

A CLOSED FORMULA FOR THE DENSITY IN ARTIN'S PRIMITIVE ROOT CONJECTURE OVER NUMBER FIELDS

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ABSTRACT. Let K be a number field and $\alpha \in K^\times$. Artin's primitive root conjecture concerns the proportion of prime ideals \mathfrak{p} of K for which $(\alpha \bmod \mathfrak{p})$ is well-defined, non-zero, and generates the multiplicative group at \mathfrak{p} . The conjecture states that the set of those prime ideals admits a natural density and provides a formula for it. In 1967, assuming the Generalized Riemann Hypothesis, Hooley proved Artin's conjecture over \mathbb{Q} , and in 1975 Cooke and Weinberger proved the analogue result over number fields. Hooley has also given a closed formula for the density over \mathbb{Q} , describing the rational number that is the ratio between the density and the Artin constant. We provide a closed formula for the Artin density over any number field.

1. INTRODUCTION

Artin's conjecture on primitive roots dates back 1927 and it has been proven by Hooley assuming the Generalized Riemann Hypothesis (GRH) for the Dedekind zeta function of number fields. Indeed, in 1976, Hooley [2] proved the following result (where we work within a fixed algebraic closure of \mathbb{Q} , ζ_n denotes a root of unity of order n , and μ is the Möbius function). We consider the Artin constant

$$A := \prod_{\ell \text{ prime}} \left(1 - \frac{1}{\ell(\ell-1)}\right) \sim 0.37$$

and, for an odd squarefree integer h , its positive rational multiple

$$A(h) := \prod_{\ell \text{ prime}} \left(1 - \frac{1}{\ell^{1-v_\ell(h)}(\ell-1)}\right).$$

Theorem 1 (Hooley). *Assume GRH. Let a be an integer that is neither $0, \pm 1$ nor a square and consider the set of primes p such that $(a \bmod p)$ is non-zero and generates $(\mathbb{Z}/p\mathbb{Z})^\times$.*

- (i) *This set admits a natural density, which we call $\text{dens}(a)$.*
- (ii) *We have*

$$(1) \quad \text{dens}(a) = \sum_{n \geq 1} \frac{\mu(n)}{[\mathbb{Q}(\zeta_n, \sqrt[n]{a}) : \mathbb{Q}]}$$

and $\text{dens}(a)$ is also the natural density of the set of primes that, for any prime ℓ , do not split completely in the number field $\mathbb{Q}(\zeta_\ell, \sqrt[\ell]{a})$.

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(iii) Call h the largest (odd) squarefree integer for which a is an h -th power and let Δ be the discriminant of $\mathbb{Q}(\sqrt{a})$. Then we have

$$(2) \quad \frac{\text{dens}(a)}{A(h)} = \begin{cases} 1 & \text{if } \Delta \not\equiv 1 \pmod{4}, \\ 1 - \mu(|\Delta|) \prod_{\substack{\ell \text{ prime} \\ \ell | \Delta}} \frac{1}{\ell^{1-v_\ell(h)}(\ell-1)-1} & \text{otherwise.} \end{cases}$$

Almost a decade later, in 1975, Hooley's result was generalized by Cooke and Weinberger [1] who proved the following result.

Theorem 2 (Cooke and Weinberger). *Assume GRH. Let K be a number field and let $\alpha \in K^\times$ be not a root of unity. Consider the set of primes \mathfrak{p} of K such that $(\alpha \bmod \mathfrak{p})$ is well-defined, non-zero, and generates the multiplicative group at \mathfrak{p} .*

- (i) *This set admits a natural density, which we call $\text{dens}(\alpha)$.*
- (ii) *We have*

$$(3) \quad \text{dens}(\alpha) = \sum_{n \geq 1} \frac{\mu(n)}{[K(\zeta_n, \sqrt[n]{\alpha}) : K]}$$

and $\text{dens}(\alpha)$ is also the natural density of the set of primes of K that, for any prime ℓ , do not split completely in the number field $K(\zeta_\ell, \sqrt[\ell]{\alpha})$.

