L^p-Boundedness of the Covariant Riesz Transform on Differential Forms for $p > 2$

Li-Juan Cheng[1], Anton Thalmaier[2], and Feng-Yu Wang[3]

[1]School of Mathematics, Hangzhou Normal University,
Hangzhou 311121, People's Republic of China
lijuan.cheng@hznu.edu.cn

[2]Department of Mathematics, University of Luxembourg, Maison du Nombre,
L-4364 Esch-sur-Alzette, Luxembourg
anton.thalmaier@uni.lu

[3]Center for Applied Mathematics and KL-AAGDM, Tianjin University,
Tianjin 300072, People's Republic of China
wangfy@tju.edu.cn

November 13, 2025

Abstract

The L^p-boundedness for $p > 2$ of the covariant Riesz transform on differential forms is proved for a class of non-compact weighted Riemannian manifolds under certain curvature and volume growth conditions, which in particular settles a conjecture of Baumgarth, Devyver and Güneysu [6]. As an application, the Calderón-Zygmund inequality for $p > 2$ is derived on weighted manifolds, which extends the recent work [7] on manifolds without weight.

Contents

1 Introduction

Let (M, g) be a complete geodesically connected m-dimensional Riemannian manifold, ∇ the Levi-Civita covariant derivative, and Δ the Laplace-Beltrami operator. The operator Δ is understood as

2010 *Mathematics Subject Classification*. Primary: 35K08; Secondary: 58J65, 35J10, 47G40.

Keywords and phrases. Covariant Riesz transform, Heat kernel, Bochner formula, Calderón-Zygmund inequality, Hardy-Littlewood maximal function, Kato inequality.

This work has been supported in part by the National Key R&D Program of China (2022YFA1006000), NSFC (12531007) and Natural Science Foundation of Zhejiang Provincial (Grant No. LGJ22A010001).

a self-adjoint positive operator on $L^2(M)$. The Riesz transform on the space of smooth functions on Euclidean space, defined by $\mathbf{T}^{(0)} := \nabla\Delta^{-1/2}$, was first introduced by Strichartz [27]. The L^p-boundedness of $\mathbf{T}^{(0)}$ and its extension to manifolds have been the subject of extensive research; see [2–5, 9, 11–13, 22, 31] and the references therein.

In this paper, we investigate the L^p-boundedness of the covariant Riesz transform on the space $\Omega^{(k)} := \Gamma(\Lambda^k T^*M)$ of smooth differential k-forms for $k \in \{1, \dots, m\}$:

$$\mathbf{T}^{(k)}_\sigma := \nabla(\Delta^{(k)} + \sigma)^{-1/2}, \quad \sigma \in (0, \infty), \tag{1.1}$$

where ∇ denotes the Levi-Civita covariant derivative, and $\Delta^{(k)}$ the Hodge Laplacian on $\Omega^{(k)}$.

For $p \in (1, 2)$, the L^p-boundedness of $\mathbf{T}^{(k)}_\sigma$ was established by F.-Y. Wang and A. Thalmaier [31], following the approach of Coulhon and Duong [11] by verifying the doubling volume property, Li–Yau heat kernel upper bounds, and heat kernel derivative estimates. This result was later improved by Baumgarth, Devyver, and Güneysu [6], who relaxed the boundedness condition on the derivatives of the curvature, and further in [8], where the curvature derivative condition was entirely removed. However, as explained in [31], the argument developed in [11] does not apply to the case $p > 2$. The L^p-boundedness of $\mathbf{T}^{(k)}_\sigma$ in this regime remained an open problem for some time and was formulated as a conjecture by Baumgarth, Devyver, and Güneysu [6].

Conjecture [6]. *Assume that the Riemannian curvature tensor* Riem *satisfies*

$$\max\{\|\operatorname{Riem}\|_\infty, \|\nabla \operatorname{Riem}\|_\infty\} \leqslant A$$

for some constant A. Then there exists a constant $\sigma_0 \in (0, \infty)$ depending only on A and m, such that for any $\sigma \in [\sigma_0, \infty)$ and $p \in (1, \infty)$,

$$\sup_{1\leqslant k\leqslant m} \left\|\mathbf{T}^{(k)}_\sigma\right\|_{p\to p} \leqslant B$$

holds for some constant $B \in (0, \infty)$ depending only on A, σ and m, where $\|\cdot\|_p$ denotes the L^p-norm on M with respect to the volume measure.

We note that when ∇ is replaced by the exterior differential $\mathrm{d}^{(k)}$ or its L^2-adjoint $\delta^{(k)}$, the L^p-boundedness of $\mathrm{d}^{(k)}(\Delta^{(k)}+\sigma)^{-1/2}$ and $\delta^{(k-1)}(\Delta^{(k)}+\sigma)^{-1/2}$ has been derived in [5, 23], but the techniques developed therein do not apply to the covariant Riesz transform $\mathbf{T}^{(k)}_\sigma$.

The main goal of this paper is to confirm the above conjecture by proving the L^p-boundedness of $\mathbf{T}^{(k)}_\sigma$ for $p \in (2, \infty)$, since the case $1 < p \leqslant 2$ has already been settled in [8]. According to Güneysu and Pigola [19], the L^p-boundedness of $\mathbf{T}^{(1)}_\sigma$ and $\mathbf{T}^{(0)}_\sigma$ implies that of $\operatorname{Hess}(\Delta + \sigma)^{-1}$, since

$$\operatorname{Hess}(\Delta + \sigma)^{-1} = \nabla(\Delta^{(1)} + \sigma)^{-1/2} \circ \mathrm{d}(\Delta + \sigma)^{-1/2}.$$

The L^p-boundedness of $\operatorname{Hess}(\Delta + \sigma)^{-1}$, known as the Calderón–Zygmund inequality, was recently established for $p > 2$ by Cao, Cheng, and Thalmaier [7]. This provides positive evidence for the conjecture when $k = 1$.

In this paper, under certain curvature conditions, we establish the $L^p(\mu)$-boundedness of the covariant Riesz transform on the space $\Omega^{(k)}$ over a weighted Riemannian manifold:

$$\mathbf{T}^{(k)}_{\mu,\sigma} := \nabla(\Delta^{(k)}_\mu + \sigma)^{-1/2}, \quad 1 \leqslant k \leqslant m,$$

where $\mu(\mathrm{d}x) := \mathrm{e}^{h(x)}\,\mathrm{vol}(\mathrm{d}x)$ for some $h \in C^2(M)$ and the volume measure vol. The weighted Hodge Laplacian is defined as

$$\Delta_\mu^{(k)} := \delta_\mu^{(k+1)}\mathrm{d}^{(k)} + \mathrm{d}^{(k-1)}\delta_\mu^{(k)} \tag{1.2}$$

with $\delta_\mu^{(k+1)}: \Omega^{(k+1)} \to \Omega^{(k)}$ being the $L^2(\mu)$-adjoint of $\mathrm{d}^{(k)}$. In particular, when $h = 0$, we have $\mu(\mathrm{d}x) = \mathrm{vol}(\mathrm{d}x)$ and $\mathbf{T}_{\mu,\sigma}^{(k)} = \mathbf{T}_\sigma^{(k)}$, thereby confirming the above conjecture. For $k = 0$ we write $\mathrm{d} = \mathrm{d}^{(0)}$ and

$$\Delta_\mu = \Delta_\mu^{(0)} := \delta_\mu^{(1)}\mathrm{d} = -(\Delta + \nabla h),$$

where Δ is the Laplacian on M.

The remainder of the paper is organized as follows. In Section 2 we present our main results and their consequences. The proofs are given in Section 3 and Section 4, respectively.

Acknowledgements. The authors are indebted to Batu Güneysu, Stefano Pigola and Giona Veronelli for helpful comments on the topics of this paper.

2 Main results and consequences

We first introduce a general criterion on the L^p-boundedness ($p > 2$) of $\mathbf{T}_{\mu,\sigma}^{(k)}$ in terms of estimates on heat kernels and their gradients. Then we verify this criterion by exploiting curvature conditions, which in turn provides a positive answer to **Conjecture [6].** As a consequence, the Calderón-Zygmund inequality is presented for $p > 2$.

Let P_t be the diffusion semigroup on M generated by the weighted Laplacian $\Delta + \nabla h$, and p_t be the heat kernel of P_t with respect to μ. We introduce below the *contractive Dynkin class* of functions, which is also called generalized or extended Kato class, and has been systematically studied first by P. Stollmann and J. Voigt in [28].

Definition 2.1. (Contractive Dynkin class) We say that a function f on M belongs to the class $\hat{\mathcal{K}}$ (in short: $f \in \hat{\mathcal{K}}$) if

$$\lim_{\alpha\downarrow 0}\sup_{x\in M}\int_M\int_0^\alpha p_s(x,y)|f(y)|\,\mathrm{d}s\,\mu(\mathrm{d}y) < 1.$$

Note that $\hat{\mathcal{K}}$ contains the usual Kato class $\mathcal{K}$, defined as the set of functions f such that

$$\lim_{\alpha\downarrow 0}\sup_{x\in M}\int_M\int_0^\alpha p_s(x,y)|f(y)|\,\mathrm{d}s\,\mathrm{d}\mu(y) = 0.$$

The Kato class plays an important role in the study of Schrödinger operators and their semigroups, see Simon [26] and the reference therein. It is straight-forward that $f \in \hat{\mathcal{K}}$ if f is bounded.

To state the main result, we first introduce the weighted volume on M and the weighted curvature operator on $\Omega^{(k)}$. For $x \in M$ and $r > 0$, let $B(x,r)$ be the open geodesic ball centered at x of radius r, and

$$\mu(x,r) := \mu(B(x,r)) = \int_{B(x,r)} \mathrm{e}^{h(y)}\,\mathrm{vol}(\mathrm{d}y).$$

The weighted curvature operator $\mathscr{R}_h^{(k)}$ on $\Omega^{(k)}$ is defined as

$$\mathscr{R}_h^{(k)}(\eta) := \mathscr{R}^{(k)}(\eta) - (\mathrm{Hess}\,h)^{(k)}(\eta),$$

where for an orthonormal frame $(e_i)_{1\leqslant i\leqslant m} \in O(M)$ with dual frame $(\theta^j)_{1\leqslant j\leqslant m}$,

$$\begin{aligned}
&\mathscr{R}^{(k)} := -\sum_{i,j=1}^{m} \theta^j \wedge (e_i \lrcorner \mathrm{R}(e_j, e_i)),\\
&(\mathrm{Hess}\, h)^{(k)} := \sum_{i,j=1}^{m} e_i(e_j(h))\Big(\theta^j \wedge (e_i \lrcorner \cdot)\Big),\\
&X \lrcorner \eta\, (X_1, \ldots, X_{k-1}) := \eta(X, X_1, \ldots, X_{k-1}), \quad \eta \in \Omega^{(k)},\ X, X_1, \ldots, X_{k-1} \in TM.
\end{aligned}$$

When $k = 1$, we have

$$\mathscr{R}_h^{(1)} = \mathrm{Ric}_h := \mathrm{Ric} - \mathrm{Hess}\, h,$$

where Ric is the Ricci curvature of M. By the Weitzenböck formula, we have the decomposition

$$\Delta_\mu^{(k)} = \square_\mu + \mathscr{R}_h^{(k)},$$

with respect to the Bochner Laplacian $\square_\mu := \nabla_\mu^* \nabla$, where ∇_μ^* denotes the $L^2(\mu)$-adjoint operator of ∇.

Moreover, let $R^{(k)}$ be the curvature tensor on $\Omega^{(k)}$. For any $\eta \in \Omega^{(k)}$ and $v \in TM$, define

$$\begin{aligned}
&(R^{(k)} \cdot \eta)(v) := \sum_{i=1}^{n} R^{(k)}(v, e_i)\eta(e_i),\\
&(\nabla \cdot R^{(k)})(v)\eta := \sum_{i=1}^{n} (\nabla_{e_i} R^{(k)})(e_i, v)\eta,\\
&(R^{(k)}(\nabla h))(v)\eta := R^{(k)}(v,\, \nabla h)\eta.
\end{aligned}$$

For any $1 \leqslant k \leqslant m$, let

$$P_t^{(k)} := \mathrm{e}^{-t\Delta_\mu^{(k)}}, \quad t \geqslant 0$$

be the semigroup on $\Omega^{(k)}$ generated by $\Delta_\mu^{(k)}$ with $\Delta_\mu^{(k)}$ defined in (1.2). Finally, denote by $\Omega_{b,1}^{(k)}$ the class of differential forms $\eta \in \Omega^{(k)}$ for which $|\eta| + |\nabla\eta|$ is bounded.

