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INDUCTIVE REASONING IN FIRST-ORDER LOGICS

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Extending classical first order logic with the capability of representing object-level statements expressing a default relationship — i.e. facts linked by plausible implication — as opposed to one of necessity, has remained an open problem in the context of formalising common-sense defeasible statements. Additionally, defining meta-level defeasible inference relations over the default quantifier language is the other necessary component for a full defeasible first order logic. Conditional connectives, that have been investigated extensively in conditional propositional logics, do not seem to behave sufficiently well in the first order context. Just using universally quantified variables with default conditionals results in contradictions with exceptional instances due to specificity. Additionally, there are many ranking-based defeasible inference relations that have not been formalised over a defeasible first-order logic. The solution investigated in this thesis is that of extending first order logic with default quantifiers: variable binding binary connectives encoding a notion of typicality, quantified over free variables in the two formulae in question. To interpret default quantifier statements, a ranking measure semantics, only well defined for propositional defeasible reasoning, is chosen to extend regular first order model theory. Then, the aforementioned nonmonotonic entailment relations may be defined over this logic of default quantification. The specific case of defeasible description logics is also presented, along with a novel approach for extending these logics with defeasibility via the syntactic addition of a hypothesis box, which informs a range of speculative inference techniques. Benchmark examples from the literature are here presented and analysed to showcase the efficacy of this general approach to default reasoning with first-order information.

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Chapter 1

Introduction

Within the field of knowledge representation and reasoning (KR), defeasible reasoning constitutes the set of approaches and techniques for addressing the general problem in artificial intelligence (AI) of reasoning under uncertainty, or common-sense reasoning. These approaches amount to formulating extensions of existing logics with the capability of representing and reasoning about default conditional statements, "if x then typically y", or some notion of typicality, "normal x's are in y". Traditional, classical logics are defined with Tarskian inference relations, which consequently means that they are monotonic. Monotonicity mandates that adding information to a knowledge base cannot result in the retraction of an inference, i.e. there are no defeasible conclusions possible. This has motivated the field of research referred to as nonmonotonic logics: formalising extensions of classical logic that represent defeasible information, with corresponding nonmonotonic inference relations, for the goal of achieving patterns of uncertain and common-sense reasoning (Booth et al., 2019).

Some of the foundational formalisms for uncertain reasoning in KR were first described as extensions to first-order logic (Reiter, 1980; McCarthy, 1980; Etherington, 1986), however the resultant complexity, both computational and conceptual, motivated these and subsequent approaches to limit the investigations to extending propositional logic (Kraus et al., 1990; Weydert, 1998). The project of lifting these frameworks, that are now well defined and understood in the propositional logic context, to more expressive logics has been limited largely, with exceptions, to specific fragments of description logics and modal logics (Bonatti et al., 2006; Britz and Varzinczak, 2017). These exceptions do generally define a syntax and semantics for first-order conditional logic (Kern-Isberner and Thimm, 2012; Delgrande, 1998), however they generally do not extend to nonmonotonic inference relations over these languages, and contain representational decisions that may not fit all first-order default statements that could be desirable to encode.

The primary goal of this thesis is to take the insights and developments from propositional default reasoning, specifically from the framework utilising ranking measures (Spohn, 2012; Weydert, 1998), and the syntactic construct of default quantifiers (Weydert, 1997; Schlechta, 1995), which together will provide the initial monotonic logic. This provides a flexible and general first-order representation scheme for default statements. Interpreting these default quantifiers with ranking measures also provides a general defeasible semantic extension to first-order model theory. Once this is well defined, the goal will be to lift various defeasible inference relations, which will amount to reconstructing the procedure of ranking choice and constructible inference schemes (Weydert, 1995, 1998) from propositional default reasoning with ranking measure semantics.

Then, attention will be turned to the specific case of the first-order fragment that is the description logic $\mathcal{ALC}$. Defeasible extensions of $\mathcal{ALC}$ (Britz et al., 2020; Bonatti, 2019) have tended to focus on extending the concept language, which encodes abstract unary predicate inclusion axioms, with a set of conditional statements of the same variety: abstract unary predicate default inclusion axioms. Less studied has been the case of analysing defeasible inferences of assertions: existential facts regarding the membership of constants in unary predicates, and pairs of constants in binary relations. A defeasible $\mathcal{ALC}$ will be defined, with a semantics based on ranking measures. Then defeasible inference over this logic will be described, along with the semantic shortcomings. Finally, speculative inference extensions over this defeasible logic will also be defined, as a proposed remedy to these shortcomings.

In both cases, examples from the literature provide useful stimuli for exploring various solutions. Providing solutions to known problematic examples is one major evaluation metric.

1.1 Background

The literature of extending logics more expressive than propositional logic with defeasible reasoning has a long history. Here, the focus will be on extensions of the same classical logics at the core of this thesis: the standard first-order logic and description logics. Two broad major approaches to formalising uncertain reasoning are extending classical logics with probabilistic statements and semantics, and extending logics with connectives and semantics encoding a qualitative notion of typicality and defeasibility.

1.1.1 Probabilistic and defeasible description logics

Description logics were targeted often as a base with which to extend with defeasible connectives and inference relations, due to their status as the logical foundations for semantic technologies. Early work primarily dealt with extending fragments such as $\mathcal{ALC}$ or $\mathcal{EL}$ with a defeasible inclusion relation, and defining defeasible algorithms to compute nonmonotonic concept inclusion inferences. Recent approaches have also tackled the more expressive fragments that contain more restrictions on role relations, such as $\mathcal{SROIQ}$. Many significant approaches aimed at extending the syntax with a default conditional connective defined as a defeasible relation between concept names, interpreted with a preference ordering over the domain of the structure. This approach corresponds to a lifting of the KLM framework (Kraus et al., 1990; Lehmann and Magidor, 1992).

Pensel and Turhan (Pensel and Turhan, 2017a, 2018, 2017b) extend the restricted description logic $\mathcal{EL}_\bot$ with a defeasible inclusion relation between concept names. $\mathcal{EL}_\bot$ is characterized as the description logic built from concept names $A, B, ...$, role names $r, s, t, ...$, the tautology and falsum concepts (referred to as top and bottom) $\top, \bot$, conjunction $\sqcap$, and existential role concept restrictions $\exists r.C$. Then terminological information is the result of defining general concept inclusion axioms among the set of complex concepts constructed from the aforementioned conjunction and existential role concept restrictions. A defeasible counterpart to this strict inclusion relation is introduced over the set of complex concepts, with the corresponding semantic interpretation based on a KLM style preference ordering over the individuals in the domain of the interpretation.