A comparison between the above results shows that the closed formula (2) was not generalized to number fields. In the second case of (2), the correction factor is due to *entanglement* (namely, lack of linear disjointness) for the extensions $\mathbb{Q}(\zeta_\ell, \sqrt[\ell]{a})/\mathbb{Q}$ and indeed $\mathbb{Q}(\sqrt{a})$ is contained in the compositum of the fields $\mathbb{Q}(\zeta_\ell)$ for the primes $\ell \mid \Delta$. Probably because of the feared phenomenon of entanglement for the extensions $K(\zeta_\ell, \sqrt[\ell]{\alpha})/K$, it was not even considered that there could be a closed formula like (2) over any number field. More than fifty years later, we present a closed formula which also provides a bound for the minimal denominator of $\text{dens}(\alpha)/A$. Our method to obtain the closed formula is relying on a suitable finite case distinction and describing in a convenient way, through characters, the entanglement of the extensions $K(\zeta_\ell, \sqrt[\ell]{\alpha})/K$ for ℓ prime.

We observe that Hooley obtained his closed formula over \mathbb{Q} by computing (3) and that the correction factor in (2) has also an interpretation with characters, see [4]. We refer the reader to the survey by Moree [5] for an extensive introduction to Artin's conjecture.

A closed formula for the Artin density over number fields. Fix a number field K and let $\alpha \in K^\times$. If α is a root of unity or a q -th power in K^\times for some prime q such that $\zeta_q \in K$, then $\text{dens}(\alpha) = 0$, so we now exclude this case and call h the largest (odd) squarefree integer such that $\alpha \in K^{\times h}$. We prove in particular the following result.

Theorem 3. *Assume GRH. Then $\text{dens}(\alpha)$ is a rational multiple of A . Moreover, for any fixed K , there is a closed formula for the rational number $\text{dens}(\alpha)/A$ which can be written down with a finite case distinction on α .*

To ease notation, for any positive integer n we write $K_n := K(\zeta_n, \sqrt[n]{\alpha})$. As a consequence of Schinzel's theorem on abelian radical extension [7, Theorem 2], for any prime number ℓ we have

$$[K_\ell : K] = [K(\zeta_\ell) : K] \cdot \ell^{1-v_\ell(h)}.$$

In particular, this degree depends on α only through h . For practicality, we work (in place of A) with a constant that depends on h and on the field K :

$$(4) \quad C(h) := \prod_{\ell \text{ prime}} \left(1 - \frac{1}{\delta_\ell}\right) \quad \delta_\ell := [K_\ell : K].$$

We observe that $C(h)$ is a rational multiple of A and, because of our assumption on α , it is strictly positive (and $C(h) = A(h)$ if $K = \mathbb{Q}$ or, more generally, if $K \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ holds for every positive squarefree integer n).

We call Ω the smallest even positive integer such that $K \cap \mathbb{Q}(\zeta_\infty)$ is contained in $\mathbb{Q}(\zeta_\Omega)$ and denote by Ω_0 the square-free part of Ω . To tackle K_ℓ for the primes $\ell \mid \Omega_0$ we make a finite case distinction for α .

The primes ℓ that are not problematic are those for which K_ℓ is linearly disjoint from K_n for any positive squarefree integer n coprime to ℓ . By Schinzel's theorem on abelian radical extensions, for any prime $\ell \nmid \Omega_0$ the entanglement for K_ℓ/K is only due to the following: for the primes q such that $\zeta_q \in K$ it is possible that $\sqrt[q]{\alpha}$ is contained in $K(\zeta_{\Omega_0 m_q})$ for some positive squarefree integer m_q (which we take to be minimal) coprime to Ω_0 and with $\ell \mid m_q$. We call F the finite set of primes q for which the above entanglement occurs. For $q \in F$ we let S_q be the set of prime divisors of m_q , and we let M be the product of all primes in $\cup_q S_q$.

Call d_{Ω_0} the proportion in the Galois group of K_{Ω_0}/K of those automorphisms such that for any prime $\ell \mid \Omega_0$ they are not the identity on K_ℓ . We observe that d_{Ω_0} is a rational number that can only take finitely many values by varying α (its minimal denominator divides $\prod_{\ell \mid \Omega_0} \delta_\ell$). With the above notation, we have the following special case.

Theorem 4. *Assume GRH. If $F = \emptyset$, then we have*

$$\frac{\text{dens}(\alpha)}{C(h)} = d_{\Omega_0} \cdot \prod_{\ell \mid \Omega_0} \left(1 - \frac{1}{\delta_\ell}\right)^{-1}$$

and the minimal denominator of $\text{dens}(\alpha)/C(h)$ divides $\prod_{\ell \mid \Omega_0} (\delta_\ell - 1)$.