We are going to prove $L^p(\mu)$ boundedness of $\mathbf{T}_{\mu,\sigma}^{(k)}$ for $p > 2$ under the following assumptions.

(A) There exist a constant $A \in (0, \infty)$ and a positive function $V_k \in \hat{\mathcal{K}}$ such that the following conditions hold:

$$\mu(x, \alpha r) \leqslant A\mu(x, r)\, \alpha^m \exp(A(\alpha - 1)r), \quad x \in M,\ \alpha > 1,\ r > 0, \tag{\textbf{LD}}$$

$$p_t(x, x) \leqslant \frac{A\mathrm{e}^{At}}{\mu(x,\, \sqrt{t})}, \quad x \in M,\ t > 0, \tag{\textbf{UE}}$$

$$\langle \mathscr{R}_h^{(k)}(\eta), \eta\rangle \geqslant -V_k|\eta|^2, \quad \eta \in \Omega^{(k)}, \tag{\textbf{Kato}}$$

$$|\nabla P_t^{(k)}\eta| \leqslant \min\Big\{t^{-1/2}\mathrm{e}^{A+At}(P_t|\eta|^2)^{1/2},\ \mathrm{e}^{At}(P_t|\nabla\eta| + AtP_t|\eta|)\Big\}, \quad \eta \in \Omega_{b,1}^{(k)},\ t > 0. \tag{\textbf{GE}}$$

Theorem 2.2. *Assume that* **(A)** *holds for $k \in \mathbb{N}$. Then there exists a constant $\sigma_0 \in (0, \infty)$ depending only on A such that for any $p \in (2, \infty)$,*

$$\sup_{\sigma\in[\sigma_0,\infty)} \|\mathbf{T}_{\mu,\sigma}^{(k)}\|_{p\to p} \leqslant B \tag{2.1}$$

holds for some constant $B \in (0, \infty)$ depending on p, m, A and V_k.

For the convenience of applications, we present below explicit curvature conditions which ensure hypothesis **(A)**. To this end, for $f \in C^\infty(M)$, let $\Gamma_2(f, f) := -\frac{1}{2}\Delta_\mu|\nabla f|^2 + (\nabla\Delta_\mu f, \nabla f)_g$.

(C) There exist constants $N \geqslant m$ and $K > 0$ such that

$$\Gamma_2(f, f) \geqslant -K|\nabla f|^2 + \frac{1}{N}(\Delta_\mu f)^2, \quad f \in C^\infty(M), \tag{2.2}$$

$$\left|\mathscr{R}_h^{(k)}\right| + \left|R^{(k)}\boldsymbol{\cdot}\right| + \left|\nabla \cdot R^{(k)} + R^{(k)}(\nabla h) + \nabla\mathscr{R}_h^{(k)}\right| \leqslant K. \tag{2.3}$$

The next theorem is then a consequence of Theorem 2.2.

Theorem 2.3. *Assume that* **(C)** *holds for $k \in \mathbb{N}$. Then there exists a constant $\sigma_0 \in (0, \infty)$ depending only on K and N such that for any $p \in (2, \infty)$,*

$$\sup_{\sigma\in[\sigma_0,\infty)} \left\|\mathbf{T}^{(k)}_{\mu,\sigma}\right\|_{p\to p} \leqslant B$$

holds for some constant $B \in (0, \infty)$ depending on p, K and N.

In particular, if **(C)** *holds for $k = 1$, then there exists a constant $\sigma_0 \in (0, \infty)$ depending only on K and N, such that $\|\operatorname{Hess}(\Delta_\mu + \sigma)^{-1}\|_{L^p(\mu)} < \infty$ for all $\sigma \geqslant \sigma_0$ and $p > 2$. As a consequence, the Calderón–Zygmund inequality holds for some constant $C \in (0, \infty)$:*

$$\left\|\operatorname{Hess} f\right\|_{L^p(\mu)} \leqslant C\left(\|f\|_{L^p(\mu)} + \|\Delta_\mu f\|_{L^p(\mu)}\right), \quad f \in C_0^\infty(M). \tag{2.4}$$

Note that on a geodesically complete manifold with Riemann curvature tensor Riem satisfying $\|\operatorname{Riem}\|_\infty < \infty$, there exists a sequence of Hessian cut-off functions (see [19], p. 362), such that inequality (2.4) extends from $C_0^\infty(M)$ to $f \in C^\infty(M) \cap L^p(\mu)$ with $\|\Delta_\mu f\|_\infty < \infty$.

3 Proof of the Main Theorem

To prove our main result (Theorem 2.2), we shall need the following lemma, which is due to [11].

Lemma 3.1 ([11])**.** *If* (**LD**) *holds, then there exist a constant $c \in (0, \infty)$ and a function $C\colon (0, \infty) \to (0, \infty)$ depending only on A and m, such that*

$$\int_{B(x,\sqrt{t})^c} \mathrm{e}^{-\gamma\rho^2(x,y)/s}\,\mu(\mathrm{d}y) \leqslant C_\gamma\,\mu(x, \sqrt{s})\,\mathrm{e}^{cs/\gamma-\gamma t/s}, \quad s, t, \gamma > 0,\ x \in M, \tag{3.1}$$

where $B(x, \sqrt{t})^c := \left\{y \in M : \rho(x, y) \geqslant \sqrt{t}\right\}$. In particular, $t \to 0$ yields

$$\int_M \frac{\mathrm{e}^{-cs/\gamma}}{C_\gamma\,\mu(x, \sqrt{s})}\mathrm{e}^{-\gamma\rho^2(x,y)/s}\,\mu(\mathrm{d}y) \leqslant 1, \quad s, \gamma > 0,\ x \in M. \tag{3.2}$$

3.1 Heat kernel estimates

By the usual abuse of notation, the corresponding self-adjoint realizations of Δ_μ and $\Delta_\mu^{(k)}$ will again be denoted by the same symbol. By local parabolic regularity, for all square-integrable k-forms $a \in L^2(\Omega^{(k)}, \mu)$, the time-dependent k-form

$$(0, \infty) \times M \ni (t, x) \mapsto P_t^{(k)}a(x) \in \Omega_x^{(k)} := \Lambda^k T_x^* M$$

has a smooth representative which extends smoothly to $[0,\infty)\times M$ if a is smooth. In addition, there exists a unique smooth heat kernel $p_t^{(k)}$ to P_t^k with respect to the measure μ, understood as a map

$$(0,\infty)\times M\times M \ni (t,x,y)\mapsto p_t^{(k)}(x,y)\in \mathrm{Hom}(\Omega_y^{(k)},\,\Omega_x^{(k)})$$

such that

$$P_t^{(k)}a(x)=\int_M p_t^{(k)}(x,y)a(y)\,\mu(\mathrm{d}y).$$

Let $P_t^{V_k}$ be the heat semigroup associated to $\Delta_\mu+V_k$ and $p_t^{V_k}(x,y)$ be the corresponding heat kernel. If condition (**Kato**) in **(A)** holds, then

$$|p_t^{(k)}(x,y)|\leqslant p_t^{V_k}(x,y).$$

Combining this inequality with [24, Lemma 2.2] for the upper bound estimate on $p_t^{V_k}(x,y)$, we obtain the following result; see [14, 29, 33, 34] for earlier results on Schödinger heat kernel estimates.

Lemma 3.2. *Let M be a complete non-compact Riemannian manifold satisfying* (**LD**)*,* (**UE**) *and* (**Kato**)*. There exists a function $C\colon (0,1/4)\to(0,\infty)$, depending only on A, m, and V_k, such that for all $x,y\in M$, $t>0$, and $\gamma\in(0,1/4)$,*

$$\left|p_t^{(k)}(x,y)\right|\leqslant \frac{C_\gamma e^{C_\gamma t}}{\mu(y,\sqrt{t})}\exp\left(-\frac{\gamma\rho(x,y)^2}{t}\right),\quad \forall x,y\in M,\ t>0,\ 0<\gamma<1/4, \tag{3.3}$$

where we write $C_\gamma=C(\gamma)$ for notational simplicity. This estimate, combined with (3.2)*, yields*

$$\sup_{t\in(0,1],\,x\in M}\int_M \left|p_t^{(k)}(x,y)\right|\mu(\mathrm{d}y)<\infty. \tag{3.4}$$

We are now ready to present the following estimate.

Theorem 3.3. *Let M be a complete non-compact Riemannian manifold satisfying the condition* **(A)***. There exists $C\colon (0,1/4)\to(0,\infty)$, depending only on A,m and V_k, such that*

$$\int_M\left(t|\nabla p_t^{(k)}(z,y)|^2+|p_t^{(k)}(z,y)|^2\right)e^{2\gamma\rho^2(z,y)/t}\,\mu(\mathrm{d}z)\leqslant\frac{C_\gamma e^{C_\gamma t}}{\mu(y,\sqrt{t})},\quad y\in M,\ t>0,\ 0<\gamma<1/4.$$

Proof. By [24, Lemma 2.2], if $V_k\in\hat{\mathcal{K}}$, then there exist constants $\kappa\in[0,1)$ and $c_1>0$, depending only on V_k, such that

$$\int_M V_k|f|^2\,\mathrm{d}\mu\leqslant\kappa\,\|\,|\nabla f|\,\|_2^2+c_1\|f\|_2^2 \tag{3.5}$$

for all $f\in W^{1,2}(M)$. It means in particular that the operator $\Delta-V_k+c_1$ is strongly positive. Combining this with the Gaussian upper bound (3.3) in Lemma 3.2, we find that the proof of [8, Theorem 2.6] remains valid under the present assumptions. As a consequence,

$$\int_M t|\nabla p_t^{(k)}(z,y)|^2 e^{2\gamma\rho^2(z,y)/t}\,\mu(\mathrm{d}z)\leqslant\frac{\tilde{C}_\gamma e^{\tilde{C}_\gamma t}}{\mu(y,\sqrt{t})},\quad y\in M,\ t>0,\ \gamma\in(0,1/4)$$

for some $\tilde{C}\colon (0,1/4)\to(0,\infty)$ depending only on A,m and V_k. Combined with [8, Lemma 2.5], this yields the desired estimate for some function $C\colon(0,1/4)\to(0,\infty)$. □

The following is a direct consequence of Theorem 3.3 and extends [6, Theorem 1.2] to the case of weighted manifolds.