Britz et al (Britz et al., 2020, 2019) have a well defined framework for defeasible description logics, specifically the fragment $\mathcal{ALC}$, strongly based on the KLM framework for defeasible propositional logic (Kraus et al., 1990; Lehmann and Magidor, 1992). A defeasible inclusion relation defined over concept names, $C \sqsubsetsim D$ is extends the syntax with the interpretation "elements in C are defeasibly assumed to be also in D". This relation is interpreted in the semantics by a total pre-order over individuals in the first-order structure, with satisfaction in a structure based on the minimal individuals satisfying the conditional statements. Reasoning, however, is primarily restricted to defeasible inclusion statements between concept names — that is, inferring which concepts are defeasibly assumed to be in which other concepts.

In (Britz et al., 2018), this framework defines an algorithm for determining assertions entailed by a default base including an ABox, by extending the ABox with assertions until it is maximal while still being consistent. This approach is not canonical, however, as the order in which assertions are added matters, and therefore any given default base may have more than one possible extensions of the ABox. This approach is also purely algorithmic, without a semantic understanding of the entailment relation.

(Giordano et al., 2020) represents default information with a unary typicality operator, similar to, for instance (Booth et al., 2012). However the typicality operator here is only permitted in the antecedent of a general concept subsumption statement. This means the representational strength of this approach is similar to that of approaches using a default binary connective representing a defeasible subsumption. These typicality statements are interpreted with a first-order structure with a total pre-order over the domain individuals. This approach is then extended with probabilistic statements: the subsumption relation between typical concepts and general concepts has a value between 0 and 1 attached to it, representing the probability that elements belonging to the antecedent are also elements in the consequent. This also builds on much previous work (Giordano et al., 2012, 2015) on extending description logics with default conditionals interpreted with KLM-style semantic extensions over the domains of the first-order structures, and defining rational closure over the concept language.

(Britz and Varzinczak, 2016) defined defeasibility over role relations, syntactically represented with defeasible value restriction in role concept constructors, interpreted with a preference order over roles. More expressive description logics such as $\mathcal{SROIQ}$ have had defeasible extensions as well, notably by having a defeasible role inclusion syntactic extension proposed by (Britz and Varzinczak, 2017). In classical $\mathcal{SROIQ}$ role axioms, statements about binary relations specifying what kind of properties they satisfy are a part of the language itself.

(Bonatti, 2019) proposes a framework around "stable rational closure", semantically characterized by upward closed models. This work focuses solely on rational closure, also referred to as System Z (Pearl, 1990) for any given description logic fragment. The major issue targeted here is to extend defeasible KLM-style semantics for default conditionals and the associated rational closure entailment relation to description logic knowledge bases that do not satisfy the disjoint model union property. These semantics interpret defaults conditional statements between concepts with a ranking function over the domain of an interpretation. Upward closure requires that for every complex concept that is non-empty in a given domain, that there is an individual in that domain with the same rank as the concept. Also discussed here is the issue of ABox reasoning, with the same observation as (Britz et al., 2018) regarding multiple extensions of a given default knowledge base, however this time from a semantic perspective.

Description logics extended with pointwise circumscription (Lifschitz, 1986) has also been formalised by (Di Stefano et al., 2023) using $\mathcal{ALCIO}$ as the base classical logic. $\mathcal{ALCIO}$ is the description logic that is equivalent to $\mathcal{ALC}$ with inverse roles and nominals (concepts constructed from a constant names).

Jaeger defined a probabilistic extension to description logics (Jaeger, 1994) using as the base description logic $\mathcal{PCL}$, where terminological information is restricted such that the left hand side of the general concept inclusion statements are concept names, and the right hand side are complex concepts. Then, a set of probabilistic statements of the form $P(C \mid D) = p$ where C and D are complex concepts, representing the statement "the probability of a member of the concept D also being in C is p", along with the associated probabilistic assertions $P(a \in C) = p$ representing "the probability of the constant name a being in the concept C is p". A significant aspect to note of this logic is how the semantics are extended in order to interpret these probabilistic statements: a probability measure is defined, with the Lindenbaum algebra used as the probability space. (Jaeger, 1994) ends with directions as to how one would lift this probabilistic approach to extend $\mathcal{ALC}$ with the same syntax and semantics.

1.1.2 Probabilistic and defeasible first order logics

(Halpern, 1990) studied first order probabilistic logics, where probability distributions are examined both over the domain and over interpretations. The key question examined in that work is to examine the different semantics around statements such as "80% of birds fly", and "I am 80% sure this bird flies". The former is reasoning about quantitative, statistical information, whereas the latter is representing a degree of belief. This distinction is important since there are implicitly different statistical quantifiers: one over the domain of objects, and one over all possible worlds. (Friedman et al., 2000) builds upon this work and analyses subjective conditional in first-order logics interpreted by plausibility structures: enriching first-order structures with a plausibility space. Satisfaction of a default conditional is defined if the plausibility of the worlds satisfying the default is higher than the plausibility of the non-satisfying worlds (or if the antecedent of the conditional is impossible, the conditional is also satisfied). This is also notable for providing a sound and complete axiomatization of this logic.

(Delgrande, 1998) has a qualitative approach that attempts to represent the same ideas as (Halpern, 1990): plausible statements about the individuals in a domain, and plausible statements about the world in general. Whereas (Halpern, 1990) formalises these ideas in a probabilistic extension, Delgrande uses a preference ordering instead, to encode a qualitative, rather than quantitative, reading of uncertainty. What is notable here is the attention given to analysing a semantics that orders the domain of the interpretations, to read the conditional as specifying normal individuals, and then also a semantics that orders the interpretations themselves, to read the conditional as specifying normal worlds, or models. While this approach successfully addressed representational problems in default first-order logics, it did not address the open problem of default inference over such a logic,

(Lehmann and Magidor, 1990) extended their work in defeasible propositional logic (Kraus et al., 1990; Lehmann and Magidor, 1992) to the case of first-order logic by introducing a default consequence relation over first-order formulae. This is characterized by preferential models, as in (Kraus et al., 1990), by defining a set of worlds mapped to first-order structures, with a partial order defined over these worlds. These preferential models then define preferential consequence relations over the first-order logic. This work parallels closely the preferential consequence relations defined over propositional logic. Notably, the first-order language is not extended with a default conditional, and so representational issues in defeasible first-order logic are not addressed, and the existence of the rational closure of an arbitrary first-order knowledge base is conjectured but not proven.

