Some naturally defined star products for Kähler manifolds

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Gauge Theories and Poisson Geometry
24-28 November 2025, Arpino, Italy

Report on work partly done in joint with M. Bordemann, E. Meinrenken, A. Karabegov. But also results by others will appear.

All the best to you, Thomas!!

- One mathematical aspect of quantization is the passage from the commutative world to the non-commutative world.
- one way is by deformation quantization (also called star product)
- deform the Poisson algebra of functions on the phase space
- can only be done on the level of formal power series over the algebra of functions
- after some approaches, it was pinned down in a mathematically satisfactory manner by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer.

Outline

- give an overview of some naturally defined star products in the case that our "phase-space manifold" is a (compact) Kähler manifold
- here we have additionally a complex structure and search for star products respecting it
- yield star products of separation of variables type (Karabegov) resp. Wick or anti-Wick type (Bordemann and Waldmann)
- both constructions are quite different, but there is a 1:1 correspondence (Neumaier)
- still quite a lot of them

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- single out certain naturally given ones
- restrict to quantizable Kähler manifolds
- Berezin-Toeplitz star product, Berezin transform, Berezin star product
- a side result: star product of geometric quantization
- all of the above are different star products, but nevertheless are equivalent as star products.
- We give the classifying Deligne-Fedosov class and the Karabegov forms.
- Moreover, we give the equivalence transformations.

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Geometric set-up

- (M, ω) a pseudo-Kähler manifold.
  M a complex manifold, and ω, a non-degenerate closed $(1,1)$-form
- if ω is a positive form then (M, ω) is a honest Kähler manifold
- $C^\infty(M)$ the algebra of complex-valued differentiable functions with associative product given by point-wise multiplication
- define the Poisson bracket

$$\{f, g\} := \omega(X_f, X_g) \qquad \omega(X_f, \cdot) = df(\cdot)$$

- $C^\infty(M)$ becomes a Poisson algebra.

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Star product

star product for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, such

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

Also

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M),$$

differential (or local) if $C_k(\ ,\ )$ are bidifferential operators.
Usually: $1 \star f = f \star 1 = f$.

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Equivalence of star products

$\star$ and $\star'$ (of the same Poisson structure) are *equivalent* means

there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ and $B(f) \star' B(g) = B(f \star g)$.

To every equivalence class of a differential star product one assigns its Deligne-Fedosov class

$$cl(\star) \in \frac{1}{\mathrm{i}}\left(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M, \mathbb{C})[[\nu]]\right).$$

Gives a 1:1 correspondence. between equivalence classes of star products and such formal forms.

Existence of star products for Poisson manifolds, resp. for symplectic manifolds:
by DeWilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, ..., Kontsevich.

For manifolds with additional structure (e.g. complex structure one is searching for star products respecting it (in a certain sense).

Separation of Variables type

- pseudo-Kähler case: we look for star products (always differential ones) adapted to the complex structure
- separation of variables type (Karabegov)
- Wick and anti-Wick type (Bordemann - Waldmann)
- Karabegov convention: of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument is differentiated in anti-holomorphic and the second argument in holomorphic directions.
- we call this convention separation of variables (anti-Wick) type and call a star product of separation of variables (Wick) type if the role of the variables is switched
- we need both conventions

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- Our star products are globally defined. But as they are local star products they define star products also for local functions.
- Moreover the global star product is fixed by its local forms.
- $\star$ of anti-Wick type is equivalent to : for every given $U \subset M$, open non-empty, and local antiholomorphic functions a, holomorphic functions b, and f differentiable function on $U \subset M$ we have the relations

$$b \star f = b \cdot f, \qquad f \star a = f \cdot a\,.$$

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Karabegov construction (sketch of a sketch)

- (M, ω_{-1}) the pseudo-Kähler manifold
- a formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\,\omega_1 + \dots$$

$\omega_r,\ r \geq 0$, closed (1,1)-forms on M.
- Karabegov: to every such $\widehat{\omega}$ there exists a star product $\star$ of anti-Wick type
- and vice-versa
- Karabegov form of the star product $\star$ is $kf(\star) := \widehat{\omega}$,
- the star product $\star_K$ with classifying Karabegov form $(1/\nu)\omega_{-1}$ is Karabegov's standard star product.

Formal Berezin transform

- $\star$ of anti-Wick type. For local antiholomorphic functions a and holomorphic b on $U \subset M$ we have the relation $b \star a = b \cdot a$.
- The Berezin transform $I_\star$ associated to $\star$ is given by the local relation

$$a \star b = I_\star(b \star a) = I_\star(b \cdot a),$$

- It can be written as

$$I_\star = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M),$$

$$I_0 = id, \quad I_1 = \Delta.$$

- The I_k are differential operators of type (k, k).
- The formal Berezin transform $I_\star$ determines the $\star$ uniquely.
- $\star$ can be obtained back from $I_\star$ by polarization.

