UNIVERSITÉ DU LUXEMBOURG

Little Bernstein Theorem: New Perspectives

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Some context

Bohr-Mollerup Theorem (1922)

The log-gamma function $f = \ln \circ \Gamma$ is the unique convex solution vanishing at $x = 1$ to the equation

$$f(x+1) - f(x) = \ln(x) \quad (x > 0).$$

This result can actually be slightly generalized as follows: *All eventually convex solutions to the equation* $\Delta f(x) = \ln x$ *on* $\mathbb{R}_+$ *are of the form* $f(x) = c + \ln \Gamma(x)$*, where* $c \in \mathbb{R}$*.*

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Theorem (Krull (1948)-Webster (1997))

For any eventually concave function $g : \mathbb{R}_+ \to \mathbb{R}$ having the asymptotic property that the sequence $n \mapsto \Delta g(n)$ converges to zero, there exists exactly one (up to an additive constant) eventually convex solution $f : \mathbb{R}_+ \to \mathbb{R}$ to the equation $\Delta f = g$.

Completely monotone functions

A real-valued function $f\colon\ I \to \mathbb{R}$, defined on an open interval I, is called *completely monotone* if it is infinitely differentiable and satisfies

$$(-1)^n f^{(n)}(x)\ \geq\ 0 \quad \text{for all } x \in I \text{ and all } n \in \mathbb{N}.$$

Bernstein's little Theorem (1928)

Any completely monotone function $f\colon\ I \to \mathbb{R}$, defined on an open interval I, is real analytic. That is, for any point $a \in I$, there exists an open neighborhood $U \subset I$ of a such that

$$f(x)\ =\ \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}\,(x-a)^k \qquad (x \in U),$$

which means that f coincides with its Taylor series expansion about a.

Completely monotone functions

Bernstein's Theorem on monotone functions (1928)

A function $f: \mathbb{R}_+ \to \mathbb{R}$ is completely monotone if and only if it is representable as a Laplace type integral of the form

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t) \qquad (x > 0),$$

where μ is the Lebesgue-Stieltjes measure induced by an increasing function from the interval $[0, \infty)$ into itself.

A real-valued function $f: I \to \mathbb{R}$, defined on an open interval I, is called *absolutely monotone* if it is infinitely differentiable and satisfies

$$f^{(n)}(x) \geq 0 \quad \text{for all } x \in I \text{ and all } n \in \mathbb{N}.$$

Remark that f is completely monotone on I if and only if the function $x \mapsto f(-x)$ is absolutely monotone on the reflected interval $-I = \{-x : x \in I\}$.

Bernstein's little Theorem

Bernstein's little Theorem

Let I be a real open interval. Suppose that a function $f : I \to \mathbb{R}$ is infinitely differentiable and that $f^{(q)}$ is absolutely monotone for some $q \in \mathbb{N}$. Then f is real analytic on I.

Proof.

For any $a, x \in I$ and any $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, with $n \geq q$,

$$f(x) - \sum_{k=0}^{q-1} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=q}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x),$$

where

$$R_n(x) = \int_a^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}\,dt = \frac{(x-a)^n}{(n-1)!}\int_0^1 f^{(n)}(a+s(x-a))\,(1-s)^{n-1}\,ds.$$

Now, fix $a \in I$ and let us prove that f is real analytic at a. To this extent, we fix $\varepsilon > 0$ and $x \in I$ such that $x \in (a-\varepsilon, a+\varepsilon) \subset I$. Let also $b \in I$ with $x < b$ and $|x-a| < b-a$. Since $f^{(n+1)} \geq 0$, it follows that $f^{(n)}$ is increasing on I.

Continuation of the proof.

Therefore, we obtain the estimate

$$|R_n(x)| \le \frac{|x-a|^n}{(n-1)!} \int_0^1 f^{(n)}\big(a+s(b-a)\big)\,(1-s)^{n-1}\,ds,$$

or equivalently,

$$|R_n(x)| \le \left|\frac{x-a}{b-a}\right|^n R_n(b).$$

We obtain:

$$f(b) - \sum_{k=0}^{q-1} \frac{f^{(k)}(a)}{k!}\,(b-a)^k - R_n(b) \ge 0,$$

and hence

$$0 \le |R_n(x)| \le \left|\frac{x-a}{b-a}\right|^n \left(f(b) - \sum_{k=0}^{q-1} \frac{f^{(k)}(a)}{k!}\,(b-a)^k\right).$$