For $q \in F$ and $t \in \mathbb{Z}/q\mathbb{Z}$ we define $d_{\Omega_0, t}$ similarly to d_{Ω_0} , restricting to the automorphisms for which t is the value of a certain q -character related to $\sqrt[q]{\alpha}$. Moreover, if Z is a finite set of prime numbers, we define a certain combinatorial quantity $n_{Z, t}$.

Theorem 5. *Assume GRH. If $F = \{q\}$, then we have*

$$\frac{\text{dens}(\alpha)}{C(h)} = \sum_{Z \subseteq S_q} \sum_{t \in \mathbb{Z}/q\mathbb{Z}} d_{\Omega_0, t} \cdot n_{S_q \setminus Z, t} \cdot \frac{\prod_{\ell \mid \Omega_0 m_q, \ell \notin Z} \delta_\ell \cdot \prod_{\ell \in Z} (\delta_\ell - q)}{q^{(\#S_q)-1} \prod_{\ell \mid \Omega_0 m_q} (\delta_\ell - 1)}.$$

For a general F we work with a tuple $T \in \prod_{q \in F} \mathbb{Z}/q\mathbb{Z}$ and make use of analogously defined rational numbers $d_{\Omega_0, T}$ (which only depend on α via a finite case distinction) and combinatorial quantities $n_{Z, T}$. The formula in the general case (see Theorem 7) is similar to the one of Theorem 5. In particular, we obtain the following result.

Theorem 6. *Assume GRH. Up to a finite case distinction on α , the rational number $\text{dens}(\alpha)/C(h)$ depends on α only through h , F , and the sets S_q for $q \in F$.*

Moreover, the minimal denominator of $\text{dens}(\alpha)/C(h)$ divides

$$\prod_{\ell \mid \Omega_0 M} (\delta_\ell - 1) \cdot \prod_{q \in F} q^{(\#S_q)-1}.$$

We remark that if K is an abelian extension of \mathbb{Q} (for example, if K is a multiquadratic field) and $\alpha \in \mathbb{Q}$, then by Schinzel's theorem on abelian radical extensions the set F consists at most of the prime 2. The same holds (for $\alpha \in K$) if the only roots of unity contained in K have degree a power of 2 (for example, if the degree of K/\mathbb{Q} is odd). This shows that the special cases considered in Theorems 4 and 5 cover in particular all multiquadratic fields such that $\zeta_3 \notin K$ (or $\alpha \in \mathbb{Q}$) and all number fields of odd degree.

2. THE PROOF OF THE CLOSED FORMULA

Let K be a number field and let $\alpha \in K^\times$ be not a root of unity. To ease notation, for every positive integer n we write $K_n := K(\zeta_n, \sqrt[n]{\alpha})$. We let h be the largest positive squarefree integer such that $\alpha \in K^{\times h}$. We assume that there is no prime number $q \mid h$ such that $\zeta_q \in K$ (because in that case we would have $K_q = K$ hence $\text{dens}(\alpha) = 0$). In particular, the constant $C(h)$ defined in (4) is strictly positive.

We call Ω the smallest even positive integer such that $K \cap \mathbb{Q}(\zeta_\infty)$ is contained in $\mathbb{Q}(\zeta_\Omega)$ and denote by Ω_0 the square-free part of Ω . We call $\text{dens}_{\Omega_0}(\alpha)$ the proportion of the Galois automorphisms of K_{Ω_0}/K such that for any prime ℓ they are not the identity on K_ℓ (in other words, dens_{Ω_0} is the natural density of the primes \mathfrak{p} of K such that the index of $(\alpha \bmod \mathfrak{p})$ in the multiplicative group at \mathfrak{p} is well-defined and coprime to Ω_0).

We then consider the finite set F consisting of the primes q having the following properties: $\zeta_q \in K$; $\sqrt[q]{\alpha} \notin K(\zeta_{\Omega_0})$; $\sqrt[q]{\alpha} \in K(\zeta_{\Omega_0 m_q})$ for some square-free positive integer m_q coprime to Ω_0 and which we suppose to be minimal. For every $q \in F$ we let S_q be the set of prime divisors of m_q and we call M the product of the primes in $\cup_{q \in F} S_q$. For every prime $\ell \mid M$ we call H_ℓ the set of primes $q \in F$ such that $\ell \in S_q$.