Corollary 3.4. *Let M be a complete non-compact Riemannian manifold satisfying the condition* **(A)**. *There exists $C\colon (0,1/8)\to(0,\infty)$ depending only on A, m and V_k, such that*

$$|\nabla p_t^{(k)}(\cdot,y)(x)| \leqslant \frac{C_\gamma \mathrm{e}^{C_\gamma t}}{\sqrt{t}\,\mu(y,\sqrt{t})}\exp\Big(-\frac{\gamma\rho^2(x,y)}{t}\Big),\quad \forall x,y\in M,\ t>0,\ 0<\gamma<1/8. \tag{3.6}$$

Proof. Let $x,y\in M$. It is easy to see that

$$\nabla p_{2t}^{(k)}(\cdot,y)(x) = \nabla P_t^{(k)}\left(p_t^{(k)}(\cdot,y)\right)(x).$$

Using condition (**GE**), we have

$$|\nabla P_t^{(k)}\eta| \leqslant \frac{\mathrm{e}^{A+At}}{\sqrt{t}}(P_t|\eta|^2)^{1/2},$$

for $\eta\in\Omega^{(k)}$ with $P_t(|\eta|^2)<\infty$. We use this inequality with $\eta(z)=p_t^{(k)}(\cdot,y)(z)$ to obtain

$$\left|\nabla P_t^{(k)}\left(p_t^{(k)}(\cdot,y)\right)\right|(x) \leqslant \frac{\mathrm{e}^{A+At}}{\sqrt{t}}\left(\int_M p_t(x,z)\left|p_t^{(k)}(z,y)\right|^2\mu(\mathrm{d}z)\right)^{1/2}.$$

By Theorem 3.3, this implies that for any $\gamma\in(0,1/4)$,

$$\begin{aligned}|\nabla p_{2t}^{(k)}(\cdot,y)(x)| &\leqslant \frac{\mathrm{e}^{A+At}}{\sqrt{t}}\left(\int_M |p_t^{(k)}(z,y)|^2\mathrm{e}^{\frac{2\gamma\rho^2(z,y)}{t}-\frac{2\gamma\rho^2(z,y)}{t}}p_t(x,z)\,\mu(\mathrm{d}z)\right)^{1/2}\\ &\leqslant \frac{C_\gamma\mathrm{e}^{A+(A+C_\gamma)t}}{\sqrt{t\mu(y,\sqrt{t})}}\sup_{z\in M}\left\{\mathrm{e}^{-\frac{2\gamma\rho^2(z,y)}{t}}p_t(x,z)\right\}^{1/2}.\end{aligned} \tag{3.7}$$

Since $p_t(x,x)$ satisfies the diagonal estimate **(UE)**, from the proof of [24, Lemma 3.2], there exists a function $\tilde{C}\colon (0,1/4)\to(0,\infty)$ depending only on A and m such that

$$p_t(x,z)\leqslant\frac{\tilde{C}_\gamma\mathrm{e}^{\tilde{C}_\gamma t}}{\mu(x,\sqrt{t})}\exp\left(-\frac{2\gamma\rho(x,z)^2}{t}\right),\ \ 0<\gamma<1/8,\ t>0,\ x,y\in M. \tag{3.8}$$

By (**LD**), there exists a decreasing function $c\colon (0,1)\to(0,\infty)$ depending only on A and m such that

$$\begin{aligned}\mu\big(y,\sqrt{t}\big) &\leqslant \mu\big(x,\sqrt{t}(1+t^{-1/2}\rho(x,y))\big)\leqslant A\mu\big(x,\sqrt{t}\big)(1+t^{-1/2}\rho(x,y))^m\mathrm{e}^{A\rho(x,y)}\\ &\leqslant c_\varepsilon\mu\big(x,\sqrt{t}\big)\exp\left(\frac{\varepsilon\rho(x,y)^2}{t}+c_\varepsilon t\right),\quad \varepsilon\in(0,1),\ t>0,\ x,y\in M.\end{aligned}$$

Combining this with (3.8) and

$$2\rho(x,z)^2+2\rho(y,z)^2\geqslant\rho(x,y)^2,$$

we find $\hat{C}\colon \{(\gamma,\varepsilon)\colon\ 0<\varepsilon<\gamma<1/8\}\to(0,\infty)$ depending only on A, m and V_k, such that

$$\sup_{z\in M}\left\{\mathrm{e}^{-\frac{2\gamma\rho^2(z,y)}{t}}p_t(x,z)\right\}\leqslant\frac{\hat{C}_{\gamma,\varepsilon}\mathrm{e}^{\hat{C}_{\gamma,\varepsilon}t}}{\mu(y,\sqrt{t})}\exp\left(-\frac{(\gamma-\varepsilon)\rho^2(x,y)}{t}\right),\quad x,y\in M,\ t>0,\ 0<\varepsilon<\gamma<1/8.$$

Combining this with (3.7) yields

$$|\nabla p_{2t}^{(k)}(\cdot,y)(x)| \leqslant \frac{\sqrt{\hat{C}_{\gamma,\varepsilon}}C_\gamma \mathrm{e}^{A+(A+C_\gamma)t+\hat{C}_{\gamma,\varepsilon}t/2}}{\sqrt{t}\mu(y,\ \sqrt{t})} \exp\Big(-\frac{(\gamma-\varepsilon)\rho^2(x,y)}{2t}\Big).$$

By this and (**LD**), we obtain the desired estimate for some $C\colon (0,1/8)\to(0,\infty)$. □

As a consequence of the pointwise estimates in Corollary 3.4 and the local volume doubling property (**LD**), we have the following result which extends [6, Corollary 1.3] to the case of a weighted L^p-estimates of $|\nabla p_t^{(k)}|$.

Theorem 3.5. *Let M be a complete non-compact Riemannian manifold satisfying condition* (**A**)*. Then for any $p\in[1,\infty)$ there exists a function $C\colon (0,1/8)\to(0,\infty)$ depending only on p, A, m and V_k, such that*

$$\int_M \Big|\sqrt{t}\nabla p_t^{(k)}(x,y)\Big|^p \mathrm{e}^{\gamma p\rho^2(x,y)/t}\mu(\mathrm{d}x) \leqslant \frac{C_\gamma \mathrm{e}^{C_\gamma t}}{\big(\mu(y,\ \sqrt{t})\big)^{p-1}},\quad y\in M,\ t>0,\ 0<\gamma<1/8.$$

Proof. According to inequality (3.2), we find a function $h\colon (0,\infty)\to(0,\infty)$ depending only on A, m, such that

$$\int_M \mathrm{e}^{-\gamma\rho^2(x,y)/t}\mu(\mathrm{d}x) \leqslant h_\gamma\,\mu(y,\ \sqrt{t})\,\mathrm{e}^{h_\gamma t},\quad t,\gamma>0.$$

By Corollary 3.4, there exists $C\colon (0,1/8)\to(0,\infty)$ depending on A, m and V_k such that

$$\int_M \Big|\sqrt{t}\nabla p_t^{(k)}(x,y)\Big|^p \mathrm{e}^{(1-\varepsilon)\gamma p\rho^2(x,y)/t}\mu(\mathrm{d}x) \leqslant \frac{C_\gamma^p \mathrm{e}^{pC_\gamma t}}{\mu(y,\ \sqrt{t})^p}\int_M \mathrm{e}^{-(p\gamma-p(1-\varepsilon)\gamma)\rho^2(x,y)/t}\mu(\mathrm{d}x).$$

Then by Lemma 3.1, we find $C, c\colon (0,1/8)\times(0,1)\to(0,\infty)$ depending only on p, A, m and V_k, such that

$$\int_M \Big|\sqrt{t}\nabla p_t^{(k)}(x,y)\Big|^p \mathrm{e}^{(1-\varepsilon)\gamma p\rho^2(x,y)/t}\mu(\mathrm{d}x) \leqslant \frac{C_{\gamma,\varepsilon}\mathrm{e}^{c_{\gamma,\varepsilon}t}}{\mu(y,\ \sqrt{t})^{p-1}},\quad (\gamma,\varepsilon)\in(0,1/8)\times(0,1),\ t>0,$$

which completes the proof. □

We now introduce L^2-Davies-Gaffney bounds under condition (**A**) which extend the L^2-Davies-Gaffney bound in [6, Theorem 1.9]. Recall that the distance between two non-empty subsets E, F of M is defined as

$$\rho(E,F) := \max\Big\{\sup_{x\in E}\inf_{y\in F}\rho(x,y),\ \sup_{y\in F}\inf_{x\in E}\rho(x,y)\Big\}.$$

Lemma 3.6. *Assume that the conditions* (**LD**), (**UE**) *and* (**Kato**) *hold. Then there exist constants $c_1, c_2>0$ depending only on A, m and V_k, such that for all non-empty relatively compact subsets $E, F\subset M$,*

$$\Big\|\mathbb{1}_F\ \sqrt{t}|\nabla P_t^{(k)}\alpha|\Big\|_2 \leqslant c_1(1+\ \sqrt{t})\,\mathrm{e}^{-c_2\rho(E,F)^2/t}\Big\|\mathbb{1}_E|\alpha|\Big\|_2,\quad t>0,\ \alpha\in L^p(\Omega^{(k)},\mu)\ \textit{with}\ \operatorname{supp}(\alpha)\subset E.$$

Proof. All constants below depend only on A and V_k. By Lemma 3.2, the L^2-Gaffney off-diagonal estimates for $P_t^{(k)} f$ and $\Delta^{(k)} P_t^{(k)} f$ are obtained as in [6, Theorem 1.9], i.e. there exist constants $C_1, C_2 > 0$ such that

$$\left\| \mathbb{1}_F |P_t^{(k)}(\alpha)| \right\|_2 + t \left\| \mathbb{1}_F \, |\Delta_\mu^{(k)} P_t^{(k)}(\alpha)| \right\|_2 \leqslant C_1 \mathrm{e}^{-C_2 \rho(E,F)^2/t} \left\| \mathbb{1}_E |\alpha| \right\|_2, \quad t > 0. \tag{3.9}$$

Combined with (3.5), there exist constants $\kappa \in (0,1)$ and $C > 0$, such that for any $\phi \in C_0^\infty(M)$ with $F \subset \operatorname{supp}(\phi)$ and $\phi = 1$ on F, we have

$$\begin{aligned}
\int_F | \sqrt{t} \nabla P_t^{(k)} \alpha|^2(x)\, \mu(\mathrm{d}x) &\leqslant \int_M \phi^2 \left| \sqrt{t} \nabla P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) \\
&\leqslant \int_M \phi^2 t \langle \Delta_\mu^{(k)} P_t^{(k)} \alpha, P_t^{(k)} \alpha \rangle (x)\, \mu(\mathrm{d}x) + \int_M V_k \phi^2 t \left| P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) \\
&\quad + 2 \int_M \phi t \langle \nabla P_t^{(k)} \alpha, d\phi \otimes P_t^{(k)} \alpha \rangle (x)\, \mu(\mathrm{d}x) \\
&\leqslant \int_M t \phi^2 \langle \Delta_\mu^{(k)} P_t^{(k)} \alpha, P_t^{(k)} \alpha \rangle (x)\, \mu(\mathrm{d}x) + \kappa t \int_M \phi^2 \left| \nabla P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) \\
&\quad + \kappa t \int_M |\nabla \phi|^2 \cdot \left| P_t^{(k)} \alpha \right|^2 \mu(\mathrm{d}x) + 2\kappa t \int_M \phi\, |\nabla \phi| \cdot \left| P_t^{(k)} \alpha | \cdot |\nabla P_t^{(k)} \alpha \right| \mu(\mathrm{d}x) \\
&\quad + C \int_M t \phi^2 |P_t^{(k)} \alpha|^2 (x)\, \mu(\mathrm{d}x) + 2t \int_M \phi\, |\nabla \phi| \cdot \left| \nabla P_t^{(k)} \alpha \right| \cdot \left| P_t^{(k)} \alpha \right| \mu(\mathrm{d}x).
\end{aligned}$$

As $\kappa < 1$ and

$$\begin{aligned}
&4t \int_M \phi\, |\nabla \phi| \cdot \left| \nabla P_t^{(k)} \alpha \right| \cdot \left| P_t^{(k)} \alpha \right| \mu(\mathrm{d}x) \\
&\quad \leqslant \frac{t(1-\kappa)}{2} \int_M \phi^2 \left| \nabla P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) + \frac{8t}{1-\kappa} \int_M \left| \nabla \phi |^2 \cdot |P_t^{(k)} \alpha \right|^2 \mu(\mathrm{d}x),
\end{aligned}$$

we arrive at

$$\begin{aligned}
\int_M \phi^2 \left| \sqrt{t} \nabla P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) \leqslant & \int_M t \phi^2 \langle \Delta_\mu^{(k)} P_t^{(k)} \alpha, P_t^{(k)} \alpha \rangle (x)\, \mu(\mathrm{d}x) \\
& + \frac{\kappa + 1}{2} \int_M \phi^2 t \left| \nabla P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x) \\
& + t \left(\frac{8}{1-\kappa} + \kappa \right) \int_M \left| \nabla \phi |^2 \cdot |P_t^{(k)} \alpha \right|^2 \mu(\mathrm{d}x) + C \int_M t \phi^2 \left| P_t^{(k)} \alpha \right|^2 (x)\, \mu(\mathrm{d}x).
\end{aligned}$$

The rest of the proof is identical to the proof of [6, Theorem 1.9]. The details are omitted here. □

3.2 Proof of Theorem 2.2

To begin our discussion, we need the following lemma taken from [2, Section 4].