(Bacchus et al., 1996, 1994) have a very general approach to formalising probabilistic first-order logic, limited partly by considering just finite domains. This work attempted to combine representing and reasoning with statistical information, first-order statements, and default conditionals, into a general approach referred to as the random-worlds method. Probabilistic and default statements are represented as conditional formula with an associated approximate probability. Then, truth is assigned to these probabilistic statements based on the structure of a particular domain in a given interpretation with an error tolerance parameter so as to better satisfy the exact numbers represented by probabilistic statements, and it assigns degrees of belief based on a measure over all the possible worlds. The final degree of belief in a formula, conditioned with a knowledge base, is calculated by computing the limit as the size of the domain grows arbitrarily large, and the error tolerance shrinks arbitrarily small.

Circumscription (McCarthy, 1980, 1986) is a general form of closed-world reasoning where predicates that are labelled as being abnormal are minimized as much as possible in the semantics. This form of reasoning has in mind particular issues in default reasoning, specifically the frame problem: how to exclude a default inference process from considering extraneous, irrelevant information. In particular, circumscription has many built-in assumptions regarding existence, and, correspondingly, emptiness, of predicates that may not be preferable depending on the specific application.

Reiter's default logics (Reiter, 1980; Reiter and Criscuolo, 1983) is an approach to default reasoning where an underlying classical logic is enriched with a set of default rules, collected in a default base, which give permission to infer some formulae by default if some preconditions are satisfied. Since these default rules may provide conclusions that conflict with the preconditions of other default rules, default bases typically result in various distinct extensions of a classical theory. While the original definition was purely rule-based, later work (Etherington, 1986) provided a semantics for the extensions that can be computed from a default base.

An early approach that used default quantifiers was proposed by (Weydert, 1997). This approach interpreted default quantification with ranking measures over the domain of the first-order structure: default quantifier sentences such as $\phi \Rightarrow_x \psi$ are satisfied in a first order ranking measure structure if the rank of the satisfying tuples of individuals in the domain is lower than the rank of the violating tuples of individuals. This work, however, did not have a defeasible inference relation over the logic.

(Delgrande and Rantsoudis, 2020) defined an approach characterized by first order structures enriched with a preference relation over the domain $\mathcal{M} = \langle D, I, O \rangle$ and makes use of predicate-forming constructs of the form $\{P(\overrightarrow{y}, \phi(\overrightarrow{y})\}(\overrightarrow{x})$ which states that the ordering of the domain when evaluating the formula is with respect to P and whether or not ϕ holds varies. This approach has a very strange syntax. The predicate forming constructs obscure, rather than clarify, the language, and generally

Casini et al (Casini et al., 2022) propose a KLM-style approach. The semantics use a Herbrand base enriched with typicality constants, that are used to satisfy default information. The models are ranked interpretations that give ranks to different Herbrand universes, much like Kern-Isberner and Thimm did (Kern-Isberner and Thimm, 2012) that are an OCF, but without "gaps" - if $x \neq \infty$ is mapped to a Herbrand universe, then every $0 \geq y \geq x$ is also mapped to a Herbrand universe. The main issues with this approach are strongly related to the problems with Kern-Isberner and Thimm's approach (Kern-Isberner and Thimm, 2012): too much a propositional style approach (rankings over worlds), and not quite general enough, it deals only with the finite case. This approach is very similar to Kern-Isberner and Thimm's approach except there are no universal or existential quantifiers.

1.2 Problem statement

The central problem at the core of this thesis is first-order default reasoning, specifically to formalise a coherent framework for representing and reasoning first-order defeasible information. Therefore, the primary task is to propose a coherent semantic framework that extends classical first-order logic for defeasible reasoning, both for satisfying defeasible knowledge bases, as well as lifting defeasible inference relations from propositional logic to reason over it.

The syntactic construction of default quantifiers (Weydert, 1997) presents as a promising syntactic extension for formalising default conditional information, as the ability to bind variables distinguishes it as successfully encoding information about typical generic individuals. Properly investigating and formalising a logic of default quantifiers that successfully represents and reasons about first-order information, interpreted with the ranking measure semantics is one first issue. The major open problem addressed here will be to lift a general framework for defeasible inference over this logic, define specific defeasible inference relations within it, and analyse their behaviours.

The aim is to extend a specific fragment of first-order logic, the description logic $\mathcal{ALC}$, with defeasible syntax and semantic notions. This logic has been commonly used as a baseline description logic to introduce defeasible connectives and semantics in part due to the relatively low expressivity compared to $\mathcal{SROIQ}$, the description logic that forms the theoretical foundation of the Web Ontology Language OWL 2 (Grau et al., 2008). This thesis nevertheless took as inspiration the papers extending $\mathcal{ALC}$ with defeasible syntax and semantics (Britz et al., 2018, 2019, 2020) and chose the open problem presented here regarding defining a semantically grounded defeasible inference relation for the assertion language.

1.2.1 Issues in first-order defeasible reasoning

There are particular challenges to lifting propositional default reasoning to defeasible first-order logics, as well as certain considerations to keep in mind.

Probabilistic and qualitative semantics

Many of the approaches surveyed in order to extend logic with uncertain reasoning capabilities use probabilities to extend the syntax and semantics. This is due to the robust literature regarding probability theory, and the large literature on which one can draw. However, probabilities are not always appropriate for representing the kind of uncertainty desired depending on the domain (Halpern, 2017). Not all situations may justify the principle of indifference, and the exact numbers, or even ranges of numbers, may not be easily deduced (Weydert, 1995).

The strain of uncertain reasoning to be lifted in this thesis is that of qualitative default reasoning. While it takes inspiration from probabilistic reasoning, the nature of the uncertainty encoded in these semantics is distinct from that found in probabilistic variants. The benefits of this approach are primarily that this alternative method for modelling belief states is better suited to representing subjective beliefs: it is possible to conceptualise the beliefs a person has about the world as an ordering over what situations that person considers most to least likely.

Issues with first-order default conditionals

The majority of approaches to first-order default reasoning covered in section 1.1 represent default knowledge with a default conditional connective much like in the propositional case. However, extending first-order logic syntax with a propositional connective runs into issues regarding contradiction. Consider the following example from (Weydert, 1997) and (Delgrande, 1998):

Example 1 (Simple conditionals)**.** *Let the following be a conditional knowledge base:*

- $\forall x(\texttt{Bird}(x) \Rightarrow \texttt{CanFly}(x))$
- $\forall x(\texttt{Blurb}(x) \wedge \texttt{Bird}(x) \wedge \texttt{CanFly}(x)) \Rightarrow \bot$

The expectation in default reasoning is that these default statements should be consistent with the statement $Bird(Tweety) \wedge Blurb(Tweety)$. However, from the above statements, $\forall x(\top \Rightarrow \neg(Bird(x) \wedge Blurb(x))$ is inferred, from which we may instantiate $\top \Rightarrow \neg(Bird(Tweety) \wedge Blurb(Tweety))$. Since the fact $Bird(Tweety) \wedge Blurb(Tweety)$ may be restated in conditional logic as $\neg(Bird(Tweety) \wedge Blurb(Tweety)) \Rightarrow \bot$, $\top \Rightarrow \bot$ is straightforwardly obtained.