Theorem 7. *Assume GRH.*

- (i) *Up to a finite case distinction on α , the ratio $\text{dens}(\alpha)/C(h)$ is a rational number that depends on α only through h , F and the sets S_q for $q \in F$.*
- (ii) *For $F = \emptyset$ we have*

$$\frac{\text{dens}(\alpha)}{C(h)} = \text{dens}_{\Omega_0}(\alpha) \cdot \prod_{\ell \mid \Omega_0} \frac{\delta_\ell}{\delta_\ell - 1}$$

and the minimal denominator of $\text{dens}(\alpha)/C(h)$ divides $\prod_{\ell \mid \Omega_0} (\delta_\ell - 1)$.

- (iii) *For $F \neq \emptyset$ we have*

$$\frac{\text{dens}(\alpha)}{C(h)} = \sum_{Z \subseteq \cup S_q} w_Z \cdot \prod_{\ell \in Z} \left(\delta_\ell - \prod_{q \in H_\ell} q \right)$$

where the non-negative rational numbers w_Z are defined in (8). The minimal denominator of $\text{dens}(\alpha)/C(h)$ divides

$$\prod_{\ell \mid \Omega_0 M} (\delta_\ell - 1) \cdot \prod_{q \in F} q^{(\#S_q)-1}.$$

Proof. For every prime $\ell \nmid \Omega_0 M$ we have (see [6, Section 4]) that $K_\ell \cap K_n = K$ for every positive integer n that is coprime to ℓ . We deduce from the Master theorem of Artin type problems [3, Theorem 4.1] that

$$\text{dens}(\alpha) = \text{dens}'(\alpha) \cdot \prod_{\ell \nmid \Omega_0 M} \left(1 - \frac{1}{\delta_\ell}\right)$$

and hence

$$(5) \quad \frac{\text{dens}(\alpha)}{C(h)} = \text{dens}'(\alpha) \cdot \prod_{\ell \nmid \Omega_0 M} \left(1 - \frac{1}{\delta_\ell}\right)^{-1},$$

where $\text{dens}'(\alpha)$ is the natural density of the primes \mathfrak{p} of K such that the index of $(\alpha \bmod \mathfrak{p})$ is coprime to $\Omega_0 M$.

In case $F = \emptyset$, we have $M = 1$ so, up to a finite case distinction, we can treat $\text{dens}_{\Omega_0}(\alpha)$ as a constant and we easily conclude, observing that this density is a rational number whose minimal denominator divides $[K_{\Omega_0} : K]$ and hence $\prod_{\ell \nmid \Omega_0} \delta_\ell$.

Now suppose that $F \neq \emptyset$. By Kummer theory, for every $q \in F$ we can write

$$\alpha = \alpha_{0,q} \prod_{\ell \in S_q} \alpha_{\ell,q}$$

such that $\alpha_{i,q} \in K^\times$ and $\sqrt[q]{\alpha_{\ell,q}}$ generates the subextension of degree q of $K(\zeta_\ell)/K$ and such that $\alpha_{0,q} = 1$ or $\sqrt[q]{\alpha_{0,q}}$ generates a subextension of degree q of $K(\zeta_{\Omega_0})/K$. Thus, understanding whether a Galois automorphism $\sigma \in \text{Gal}(\overline{K}/K)$ fixes $\sqrt[q]{\alpha}$ can be done by inspecting the Galois action of σ on $\sqrt[q]{\alpha_{i,q}}$. Such a Galois action is, by Kummer theory, encoded by the integers $\chi_{i,q}(\sigma)$ modulo q such that

$$\zeta_q^{\chi_{i,q}(\sigma)} = \frac{\sigma(\sqrt[q]{\alpha_{i,q}})}{\sqrt[q]{\alpha_{i,q}}}.$$

Setting $\chi_q = \sum_i \chi_{i,q}$, we have

$$\zeta_q^{\chi_q(\sigma)} = \frac{\sigma(\sqrt[q]{\alpha})}{\sqrt[q]{\alpha}}.$$

To ease notation, for $q \in F \setminus H_\ell$ we define $\chi_{\ell,q}(\sigma) = 0 \bmod q$ for any $\sigma \in \text{Gal}(\overline{K}/K)$.