This shows that the sequence $n \mapsto R_n(x)$ converges pointwise to zero for any $x \in (a-\varepsilon, a+\varepsilon)$, and hence the function f is real analytic at a. Since a was chosen arbitrary in I, we conclude that f is real analytic on the entire interval I. □

Newton Series Representation

We say that a function $f: I \to \mathbb{R}$, defined on a right-unbounded open interval I, *admits a Newton series expansion on I at a point $a \in I$* if the following identity holds:

$$f(x) = \sum_{k=0}^{\infty} \binom{x-a}{k} \Delta^k f(a) \qquad (x \in I).$$

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- When $x - a$ is a nonnegative integer, the identity always holds, and the series clearly reduces to a finite sum.
- when $x - a$ is not a nonnegative integer, the identity may fail for an arbitrary function $f : I \to \mathbb{R}$, even if f is real analytic: Take for instance $f : x > 0 \mapsto \sin(\pi x)$ and $a = 1$.

About Newton Series

Proposition

Let $f : I \to \mathbb{R}$ be a function defined on a real right-unbounded open interval. Let also $a, x \in I$, and let $n \in \mathbb{N}$. If $x \in \{a, a+1, \ldots, a+n-1\}$, then

$$f(x) - \sum_{k=0}^{n-1} \binom{x-a}{k} \Delta^k f(a) \;=\; 0.$$

If $x \notin \{a, a+1, \ldots, a+n-1\}$ or if f is differentiable at x, then

$$f(x) - \sum_{k=0}^{n-1} \binom{x-a}{k} \Delta^k f(a) \;=\; (x-a)^{\underline{n}} f[a, a+1, \ldots, a+n-1, x].$$

Proof.

$$(x-a)^{\underline{n+1}} f[a, a+1, \ldots, a+n, x] = (x-a)^{\underline{n}} f[a, a+1, \ldots, a+n-1, x] - (x-a)^{\underline{n}} f[a, a+1, \ldots, a+n].$$

□

p-convexity

Definition

Let I be any real interval and let $p \geq -1$ be an integer. A function $f\colon\ I \to \mathbb{R}$ is said to be *p-convex* (resp. *p-concave*) if for any system $x_0 < x_1 < \cdots < x_{p+1}$ of $p+2$ points in I the following inequality holds:

$$f[x_0, x_1, \ldots, x_{p+1}] \ \geq\ 0 \qquad (\text{resp. } f[x_0, x_1, \ldots, x_{p+1}] \ \leq\ 0).$$

We denote by $\mathcal{K}^p_1(I)$ (resp. $\mathcal{K}^p_{-1}(I)$) the set of functions $f\colon\ I \to \mathbb{R}$ that are p-convex (resp. p-concave), and we introduce the notation

$$\mathcal{K}^p(I) \ =\ \mathcal{K}^p_1(I) \cup \mathcal{K}^p_{-1}(I).$$

- $\mathcal{K}^{p+1}(I) \ \subset\ \mathcal{C}^p(I) \qquad (p \in \mathbb{N})$,
- If f lies in $\mathcal{K}^p_1(I)$, where I is right-unbounded, then Δf lies in $\mathcal{K}^{p-1}_1(I)$,
- If $f\colon\ I \to \mathbb{R}$ is differentiable, then f lies in $\mathcal{K}^p_1(I)$ if and only if f' lies in $\mathcal{K}^{p-1}_1(I)$,
- If f lies in $\mathcal{C}^p(I) \cap \mathcal{K}^p_1(I)$, then the map

  $$(z_0, z_1, \ldots, z_p) \ \mapsto\ f[z_0, z_1 \ldots, z_p]$$

  from I^{p+1} to $\mathbb{R}$ is continuous and increasing in each place.

p-convexity

A function $f\colon\ I \to \mathbb{R}$ is completely monotone if and only if

$$f \ \in\ \mathcal{C}^{\infty}(I) \quad \text{and} \quad (-1)^n f^{(n)} \ \in\ \mathcal{K}_1^{-1}(I) \quad \text{for all } n \in \mathbb{N}.$$

This latter condition exactly means that

$$f \ \in\ \mathcal{K}_{(-1)^n}^{n-1}(I) \quad \text{for all } n \in \mathbb{N}.$$