The above example is illustrates the unsuitability of propositional defaults in a first-order setting. Dealing with both constants and variables is enough to demonstrate the necessity of a syntax and semantics for default reasoning sufficiently more sophisticated than that of propositional default conditionals.

Even approaches that take this into account, such as (Delgrande, 1998), reveal the difference in their syntax compared to default quantification. Consider the following example:

Example 2 (Elephant zoo)**.** *What is the best way to translate the following natural language statements into a defeasible first-order logic:*

- *Elephants usually like their keepers.*
- *Elephants usually do not like the keeper Fred.*
- *Clyde the elephant does usually like the keeper Fred.*

Using the signature $Elephant(x)$ = "x is an elephant", $Keeper(x)$ = "x is an elephant keeper", and $Likes(x, y)$ = "x likes y" (Delgrande, 1998) formulates these statements as:

- $Elephant(x) \wedge Keeper(y) \Rightarrow_{xy} Likes(x, y)$

- $Elephant(x) \wedge Keeper(Fred) \Rightarrow_x \neg Likes(x, Fred)$
- $Elephant(Clyde) \wedge Keeper(Fred) \Rightarrow Likes(Clyde, Fred)$

The result is that the constants cause the default connective to lose the bound variables until it is a propositional connective with atomic formulae in the final statement. The issue noted with this approach is that the intended meaning becomes confused: statements with bound variables are clearly information regarding typical individuals, whereas statements with constant names seem to be about the most typical worlds, since they lose the quality of representing statements about variables.

This leads into a point regarding what default quantification really means: the vector of variables informs the size of the tuples in the domain over which to consider the satisfaction or violation of a formula.

Plausibility over worlds and plausibility over individuals

As pointed out in, for example, (Delgrande, 1998), there is an important distinction over whether default statements are interpreted as talking about what is true regarding typical individuals, versus what is true regarding typical worlds. In the former, defaults provide information about how to assign plausibility measures (whether the specific semantics are preferential, ranking measure, or possibility measures) to the individuals within each first-order structure. In the latter case, the same plausibility measures are assigned to the structures themselves.

This dichotomy is an important consideration, as it is the difference between reasoning regarding what is true about a particular generic individual, and what formulae may be true in the general, typical case. Reasoning about typical individuals allows for inference regarding what can be defeasibly inferred about a specific individual. Reasoning about worlds defeasibly infers what may be conditionally believed in the most typical situations.

Assertion reasoning in defeasible description logics

In the context of defeasible description logics, and specifically $\mathcal{ALC}$, there is a wealth of work done over extending the concept language, that is, the $\mathcal{T}$ statements, with default conditional statements, along with defeasible inference relations that entails default conditional statements from default and terminological knowledge bases (Britz et al., 2020). However, only certain formalisms attempt to define inference for assertions, notably in (Bonatti, 2019; Britz et al., 2018), and these approaches do not have a canonical set of inferences that follow from a default knowledge base, instead resulting in competing extensions.

1.2.2 Research questions

Having outlined the problem area and broad aims, the specific questions and issues to be tackled in this thesis remain. The main idea is to define an extension of classical first-order logic for common-sense, or defeasible, reasoning. To that end, the existing formalisms for conditional logics and default reasoning for propositional defeasible logics, specifically those defined with ranking measure semantics, will be lifted to the first-order context. Therefore, the primary research question is how to identify strict and defeasible reasoning formalisms over first-order logics expressing conditional default knowledge, exploiting the quasi-probabilistic ranking measure semantics.

Sub questions include:

- What are the properties of fragments of general default quantifier logics, starting from specific defeasible description logics?
- How to define defeasible inference relations over fragments of first-order logics of defaults, and what are their behaviours?
- How to handle preferences over interpretations and preferences over elements in the domains, and how these preferences interact.

In order to test the framework to be defined, comparing the behaviour of the framework with common examples from the literature will be an important evaluation metric. Representing a formalisation of these examples with the proposed syntax, such that it results in a satisfiable defeasible knowledge base. Being able to then define defeasible and monotonic inference over these defeasible knowledge bases, such that expected patterns of reasoning may be observed will also validate the approach.

1.3 Overview of the thesis

Given the outline of the problem to be addressed above, this thesis is organised as follows: in chapter 2, the fundamental classical logics to be extended, and the framework for propositional default reasoning that is to be lifted will be defined. Specifically, the description logic $\mathcal{ALC}$ and first-order logic with equality will be defined, and the ranking measure semantic framework will be covered. Following this, in chapter 3, the syntax and semantics for a first-order default quantifier logic will be presented. This is the novel approach to first-order default reasoning proposed here, centred around the syntactic extension of default quantifiers, and the semantic framework based on lifting ranking measures to first-order model theory with which to interpret them. Chapter 4 defines a general approach to defining defeasible inference relations over this logic of default quantifiers, by reconstructing the same inference relations from propositional default reasoning, starting with Z-inference, and then constructible inference relations. Chapter 5 then addresses the special case of defeasible description logics, specifically a defeasible extension of $\mathcal{ALC}$, and analyses the shortcomings of purely propositional default conditionals in this context, and proposes an extension to defeasible description logics in the form of HBox, or speculative, reasoning.

Chapter 2

Formal Preliminaries

Before discussing the central contributions of this thesis, an overview of the work upon which this thesis builds is presented. The classical description and first-order logic to be extended will be described, along with the propositional defeasible concepts which will be lifted to these extensions.

2.1 Classical logics

2.1.1 Description logics

The central role of the description logic $\mathcal{ALC}$ (Baader et al., 2007) makes it a suitable starting point for developing new techniques in the context of defeasible description logics (DDL). We will therefore begin with a little reminder on $\mathcal{ALC}$, its vocabulary, its language, its philosophy, and its semantics, which determines a monotonic entailment notion with a complete proof theory. In a next step, this framework will be exteded and refined to handle defeasible subsumption and nonmonotonic entailment by exploiting the ranking measure semantics.

Classical language

First we will introduce the class of $\mathcal{ALC}$-vocabularies. In general, a vocabulary consists of a logical core vocabulary V^{log}, which collects the domain-independent logical and auxiliary symbols, and a context-dependent domain vocabulary V^{dom}.