For any prime $\ell \mid M$ the extension K_ℓ/K is linearly disjoint from $K_{\Omega_0 M \ell^{-1}}/K$ up to the compatibility of the above q -characters for $q \in F$, which only affects the subfield of $K(\zeta_\ell)$ of degree $\prod_{q \in H_\ell} q$ (see also [6]). Let T_ℓ be a tuple of values for $(\chi_{\ell,q})_{q \in F}$. The proportion of automorphisms in $\text{Gal}(K_\ell/K)$ that are not the identity among those whose values for the characters $\chi_{\ell,q}$'s are those in T_ℓ equals

$$1 \quad \text{or} \quad 1 - \frac{\prod_{q \in H_\ell} q}{\delta_\ell}$$

according to whether $T_\ell \neq 0$ or $T_\ell = 0$. Indeed, if T_ℓ is not the zero tuple, we can take any lift of the automorphism with the prescribed character values in the subfield of $K(\zeta_\ell)$ of degree $\prod_{q \in H_\ell} q$, while if $T_\ell = 0$ we can take all the lifts except the identity.

With a finite case distinction on α , for every tuple T of integers modulo q with $q \in F$ we can treat as a constant the proportion c_T of automorphisms in

$\text{Gal}(K_{\Omega_0}/K)$ that are not the identity on any subfield K_ℓ for $\ell \mid \Omega_0$ and such that the tuple T is the value of the characters $(\chi_q - \chi_{0,q})_{q \in F}$. Notice that c_T is a non-negative rational number that is at most 1 and whose minimal denominator divides $\prod_{\ell \mid \Omega_0} \delta_\ell$ (in turn, this number divides $\prod_{\ell \mid \Omega_0} \ell[K(\zeta_\ell) : K]$, which depends only on K).

For every subset $S \subseteq \cup_{q \in F} S_q$ we let $n_{S,T}$ be the number of decompositions

$$T = \sum_{\ell \in S} T_\ell$$

such that all tuples T_ℓ are non-zero and their entries modulo q are non-zero only for $q \in S_\ell$. Moreover, let

$$N_T := \sum_S n_{S,T} = \prod_{q \in F} q^{(\#S_q)-1}$$

be the number of all decompositions (for $S = \emptyset$ we only have a trivial decomposition for the zero tuple). We may then write

$$(6) \quad \text{dens}'(\alpha) = \sum_T c_T \cdot \text{dens}''(\alpha)_T$$

where $\text{dens}''(\alpha)_T$ is the proportion, among the automorphisms of

$$\text{Gal}(K_M/K) \simeq \prod_{\ell \mid M} \text{Gal}(K_\ell/K)$$

such that $T = (\chi_q)_q$, of those that, for any $\ell \mid M$, are not the identity on K_ℓ . We have

$$(7) \quad \text{dens}''(\alpha)_T = \sum_S \frac{n_{S,T}}{N_T} \prod_{\ell \in \cup S_q \setminus S} \left(1 - \frac{\prod_{q \in H_\ell} q}{\delta_\ell}\right).$$

We may then conclude by combining (5), (6) and (7), setting $Z = \cup_{q \in F} S_q \setminus S$ and defining the rational number

$$(8) \quad w_Z := \frac{\sum_T c_T \cdot n_{(\cup_q S_q \setminus Z), T} \cdot \prod_{\ell \mid \Omega_0 M, \ell \notin Z} \delta_\ell}{\prod_{q \in F} q^{(\#S_q)-1} \prod_{\ell \mid \Omega_0 M} (\delta_\ell - 1)}.$$

□

The above result proves all statements from the introduction, with the exception of the special case Theorem 5 which will be investigated in the next section.