Lemma 3.7 ([2]). *If* (**LD**) *holds, then there exist $N_0 \in \mathbb{N}$ depending only on A and m, and a countable subset $\{x_j\}_{j \geqslant 1} \subset M$, such that*

(i) $M = \cup_{j \geqslant 1} B(x_j, 1)$*;*

(ii) $\{B(x_j, 1/2)\}_{j \geqslant 1}$ *are disjoint;*

(iii) *for every $x \in M$, there are at most N_0 balls $B(x_j, 4)$ containing x;*

(iv) *for any $c_0 \geqslant 1$, there exists a constant $C > 0$ depending only on c_0, A and m, such that for any $j \geqslant 1$ and $x \in B(x_j, c_0)$,*

$$\mu\left(B(x, 2r) \cap B(x_j, c_0)\right) \leqslant C\mu\left(B(x, r) \cap B(x_j, c_0)\right), \quad r \in (0, \infty),$$
$$\mu(B(x, r)) \leqslant C\mu\left(B(x, r) \cap B(x_j, c_0)\right), \quad r \in (0, 2c_0].$$

For $p \in (2, \infty)$ and $\sigma \in (A, \infty)$, we intend to find $C \in (0, \infty)$ depending only on p, σ, A, m and V_k such that

$$\left\| |\mathbf{T}^{(k)}_{\mu,\sigma}(\alpha)| \right\|_p \leqslant C\|\alpha\|_p, \quad \alpha \in \Omega^{(k)}. \tag{3.10}$$

To this end, let w be a C^∞ function on $[0, \infty)$ satisfying $0 \leqslant w \leqslant 1$ and

$$w(t) = \begin{cases} 1 & \text{on } [0, 1/2], \\ 0 & \text{on } [1, \infty), \end{cases}$$

and let $\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}$ be the operator defined by

$$\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha) := \int_0^\infty v(t)\nabla P^{(k)}_t \alpha \, dt \tag{3.11}$$

where $v(t) := w(t)\mathrm{e}^{-\sigma t}/\sqrt{t}$. We need the following lemma, which reduces (3.10) to a time and spatial localized version.

Lemma 3.8. *Suppose that Condition* **(A)** *holds. Let $p \in (2, \infty)$ and $\{x_j\}_{j\geqslant 1}$ be as in Lemma 3.7. If there exists a constant $c > 0$ depending only on p, σ, A, m and V_k such that*

$$\left\| |\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha)| \right\|_{L^p(B(x_j,4))} \leqslant c\|\alpha\|_{L^p(B(x_j,1))} \tag{3.12}$$

for any $\alpha \in L^p(\Omega^{(k)}, \mu)$, then inequality (3.10) *holds for some constant $C > 0$ depending also only on p, σ, A, m and V_k.*

Proof. In the sequel, $\xi \lesssim \eta$ for two positive variables ξ and η means that $\xi \leqslant \kappa\eta$ holds for some constant $\kappa > 0$ depending only on p, σ, A, m and V_k.

Since $w \equiv 1$ on $[0, 1/2]$, if $\sigma > A$, then **(GE)** implies that for any $\alpha \in L^p(\Omega^{(k)}, \mu)$,

$$\left\| \int_0^\infty (1 - w(t)) \, |\nabla P^{(k)}_t \alpha| \, \frac{\mathrm{e}^{-\sigma t}}{\sqrt{t}} \, \mathrm{d}t \right\|_p \lesssim \int_{1/2}^\infty \mathrm{e}^{(A-\sigma)t} \frac{1}{\sqrt{t}} \mathrm{d}t \, \|\alpha\|_p \lesssim \|\alpha\|_p.$$

(Note that the first inequality in condition **(GE)** extends to general $\alpha \in L^p(\Omega^{(k)}, \mu)$ by a standard approximation argument in $L^p(\mu)$). This and (3.11) imply that (3.10) follows from

$$\left\| |\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha)| \right\|_p \lesssim \|\alpha\|_p, \quad \alpha \in L^p(\Omega^{(k)}, \mu). \tag{3.13}$$

Let $\{x_j\}_{j\geqslant 1}$ be as in Lemma 3.7 and $\{\varphi_j\}$ be a subordinated C^∞ partition of the unity such that $0 \leqslant \varphi_j \leqslant 1$ and φ_j is supported in $B_j := B(x_j, 1)$. For each j, denote the characteristic function of the ball $4B_j := B(x_j, 4)$ by χ_j. For any $\alpha \in L^p(\Omega^{(k)}, \mu)$ and $x \in M$, we then may write

$$\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha(x) \leqslant \sum_{j\geqslant 1} \chi_j \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha\varphi_j)(x) + \sum_{j\geqslant 1}(1 - \chi_j)\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha\varphi_j)(x) =: \mathrm{I}(x) + \mathrm{II}(x). \tag{3.14}$$

By Lemma 3.7, we know

$$\sum_{j\geqslant 1}\left|(1-\chi_j)(x)\varphi_j(y)\right| \leqslant N_0 \mathbb{1}_{\{\rho(x,y)\geqslant 3\}}.$$

First note by Lemma 3.1, along with the volume doubling property (**LD**), there exists $C\colon (0,\infty)\to(0,\infty)$ depending only on A and m such that

$$\int_{\{\rho(x,y)\geqslant 3\}} \frac{\mathrm{e}^{-\gamma\rho^2(x,y)/t}}{\mu(y,\sqrt{t})}\,\mu(\mathrm{d}y) \leqslant C_\gamma \mathrm{e}^{-\frac{1}{C_\gamma t}}, \quad t\in(0,1],\ \gamma>0,\ x\in M. \tag{3.15}$$

By this and Hölder's inequality, we find $h_1, c\colon (0,\infty)\to(0,\infty)$ depending only on p, A and m such that

$$\begin{aligned}
\mathrm{II}(x) &\leqslant \int_0^1 v(t)\int_M \left|\nabla_x p_t^{(k)}(x,y)\right|\left(\sum_{j\in\Lambda}\left|(1-\chi_j)(x)\varphi_j(y)\right|\right)|\alpha(y)|\,\mu(\mathrm{d}y)\mathrm{d}t\\
&\leqslant N_0\int_0^1 \frac{1}{\sqrt{t}}\int_{\{\rho(x,y)\geqslant 3\}}\left|\nabla_x p_t^{(k)}(x,y)\right|\cdot|\alpha(y)|\,\mu(\mathrm{d}y)\mathrm{d}t\\
&\leqslant N_0\int_0^1 \frac{1}{\sqrt{t}}\int_{\{\rho(x,y)\geqslant 3\}}\left|\nabla_x p_t^{(k)}(x,y)\right| \mathrm{e}^{\gamma\rho^2(x,y)/pt}\mu(y,\sqrt{t})^{(p-1)/p}\,|\alpha(y)|\,\frac{\mathrm{e}^{-\gamma\rho^2(x,y)/pt}}{\mu(y,\sqrt{t})^{(p-1)/p}}\,\mu(\mathrm{d}y)\mathrm{d}t\\
&\leqslant h_1(\gamma)\int_0^1\left(\int_M\left|\sqrt{t}\nabla_x p_t^{(k)}(x,y)\right|^p \mathrm{e}^{\gamma\rho^2(x,y)/t}\left(\mu(y,\sqrt{t})\right)^{p-1}|\alpha(y)|^p\mu(\mathrm{d}y)\right)^{1/p}\frac{\mathrm{e}^{-c_\gamma/t}}{t}\,\mathrm{d}t.
\end{aligned}$$

By Theorem 3.5, there exists $h_2\colon (0,1/8)\to(0,\infty)$ depending only on p, A, m and V_k such that

$$\int_M\left|\sqrt{t}\,|\nabla_x p_t^{(k)}(x,y)|\right|^p \mathrm{e}^{\frac{\gamma\rho^2(x,y)}{t}}\mu(\mathrm{d}x) \leqslant \frac{h_2(\gamma)}{\left(\mu(y,\sqrt{t})\right)^{p-1}}, \quad 0<\gamma<1/8.$$

Taking for instance $\gamma = 1/16$, we find constants $c_0, c_1, c_2, c_3 \in (0,\infty)$ depending only on p, A, m and V_k such that

$$\begin{aligned}
&\int_M |\mathrm{II}(x)|^p\mu(\mathrm{d}x)\\
&\leqslant c_1\int_M\left(\int_0^1\left(\int_M|\sqrt{t}\nabla_x p_t^{(k)}(x,y)|^p\mathrm{e}^{\gamma\rho^2(x,y)/t}\mu(y,\sqrt{t})^{p-1}|\alpha(y)|^p\mu(\mathrm{d}y)\right)^{1/p}\frac{\mathrm{e}^{-c_0/t}}{t}\mathrm{d}t\right)^p\mu(\mathrm{d}x)\\
&\leqslant c_2\int_0^1\left(\int_M\left(\mu(y,\sqrt{t})\right)^{p-1}|\alpha(y)|^p\left(\int_M\left|\sqrt{t}\nabla_x p_t^{(k)}(x,y)\right|^p \mathrm{e}^{\gamma\rho^2(x,y)/t}\mu(\mathrm{d}x)\right)\mu(\mathrm{d}y)\right)\mathrm{d}t\\
&\leqslant c_3\int_M|\alpha(y)|^p\mu(\mathrm{d}y).
\end{aligned} \tag{3.16}$$

Next we turn to the estimate of $\mathrm{I}(x)$. According to Lemma 3.7, the balls $\{4B_j\}_{j\in\Lambda}$ form a unity overlap and hence

$$\sum_j \|\rho\chi_j\|_{p/(p-1)}^{p/(p-1)} \lesssim \|\rho\|_{p/(p-1)}^{p/(p-1)}, \quad \rho\in C_0^\infty(M).$$

Combined with assumption (3.12), since $|\alpha|\varphi_j \in C_0^\infty(B(x_j,1))$, we conclude that

$$\left|\int_M \rho(x)|\mathrm{I}(x)|\,\mu(\mathrm{d}x)\right| \leqslant \int_M|\rho(x)|\left|\sum_j \chi_j \widetilde{\mathbf{T}}_{\mu,\sigma}^{(k)}(\alpha\varphi_j)(x)\right|\mu(\mathrm{d}x)$$

$$\lesssim \sum_j \big\| |\alpha|\, \varphi_j \big\|_p \, \|\rho\chi_j\|_{p/(p-1)} \lesssim \|\alpha\|_p \|\rho\|_{p/(p-1)}.$$

This together with (3.14) and (3.16) implies (3.10), and concludes the proof. □

In the sequel, we continue to write $B_j := B(x_j, 1)$ for simplicity. By Lemma 3.8, it suffices to verify (3.12). To this end, we use the local L^p boundedness criterion via maximal functions from [2]. More precisely, we define the *local maximal function* by

$$(\mathscr{M}_{\mathrm{loc}} f)(x) := \sup_{\substack{x \in B \\ r(B) \leqslant 32}} \frac{1}{\mu(B)} \int_B f \,\mathrm{d}\mu, \quad x \in M, \tag{3.17}$$

for any locally integrable function f on M; the supremum is taken over all balls B in M, containing x and of radius at most 32. From (**LD**), it follows that $\mathscr{M}_{\mathrm{loc}}$ is bounded on $L^p(\mu)$ for all $1 < p \leqslant \infty$. For a measurable subset $E \subset M$, the *maximal function relative to* E is defined as

$$(\mathscr{M}_E f)(x) := \sup_{B \text{ ball in } M,\, x \in B} \frac{1}{\mu(B \cap E)} \int_{B \cap E} f \,\mathrm{d}\mu, \quad x \in E, \tag{3.18}$$

for any locally integrable function f on M. If in particular E is a ball of radius r, it is enough to consider balls B with radii not exceeding $2r$. It is also easy to see $\mathscr{M}_E$ is weak type $(1,1)$ and $L^p(\mu)$-bounded for $1 < p \leqslant \infty$ if E satisfies the *relative doubling property*, namely, if there exists a constant C_E (called *relative doubling constant of* E) such that for $x \in E$ and $r > 0$,

$$\mu(B(x, 2r) \cap E) \leqslant C_E\, \mu(B(x, r) \cap E). \tag{3.19}$$

Note that by Lemma 3.7 (iv), for any $j \in \Lambda$, in particular the subsets $4B_j$ satisfy the relative doubling property (3.19) with a relative doubling constant independent of j.