A RELATED STAR PRODUCT

- Start with $\star$ separation of variables type (anti-Wick) (M, ω_{-1})
- related star product (opposite of the dual)

$$f \star' g := I_\star^{-1}(I_\star(f) \star I_\star(g)).$$

  on (M, ω_{-1}), is of Wick type

- the formal Berezin transform $I_\star$ establishes an equivalence of the star products

$$(\mathcal{A}, \star) \text{ and } (\mathcal{A}, \star')$$

Classifying forms

$\star$ star product of anti-Wick type with Karabegov form $kf(\star) = \widehat{\omega}$
Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{\mathrm{i}}([\widehat{\omega}] - \frac{\delta}{2}).$$

[..] denotes the de-Rham class of the forms and δ is the canonical class of the manifold i.e. $\delta := c_1(K_M)$.

standard star product $\star_K$ (with Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$)

$$cl(\star_K) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

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Other general constructions

- Bordemann and Waldmann: modification of Fedosov's geometric existence proof.
- fibre-wise Wick product.
- by a modified Fedosov connection a star product $\star_{BW}$ of Wick type is obtained.
- Karabegov form is $-(1/\nu)\,\omega$
- Deligne class class

$$cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

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Neumaier: by adding a formal closed $(1,1)$ form as parameter each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction

Reshetikhin and Takhtajan:
formal Laplace expansions of formal integrals related to the star product.
coefficients of the star product can be expressed (roughly) by Feynman diagrams

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Berezin-Toeplitz Star product

- compact and quantizable Kähler manifold (M,ω),
- quantum line bundle (L,h,∇), L is a holomorphic line bundle over M, h a hermitian metric on L, ∇ a compatible connection (with metric and holomorphic structures)
- (M,ω) is quantizable, if there exists such (L,h,∇), with

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega = \bar{\partial}\partial \log \hat{h}.$$

  A Kähler manifold with such quantum line bundle (fixed)is called quantized Kähler manifold.
- Consider all positive tensor powers $(L^m, h^{(m)}, \nabla^{(m)})$,

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scalar product for $C^\infty(M, L^m)$ $\quad (n = \dim_{\mathbb{C}} M)$

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\, \Omega, \qquad \Omega := \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad T_f^{(m)}(s) := \Pi^{(m)}(f \cdot s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level m.

Berezin-Toeplitz operator quantization

$$f \mapsto \left(T_f^{(m)} \right)_{m \in \mathbb{N}_0} .$$

has the correct semi-classical behavior:

Theorem (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m \to \infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\, [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}|| = O(1/m)$$

Theorem (BMS, Schl., Karabegov and Schl.)

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}$$

Further properties: is of separation of variables type (Wick type)

classifying Deligne-Fedosov class $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ and Karabegov form $\frac{-1}{\nu}\omega + \omega_{can}$

possible: auxiliary hermitian line (or even vector) bundle can be added, meta-plectic correction.

Geometric quantization

Further result: The Toeplitz map of level m

$$T^{(m)} : C^\infty(M) \to End(\Gamma_{hol}(M, L^m))$$

is surjective

implies that e.g. the operator $Q_f^{(m)}$ of geometric quantization (with holomorphic polarization) can be written as Toeplitz operator of a function f_m (maybe different for every m)

indeed Tuynman relation (for compact manifolds):

$$Q_f^{(m)} = \mathrm{i}\, T_{f - \frac{1}{2m}\Delta f}^{(m)}$$

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}, \qquad P_f^{(m)} := \nabla_{X_f^{(m)}}^{(m)} + \mathrm{i} f \cdot id.$$

- star product of geometric quantization
- set $B(f) := (id - \nu\frac{\Delta}{2})f$

$$f \star_{GQ} g := B^{-1}(B(f) \star_{BT} B(g))$$

defines an equivalent star product

- can also be given by the asymptotic expansion of products of geometric quantization operators
- it is not of separation of variable type
- but equivalent to $\star_{BT}$ via B.

Where is the Berezin star product ??