Definition 1 ($\mathcal{ALC}$-vocabulary)**.** *An $\mathcal{ALC}$-vocabulary $V = V^{log} \cup V^{dom}$ consists of*

- *its core vocabulary V^{log}, which includes the auxiliary symbols $($, $)$, and the logical symbols encompassing the 0-ary operators $\top, \bot$, the unary operator $\neg$, the binary operators $\sqcap, \sqcup$, the role restriction operators $\exists, \forall$, and the subsumption relation symbol $\sqsubseteq$;*
- *a domain vocabulary $V^{dom} = \mathsf{N}_{\mathscr{C}} \cup \mathsf{N}_{\mathscr{R}} \cup \mathsf{N}_{\mathscr{I}}$, where $\mathsf{N}_{\mathscr{C}}$, $\mathsf{N}_{\mathscr{R}}$, and $\mathsf{N}_{\mathscr{I}}$ are mutually disjoint (finite) sets of names of atomic concepts, roles, and individuals, respectively.*

An $\mathcal{ALC}$-language $\mathcal{L} = \mathcal{L}_V^{alc}$ over an $\mathcal{ALC}$-vocabulary V is based on three types of expressions: concepts, subsumption statements between concepts, and concept or role assertions about individuals.

Concepts are constructed from the atomic concept names in $\mathsf{N}_{\mathscr{C}}$, the boolean operator connectives, including $\top$ and $\bot$ (tautology resp. contradiction), and the role restriction operators parametrized by role names in $\mathsf{N}_{\mathscr{R}}$. Concepts can be understood as a kind of set-terms, not to be confused with propositions. Therefore, the set of all possible concepts can be defined as:

Definition 2 (Concepts)**.** *Given an $\mathcal{ALC}$-vocabulary V, the corresponding set of concepts $\mathbb{C} = \mathbb{C}_V$ is defined to be the closure of the following construction scheme:*

$$C ::= \top \mid \bot \mid A \mid (\neg C) \mid (C \sqcap C) \mid (C \sqcup C) \mid (\exists r.C) \mid (\forall r.C)$$

for $r \in \mathsf{N}_{\mathscr{R}}$ and $A \in \mathsf{N}_{\mathscr{C}}$.

In $\mathcal{ALC}$ the only explicit relation between concepts is subsumption, which essentially models set inclusion. In description logic, subsumption statements originally have been interpreted as terminological axioms specifying the a priori structure of an ontology. Here we adopt a more liberal perspective, allowing an empirical a posteriori interpretation.

Definition 3 (Subsumption)**.** *Given an $\mathcal{ALC}$-vocabulary V, the set of subsumption statements $\mathbb{S} = \mathbb{S}_V$ over V consists of all the expressions of the form $C \sqsubseteq D$ for concepts $C, D \in \mathbb{C}$. $C \equiv D$ is an abbreviation for holding of $C \sqsubseteq D$ and $D \sqsubseteq C$.*

The assertions correspond to ground unary and binary relational statements in first-order logic, i.e. variable-free statements about specific domain elements. They allow to describe the context-dependent premises (input) and consequences (output) in the context of a given base of subsumption statements. Formally, they are defined as:

Definition 4 (Assertions)**.** *Given an $\mathcal{ALC}$-vocabulary V, the set of assertions $\mathbb{A} = \mathbb{A}_V$ consists of the concept assertions $a : C$, where a is an individual name and C a concept, and role assertions $(a, b) : r$. where a, b are individual names and r is a role name.*

The subsumption formulas together with the assertions constitute the set of basic $\mathcal{ALC}$-sentences $\mathbb{S} \cup \mathbb{A}$.

This fixes the syntax of the basic $\mathcal{ALC}$-language $\mathcal{L}_V^{alc}$ over V. In what follows we will however also consider a boolean variant $\mathcal{L}_V^{alc*}$ of $\mathcal{L}_V^{alc}$. It adds to the logical core vocabulary V^{log} the standard propositional connectives $\neg, \wedge, \vee$, and closes the assertions under these, which defines an extended set of assertions $\mathbb{A}^*$. This extended version of the $\mathcal{ALC}$-language, which is also available for defeasible derivatives of $\mathcal{L}_V^{alc}$, gives the user more flexibility to express partial knowledge and conclusions about the world.

Definition 5. *An $\mathcal{ALC}$ knowledge base $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ consists of a TBox $\mathcal{T}$ containing finitely many subsumption statements $C \sqsubseteq D$, and an ABox $\mathcal{A}$ including finitely many assertions $\alpha \in \mathbb{A}$.*

The TBox indicates subsumption relations between concepts, and the ABox states what is known about concept properties of specific individuals and role relationships between them.

Classical semantics

The semantics of $\mathcal{ALC}$ is based on first-order structures over some signature with finitely many constants, unary, and binary predicates. That $\mathcal{ALC}$-structures are such that:

Definition 6 ($\mathcal{ALC}$-structures)**.** *An $\mathcal{ALC}$-structure over an $\mathcal{ALC}$ domain vocabulary V^{dom} is a first order structure $(\Delta, \mathcal{I})$, where Δ is a non-empty domain of discourse, and $\mathcal{I}$ a function interpreting the domain symbols from V^{dom} over Δ (usual notation: $X^{\mathcal{I}} := \mathcal{I}(X)$), such that:*

- *for any atomic concept name $A \in \mathsf{N}_{\mathscr{C}}$, $A^{\mathcal{I}} \subseteq \Delta$,*

- *for any role name* $r \in \mathsf{N}_{\mathscr{R}}$, $r^{\mathcal{I}} \subseteq \Delta \times \Delta$,
- *for any individual name* $a \in \mathsf{N}_{\mathscr{I}}$, $a^{\mathcal{I}} \in \Delta$.

A characteristic of description logic is that concepts are handled as terms to which one may apply certain functors. The interpretation function $\mathcal{I}$ specified for atomic concepts then has to be lifted to an extended function, which is able to interpret the complex concepts built with these functors, and is defined as:

Definition 7 (Extended $\mathcal{ALC}$-interpretations)**.** *For an $\mathcal{ALC}$-structure $(\Delta, \mathcal{I})$, the associated extended interpretation $\hat{\mathcal{I}}$ is defined inductively as follows, for $C, D \in \mathbb{C}$:*