The following lemma will be crucial in the proof of Theorem 2.2. For any $x \in M$, let $\mathscr{B}(x)$ be the class of geodesic balls in M containing x.

Lemma 3.9. *Let $p \in (2, \infty)$ and assume* (**LD**). *Then* (3.12) *holds for some constant $c > 0$ depending only on p, σ, A, m and V_k, provided there exist an integer n and a constant $C > 0$ depending only on p, σ, A, m and V_k such that the following two items hold:*

(i) *the operator*

$$\mathscr{M}^{\#}_{4B_j, \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}, n} \alpha(x) := \sup_{B \in \mathscr{B}(x)} \left(\frac{1}{\mu(B \cap 4B_j)} \int_{B \cap 4B_j} \left| \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma} (I - P^{(k)}_{r^2})^n \alpha(y) \right|^2 \mu(\mathrm{d}y) \right)^{1/2}$$

for $x \in 4B_j$ satisfies

$$\left\| \mathscr{M}^{\#}_{4B_j, \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}, n} \alpha \right\|_{L^p(4B_j, \mu)} \leqslant C \|\alpha\|_{L^p(\Omega^{(k)}(B_j), \mu)}, \quad j \geqslant 1.$$

(ii) *for any $\ell \in \{1, 2, \ldots, n\}$, $j \geqslant 1$, and any $\alpha \in L^p(\Omega^{(k)}(B_j), \mu)$, there exists a sublinear operator S_j bounded from $L^p(\Omega^{(k)}(B_j), \mu)$ to $L^p(4B_j, \mu)$ with*

$$\|S_j\|_{L^p(\Omega^{(k)}(B_j), \mu) \to L^p(4B_j, \mu)} \leqslant C,$$

such that

$$\begin{aligned}\sup_{B\in\mathscr{B}(x)}&\left(\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}\left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(P^{(k)}_{\ell r^2}\alpha)\right|^p \mathrm{d}\mu\right)^{1/p}\\ &\leqslant C\left(\mathscr{M}_{4B_j}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2)+(S_j(\alpha))^2\right)^{1/2}(x),\quad j\geqslant 1,\ x\in 4B_j.\end{aligned}\tag{3.20}$$

Proof. We use [2, Theorem 2.4]: First note that we may take B_j and $4B_j$ for E_1 and E_2 there, respectively, as the sets B_j and $4B_j$ possess the relative volume doubling property (3.19) with relative doubling constants independent of j (see Lemma 3.7). As in [2] consider the operators $\{A_r\}_{r>0}$ given by the relation

$$I - A_r = (I - P^{(k)}_{r^2})^n,\quad r > 0,$$

for some integer n (to be chosen later). Following the proof of [2, Theorem 2.4], replacing $f \in L^p(B_j,\mu)$ by $\alpha \in L^p(\Omega^{(k)}(B_j),\mu)$, we find a constant $C' > 0$ depending only on p,σ,A,m and V_k such that

$$\|\mathscr{M}_{4B_j}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2)^{1/2}\|_{L^p(4B_j)} \leqslant C'\left(\left\|\mathscr{M}^{\sharp}_{4B_j,\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma},n}\alpha\right\|_{L^p(4B_j)} + \|S_j(\alpha)\|_{L^p(4B_j)} + \|\,\alpha\,\|_{L^p(4B_j)}\right).$$

Thus, assuming L^p-boundedness of both $\mathscr{M}^{\#}_{4B_j,\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma},n}$ and S_j, we may conclude that $\mathscr{M}_{4B_j}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2)^{1/2}$ is bounded in $L^p(4B_j,\mu)$ and thus $\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}$ bounded from $L^p(\Omega^{(k)}(B_j),\,\mu)$ to $L^p(\Omega^{(k)}(4B_j),\,\mu)$. □

Hence it suffices to check (i) and (ii) of Lemma 3.9. We establish two technical lemmas which verify (i) and (ii) respectively. To this end, observe that (**LD**) implies: for any $r_0 > 0$ there exists $C_{r_0} > 0$ depending only on A,m and r_0 such that

$$\mu(x,2r) \leqslant C_{r_0}\mu(x,r),\quad r\in(0,r_0),\ x\in M.\tag{3.21}$$

An immediate consequence of (**LD**) is that for all $y \in M$, $0 < r < 8$ and $s \geqslant 1$ satisfying $sr < 32$,

$$\mu(y,sr) \leqslant Cs^m\mu(y,r),\tag{3.22}$$

for some constants C depending only on A.

The following lemma is essential to the proof of part (i) of Lemma 3.9.

Lemma 3.10. *Assume condition* (**A**). *Then there exists an integer n depending only on m and a constant $C > 0$ depending on σ, A, m and V_k, such that*

$$\sup_{B\in\mathscr{B}(x)}\left(\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}\left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I-P^{(k)}_{r^2})^n\alpha(y)\right|^2\,\mu(\mathrm{d}y)\right)^{1/2} \leqslant C\left(\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x)\right)^{1/2}\tag{3.23}$$

holds for any $x\in 4B_j,\ j\geqslant 1$ and $\alpha\in L^2(\Omega^{(k)}(4B_j),\mu)$ where $\mathscr{M}_{\mathrm{loc}}$ is defined by (3.17).

Proof. All constants appearing below depend only on σ, A, m and V_k, and $\xi \lesssim \eta$ for positive variables ξ and η means that $\xi \leqslant \kappa\eta$ holds for such a constant $\kappa > 0$.

Viewing the left-hand side of (3.23) as maximal function relative to $4B_j$, since the radius of $4B_j$ is 4, it is sufficient to consider balls B of radii not exceeding 8. By Lemma 3.7, there exists a constant $c_0 > 0$ depending only on A, m such that

$$\mu(B) \leqslant c_0\mu(B \cap 4B_j), \quad B = B(x_0, r), \ x_0 \in 4B_j, \ r \in (0, 8), \ j \geqslant 1. \tag{3.24}$$

Hence,

$$\begin{aligned}&\frac{1}{\mu(B \cap 4B_j)} \int_{B\cap 4B_j} \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha\right|^2 \mathrm{d}\mu \leqslant \frac{c_0}{\mu(B)} \int_B \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha\right|^2 \mathrm{d}\mu,\\ &j \geqslant 1, \ B = B(x_0, r), \ x_0 \in 4B_j, \ r \in (0, 8).\end{aligned}$$

Thus, we only need to show that

$$\sup_{\substack{B=B(x_0,r)\in\mathscr{B}(x)\\ r<8}} \frac{1}{\mu(B)} \int_B \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha(y)\right|^2 \mu(\mathrm{d}y) \lesssim \mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x), \quad j \geqslant 1, \ x \in 4B_j. \tag{3.25}$$

For any $r \in (0, 8)$, we may choose $i_r \in \mathbb{Z}_+$ satisfying

$$2^{i_r} r \leqslant 8 < 2^{i_r+1} r. \tag{3.26}$$

Denote by

$$\begin{aligned}&\mathcal{D}_i := (2^{i+1}B) \setminus (2^i B) \quad \text{if } i \geqslant 2, \qquad \text{and}\\ &\mathcal{D}_1 = 4B.\end{aligned} \tag{3.27}$$

Using the fact that $\operatorname{supp}\alpha \subset 4B_j \subset 2^i B$ when $i > i_r$, we find that

$$\alpha = \sum_{i=1}^{i_r} \alpha \mathbb{1}_{\mathcal{D}_i} =: \sum_{i=1}^{i_r} \alpha_i$$

which then implies

$$\left\| \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha\right| \right\|_{L^2(B)} \leqslant \sum_{i=1}^{i_r} \left\| \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha_i\right| \right\|_{L^2(B)}. \tag{3.28}$$

For $i = 1$ we use the L^2-boundedness of $\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\left(I - P^{(k)}_{r^2}\right)^n$ to obtain

$$\left\| \left|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(I - P^{(k)}_{r^2})^n\alpha_1\right| \right\|_{L^2(B)} \leqslant \|\alpha\|_{L^2(4B)} \leqslant \mu(4B)^{1/2}(\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x))^{1/2} \tag{3.29}$$

as desired. For $i \geqslant 2$, we infer from (3.11) that

$$\begin{aligned}\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\left(I - P^{(k)}_{r^2}\right)^n \alpha_i &= \int_0^\infty v(t)\nabla\left(P^{(k)}_t(I - P^{(k)}_{r^2})^n\alpha_i\right) \mathrm{d}t\\ &= \int_0^\infty v(t) \sum_{\ell=0}^n \binom{n}{\ell}(-1)^\ell \nabla P^{(k)}_{t+\ell r^2}\alpha_i \,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \left(\sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell \mathbb{1}_{\{t>\ell r^2\}} v(t-\ell r^2) \right) \nabla P_t^{(k)} \alpha_i \, \mathrm{d}t \\
&= \int_0^\infty g_r(t) \, \nabla P_t^{(k)} \alpha_i \, \mathrm{d}t,
\end{aligned}
$$

where

$$
g_r(t) := \sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^\ell \mathbb{1}_{\{t>\ell r^2\}} v(t-\ell r^2).
$$

For g_r, according to the definition $v(t) = w(t)\mathrm{e}^{-\sigma t}/\sqrt{t}$ along with an elementary calculation (see the proof of [2, Lemma 3.1]), we observe that

$$
\begin{cases}
|g_r(t)| \lesssim \frac{1}{\sqrt{t-\ell r^2}}, & \text{for } 0 < \ell r^2 < t \leqslant (1+\ell)r^2 \leqslant (1+n)r^2, \\
|g_r(t)| \lesssim r^{2n} t^{-n-\frac{1}{2}}, & \text{for } (1+nr^2) \wedge (1+n)r^2 < t \leqslant 1+nr^2, \\
g_r(t) = 0, & \text{for } t > 1+nr^2.
\end{cases}
$$

Combined with Lemma 3.6, this gives

$$
\left\| \left| \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma} (I - P^{(k)}_{r^2})^n \alpha_i \right| \right\|_{L^2(B)} \lesssim \left(\int_0^\infty |g_r(t)| \left(1 + \sqrt{t}\right) \mathrm{e}^{-c_2 4^i r^2/t} \frac{\mathrm{d}t}{\sqrt{t}} \right) \|\alpha_i\|_{L^2(\mathcal{D}_i)}
$$

for some constant c_2 from (3.6), where by the fact that $0 < r < 8$, we have

$$
\int_0^\infty \left(1 + \sqrt{t}\right) |g_r(t)| \, \mathrm{e}^{-c_2 4^i r^2/t} \frac{\mathrm{d}t}{\sqrt{t}} \leqslant C_n \int_0^{1+nr^2} |g_r(t)| \, \mathrm{e}^{-c_2 4^i r^2/t} \frac{\mathrm{d}t}{\sqrt{t}} \leqslant C_n' 4^{-in},
$$

for some constant $C_n' > 0$. Now, since $r(2^i B) \leqslant 8$ when $1 \leqslant i \leqslant i_r$, an easy consequence of the local doubling (3.22) is that

$$
\mu(2^{i+1}B) \leqslant C 2^{(i+1)m} \mu(B),
$$

with a constant C independent of B and i. Therefore, as $\mathcal{D}_i \subset 2^{i+1}B$,

$$
\|\alpha_i\|_{L^2(\mathcal{D}_i)} \leqslant \mu(2^{i+1}B)^{1/2} \left(\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x) \right)^{1/2} \leqslant C 2^{im/2} \mu(B)^{1/2} \left(\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x) \right)^{1/2}.
$$

Using the definition of i_r, $r \leqslant 8$, and then choosing $2n > m/2$, we finally obtain

$$
\left\| \left| \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma} (I - P^{(k)}_{r^2})^n \alpha \right| \right\|_{L^2(B)} \leqslant C' \left(\sum_{i=1}^{i_r} 2^{i(m/2-2n)} \right) \mu(B)^{1/2} \left(\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x) \right)^{1/2},
$$

for some constant $C' > 0$, so that (3.25) holds. Then the proof is finished. □

The following lemma is used to prove part (ii) of Lemma 3.9.