Similarly, the function $f\colon\ I \to \mathbb{R}$ is absolutely monotone if and only if

$$f \ \in\ \mathcal{K}_1^{n-1}(I) \quad \text{for all } n \in \mathbb{N},$$

and it is regularly monotone if and only if

$$f \ \in\ \mathcal{K}^{n-1}(I) \quad \text{for all } n \in \mathbb{N}.$$

Newton Series Representation for CM functions

Theorem (L., Marichal, Zenaïdi)

Let I be a real right-unbounded open interval. Suppose that a function $f\colon\ I \to \mathbb{R}$ is infinitely differentiable and that $f^{(q)}$ is completely monotone for some $q \in \mathbb{N}$. Then, for any $a \in I$, the function f admits the Newton series expansion:

$$f(x) \ = \ \sum_{k=0}^{\infty} \binom{x-a}{k} \Delta^k f(a) \qquad (x \in I)$$

and the convergence of the series is uniform on compact subsets of I.

Proof.

Let $a, x \in I$ and let $n \in \mathbb{N}^*$, with $n \geq q$,

$$f(x) - \sum_{k=0}^{q-1} \binom{x-a}{k} \Delta^k f(a) \ = \ \sum_{k=q}^{n-1} \binom{x-a}{k} \Delta^k f(a) + R_n(x), \tag{1}$$

where $R_n(x) \ = \ (x-a)^{\underline{n}} f[a, a+1, \ldots, a+n-1, x]$.

Continuation of the proof.

Let also $b \in I$, with $b < \min\{a, x\}$. Using the latter observation, we immediately derive the following inequalities:

$$\begin{aligned} 0 \leq |R_n(x)| &= \left|(x-a)^{\underline{n}}\right| \left|f[a, a+1, \ldots, a+n-1, x]\right| \\ &\leq \left|(x-a)^{\underline{n}}\right| \left|f[a, a+1, \ldots, a+n-1, b]\right| \\ &= \left|\frac{(x-a)^{\underline{n}}}{(b-a)^{\underline{n}}}\right| |R_n(b)|, \end{aligned} \tag{2}$$

and

$$0 \leq |R_n(x)| \leq \left|\frac{(x-a)^{\underline{n}}}{(b-a)^{\underline{n}}}\right| \left|f(b) - \sum_{k=0}^{q-1} \binom{b-a}{k} \Delta^k f(a)\right|.$$

□

Examples

- We can readily see that the restriction of the reciprocal function $f(x) = 1/x$ to $\mathbb{R}_+$ is completely monotone. It follows that this function has a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$. For any $k \in \mathbb{N}$ and any $a > 0$,

$$\Delta^k f(a) = (-1)^k \frac{k!}{a(a+1)\cdots(a+k)} = \frac{(-1)^k}{a\binom{a+k}{k}}.$$

Hence, for any $a > 0$, the reciprocal function admits the Newton series expansion:

$$\boxed{\frac{1}{x} = \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k \frac{\binom{x-a}{k}}{\binom{a+k}{k}}} \qquad (x > 0).$$

Examples

- The classical *binomial theorem* states that the identity

$$(c+1)^x \;=\; \sum_{k=0}^{\infty}\binom{x}{k}\,c^k \qquad (x\in\mathbb{R}),$$

  or equivalently,

$$(c+1)^x \;=\; (c+1)^a\sum_{k=0}^{\infty}\binom{x-a}{k}\,c^k \qquad (a,x\in\mathbb{R}),$$

  holds for $-1<c<1$. When $c>1$, the latter series diverges by the ratio test (unless $x-a$ is a nonnegative integer). This result shows that the exponential function

$$f(x) \;=\; (c+1)^x \qquad (c>-1,\; x\in\mathbb{R})$$

  admits a Newton series expansion on $\mathbb{R}$ at every $a\in\mathbb{R}$ when $-1<c<1$, but not when $c>1$. Moreover, we can easily see that this function is completely monotone if $-1<c\leq 0$, and absolutely monotone if $c\geq 0$.

Examples

Consider functions f_1, f_2, f_3 defined by $f_1(x) = e^{-x}$, $f_2(x) = (e/2)^x$ and $f_3(x) = e^x$.

- The completely monotone function f_1 admits a Newton series expansion on $\mathbb{R}$ at every point $a \in \mathbb{R}$, as does the absolutely monotone function f_2.
- The absolutely monotone function f_3 does not admit such expansions.

