A CLOSED FORMULA FOR THE DENSITY IN ARTIN'S CONJECTURE OVER NUMBER FIELDS

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Abstract. In 1967, Hooley proved Artin's conjecture on primitive roots – conditionally under GRH – over $\mathbb{Q}$, and in 1975 Cooke and Weinberger proved the analogue result over number fields. Hooley computed in particular that, over $\mathbb{Q}$, the Artin density equals the Artin constant times an explicit rational correction factor. We prove a closed formula for the Artin density over any number field.

1. Introduction

Artin's conjecture on primitive roots dates back 1927 and, almost a century after its formulation, there are no known examples (among the rational numbers that are neither -1 nor a square). Nevertheless, the conjecture has been proven by Hooley in 1967 [2] conditionally under GRH. We refer the reader to the survey by Moree [5] for an extensive introduction to Artin's conjecture.

We let K be a number field, we take $\alpha \in K^\times$ that is not a root of unity, and we assume GRH as in [1]. Then the following holds:

Theorem 1 (Cooke and Weinberger, 1975)**.** *The set of primes $\mathfrak{p}$ of K such that the multiplicative index of $(\alpha \bmod \mathfrak{p})$ is well-defined and equals 1 admits a natural density $\mathrm{dens}(\alpha)$. This is also the natural density of the set of primes of K that do not split completely in any of the fields $K(\zeta_\ell, \sqrt[\ell]{\alpha})$ for ℓ prime, and we have*

$$\mathrm{dens}(\alpha) = \sum_{n\geqslant 1} \frac{\mu(n)}{[K(\zeta_n, \sqrt[n]{\alpha}) : K]}\,.$$

We exhibit a closed formula for the Artin density, describing the ratio between $\mathrm{dens}(\alpha)$ and the Artin constant

$$A := \prod_{\ell \text{ prime}} \Big(1 - \frac{1}{\ell(\ell-1)}\Big) \sim 0.37\,.$$

Let $K = \mathbb{Q}$ and suppose additionally that α is not a square (because in that case $\mathrm{dens}(\alpha)$ would be zero). Consider the largest squarefree integer h (which is odd) such that α is an h-th power in $\mathbb{Q}^\times$ and define the modified Artin constant

$$A(h) := \prod_{\ell} \left(1 - \frac{1}{\ell^{1-v_\ell(h)}(\ell-1)}\right)$$

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which is a positive rational multiple of A. Letting Δ be the discriminant of $\mathbb{Q}(\sqrt{\alpha})$, Hooley in [2] (assuming GRH) has computed a closed formula for the density:

$$\frac{\mathrm{dens}(\alpha)}{A(h)} = \begin{cases} 1 & \text{if } \Delta \not\equiv 1 \pmod 4, \\ 1 - \mu(|\Delta|) \prod_{\ell \mid \Delta} \dfrac{1}{\ell^{1-v_\ell(h)}(\ell-1)-1} & \text{otherwise}. \end{cases} \tag{1}$$

The correction factor in the latter case stems from the fact that $\mathbb{Q}(\sqrt{\alpha})$ is contained in the compositum of the fields $\mathbb{Q}(\zeta_\ell)$ for the primes $\ell \mid \Delta$.

In Theorem 2 we generalize this formula (assuming GRH as in [1]) to any number field. We observe that, over $\mathbb{Q}$, the correction factor for the Artin density already had an interpretation with characters (see [4]). It was however not expected that it would be possible to obtain a closed formula over any number field. Our method is relying on a finite case distinction (concerning the prime divisors of the smallest integer n such that $K \cap \mathbb{Q}(\zeta_\infty) \subseteq \mathbb{Q}(\zeta_n)$) and describing in a convenient way the *entanglement* (namely, the failure of linear disjointness) of the fields $K(\zeta_\ell, \sqrt[\ell]{\alpha})$ for ℓ prime.