Lemma 3.11. *In the situation of Theorem 2.2, let the integer i_r be defined by* (3.26)*, and let $n \in \mathbb{N}$ be as in Lemma* 3.10*. Then there exist constants $c, C > 0$ depending only on p, σ, A, m and V_k, such that for any $i \geqslant 1, \ell \in \{1, \ldots, n\}, r \in (0, 8)$, $B = B(x_0, r) \in \mathscr{B}(x)$, and for $\alpha \in L^2(\Omega^{(k)}, \mu)$ supported in $\mathcal{D}_i$ as in* (3.27)*,*

$$
\left(\frac{1}{\mu(B)} \int_B |\nabla P^{(k)}_{\ell r^2} \alpha|^p \, \mathrm{d}\mu \right)^{1/p} \leqslant \frac{C \mathrm{e}^{C\ell r^2 - c4^i}}{r} \left(\frac{1}{\mu(2^{i+1}B)} \int_{\mathcal{D}_i} |\alpha|^2 \, \mathrm{d}\mu \right)^{1/2}, \tag{3.30}
$$

for $\alpha \in L^2(\Omega^{(k)}, \mu)$ *supported in* $2^{i_r+2}B$,

$$\left(\frac{1}{\mu(B)}\int_B \left|\nabla P^{(k)}_{\ell r^2}(\alpha)\right|^p \mathrm{d}\mu\right)^{1/p} \leqslant C\mathrm{e}^{C\ell r^2}\sum_{i=1}^{i_r+1}\frac{\mathrm{e}^{-c4^i}}{\sqrt{\mu(2^{i+1}B)}}\left[\left(\int_{\mathcal{D}_i}|\nabla\alpha|^2\,\mathrm{d}\mu\right)^{1/2}+\left(\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right)^{1/2}\right]. \quad (3.31)$$

Proof. All constants appearing below depend only on p, A, m and V_k. We first observe from condition (**GE**) that

$$\left(\int_B \left|\nabla P^{(k)}_t\alpha\right|^p\,\mathrm{d}\mu\right)^{1/p} \leqslant \frac{1}{\sqrt{t}}\mathrm{e}^{A+At}\left(\int_B \left(P_t|\alpha|^2\right)^{p/2}(x)\,\mu(\mathrm{d}x)\right)^{1/p}. \quad (3.32)$$

We substitute $t = \ell r^2$ in estimate (3.32) for $\ell \in \{1, 2, \dots, n\}$. As $r \in (0, 8)$, there exists a positive constant $\tilde{C}$ depending on n and A such that

$$\left(\int_B |\nabla P^{(k)}_{\ell r^2}\alpha|^p\,\mathrm{d}\mu\right)^{1/p} \leqslant \frac{\tilde{C}}{r}\left(\int_B \left(P_{\ell r^2}|\alpha|^2\right)^{p/2}(x)\,\mu(\mathrm{d}x)\right)^{1/p}.$$

By the off-diagonal heat kernel upper bound of $p_t(x,y)$, see (3.8), we have

$$p_t(x,y) \leqslant \frac{C\mathrm{e}^{\tilde{\sigma}_2 t}}{\mu(y,\sqrt{t})}\exp\left(-c_0\frac{\rho^2(x,y)}{t}\right), \quad x, y \in M,$$

for some constants $C, \tilde{\sigma}_2 > 0$ and $c_0 \in (0, 1/4)$. As a consequence, since $0 < r < 8$, we obtain for $x \in B$, a positive constant $C > 0$ such that

$$\begin{aligned} P_{\ell r^2}(|\alpha|^2)(x) &\leqslant C\int_{\mathcal{D}_i}\mu\left(y,\ \sqrt{\ell}r\right)^{-1}\exp\left(-c_0\frac{\rho^2(x,y)}{\ell r^2}+\tilde{\sigma}_2\ell r^2\right)|\alpha|^2(y)\,\mu(\mathrm{d}y) \\ &\leqslant C\mathrm{e}^{\tilde{\sigma}_2\ell r^2-c_0 4^i/\ell}\int_{\mathcal{D}_i}\mu\left(y,\ \sqrt{\ell}r\right)^{-1}|\alpha|^2(y)\,\mu(\mathrm{d}y). \end{aligned}$$

Moreover, for $y \in \mathcal{D}_i$, we have $2^{i+1}B \subset B(y, 2^{i+2}r)$, and then by (**LD**), for $\ell \in \{1, 2, \dots, n\}$,

$$\frac{1}{\mu\left(y,\ \sqrt{\ell}r\right)} \leqslant \frac{2^{m(i+2)}\mathrm{e}^{C2^{i+2}}}{\mu(y, 2^{i+2}r)} \leqslant \frac{2^{m(i+2)}\mathrm{e}^{C2^{i+2}}}{\mu(2^{i+1}B)}.$$

It follows that

$$P_{\ell r^2}(|\alpha|^2)(x) \leqslant C\mathrm{e}^{\tilde{\sigma}\ell r^2-c_0 4^i/\ell}\left(\frac{2^{m(i+2)}\mathrm{e}^{C2^{i+2}}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right) \quad (3.33)$$

for all $x \in B$, and there exists $\alpha_1 < c_0/n$ such that for all $\ell \in \{1, 2, \cdots, n\}$.

$$\left(\frac{1}{\mu(B)}\int_B \left(P_{\ell r^2}(|\alpha|^2)\right)^{p/2}\mathrm{d}\mu\right)^{1/p} \leqslant C\mathrm{e}^{\tilde{\sigma}\ell r^2-\alpha_1 4^i}\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right)^{1/2}. \quad (3.34)$$

Combining (3.32) and (3.34), we complete the proof of (3.30).

We next observe that condition (**GE**) yields

$$\left(\int_B |\nabla P^{(k)}_t\alpha|^p\,\mathrm{d}\mu\right)^{1/p} \leqslant \mathrm{e}^{At}\left(\int_B \left(P_t|\nabla\alpha|^2\right)^{p/2}(x)\,\mu(\mathrm{d}x)\right)^{1/p} + At\mathrm{e}^{At}\left(\int_B \left(P_t|\alpha|^2\right)^{p/2}(x)\,\mu(\mathrm{d}x)\right)^{1/p}. \quad (3.35)$$

If α is supported in $2^{i_r+2}B := \cup_{i=1}^{i_r+1}\mathcal{D}_i$, then from (3.33), there exists $\alpha_1 > 0$ such that

$$P_{\ell r^2}(|\alpha|^2)(x) \leqslant C\sum_{i=1}^{i_r+1}\left(\frac{\mathrm{e}^{\tilde{\sigma}\ell r^2-\alpha_1 4^i}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right),$$

which implies

$$\left(\frac{1}{\mu(B)}\int_B\left(P_{\ell r^2}(|\alpha|^2)\right)^{p/2}\mathrm{d}\mu\right)^{1/p} \leqslant C\sum_{i=1}^{i_r+1}\mathrm{e}^{\tilde{\sigma}\ell r^2}\left(\frac{\mathrm{e}^{-2\alpha_1 4^i}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right)^{1/2}.$$

By the same reason, we have

$$\left(\frac{1}{\mu(B)}\int_B\left(P_{\ell r^2}(|\nabla\alpha|^2)\right)^{p/2}\mathrm{d}\mu\right)^{1/p} \leqslant C\sum_{i=1}^{i_r+1}\mathrm{e}^{\tilde{\sigma}\ell r^2}\left(\frac{\mathrm{e}^{-2\alpha_1 4^i}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\nabla\alpha|^2\,\mathrm{d}\mu\right)^{1/2}.$$

Altogether, these estimates yield

$$\left(\frac{1}{\mu(B)}\int_B\left|\nabla P_{\ell r^2}^{(k)}\alpha\right|^p\mathrm{d}\mu\right)^{1/p} \leqslant C'\sum_{i=1}^{i_r+1}\mathrm{e}^{(A+\tilde{\sigma})\ell r^2}\left[\left(\frac{\mathrm{e}^{-2\alpha_1 4^i}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\alpha|^2\,\mathrm{d}\mu\right)^{1/2}+\left(\frac{\mathrm{e}^{-2\alpha_1 4^i}}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\nabla\alpha|^2\,\mathrm{d}\mu\right)^{1/2}\right]$$

which completes the proof of (3.31). □

With the help of the Lemmas 3.9, 3.10 and 3.11, we are now in position to finish the proof of Theorem 2.2.

Proof of Theorem 2.2. For simplicity, denote by C, c positive constants depending only on p, σ, A, m and V_k, which may vary from one term to another.

By Lemma 3.9, we only need to show that under the given assumptions, items (i) and (ii) of Lemma 3.9 hold true. We first verify item (i) of Lemma 3.9. Observe from Lemma 3.10, there exists an integer n and a constant $C > 0$ such that for all $j \geqslant 1$, $\alpha \in L^2(\Omega^{(k)}(B_j),\mu)$ and $x \in 4B_j$,

$$\sup_{B\in\mathscr{B}(x)}\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}\left|\widetilde{\mathbf{T}}_{\mu,\sigma}^{(k)}(I-P_{r^2}^{(k)})^n\alpha\right|^2\mathrm{d}\mu \leqslant C\mathscr{M}_{\mathrm{loc}}(|\alpha|^2)(x).$$

Recall that $\mathscr{M}_{\mathrm{loc}}$ is bounded on $L^p(\mu)$ for $1 < p \leqslant \infty$; thus $\mathscr{M}^{\#}_{4B_j,\widetilde{\mathbf{T}}_{\mu,\sigma}^{(k)},n}$ is bounded from $L^p(\Omega^{(k)}(B_j),\mu)$ to $L^p(4B_j,\mu)$ uniformly in j, i.e., assertion (i) is proved.

Next, we prove (ii) of Lemma 3.9. Assume that $\alpha \in \Omega_0^{(k)}(B_j)$ and let $h = \int_0^\infty v(t)P_t^{(k)}\alpha\,\mathrm{d}t$ with v as in (3.11). Since $\widetilde{\mathbf{T}}_{\mu,\sigma}^{(k)}(\alpha) = \nabla h$ and inequality (3.24) holds for $B\cap 4B_j$, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}\left|\widetilde{\mathbf{T}}_{\mu,\sigma}^{(k)}P_{\ell r^2}^{(k)}\alpha\right|^p\mathrm{d}\mu\right)^{1/p}\\
&\quad=\left(\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}\left|\nabla P_{\ell r^2}^{(k)}h\right|^p\mathrm{d}\mu\right)^{1/p}\\
&\quad\leqslant C\left(\frac{1}{\mu(B)}\int_B\left|\nabla P_{\ell r^2}^{(k)}h\right|^p\mathrm{d}\mu\right)^{1/p}.
\end{aligned}$$