- $C^{\hat{\mathcal{I}}} := C^{\mathcal{I}}$, *for any* $C \in \mathsf{N}_{\mathscr{C}}$,
- $r^{\hat{\mathcal{I}}} := r^{\mathcal{I}}$, *for any* $r \in \mathsf{N}_{\mathscr{R}}$,
- $a^{\hat{\mathcal{I}}} := r^{\mathcal{I}}$, *for any* $a \in \mathsf{N}_{\mathscr{I}}$,
- $\top^{\hat{\mathcal{I}}} := \Delta, \;\; \bot^{\hat{\mathcal{I}}} := \emptyset$,
- $(\neg C)^{\hat{\mathcal{I}}} := \Delta \setminus C^{\hat{\mathcal{I}}}$
- $(C \sqcap D)^{\hat{\mathcal{I}}} := C^{\hat{\mathcal{I}}} \cap D^{\hat{\mathcal{I}}}, \;\; (C \sqcup D)^{\hat{\mathcal{I}}} := C^{\hat{\mathcal{I}}} \cup D^{\hat{\mathcal{I}}}$,
- $(\exists r.C)^{\hat{\mathcal{I}}} := \{x \in \Delta \mid \text{ there is } y \in \Delta \text{ such that } (x,y) \in r^{\hat{\mathcal{I}}} \text{ and } y \in C^{\hat{\mathcal{I}}}\}$,
- $(\forall r.C)^{\hat{\mathcal{I}}} := \{x \in \Delta \mid \text{ for every } y \in \Delta \text{ if } (x,y) \in r^{\hat{\mathcal{I}}} \text{ then } y \in C^{\hat{\mathcal{I}}}\}$.

For ease of notation, we will refer to $\hat{\mathcal{I}}$ by $\mathcal{I}$.

The satisfaction relation between $\mathcal{ALC}$-interpretations structures and $\mathcal{ALC}$-sentences is defined in the obvious way:

Definition 8 ($\mathcal{ALC}$ satisfaction relation)**.** *For any $\mathcal{ALC}$ signature V, the (general) $\mathcal{ALC}$-satisfaction relation $\models_{alc}$ specifies for any $\mathcal{ALC}$-structure $(\Delta, \mathcal{I})$ which subsumption statements $C \sqsubseteq D$ and assertions $a : C$ or $(a, b) : r$ it satisfies.*

- $(\Delta, \mathcal{I}) \models_{alc} C \sqsubseteq D$ *iff* $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

- $(\Delta, \mathcal{I}) \models_{alc} a : C$ *iff* $a^{\mathcal{I}} \in C^{\mathcal{I}}$.
- $(\Delta, \mathcal{I}) \models_{alc} (a, b) : r$ *iff* $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.

An $\mathcal{ALC}$-structure $(\Delta, \mathcal{I})$ satisfies an $\mathcal{ALC}$ knowledge base $\mathcal{T} \cup \mathcal{A}$ if and only if it satisfies all its elements. We define the associated $\mathcal{ALC}$-model set by:

$$Mod_{alc}(\mathcal{T} \cup \mathcal{A}) = \{(\Delta, \mathcal{I}) \mid (\Delta, \mathcal{I}) \models_{alc} \mathcal{T} \cup \mathcal{A}\}$$

$\mathcal{ALC}$-entailment is the Tarskian inference relation $\vdash_{alc}$ induced by $\models_{alc}$.

Definition 9 ($\mathcal{ALC}$-inference)**.** *A knowledge base $\mathcal{T} \cup \mathcal{A}$ entails an assertion $\alpha \in \mathbb{A}$, $\mathcal{T} \cup \mathcal{A} \vdash_{alc} \alpha$, if and only if every model of $\mathcal{T} \cup \mathcal{A}$ is a model of α, i.e. if $Mod_{alc}(\mathcal{T} \cup \mathcal{A}) \subseteq Mod_{alc}\{\alpha\}$.*

2.1.2 First-order logic

Extending first-order logic with the goal of defeasible reasoning is the central focus of this work, and so this necessitates specifying the precise language and semantics of interest. The following definitions are with reference to Gallier's book (Gallier, 2015).

First order language

The fundamental building block of any first-order language is the signature, providing the core parameters of the language:

Definition 10 (Signature)**.** *Let $\mathcal{L}$ be a standard first-order predicate language defined by a signature of predicate names $\mathcal{P}$, function names $\mathcal{F}$, and constant names $\mathcal{C}$. The set of variables is denoted $\mathtt{V}$.*

The language of first-order logic is two-sorted, and so first the set of terms in the language will be defined:

Definition 11 (Terms)**.** *The set of terms $\mathcal{T}$ is defined recursively such that:*

- *if $v \in \mathtt{V}$, then $v \in \mathcal{T}$.*
- *if $c \in \mathcal{C}$, then $c \in \mathcal{T}$.*
- *if $f \in \mathcal{F}$ has arity n, and $t_i \in \mathcal{T}$ for all $0 < i \leq n$ then $f(t_1, ..., t_n) \in \mathcal{T}$.*

Then, the set of well-formed formulae is defined recursively, as usual, starting with the base, atomic case:

Definition 12 (Atomic formula)**.** *For any $P \in \mathcal{P}$ having arity n, if $t_i \in \mathcal{T}$ for all $0 < i \leq n$, then $P(t_1, ..., t_n)$ is an atomic formula in $\mathcal{L}$. For any two terms $t_1, t_2 \in \mathcal{T}$, $t_1 \doteq t_2$ is an atomic formula of $\mathcal{L}$.*

Then, the language of first-order logic may be fully described:

Definition 13 (First-order syntax)**.** *The formulae of $\mathcal{L}$ are defined recursively with the set of logical connectives $\{\wedge, \vee, \neg, \rightarrow, \forall, \exists, \leftrightarrow, \doteq\}$ and the auxiliary symbols $\{(,)\}$:*

- *Every atomic formula A is a formula in $\mathcal{L}$.*
- *$\bot$, $\top$ are formulae in $\mathcal{L}$.*
- *If $A \in \mathcal{L}$, then $\neg A \in \mathcal{L}$.*
- *If $A, B \in \mathcal{L}$, then $(A * B) \in \mathcal{L}$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.*
- *If $A \in \mathcal{L}$ and $x \in \mathtt{V}$, then $\forall x A \in \mathcal{L}$, and $\exists x A \in \mathcal{L}$.*

Where it is unambiguous, the auxiliary symbols (and) may be dropped from the formula.