The function f_2 demonstrates that a function admitting a Newton series expansion at every point $a \in \mathbb{R}$ need not be completely monotone. The function f_3 illustrates that an absolutely monotone function need not admit a Newton series expansion. Together, f_1 and f_3 illustrate that, although the map $x \mapsto -x$ preserves real analyticity, it does not, in general, preserve the existence of Newton series expansions.

Applications to Principal Indefinite Sums

For any $p \in \mathbb{N}$, we also let $\mathcal{D}^p$ denote the set of functions $g\colon\ \mathbb{R}_+ \to \mathbb{R}$ such that the sequence $n \mapsto \Delta^p g(n)$ converges to zero.

Theorem (Marichal, Zenaïdi (2022))

If g lies in $\mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$, then there exists a unique solution $f \in \mathcal{K}^p$, satisfying $f(1) = 0$, to the difference equation $\Delta f = g$ on $\mathbb{R}_+$ given by

$$f(x) = \Sigma g(x) = \lim_{n\to\infty} \sum_{k=1}^{n-1} g(k) - \sum_{k=0}^{n-1} g(x+k) + \sum_{j=1}^{p} \binom{x}{j} \Delta^{j-1} g(n) \qquad (x > 0), \tag{3}$$

where and f is p-convex (resp. p-concave) on any right-unbounded subinterval of $\mathbb{R}_+$ on which g is p-concave (resp. p-convex). Furthermore, the convergence in (3) is uniform on any bounded subset of $\mathbb{R}_+$.

If $g(x) = \ln(x)$ and $p = 1$, $\Sigma \ln x \ = \ \ln \Gamma(x)$ and it provides the additive form of Gauss' well-known limit:

$$\ln \Gamma(x) \ = \ \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} \ln k - \sum_{k=0}^{n-1} \ln(x+k) + x \ln n \right) \qquad (x > 0).$$

Applications to Principal Indefinite Sums

Proposition (L., Marichal, Zenaïdi)

Let $p, q \in \mathbb{N}$ with $p < q$, and let $g\colon \mathbb{R}_+ \to \mathbb{R}$ be an infinitely differentiable function. Suppose that $g^{(p)}$ eventually tends monotonically to zero, and that $g^{(q)}$ is regularly monotone. Then, the following assertions hold:

(a) $g^{(n)}$ eventually tends monotonically to zero for every $n \geq p$.

(b) $g^{(q-1)}$ or $-g^{(q-1)}$ is completely monotone.

Consequently, g is real analytic and admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$.

Theorem (L., Marichal, Zenaïdi)

Under the assumptions of the previous Proposition, the function Σg exists and is infinitely differentiable. Moreover, the following assertions hold:

(a) $(\Sigma g)^{(n)}$ eventually tends monotonically to zero for every $n \geq p + 1$.

(b) Both $g^{(q-1)}$ and $(\Sigma g)^{(q)}$, or their negatives, are completely monotone.

Consequently, Σg is real analytic and admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$.

Applications to Principal Indefinite Sums

Corollary (L., Marichal, Zenaïdi)

Let $p, q \in \mathbb{N}$ with $p < q$, let I be a fixed right-unbounded interval of $\mathbb{R}_+$, and let $g\colon \mathbb{R}_+ \to \mathbb{R}$ be a function such that $g|_I$ is infinitely differentiable. Assume further that the following two conditions hold:

(a) $g|_I^{(p)}$ eventually tends monotonically to zero.

(b) $g|_I^{(q)}$ is regularly monotone.

Suppose also that g is real analytic (resp. admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$). Then the function Σg exists and is real analytic (resp. admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$).

Examples (Log-gamma Function)

Returning to the functions $g(x) = \ln x$ and $\Sigma g(x) = \ln \Gamma(x)$ defined on $\mathbb{R}_+$, it is straightforward to verify that the assumptions are satisfied with $p = 1$, $q = 2$, and $I = \mathbb{R}_+$. It follows that the function $\ln \Gamma(x)$ is real analytic and admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$. Taking $a = 1$ for instance, we obtain the following Newton series expansion

$$\boxed{\ln \Gamma(x) \; = \; \sum_{k=1}^{\infty} \binom{x-1}{k} \left(\Delta_t^{k-1} \ln t\right)\Big|_{t=1}} \qquad (x > 0),$$

which was already studied by Hermite (1900) and also Graham (1994).