2. Closed density formula over number fields

Let K be a number field and let $\alpha \in K^\times$ be not a root of unity. To ease notation, for every integer $n > 0$ we write $K_n := K(\zeta_n, \sqrt[n]{\alpha})$. We let h be the largest positive squarefree integer such that $\alpha \in K^{\times h}$. We assume that there is no prime number $q \mid h$ such that $\zeta_q \in K$ (because in that case we would have $K_q = K$ hence $\mathrm{dens}(\alpha) = 0$).

We call Ω the smallest even positive integer such that $K \cap \mathbb{Q}(\zeta_\infty)$ is contained in $\mathbb{Q}(\zeta_\Omega)$ and denote by Ω_0 the square-free part of Ω.

We then consider the finite set F consisting of the primes q having the following properties: $\zeta_q \in K$; $\sqrt[q]{\alpha} \notin K(\zeta_{\Omega_0})$; $\sqrt[q]{\alpha} \in K(\zeta_{m_q})$ for some square-free positive integer m_q which we may suppose to be minimal. For every prime $q \in F$ we define the finite non-empty set S_q consisting of the primes ℓ such that $\ell \mid m_q$ and $\ell \nmid \Omega_0$. We call M the product of the primes in $\cup_{q \in F} S_q$. For every prime $\ell \mid M$ we call H_ℓ the set of primes $q \in F$ such that $\ell \in S_q$.

It is convenient to define the following quantity, which is a rational multiple of the Artin constant A (observing that the factors for $\ell \nmid \Omega_0 M h$ are $1 - \frac{1}{\ell(\ell-1)}$):

$$A_\alpha := \prod_{\ell \text{ prime}} \left(1 - \frac{1}{\delta_\ell}\right) \qquad \delta_\ell := [K_\ell : K]\,.$$

For any ℓ such that $\zeta_\ell \in K$ we have by assumption $\delta_\ell = \ell$ while for ℓ such that $\zeta_\ell \notin K$ we have $\delta_\ell = [K(\zeta_\ell) : K]\ell^{1-v_\ell(h)}$, which means that δ_ℓ depends on α only through h. Over $\mathbb{Q}$, A_α is precisely $A(h)$.

Theorem 2. *Let K be a number field. Let $\alpha \in K^\times$ be not a root of unity and suppose that there is no prime number $q \mid h$ such that $\zeta_q \in K$. Assume GRH (as in [1]).*

(i) Up to a finite case distinction on α, the ratio $\mathrm{dens}(\alpha)/A_\alpha$ is a rational number that depends on α only through h and the pairs (q, S_q) for $q \in F$.

(ii) For $F = \varnothing$ we have

$$\frac{\mathrm{dens}(\alpha)}{A_\alpha} = \mathrm{dens}_{\Omega_0}(\alpha) \cdot \prod_{\ell \mid \Omega_0} \frac{\delta_\ell}{\delta_\ell - 1}$$

where dens_{Ω_0} *is the natural density of the primes* $\mathfrak{p}$ *of* K *such that the index of* $(\alpha \bmod \mathfrak{p})$ *is coprime to* Ω_0*. The minimal denominator of* $\mathrm{dens}(\alpha)/A_\alpha$ *divides*

$$\prod_{\ell \mid \Omega_0} (\delta_\ell - 1)\,.$$

(iii) For $F \neq \varnothing$ *we have*

$$\frac{\mathrm{dens}(\alpha)}{A_\alpha} = \sum_{Z \subseteq \cup S_q} w_Z \prod_{\ell \in Z} \Big(\delta_\ell - \prod_{q \in H_\ell} q \Big)$$

where we refer to the proof for the definition of the non-negative rational numbers w_Z*. The minimal denominator of* $\mathrm{dens}(\alpha)/A_\alpha$ *divides*

$$\prod_{\ell \mid \Omega_0 M} (\delta_\ell - 1) \cdot \prod_{q \in F} q^{(\# S_q) - 1}\,.$$