3. EXAMPLES

3.1. The closed formula over the rationals. Let $K = \mathbb{Q}$, thus $\Omega_0 = 2$, and recall the notation of Theorem 1. Observe that $C(h) = A(h)$. If $\Delta \not\equiv 1 \pmod{4}$, then $F = \emptyset$ hence $\text{dens}(\alpha)/C(h) = \text{dens}_2(\alpha) \cdot 2 = 1$. In the remaining case, we have $F = \{2\}$. We have $c_T = \frac{1}{2}$ for $T = (1 \pmod{2})$ and $c_T = 0$ for $T = (0 \pmod{2})$, so we may consider only $\bar{T} = (1 \pmod{2})$. We have $n_{S,T} = 1$ if $\#S$ is odd and $n_{S,T} = 0$ if $\#S$ is even and $N_T = 2^{(\#S_2)-1}$. Our formulas (6) and (7) give

$$\text{dens}'(\alpha) = \frac{1}{2} \sum_{\substack{S \subseteq S_2 \\ \#S \text{ odd}}} 2^{1-\#S_2} \prod_{\ell \in S_2 \setminus S} \left(1 - \frac{2}{\delta_\ell}\right)$$

and hence, setting $x_\ell = \delta_\ell/2$

$$(9) \quad \text{dens}'(\alpha) \prod_{\ell \in S_2} \delta_\ell = \sum_{\substack{S \subseteq S_2 \\ \#S \text{ odd}}} \prod_{\ell \in S} x_\ell \prod_{\ell \in S_2 \setminus S} (x_\ell - 1).$$

Hooley's formula (2) gives

$$\frac{\text{dens}(\alpha)}{C(h)} = 1 - (-1)^{\#S_2} \prod_{\ell \in S_2} \frac{1}{\delta_\ell - 1}$$

thus by (5) we have

$$\begin{aligned} \text{dens}'(\alpha) &= \frac{\text{dens}(\alpha)}{C(h)} \cdot \frac{1}{2} \prod_{\ell \in S_2} \frac{\delta_\ell - 1}{\delta_\ell} \\ &= \frac{1}{2} \left(\left(\prod_{\ell \in S_2} (\delta_\ell - 1) \right) - (-1)^{\#S_2} \right) \prod_{\ell \in S_2} \frac{1}{\delta_\ell} \end{aligned}$$

and hence

$$(10) \quad \text{dens}'(\alpha) \prod_{\ell \in S_2} \delta_\ell = \sum_{\emptyset \neq Z \subseteq S_2} 2^{\#Z-1} \cdot (-1)^{\#S_2 \setminus Z} \prod_{\ell \in Z} x_\ell.$$

To conclude that the formula by Hooley is equivalent to our formula over \mathbb{Q} we show that the polynomial expressions in (9) and (10) are the same. There is no constant term and each variable x_ℓ appears at most in degree 1. If Z is the set of indices for the variables appearing in a monomial, then by (10) the coefficient of the monomial is $2^{\#Z-1} \cdot (-1)^{\#S_2 \setminus Z}$. We find this monomial in (9) by considering all partitions $Z = S \cup (Z \setminus S)$ where $\#S$ is odd, putting aside $(-1)^{\#S_2 \setminus Z}$ from the product over $\ell \in S_2 \setminus S$. We may conclude because the number of odd subsets of Z is $2^{\#Z-1}$.

3.2. The special case $\#F = 1$. Let K be a number field. Let $\alpha \in K^\times$ be not a root of unity and suppose that there is no prime number $q \mid h$ such that $\zeta_q \in K$. We suppose that $F = \{q\}$. Theorem 5 is a special case of Theorem 7 because we have

$$\frac{\text{dens}(\alpha)}{C(h)} = \sum_{Z \subseteq S_q} w_Z \prod_{\ell \in Z} (\delta_\ell - q)$$

where

$$w_Z := \frac{\sum_T c_T \cdot n_{S_q \setminus Z, T} \cdot \prod_{\ell \in \Omega_0 m_q, \ell \notin Z} \delta_\ell}{q^{(\#S_q)-1} \prod_{\ell \in \Omega_0 m_q} (\delta_\ell - 1)}.$$

In this case, the tuple T is an integer modulo q and for $S = S_q \setminus Z$ we can write (introducing functions f and g)

$$n_{S, T} := \begin{cases} f(\#S) & \text{if } T \neq 0 \\ g(\#S) & \text{if } T = 0. \end{cases}$$

We have $f(0) = 0$ and $f(1) = 1$, while $g(0) = 1$ and $g(1) = 0$. Moreover, we have $f(2) = q - 2$ and $g(2) = q - 1$. For every $a \geq 2$ we have

$$f(a) = g(a - 1) + (q - 2)f(a - 1) \quad \text{and} \quad g(a) = (q - 1)f(a - 1).$$

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