- It is an important star product: Berezin, Cahen-Gutt-Rawnsley, etc.
- The original definition is limited in applicability.
- We will give a definition for quantizable Kähler manifold.
- Clue: define $\star_B$ so that the opposite of its dual is $\star_{BT}$, e.g.
-

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$$

- Problem: How to determine I?
- describe the formal I by asymptotic expansion of some geometrically defined $I^{(m)}$

COHERENT STATES - BEREZIN SYMBOLS

- assume the bundle L is very ample (i.e. has enough global sections)
- pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric k
- inside of the total space U, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection,

coherent vectors/states in the sense of Berezin-Rawnsley-Cahen-Gutt: (with a slight twist)

$$\alpha \in U \setminus 0, \quad s \in \Gamma_{hol}(M, L^m)$$

$$x = \tau(\alpha) \quad \in M$$

$$s(\tau(\alpha)) \quad \in L^m$$

$$\alpha^{\otimes m}(s(\tau(\alpha))) \quad \in \mathbb{C}$$

$$s \mapsto \alpha^{\otimes m}(s(\tau(\alpha))).$$

this is a linear form on $\Gamma_{hol}(M, L^m)$.

This linear form defines defines the coherent vector $e_\alpha^{(m)}$ by

$$\langle e_\alpha^{(m)}, s\rangle = \alpha^{\otimes m}(s(\tau(\alpha))) \ .$$

Starting with x we have to chose α above x

$$x \in M \mapsto \alpha = \tau^{-1}(x) \in U \setminus 0 \mapsto e_\alpha^{(m)} \in \Gamma_{hol}(M, L^m).$$

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \ .$$

We obtain the coherent state

$$x \in M \mapsto \mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)).$$

Applications

- Bergman projectors $\Pi^{(m)}$, Bergman kernels, ...
- Covariant Berezin symbol $\sigma^{(m)}(A)$ (of level m) of an operator $A \in \mathbf{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\sigma^{(m)}(A) : M \to \mathbb{C},$$

$$x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} = \mathbf{Tr}(AP_x^{(m)})$$

Berezin transform (geometric)

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad \boxed{f \mapsto \sigma^{(m)}(T_f^{(m)}) =: I^{(m)}(f)}$$

Theorem: (Karabegov - Schl.)
$I^{(m)}(f)$ has a complete asymptotic expansion as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

$$I_i : C^\infty(M) \to C^\infty(M), \qquad I_0(f) = f, \qquad I_1(f) = \Delta f.$$

- Δ is the Laplacian with respect to the metric given by the Kähler form ω

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Berezin star product

- from asymptotic expansion of the Berezin transform get formal expression

$$I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M)$$

- set $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- $\star_B$ is called the Berezin star product
- I gives the equivalence from $\star_B$ to $\star_{BT}$ ($I_0 = id$). Hence, the same Deligne-Fedosov classes

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- $\star_B$ is of separation of variables type (but now of anti-Wick type).
- Karabegov form is $\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$
- u_m is the Bergman kernel $\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle$, associated to the Bergman projector $\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, evaluated along the diagonal
- $\mathbb{F}$ means: take asymptotic expansion in $1/m$ as formal series in ν

- $I = I_{\star_B}$, the geometric Berezin transform equals the formal Berezin transform of Karabegov for $\star_B$
- both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other
- if the covariant symbol star product works, (see later) it will coincide with the star product $\star_B$.

- Berezin transform is not only the equivalence relating $\star_{BT}$ with $\star_B$
- also it (resp. the Karabegov form, resp. the Bergman kernel) can be used to calculate the coefficients of these naturally defined star products,
- either directly
- or with the help of the certain type of graphs (see the very interesting work of Gammelgaard and Hua Xu).

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Integral representation of the Berezin transform

$\tau(\alpha) = x$, $\tau(\beta) = y$ with $\alpha, \beta \in Q$

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$
$$= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \int_M \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle f(y)\Omega(y)\ .$$

Note that :

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle,$$
$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta)\cdot\mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle$$

are well-defined on M and on $M \times M$ respectively.

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Original Berezin star product

- Construction of the Berezin star product, only for limited classes of manifolds (see Berezin, Cahen-Gutt-Rawnsley)
- $\mathcal{A}^{(m)} \le C^\infty(M)$, of level m covariant symbols (they are functions).
- symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed

- $$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$
- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level m ?

Summary of naturally defined star product

	name	Karabegov form	Deligne Fedosov class
$\star_{BT}$	Berezin-Toeplitz	$\frac{-1}{\nu}\omega + \omega_{can}$ (Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{B}$	Berezin	$\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\bar{\partial}\log u_m)$ (anti-Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{GQ}$	geometric quantization	(—)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{K}$	standard product	$(1/\nu)\,\omega$ (anti-Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$.
$\star_{BW}$	Bordemann-Waldmann	$-(1/\nu)\,\omega$ (Wick)	$\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$.

u_m Bergman kernel evaluated along the diagonal in $Q \times Q$
δ the canonical class of the manifold M

Further readings

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