Proof. For every prime $\ell \nmid \Omega_0 M$ we have (see [6, Section 4]) $K_\ell \cap K_n = K$ for every positive integer n that is coprime to ℓ. We deduce from the Master theorem of Artin type problems [3, Theorem 4.1] that

$$\mathrm{dens}(\alpha) = \mathrm{dens}'(\alpha) \cdot \prod_{\ell \nmid \Omega_0 M} \Big(1 - \frac{1}{\delta_\ell} \Big)$$

and hence

$$\frac{\mathrm{dens}(\alpha)}{A_\alpha} = \mathrm{dens}'(\alpha) \cdot \prod_{\ell \mid \Omega_0 M} \Big(1 - \frac{1}{\delta_\ell} \Big)^{-1}, \tag{2}$$

where $\mathrm{dens}'(\alpha)$ is the natural density of the primes $\mathfrak{p}$ of K such that the index of $(\alpha \bmod \mathfrak{p})$ is coprime to $\Omega_0 M$.

In case $F = \varnothing$, we have $M = 1$ so, up to a finite case distinction, we can treat $\mathrm{dens}_{\Omega_0}(\alpha)$ as a constant and we easily conclude, observing that this density is a rational number whose minimal denominator divides $[K_{\Omega_0} : K]$ and hence $\prod_{\ell \mid \Omega_0} \delta_\ell$.

Now suppose that $F \neq \varnothing$. By Kummer theory, for every $q \in F$ we can write

$$\alpha = \alpha_{0,q} \prod_{\ell \in S_q} \alpha_{\ell,q}$$

such that $\alpha_{i,q} \in K^\times$ and $\sqrt[q]{\alpha}_{\ell,q}$ generates the subextension of degree q of $K(\zeta_\ell)/K$ and such that $\alpha_{0,q} = 1$ or $\sqrt[q]{\alpha}_{0,q}$ generates a subextension of degree q of $K(\zeta_{\Omega_0})/K$. Then understanding whether a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{K}/K)$ fixes $\sqrt[q]{\alpha}$ can be done by inspecting the Galois action of σ on $\sqrt[q]{\alpha}_{i,q}$. Such a Galois action is, by Kummer theory, encoded by the integers $\chi_{i,q}(\sigma)$ modulo q such that

$$\zeta_q^{\chi_{i,q}(\sigma)} = \frac{\sigma\big(\sqrt[q]{\alpha_{i,q}}\big)}{\sqrt[q]{\alpha_{i,q}}}$$

and, setting $\chi_q = \sum_i \chi_{i,q}$, we have

$$\zeta_q^{\chi_q(\sigma)} = \frac{\sigma\big(\sqrt[q]{\alpha}\big)}{\sqrt[q]{\alpha}}\,.$$

To ease notation, for $q \in F \backslash H_\ell$ we also define $\chi_{\ell,q}(\sigma) = 0 \bmod q$ for any $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

For any prime $\ell \mid M$ the extension K_ℓ/K is linearly disjoint from $K_{\Omega_0 M\ell^{-1}}/K$ up to the compatibility of the above q-characters for $q \in F$, which only affects the subfield of $K(\zeta_\ell)$ of degree $\prod_{q\in H_\ell} q$ (we refer to [6]). Let T_ℓ be a tuple of values for $(\chi_{\ell,q})_{q\in F}$. The proportion of automorphisms in $\mathrm{Gal}(K_\ell/K)$ that are not the identity among those whose values for the characters $\chi_{\ell,q}$'s are those in T_ℓ equals

$$1 \qquad \text{or} \qquad 1-\frac{\prod_{q\in H_\ell} q}{\delta_\ell}$$

according to whether $T_\ell \neq 0$ or $T_\ell = 0$. Indeed, if T_ℓ is not the zero tuple we can take any lift of the automorphism with the prescribed characters in the subfield of $K(\zeta_\ell)$ of degree $\prod_{q\in H_\ell} q$ while if T_ℓ is the zero tuple we can take all the lifts except the identity.