Examples (Stern's Series)

The restriction of the function $g(x) = 1/x$ to $\mathbb{R}_+$ clearly satisfies the assumptions with $p = 0$, and hence the corresponding principal indefinite sum (up to an additive constant) is given by

$$\Sigma g(x) \;=\; -\frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right) \qquad (x > 0).$$

It follows that

$$\Sigma g(x) \;=\; H_{x-1} \;=\; \psi(x) + \gamma \qquad (x > 0),$$

where $x \mapsto H_x$ denotes the *harmonic number function*, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the *digamma function*, and γ is *Euler's constant*.

The function $g(x) = 1/x$ on $\mathbb{R}_+$ also satisfies the assumptions with $p = 0$, $q = 1$, and $I = \mathbb{R}_+$. It follows that $\Sigma g(x)$ is real analytic and admits a Newton series expansion on $\mathbb{R}_+$ at every point $a > 0$. Taking $a = 1$ for instance, we obtain the following series representation

$$\boxed{\Sigma g(x) \;=\; \sum_{k=1}^{\infty}\binom{x-1}{k}\,\Delta_x^{k-1}\frac{1}{x}\Big|_{x=1} \;=\; \sum_{k=1}^{\infty}\binom{x-1}{k}\frac{(-1)^{k-1}}{k}} \qquad (x > 0),$$

which is commonly referred to as *Stern's series*.

Conclusions

Define

$$\begin{aligned}
\mathrm{AM}^p_{\pm 1}(I) &= \{f \in \mathcal{C}^\infty(I) : \pm f^{(p+n)} \geq 0, \text{ for all } n \in \mathbb{N}\}, \\
\mathrm{CM}^p_{\pm 1}(I) &= \{f \in \mathcal{C}^\infty(I) : \pm(-1)^n f^{(p+n)} \geq 0, \text{ for all } n \in \mathbb{N}\}, \\
\mathrm{RM}^p(I) &= \{f \in \mathcal{C}^\infty(I) : f^{(p+n)} \geq 0 \text{ or } f^{(p+n)} \leq 0, \text{ for all } n \in \mathbb{N}\}.
\end{aligned}$$

Remark that

$$\mathrm{AM}^p_{\pm 1}(I) = \bigcap_{n \geq p} \mathcal{K}^{n-1}_{\pm 1}(I), \quad \mathrm{CM}^p_{\pm 1}(I) = \bigcap_{n \geq p} \mathcal{K}^{n-1}_{\pm(-1)^{n-p}}(I), \quad \mathrm{RM}^p(I) = \bigcap_{n \geq p} \mathcal{K}^{n-1}(I)$$

and

$$\begin{aligned}
\liminf_{n \to \infty} \mathcal{K}^{n-1}_{\pm 1}(I) &= \bigcup_{p \geq 0} \mathrm{AM}^p_{\pm 1}(I) \subset \mathcal{C}^\omega(I), \\
\liminf_{n \to \infty} \mathcal{K}^{n-1}(I) &= \bigcup_{p \geq 0} \mathrm{RM}^p(I) \subset \mathcal{C}^\omega(I).
\end{aligned}$$

Conclusions

Suppose now that I is a right-unbounded open interval of $\mathbb{R}_+$. Then,

$$\mathcal{D}^p \cap \mathrm{RM}^q(I) \subset \mathrm{CM}_1^{q-1}(I) \cup \mathrm{CM}_{-1}^{q-1}(I) \qquad (p, q \in \mathbb{N},\ p < q).$$

Let also $\mathcal{N}(I)$ denote the class of functions $f\colon I \to \mathbb{R}$ that admit a Newton series expansion on I at every point $a \in I$ so that

$$\liminf_{n\to\infty} \mathcal{K}^{n-1}_{\pm(-1)^n}(I) = \bigcup_{p\geq 0} \mathrm{CM}^p_{\pm(-1)^p}(I) \subset \mathcal{N}(I) \subset \mathcal{C}^\omega(I),$$

We have also observed the following significant fact:

$$\mathrm{RM}^0(I) \not\subset \bigcup_{a\in I} \mathcal{N}_a(I),$$

where, for any $a \in I$, the notation $\mathcal{N}_a(I)$ stands for the class of functions $f\colon I \to \mathbb{R}$ that admit a Newton series expansion on I at a.

References

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Thank you for your attention !