Let φ_0 be a C^∞ function supported in $2^{i_r+2}B$ with $\varphi_0(x)=1$ on $2^{i_r+1}B$ and $|\nabla\varphi_0|\leqslant 1/8$ as $8\leqslant 2^{i_r+1}r\leqslant 16$. We write

$$\nabla P^{(k)}_{\ell r^2}h=\nabla P^{(k)}_{\ell r^2}g_0+\sum_{i=i_r+1}^{\infty}\nabla P^{(k)}_{\ell r^2}g_i,$$

where $g_0=h\varphi_0$ and $g_i=h(1-\varphi_0)\mathbb{1}_{\mathcal{D}_i}$. Next, we distinguish the two cases $i=0$ and $i>i_r$ where i_r is defined in (3.26). For the case $i=0$, since $g_0\in\Omega^{(k)}$ is supported in $2^{i_r+1}B$, by the inequality (3.31) in Lemma 3.11 and the definition of φ_0, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(B)}\int_B\left|\nabla P^{(k)}_{\ell r^2}g_0\right|^p\mathrm{d}\mu\right)^{1/p}\\
&\leqslant C\sum_{i=1}^{i_r+1}\mathrm{e}^{-c4^i}\left(\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\nabla g_0|^2\,\mathrm{d}\mu\right)^{1/2}+\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|g_0|^2\,\mathrm{d}\mu\right)^{1/2}\right)\\
&\leqslant C\sum_{i=1}^{i_r+1}\mathrm{e}^{-c4^i}\left(\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|\nabla h|^2\,\mathrm{d}\mu\right)^{1/2}+\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|h|^2\,\mathrm{d}\mu\right)^{1/2}\right)\\
&\leqslant C\sum_{i=1}^{i_r+1}\mathrm{e}^{-c4^i}\left(\left(\mathscr{M}_{\mathrm{loc}}(|\nabla h|^2)(x)\right)^{1/2}+\left(\mathscr{M}_{\mathrm{loc}}(|h|^2)(x)\right)^{1/2}\right).
\end{aligned}\tag{3.36}$$

For the second regime $i>i_r$, we proceed with inequality (3.30) in Lemma 3.11 such that

$$\left(\frac{1}{\mu(B)}\int_B|\nabla P^{(k)}_{\ell r^2}g_i|^p\,\mathrm{d}\mu\right)^{1/p}\leqslant\frac{C\mathrm{e}^{-c4^i}}{r}\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|h|^2\,\mathrm{d}\mu\right)^{1/2}.\tag{3.37}$$

On the other hand, since $i>i_r$, it is easy to see that $4B_j\subset 2^{i+1}B$, thus

$$\begin{aligned}
\left(\frac{1}{\mu(2^{i+1}B)}\int_{\mathcal{D}_i}|h|^2\,\mathrm{d}\mu\right)^{1/2}&\leqslant\left(\frac{1}{\mu(2^{i+1}B)}\int_0^1 v(t)\int_{\mathcal{D}_i}|P^{(k)}_t\alpha|^2\,\mathrm{d}\mu\,\mathrm{d}t\right)^{1/2}\\
&\leqslant C\left(\frac{1}{\mu(4B_j)}\int_{B_j}|\alpha|^2\,\mathrm{d}\mu\right)^{1/2}\\
&\leqslant C\Big(\mathscr{M}_{4B_j}(|\alpha|^2)(x)\Big)^{1/2}.
\end{aligned}\tag{3.38}$$

Thus the contribution of the terms in the second regime $i>i_r$ is bounded by combining (3.37) and (3.38),

$$\sum_{i>i_r}\left(\frac{1}{\mu(B)}\int_B|\nabla P^{(k)}_{\ell r^2}g_i|^p\,\mathrm{d}\mu\right)^{1/p}\leqslant\sum_{i>i_r}\frac{C\mathrm{e}^{-c4^i}}{r}(\mathscr{M}_{4B_j}(|\alpha|^2)(x))^{1/2}\tag{3.39}$$

and it remains to recall that $1/r\leqslant 2^{i+1}/8$ when $i>i_r$.

We conclude from (3.36) and (3.39) that for any $p>2$ and $\ell\in\{1,2,\dots,n\}$, there exists a constant C independent of j such that

$$\left(\frac{1}{\mu(B\cap 4B_j)}\int_{B\cap 4B_j}|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}P^{(k)}_{\ell r^2}\alpha|^p\,\mathrm{d}\mu\right)^{1/p}\leqslant C\Big(\mathscr{M}_{4B_j}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2)+(S_j\alpha)^2\Big)^{1/2}(x)$$

for all $\alpha \in L^2(\Omega^{(k)}(B_j), \mu)$, all balls B in M and all $x \in B \cap 4B_j$, where the radius r of B is less than 8, and where

$$(S_j\alpha)^2 := \mathscr{M}_{\mathrm{loc}}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2 \mathbb{1}_{M\setminus 4B_j}) + \mathscr{M}_{\mathrm{loc}}(|h|^2)(x) + \mathscr{M}_{4B_j}(|\alpha|^2). \tag{3.40}$$

Our last step is to show that the operator S_j defined in (3.40) is bounded from $L^p(\Omega^{(k)}(B_j), \mu)$ to $L^p(4B_j, \mu)$ for any $p \in (2, \infty)$ with operator norm independent of j. By (3.40), we only need to show that the operators

$$(\mathscr{M}_{\mathrm{loc}}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2 \mathbb{1}_{M\setminus 4B_j}))^{1/2}, \ \ (\mathscr{M}_{\mathrm{loc}}(|h|^2))^{1/2} \ \text{ and } \ (\mathscr{M}_{4B_j}(|\alpha|^2))^{1/2}$$

respectively are bounded from $L^p(B_j)$ to $L^p(4B_j)$. Indeed, for any $\alpha \in L^p(4B_j)$, by Lemma 3.7 we know that $4B_j$ satisfies the doubling property (**LD**), which for $p > 2$ implies that $(\mathscr{M}_{4B_j}(|\alpha|^2))^{1/2}$ is bounded from $L^p(B_j)$ to $L^p(4B_j)$ by a constant depending only on the doubling property (**LD**). On the other hand, using the local estimate of $p_t^{(k)}(x, y)$ ([14]), we see that

$$|p_t^{(k)}(x, y)| \leqslant \frac{C}{\mu(x, \sqrt{t})} \mathrm{e}^{-\frac{\gamma\rho(x,y)^2}{t}}, \quad t \in (0, 1], \ \gamma < 1/4,$$

which together with (3.4) and Cauchy's inequality implies

$$\left\| P_t^{(k)}\alpha \right\|_p \leqslant C\|\alpha\|_p, \quad t \in (0, 1].$$

This, together with (**LD**) and the $L^{p/2}$-boundedness of $\mathscr{M}_{\mathrm{loc}}(\cdot)$, further implies

$$\left\| \left(\mathscr{M}_{\mathrm{loc}}(|h|^2)\right)^{1/2} \right\|_p \leqslant C \left\| \int_0^1 v(t) P_t^{(k)}\alpha \,\mathrm{d}t \right\|_p \leqslant C \left(\int_0^1 \frac{w(t)\,\mathrm{e}^{-\sigma t}}{\sqrt{t}} \,\mathrm{d}t \right) \|\alpha\|_{L^p(B_j)} \leqslant C\|\alpha\|_{L^p(B_j)},$$

for $p > 2$ and $\sigma > 0$. Finally, the L^p-boundedness of

$$(\mathscr{M}_{\mathrm{loc}}(|\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha|^2 \mathbb{1}_{M\setminus 4B_j}))^{1/2}$$

follows from the $L^{p/2}$-boundedness of $\mathscr{M}_{\mathrm{loc}}(\cdot)$ and an argument similar to the L^p boundedness of II in (3.16) since $\alpha \in \Omega^{(k)}(B_j)$ and

$$\mathbb{1}_{M\setminus 4B_j} \widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}\alpha = (1 - \chi_j)\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}(\alpha\varphi_j).$$

This implies that the operator S_j is bounded from $L^p(B_j)$ to $L^p(4B_j)$ with an upper bound independent of j.

We infer that the requirements (i) and (ii) in Lemma 3.9 both hold true under the assumptions (**LD**), (**UE**) and (**GE**). Thus, the operator $\widetilde{\mathbf{T}}^{(k)}_{\mu,\sigma}$ is bounded from $L^p(\Omega^{(k)}(B_j), \mu)$ to $L^p(\Omega^{(k)}(4B_j), \mu)$ for $p > 2$ with a constant independent of j. Therefore, by Lemma 3.8, the operator $\mathbf{T}^{(k)}_{\mu,\sigma}$ is strong type (p, p) for $p > 2$. This concludes the proof of Theorem 2.2. □

4 L^p-boundedness under curvature conditions

4.1 Proof of Theorem 2.3

By Theorem 2.2, it suffices to verify conditions (**LD**), (**UE**) and (**GE**) by using (**C**). By the Laplacian comparison theorem presented in [25] and Lemmas 2.1-2.2 in [16], (**LD**) follows from the curvature-dimension condition (2.2). Moreover, according to [16], (**UE**) is a consequence of (2.2) as well. Thus, it remains to prove (**GE**), which is Proposition 4.1 below.

4.2 Derivative formulas

Let $X_t(x)$ be diffusion process on M generated by $L := \Delta + \nabla h$ with a fixed initial value $x \in M$, and let $u_t(x)$ be the horizontal lift of $X_t(x)$ to $O(M)$, such that

$$\mathrm{d}X_t(x) = \nabla h(X_t(x))\,\mathrm{d}t + \sqrt{2}\,u_t(x) \circ \mathrm{d}B_t, \quad t \geqslant 0,\ X_0(x) = x,$$

where B_t is an m-dimensional Brownian motion on $\mathbb{R}^m$. Then the associated stochastic parallel displacement is defined as

$$//_{t,x} := u_t(x)\,u_0(x)^{-1} \colon T_xM \to T_{X_t(x)}M,$$

where as usual orthonormal frames u at a point x are read as isometries $u\colon \mathbb{R}^m \to T_xM$. We are now in position to introduce the derivative formula for $P_t^{(k)}$. To this end, let

$$\tilde{\mathscr{R}}_h^{(k+1)} = (\mathrm{Ric} - \mathrm{Hess}\ h)^{\mathrm{tr}} \otimes 1_{\Omega^{(k)}} - 2R^{(k)}\bullet + 1_{T^*M} \otimes \mathscr{R}_h^{(k)} \in \mathrm{End}\left(T^*M \otimes \Omega^{(k)}\right),$$

where $(\mathrm{Ric} - \mathrm{Hess}\,h)^{\mathrm{tr}}$ is the transpose of the Bakry-Émery Ricci curvature tensor $\mathrm{Ric} - \mathrm{Hess}\ h \in \Gamma(\mathrm{End}\,TM)$. Let $Q_t \in \mathrm{End}(\Omega_x^{(k)})$ and $\tilde{Q}_t \in \mathrm{End}(T_x^*M \otimes \Omega_x^{(k)})$ denote the solutions to the ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Q_t &= -\frac{1}{2} Q_t (\mathscr{R}_h^{(k)})_{//_{t,x}}, \qquad t \geqslant 0, \quad Q_0 = \mathrm{id}_{\Omega_x^{(k)}},\\
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{Q}_t &= -\frac{1}{2} \tilde{Q}_t (\tilde{\mathscr{R}}_h^{(k+1)})_{//_{t,x}}, \quad t \geqslant 0, \quad \tilde{Q}_0 = \mathrm{id}_{T_x^*M \otimes \Omega_x^{(k)}},
\end{aligned}$$

where

$$(\mathscr{R}_h^{(k)})_{//_{t,x}} = //_{t,x}^{-1} \circ \mathscr{R}_h^{(k)} \circ //_{t,x}, \quad \text{and} \quad (\tilde{\mathscr{R}}_h^{(k+1)})_{//_{t,x}} = //_{t,x}^{-1} \circ \tilde{\mathscr{R}}_h^{(k+1)} \circ //_{t,x}.$$

Let $\mathscr{Q}_\bullet$ and $\tilde{\mathscr{Q}}_\bullet$ be the transpose of $Q_\bullet$ and $\tilde{Q}_\bullet$ respectively.

Moreover, we have the commutation relation (see [15, Proposition 2.15])

$$\nabla \Delta_\mu^{(k)} = \tilde{\Delta}_\mu^{(k+1)} \nabla - H^{(k)},$$

where $\tilde{\Delta}_\mu^{(k+1)} := \tilde{\square}_\mu - \tilde{\mathscr{R}}_h^{(k+1)}$ with $\tilde{\square}_\mu$ the Bochner Laplacian on $T^*M \otimes \Omega^{(k)}$ with respect to the induced connection on $T^*M \otimes \Omega^{(k)}$ and

$$H^{(k)} := \nabla \cdot R^{(k)} + R^{(k)}(\nabla h) + \nabla \mathscr{R}_h^{(k)} \in \Gamma(\mathrm{Hom}(\Omega^{(k)},\ T^*M \otimes \Omega^{(k)}).$$

Let $H^{(k),\mathrm{tr}}$ be the transpose of the tensor $H^{(k)}$. Finally let

$$\tilde{P}_t^{(k+1)} := \mathrm{e}^{-t\tilde{\Delta}_\mu^{(k+1)}}, \quad t \geqslant 0.$$

For $\eta \in \Omega^{(k)}$, we define $\nabla\eta \in T^*M \otimes \Omega^{(k)}$ by letting

$$\nabla\eta(v) := \nabla_v \eta, \quad v \in TM.$$

We have the following result.