With a finite case distinction on α, for every tuple T of integers modulo q with $q \in F$ we can treat as a constant the proportion c_T of automorphisms in $\mathrm{Gal}(K_{\Omega_0}/K)$ that are not the identity on any subfield K_ℓ for $\ell \mid \Omega_0$ and such that the tuple T is the value of the characters $(\chi_q - \chi_{0,q})_{q\in F}$. Notice that c_T is a non-negative rational number that is less than 1 and whose minimal denominator divides $\prod_{\ell\mid\Omega_0} \delta_\ell$ (in turn, this number divides $\prod_{\ell\mid\Omega_0} \ell[K(\zeta_\ell):K]$, which depends only on K).

For every subset $S \subseteq \cup_{q\in F} S_q$ we let $n_{S,T}$ be the number of decompositions

$$T = \sum_{\ell\in S} T_\ell$$

such that all tuples T_ℓ are non-zero and their entries modulo q are non-zero only for $q \in S_\ell$. Moreover, let

$$N_T := \sum_S n_{S,T} = \prod_{q\in F} q^{(\# S_q)-1}$$

be the number of all decompositions (for $S = \varnothing$ we only have a trivial decomposition for the zero tuple). We may then write

$$\mathrm{dens}'(\alpha) = \sum_T c_T \cdot \mathrm{dens}''(\alpha)_T \tag{3}$$

where $\mathrm{dens}''(\alpha)_T$ is the proportion, among the Galois automorphisms of $\prod_{\ell\mid M} \mathrm{Gal}(K_\ell/K)$ such that $T = (\chi_q)_q$, of those that are not the identity on $\mathrm{Gal}(K_\ell/K)$ for any $\ell \mid M$. We have

$$\mathrm{dens}''(\alpha)_T = \sum_S \frac{n_{S,T}}{N_T} \prod_{\ell\in\cup S_q\setminus S} \left(1-\frac{\prod_{q\in H_\ell} q}{\delta_\ell}\right). \tag{4}$$

We may then conclude by combining (2), (3) and (4), setting $Z = \cup S_q \setminus S$ and defining the rational number

$$w_Z := \Big(\prod_{q\in F} q^{(\# S_q)-1} \prod_{\ell\mid\Omega_0 M} (\delta_\ell - 1)\Big)^{-1} \sum_T \Big(c_T \prod_{\ell\mid\Omega_0 M, \ell\notin Z} \delta_\ell\Big) n_{(\cup S_q\setminus Z),T}\,.$$

□

3. Examples

3.1. **The closed formula over $\mathbb{Q}$.** Let $K=\mathbb{Q}$, thus $\Omega_0=2$, and recall the notation of Theorem 1. Observe that $A_\alpha=A(h)$. If $\Delta\not\equiv 1 \bmod 4$, then $F=\varnothing$ hence $\mathrm{dens}(\alpha)/A_\alpha=\mathrm{dens}_2(\alpha)\cdot 2=1$. In the remaining case, we have $F=\{2\}$. We have $c_T=\frac{1}{2}$ for $T=1(\bmod 2)$ and $c_T=0$ for $T=0(\mathrm{mod}2)$, so we may consider only $T=1(\mathrm{mod}2)$. We have $n_{S,T}=1$ if $\#S$ is odd and $n_{S,T}=0$ if $\#S$ is even and $N_T=2^{(\#S_2)-1}$. Our formulas (3) and (4) give

$$\mathrm{dens}'(\alpha)=\frac{1}{2}\sum_{\substack{S\subseteq S_2\\ \#S\text{ odd}}}2^{1-\#S_2}\prod_{\ell\in S_2\setminus S}\Big(1-\frac{2}{\delta_\ell}\Big)$$

and hence, setting $x_\ell=\delta_\ell/2$

$$\mathrm{dens}'(\alpha)\prod_{\ell\in S_2}\delta_\ell=\sum_{\substack{S\subseteq S_2\\ \#S\text{ odd}}}\prod_{\ell\in S}x_\ell\prod_{\ell\in S_2\setminus S}(x_\ell-1)\,. \tag{5}$$