The set of free variables in a given formula is defined recursively, as usual:

Definition 14 (Free variables)**.** *If t is a term, the set $FV(t)$ of the free variables of t is defined recursively such that:*

- *$FV(x) = \{x\}$, for a variable x*
- *$FV(c) = \emptyset$, for a constant c*
- *$FV(f(t_1, ..., t_n)) = FV(t_1) \cup ... \cup FV(t_n)$ for a function f of arity n*

If A is a formula, the set $FV(A)$ is defined by:

- *$FV(P(t_1, ..., t_n)) = FV(t_1) \cup ... \cup FV(t_n)$ for a predicate symbol P of arity n*
- *$FV(t_1 \doteq t_2) = FV(t_1) \cup FV(t_2)$*
- *$FV(\neg A) = FV(A)$*

- $FV(\bot) = FV(\top) = \emptyset$

- $FV((A * B)) = FV(A) \cup FV(B)$ *where* $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

- $FV(\forall x A) = FV(A) - \{x\}$

- $FV(\exists x A) = FV(A) - \{x\}$

A closed formula, also referred to as a sentence, is defined, as normal, as any formula without free variables:

Definition 15 (Closed formula)**.** *A formula* $\phi \in \mathcal{L}$ *is closed if* $FV(\phi) = \emptyset$*. Otherwise it is open.*

Definition 16 (Substitution)**.** *If* s *and* t *are terms, then substituting* t *for a variable* x *in* s*, denoted* $s[t/x]$ *is defined recursively such that:*

- $y[t/x] = y$ *if* $y \neq x$ *else* t*, when* s *is a variable* y

- $c[t/x] = c$*, when* s *is a constant* c

- $f(t_1, ..., t_n)[t/x] = f(t_1[t/x], ..., t_n[t/x])$*, when* s *is a term* $f(t_1, ..., t_n)$

Given a formula $A \in \mathcal{L}$*,* $A[t/x]$ *is defined recursively as follows:*

- $\bot[t/x] = \bot$ *when* A *is* $\bot$

- $P(t_1, ..., t_n)[t/x] = P(t_1[t/x], ..., t_n[t/x])$*, when* $A = P(t_1, ..., t_n)$

- $(t_1 \doteq t_2)[t/x] = (t_1[t/x] \doteq t_2[t/x])$ *when* $A = (t_1 \doteq t_2)$

- $(B * C)[t/x] = (B[t/x] * C[t/x])$ *when* $A = (B * C)$*,* $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

- $(\neg B)[t/x] = \neg B[t/x]$ *when* $A = \neg B$

- $(\forall y B)[t/x] = \forall y B[t/x]$ *if* $x \neq y$*, else* $\forall y B$*, when* $A = \forall y B$

- $(\exists y B)[t/x] = \exists y B[t/x]$ *if* $x \neq y$*, else* $\exists y B$*, when* $A = \exists y B$

Now that the first-order language to be extended has been defined, the first-order model theory to be extended will now be described.

First order semantics

The classical semantics to extend here will be the regular Tarskian set-theoretic model theory. To interpret variables, assignment functions that map variables to domain individuals will be employed. First-order structures are the foundation of these semantics:

Definition 17 (First-order structure)**.** *An $\mathcal{L}$-structure is a tuple $\mathcal{I} := \langle D, I \rangle$, where D is a non-empty, possibly infinite, set referred to as the domain of discourse, and I is a function called the interpretation function which assigns individuals, functions and predicates over D to the symbols in $\mathcal{L}$ such that:*

- *For every constant c, $c^{\mathcal{I}} \in D$ is an element in the domain.*
- *For every function symbol f of arity $n > 0$, $f^{\mathcal{I}} : D^n \longrightarrow D$ is an n-ary function.*
- *For every predicate symbol P of arity $n \geq 0$, $P^{\mathcal{I}} : D^n \longrightarrow \{\top, \bot\}$ is an n-ary predicate. Predicate symbols of arity 0 are interpreted as truth values.*

Variable assignment functions are used in order to interpret formulae with variables, and so is defined as:

Definition 18 (Variable assignments)**.** *Given an $\mathcal{L}$-structure, $\langle D, I \rangle$, a variable assignment s is any function that maps variables to elements in the domain of discourse: $s : \mathtt{V} \longrightarrow D$. The set of all such assignments is denoted by S^D.*

In order to assign truth to formulae in a first-order structure, first terms need to assigned a corresponding interpretation in a given structure:

Definition 19 (Value of terms)**.** *If t is a term in $\mathcal{L}$, and $\mathcal{I}$ is an $\mathcal{L}$-structure, and s is a variable assignment, then the value $Val_s^{\mathcal{I}}(t)$ of t in $\mathcal{I}$ with s is:*

- *if t is a constant c then $Val_s^{\mathcal{I}}(t) = c^{\mathcal{I}}$*
- *if t is a variable x then $Val_s^{\mathcal{I}}(t) = s(x)$*
- *if t is a function expression $f(t_1, ..., t_n)$ then $Val_s^{\mathcal{I}}(t) = f^{\mathcal{I}}(Val_s^{\mathcal{I}}(t_1), ..., Val_s^{\mathcal{I}}(t_n))$*

In order to interpret the truth of a formulae with variables bound by existential or universal quantification, the following convenience needs to be defined:

Definition 20. *For any structure $\mathcal{I} = \langle D, I \rangle$, and any variable assignment $s \in S^D$, then given any element $d \in D$, $s[x \coloneqq d]$ denotes the new assignment s' such that $s'(y) = s(y)$ for every $y \neq x$ and $s'(x) = d$.*

Finally, the satisfaction relation between first-order structures and first-order formulae may be defined:

Definition 21 (Satisfaction)**.** *Let $\phi, \alpha, \beta \in \mathcal{L}$, $\mathcal{I}$ be a first-order structure, and $s \in S$ a variable assignment function. Then $\mathcal{I}$ and s satisfy a first-order sentence ϕ, denoted $\mathcal{I}, s \models \phi$, for each of the following cases:*

- *if ϕ is $\top$, then $\mathcal{I}, s \models \phi$*
- *if ϕ is $\bot$, then $\mathcal{I}, s \not\models \phi$*
- *if ϕ is an atomic formula $P(t_1, ..., t_n)$, $\mathcal{I}, s \models \phi$ if and only if $\langle Val_s^{\mathcal{I}}(t_1), ..., Val_s^{\mathcal{I}}(t_n) \rangle \in P^{\mathcal{I}}$*
- *if ϕ is $t_1 \doteq t_2$ then $\mathcal{I}, s \models \phi$ if and only if $Val_s^{\mathcal{I}}(t_1) = Val_s^{\mathcal{I}}(t_n)$*
- *if ϕ is $\neg\alpha$ then $\mathcal{I}, s \models \phi$ if and only if $\mathcal{I}, s \not\models \alpha$*
- *if ϕ is $\alpha \wedge \beta$, then $\mathcal{I}, s \models \phi$ if and only if $\mathcal{I}, s \models \alpha$ and $\mathcal{I}, s \models \beta$*
- *if ϕ is $\alpha \vee \beta$, then $\mathcal{I}, s \models \phi$ if and only if $\mathcal{I}, s \models \alpha$ or $\mathcal{I}, s \models \beta$*
- *if ϕ is $\alpha \rightarrow \beta$, then $\mathcal{I}, s \models \phi$ if and only if $\mathcal{I}, s \not\models \alpha$ or $\mathcal{I}, s \models \beta$*
- *if ϕ is $\alpha \leftrightarrow \beta$, then $\mathcal{I}, s \models \phi$ if and only if $\mathcal{I}, s \models \alpha \rightarrow \beta$ and $\mathcal{I}, s \models \beta \rightarrow \alpha$*
- *if ϕ is $\forall x\alpha$, then $\mathcal{I}, s \models \phi$ if and only if for every $d \in D^{\mathcal{I}}$, $\mathcal{I}, s[x \coloneqq d] \models \alpha$*
- *if ϕ is $\exists x\alpha$, then $\mathcal{I}, s \models \phi$ if and only if there exists a $d \in D^{\mathcal{I}}$ such that $\mathcal{I}, s[x \coloneqq d] \models \alpha$*

