S}^2 \times \mathbb{S}^1) = 4\pi$, which is what we used in the last line of the previous computation. We deduce our final formula of Theorem 4, derived from Theorem 39:

$$\left(2r^2\left(H_2(t) + \frac{1}{r^2}\right) \cdot (2\pi)^{\frac{1}{2}}\phi(t)\right)^{-1} \mathbb{E}[\mu_t(r\mathbb{S}^2, f_2)] = 1 + O\left(\exp\left(-\delta t^2\right)\right) . \tag{6.21}$$

6.1.4. *Asymptotic Equivalences.* Let us also define the *total Betti number* of the excursion set as $b(\varphi \geq t) = \sum_{i=0}^m b_i(\varphi \geq t)$ and let us denote the total number of critical points as $C(\varphi \geq t) := \sum_{i=0}^m C_i(\varphi \geq t)$. Putting together the asymptotics in the previous two remarks, we deduce that as $t \to +\infty$ we have

$$\begin{aligned}
\mathbb{E}\#\left(C_0(\varphi \geq t)\right) &\sim \mathbb{E}C(\varphi \geq t) \sim \mathbb{E}b(\varphi \geq t) \sim \mathbb{E}\chi(\varphi \geq t) \\
&\sim \mathbb{E}b_0(\varphi \geq t) \sim \mathbb{E}b_0(\varphi \geq t; \mathbb{B}) \\
&\sim \mathbb{P}\left(\max_{M \times \mathbb{S}^{k-1}} \varphi \geq t\right) \\
&\sim \sum_{j=0}^{m+k-1} \frac{\mathcal{L}_j(\mathbb{S}^{k-1} \times M)}{(2\pi)^{\frac{j}{2}}} H_{j-1}(t)\phi(t)
\end{aligned} \tag{6.22}$$

with an error of $O\left(\exp\left(-(\frac{1}{2} + \delta)t^2\right)\right)$. The last line being Adler and Taylor (2007, Th. 12.4.1). Observe also that the set $\{\varphi \geq t\}$ can be homotopically retracted to $\{(p, Y(p)) \colon |Y(p)| \geq t\}$ in $\mathbb{S}^{k-1} \times M$, which is diffeomorphic (being a graph) to the set $\{f_k \geq t\} \subset M$. This implies that the two sets have the same Betti numbers. Moreover, a connected component of the former is heomeomorphic to a ball if and only if the corresponding connected component of $\{f_k \geq t\}$ is. Finally, in the setting of Theorem 34, it is easy to show that critical points of φ correspond with critical points of f_k so that $C(\varphi \geq t) = \#C_t$ and we already observed that $C(\varphi \geq t) = \#\left(C_t^{\mathrm{Max}}\right)$.

Therefore, up to an error $O\left(\exp\left(-(\frac{1}{2}+\delta)t^2\right)\right)$, we have the same asymptotic equivalences for f_k:

$$\begin{aligned}\mathbb{E}(\#C_t^{\mathrm{Max}}) &\sim \mathbb{E}\#C_t \sim \mathbb{E}b(f_k \geq t) \sim \mathbb{E}\chi(f_k \geq t) \\ &\sim \mathbb{E}b_0(f_k \geq t) \sim \mathbb{E}b_0(f_k \geq t; \mathbb{B}) \\ &\sim \mathbb{P}\left(\max_M f_k \geq t\right) \\ &\sim \sum_{j=0}^{m+k-1} \frac{\mathcal{L}_j(\mathbb{S}^{k-1}\times M)}{(2\pi)^{\frac{j}{2}}} H_{j-1}(t)\phi(t) \\ &\overset{(M=\mathbb{S}^2,\, k=2)}{\sim} (2+2r^2H_2(t))\sqrt{2\pi}\phi(t).\end{aligned} \tag{6.23}$$

The latter asymptotics conclude the proof of Theorem 6.

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