Hooley's formula (1) gives

$$\frac{\mathrm{dens}(\alpha)}{A_\alpha}=1-(-1)^{\#S_2}\prod_{\ell\in S_2}\frac{1}{\delta_\ell-1}$$

thus by (2) we have

$$\mathrm{dens}'(\alpha)=\frac{\mathrm{dens}(\alpha)}{A_\alpha}\cdot\frac{1}{2}\prod_{\ell\in S_2}\frac{\delta_\ell-1}{\delta_\ell}=\frac{1}{2}\Big(\Big(\prod_{\ell\in S_2}(\delta_\ell-1)\Big)-(-1)^{\#S_2}\Big)\prod_{\ell\in S_2}\frac{1}{\delta_\ell}$$

and hence

$$\mathrm{dens}'(\alpha)\prod_{\ell\in S_2}\delta_\ell=\sum_{\varnothing\neq Z\subseteq S_2}2^{\#Z-1}\cdot(-1)^{\#S_2\setminus Z}\prod_{\ell\in Z}x_\ell\,. \tag{6}$$

To conclude that the formula by Hooley is equivalent to our formula we show that the polynomial expressions in (5) and (6) are the same. There is no constant term and each variable x_ℓ appears at most in degree 1. If Z is the set of indices for the variables appearing in a monomial, then by (6) the coefficient of the monomial is $2^{\#Z-1}\cdot(-1)^{\#S_2\setminus Z}$. We find this monomial in (5) by considering all partitions $Z=S\cup(Z\backslash S)$ where $\#S$ is odd, putting aside $(-1)^{\#S_2\setminus Z}$ from the product over $\ell\in S_2\backslash S$. We may conclude because the number of odd subsets of Z is $2^{\#Z-1}$.

3.2. **The special cases $\#F\leqslant 1$.** Let K be a number field, let $\alpha\in K^\times$ be not a root of unity and suppose that there is no prime number $q\mid h$ such that $\zeta_q\in K$. If $F=\varnothing$, the closed formula for the density is very easy by Theorem 2, so suppose that $F\neq\varnothing$.

We consider the case $F=\{q\}$. Then we have

$$\frac{\mathrm{dens}(\alpha)}{A_\alpha}=\sum_{Z\subseteq S_q}w_Z\prod_{\ell\in Z}\Big(\delta_\ell-q\Big)$$

where

$$w_Z:=\Big(q^{(\#S_q)-1}\prod_{\ell\mid\Omega_0M}(\delta_\ell-1)\Big)^{-1}\sum_T\Big(c_T\prod_{\ell\mid\Omega_0M,\ell\notin Z}\delta_\ell\Big)n_{S_q\setminus Z,T}\,.$$

In this case, the tuple T is an integer modulo q and for $S = S_q \backslash Z$ we can write (introducing functions f and g)

$$n_{S,T} := \begin{cases} f(\#S) & \text{if } T \neq 0 \\ g(\#S) & \text{if } T = 0 \,. \end{cases}$$

We have $f(0) = 0$ and $f(1) = 1$, while $g(0) = 1$ and $g(1) = 0$. Moreover, we have $f(2) = q-2$ and $g(2) = q - 1$. For every $a \geqslant 2$ we have

$$f(a) = g(a-1) + (q-2)f(a-1) \quad \text{and} \quad g(a) = (q-1)f(a-1)\,.$$

We remark that if K is an abelian extension of $\mathbb{Q}$ (for example, if K is a multiquadratic field) and $\alpha \in \mathbb{Q}$, then by Schinzel's theorem on abelian radical exensions [7, Theorem 2] the set F consists at most of the prime 2. The same holds (for $\alpha \in K$) if the only roots of unity contained in K have degree a power of 2 (for example, if the degree of $K/\mathbb{Q}$ is odd). This shows that the special cases considered above cover in particular all multiquadratic fields such that $\zeta_3 \notin K$ (or $\alpha \in \mathbb{Q}$) and all number fields of odd degree.

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