Proposition 4.1. *Assume condition* **(C)** *holds for some $k \in \mathbb{N}^+$. Then for any bounded $\eta \in \Omega_{b,1}^{(k)}$, there exists a constant $A > 0$ such that for any $t > 0$,*

$$|\nabla P_t^{(k)} \eta| \leqslant \mathrm{e}^{At} \min\left\{\left(t^{-1/2} + At\right)\left(P_t|\eta|^2\right)^{1/2},\ \left(P_t|\nabla\eta| + AtP_t|\eta|\right)\right\}. \tag{4.1}$$

Proof. Consider for $s \in [0,t]$:

$$\begin{aligned} N_s &:= Q_s //_{s,x}^{-1} P_{t-s}^{(k)} \eta(X_s(x)), \\ \tilde N_s &:= \tilde Q_s //_{s,x}^{-1} \nabla P_{t-s}^{(k)} \eta\,(X_s(x))\,. \end{aligned}$$

The crucial observation [15, Theorem 3.7] is that

$$Z_s^{(k)} := \langle \tilde N_s, \xi_s\rangle - \langle N_s, U_s^{(k)}\rangle \tag{4.2}$$

is a local martingale where

$$U_s^{(k)} := \int_0^s \mathscr{Q}_r^{-1}\tilde{\mathscr{Q}}_r \dot\xi_r \,\mathrm{d}B_r + \int_0^s \mathscr{Q}_r^{-1} H_{//_{r,x}}^{(k),\mathrm{tr}} \tilde{\mathscr{Q}}_r \xi_r \,\mathrm{d}s$$

and where ξ_s may be any adapted process with absolutely continuous paths, taking values in $T_x^*M \otimes \Omega_x^{(k)}$. For simplicity, in the sequel, we always take $\xi_s = \ell_s \xi$ for some fixed vector $\xi \in T_x^*M \otimes \Omega_x^{(k)}$ and ℓ_s real-valued with absolutely continuous paths. This leads to the local martingale

$$\begin{aligned} Z_s^{(k)} &:= \ell_s \langle \tilde Q_s //_{s,\,x}^{-1} \nabla P_{t-s}^{(k)} \eta(X_s(x)), \xi\rangle \\ &\quad - \left\langle //_{s,\,x}^{-1} P_{t-s}^{(k)} \eta(X_s(x)), \mathscr{Q}_s \int_0^s \dot\ell_r \mathscr{Q}_r^{-1}\tilde{\mathscr{Q}}_r \xi \,\mathrm{d}B_r + \mathscr{Q}_s \int_0^s \ell_r \mathscr{Q}_r^{-1} H_{//_{r,x}}^{(k),\mathrm{tr}} \tilde{\mathscr{Q}}_r \xi \,\mathrm{d}r \right\rangle. \end{aligned} \tag{4.3}$$

When exploiting the martingale property of (4.3), there are different strategies for the choice of ℓ_s leading to different types of stochastic formulas for the covariant derivative $\nabla P_t^{(k)} \eta$.

(a) (*First upper bound in* (4.1)) If ℓ is a bounded adapted process with paths in the Cameron-Martin space $L^2([0,t];[0,1])$ such that $\ell(0) = 1$ and $\ell(r) = 0$ for $r \geqslant \tau \wedge t$, where $\tau = \tau_D(x)$ is the first exit time of $X_s(x)$ from some relatively compact neighborhood D of x, then trivially the local martingale (4.3) is a true martingale and by taking expectations (see [15, Section 4]) the local covariant Bismut formula holds,

$$\begin{aligned} &\left\langle \nabla P_t^{(k)} \eta, \xi \right\rangle (x) \\ &= -\mathbb{E}\left[\left\langle //_{t\wedge\tau,\,x}^{-1} P_{t-t\wedge\tau}^{(k)} \eta(X_{t\wedge\tau}(x)), \mathscr{Q}_{t\wedge\tau} \int_0^{t\wedge\tau} \dot\ell_s \mathscr{Q}_s^{-1}\tilde{\mathscr{Q}}_s \xi \,\mathrm{d}B_s + \mathscr{Q}_{t\wedge\tau} \int_0^{t\wedge\tau} \ell_s \mathscr{Q}_s^{-1} H_{//_{s,x}}^{(k),\mathrm{tr}} \tilde{\mathscr{Q}}_s \xi \,\mathrm{d}s \right\rangle\right]. \end{aligned} \tag{4.4}$$

Under the condition **(C)**, one observes that $H^{(k)}$, $\mathscr{R}_h^{(k)}$, and $\mathscr{R}_h^{(k+1)}$ are all bounded, and one easily derives the estimate

$$|\nabla P_t^{(k)} \eta|(x) \leqslant \mathrm{e}^{At} (P_t|\eta|^2)^{1/2} \left[\left(\mathbb{E}\int_0^{t\wedge\tau} |\dot\ell_s|^2 \,\mathrm{d}s\right)^{1/2} + At\right].$$

To make this estimate more explicit, we choose a geodesic ball D of radius δ_x centered at x. It has been shown in [30] that there exists a constant $c(f) := \sup_D \left\{-2fLf + 3|\nabla f|^2\right\} < \infty$ such that

$$\mathbb{E}\left(\int_0^{t\wedge\tau} |\dot\ell_s|^2 \,\mathrm{d}s\right) \leqslant \frac{c(f)}{1 - \mathrm{e}^{-c(f)t}},$$

where $f \in C^2(D)$ such that $f(x) = 1$ and $f|_{\partial D} = 0$. Specifically we may take

$$f(p) = \cos\left(\frac{\pi\rho(x,\, p)}{2\delta_x}\right).$$

Then using the comparison theorem in [16, Theorem 1], it is easy to see that there exist positive constants $c_1(K,N)$ and $c_2(N)$ such that

$$c(f) \leqslant \frac{c_1(K,N)}{\delta_x} + \frac{c_2(N)}{\delta_x^2}.$$

Letting δ_x tend to ∞, we prove that

$$\left|\nabla P_t^{(k)}\eta\right| \leqslant \mathrm{e}^{At}\Big(t^{-1/2} + At\Big)(P_t|\eta|^2)^{1/2}. \tag{4.5}$$

(b) (*Second upper bound in* (4.1)) We first prove the remaining claim of Proposition 4.1 for compactly supported η, i.e., for $\eta \in \Omega_0^{(k)}$. To this end, we establish an estimate for $|\nabla P_t^{(k)}\eta|$ which is uniform in the time variable for small values of t. For $\eta \in \Omega_0^{(k)}$, the Kolmogorov equation gives

$$P_t^{(k)}\eta = \eta - \int_0^t P_s^{(k)}\Delta_\mu^{(k)}\eta\,\mathrm{d}s,$$

which by (4.5) implies

$$\begin{aligned}
|\nabla P_t^{(k)}\eta| &\leqslant |\nabla\eta| + \int_0^t |\nabla P_s^{(k)}\Delta_\mu^{(k)}\eta|\,\mathrm{d}s\\
&\leqslant |\nabla\eta| + c\int_0^t \mathrm{e}^{As}s^{-1/2}\left(P_s|\Delta_\mu^{(k)}\eta|^2\right)^{1/2}\,\mathrm{d}s\\
&\lesssim \|\nabla\eta\|_\infty + \sqrt{t}\mathrm{e}^{At}\|\Delta_\mu^{(k)}\eta\|_\infty.
\end{aligned} \tag{4.6}$$

Hence, $\sup_{s\in[0,t]}|\nabla P_s^{(k)}\eta| < \infty$. Also note that there exists $A > 0$ such that

$$\sup_{s\in[0,t]}\left|\tilde{Q}_s /\!/_{s,x}^{-1}\nabla P_{t-s}^{(k)}\eta(X_s(x))\right| \leqslant \mathrm{e}^{A+At}\left(\|\nabla\eta\|_\infty + \|\Delta_\mu^{(k)}\eta\|_\infty\right) < \infty$$

for all $\eta \in \Omega_0^{(k)}$. As a consequence of these bounds, we conclude that the local martingale (4.3) is a true martingale for the constant function $\ell_s \equiv 1$ as well. Taking expectations at the endpoints 0 and t, we derive the following global Bismut formula, i.e.,

$$\left\langle\nabla P_t^{(k)}\eta,\xi\right\rangle(x) = -\mathbb{E}\left\langle /\!/_{t,x}^{-1}\nabla\eta\left(X_t(x)\right), \tilde{\mathscr{Q}}_t\xi\right\rangle - \mathbb{E}\left[\left\langle /\!/_{t,x}^{-1}\eta(X_t(x)), \mathscr{Q}_t\int_0^t \mathscr{Q}_s^{-1}H_{/\!/_{s,x}}^{(k),\mathrm{tr}}\tilde{\mathscr{Q}}_s\xi\,\mathrm{d}s\right\rangle\right], \tag{4.7}$$

holds for $\eta \in \Omega_0^{(k)}$. Note that under condition **(C)**, it follows from (4.7) that there exists a constant $A > 0$ such that

$$|\nabla P_t^{(k)}\eta| \leqslant \mathrm{e}^{At}\left(P_t|\nabla\eta| + AtP_t|\eta|\right), \quad \eta \in \Omega_0^{(k)}. \tag{4.8}$$

It remains to show that estimate (4.8) extends from $\Omega_0^{(k)}$ to $\Omega_{b,1}^{(k)}$. This can be done by a standard approximation argument. As M is geodesically complete, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of first order cut-off functions (e.g. [18, Theorem III.3 a)]) with the properties

(i) $0 \leqslant \varphi_n \leqslant 1$ for all $n \in \mathbb{N}_+$;
(ii) for each compact $K \subset M$ there is $n_0(K) \in \mathbb{N}_+$ such that $\varphi_n|K \equiv 1$ for all $n \geqslant n_0(K)$;
(iii) $\|\nabla\varphi_n\|_\infty \to 0$ as $n \to \infty$.

We replace η by $\eta_n := \varphi_n\eta$ and then pass to the limit in the estimate as $n \to \infty$. From the local Bismut formula (4.4) it is then easy to see that $\nabla P_t^{(k)}\eta_n \to \nabla P_t^{(k)}\eta$ as $n \to \infty$. For the right-hand-side, we trivially have $P_t|\nabla\eta_n| + AtP_t|\eta_n| \to P_t|\nabla\eta| + AtP_t|\eta|$, as $n \to \infty$. □

Remark 4.2. Since the estimates (4.6) are uniform on compact time intervals, it also follows that (4.3) is a true martingale for any $\eta \in \Omega_{b,1}^{(k)}$ and $\ell \in C^1([0,t])$, establishing the following global version of Bismut's formula:

$$\left\langle \nabla P_t^{(k)}\eta, \xi \right\rangle(x) = -\mathbb{E}\left[\left\langle //_{t,x}^{-1}\eta(X_t(x)), \mathscr{Q}_t\int_0^t \dot{\ell}_s \mathscr{Q}_s^{-1}\tilde{\mathscr{Q}}_s\xi \,\mathrm{d}B_s + \mathscr{Q}_t\int_0^t \ell_s\mathscr{Q}_s^{-1}H^{(k),\mathrm{tr}}_{//_{s,x}}\tilde{\mathscr{Q}}_s\xi\,\mathrm{d}s\right\rangle\right]. \tag{4.9}$$

for a general deterministic $\ell \in C^1([0,t])$ with $\ell_t = 0$ and $\ell_0 = 1$ as well. A standard choice for ℓ_s is $\ell_s := (t-s)/t$, so that $\dot{\ell}_s = -1/t$.

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