A notion of validity, following from satisfaction, may also be defined:

Definition 22 (Validity)**.** *A formula $\phi \in \mathcal{L}$ is valid in an $\mathcal{L}$-structure $\mathcal{I}$, denoted $\mathcal{I} \models \phi$ if and only if $\mathcal{I}, s \models \phi$ for every $s \in S$.*

This definition of validity is extended to sets of formulae in the usual way: a set of formulae Γ is valid in a structure if every formula in Γ is valid.

The abbreviation $[\phi]$, defined such that $[\phi] \coloneqq \{\mathcal{I} \mid \mathcal{I} \models \phi\}$ for any $\phi \in \mathcal{L}$ will be used throughout this thesis: that is $[\phi]$ is shorthand for the model set satisfying ϕ.

Naturally, from this semantics the regular Tarskian classical inference relation is defined as usual:

Definition 23 (Classical entailment)**.** *A set of formulae $\Gamma \subseteq \mathcal{L}$ entails a formula ϕ, $\Gamma \vdash \phi$, if and only if for every structure $\mathcal{I}$ and variable assignment s such that $\mathcal{I}, s \models \Gamma$, it is also true that $\mathcal{I}, s \models \phi$.*

The abbreviation $\phi \vdash \psi$ may be used as a shorthand for $\{\phi\} \vdash \psi$, and similarly $\phi \dashv\vdash \psi$ is an abbreviation for $\phi \vdash \psi$ and $\psi \vdash \phi$.

It follows directly that classical entailment satisfies the following property:

- $\Gamma \vdash \phi$ implies $\Gamma \cup \{\psi\} \vdash \phi$ for any $\psi \in \mathcal{L}$ (Monotonicity)

Monotonicity, in general, is undesirable for defeasible reasoning, as has been discussed widely in the literature, for example by Kraus et al. (1990); Shoham (1988), as it ensures that the addition of information to a knowledge base never results in a previous consequence no longer being inferred.

Monotonic entailment is therefore inherently not defeasible, and most approaches to extending classical logics with uncertain, or common-sense, reasoning capabilities attempt to formalise alternative semantics and inference relations so as to be defeasible, and therefore nonmonotonic.

2.2 Ranking measure semantics

One of the approaches to extend the semantics of propositional logic to interpret default conditional logic and define defeasible inference relations is ranking measures (Spohn, 1988, 1990; Weydert, 1994, 2003). Ranking measures are coarse-grained quasi-probabilistic valuations. The basic strategy is to assign ranks from elements in a boolean algebra to a ordered group, the values of which represent a degree of belief in each element. In the context of propositional conditional logic, the default statements in the knowledge base provide the relative degrees of belief the ranking measure semantics should assign to the propositional valuations. Inference is then a matter of selecting a distinguished set of ranking measures over these valuations that guides the inference procedure.

First, ranking measure theory over an abstract boolean algebra will be defined (Weydert, 1994), before the application of this theory to propositional default reasoning will be covered (Weydert, 1996).

2.2.1 General Belief measures

The domain of a ranking measure is an arbitrary boolean algebra defined such that:

Definition 24 (Boolean algebra)**.** *Given a boolean algebra* $\mathcal{B} := (\mathbb{B}, \cup, \cap, -, \top, \bot)$ *consisting of a non-empty set* $\mathbb{B}$*, distinct elements* $\top, \bot \in \mathbb{B}$*, binary operations over* $\mathbb{B}$*,* $\cup$ *and* $\cap$*, and a unary operation,* $-$*, over* $\mathbb{B}$*.*

In the case of propositional default reasoning, this boolean algebra is the boolean sub-algebra of the propositional logic itself. When considering the particular complexities of first-order default reasoning, an important element of consideration will be how to rank tuples constructed from multiple boolean algebras. This will be achieved via product measures, the domain of which is the cross product of boolean algebras:

Definition 25 (Boolean algebra cross product)**.** *Let* $\mathcal{B}_1 := (\mathbb{B}_1, \cup, \cap, -, \top, \bot)$ *and* $\mathcal{B}_2 := (\mathbb{B}_2, \cup, \cap, -, \top, \bot)$ *be arbitrary boolean algebras, then* $\mathcal{B}_1 \otimes \mathcal{B}_2$ *is the Cartesian, or cross, product such that for every* $b_1 \in \mathcal{B}_1$*,* $b_2 \in \mathcal{B}_2$*,* $(b_1, b_2) \in \mathcal{B}_1 \otimes \mathcal{B}_2$*.*

Having defined the domain of a ranking measure, now attention is turned to the range. Ranking measures map elements of a boolean algebra to a ranking algebra, the requirements of which are:

Definition 26 (Ranking algebra)**.** *A ranking algebra* $\mathcal{V} = (\mathbb{V}, \oplus, \otimes, <, n, e)$ *is such that* $\mathbb{V}$ *is the non-negative half of a totally* $<$*-ordered commutative group with distinct elements* e *and* n*, binary operations* $\oplus$ *and* $\otimes$ *over* $\mathbb{V}$ *and such that* e *is the identity element with respect to* $\otimes$*, and* n *is* $<$*-minimal and absorptive for* $\otimes$*.*

The ordered group, or ranking algebra, may satisfy various properties depending on the expressivity and operations required from the ultimate model theory, but does require distinguished elements representing total belief and total disbelief. In general, it may be seen as an abstract model of the non-negative rational numbers extended by an infinite element. In practice, in this thesis the non-negative real numbers (extended by infinity) will be used as the ranking algebra of choice, offering the distinguished element 0 used for the absence of disbelief, an infinite element used for impossibility, an additive structure, and divisibility.

Now, the general structure of a ranking measure may be defined:

Definition 27 (Real-valued ranking measures)**.** *A real-valued ranking measure is a function $R : \mathcal{B} \longrightarrow \mathcal{V}$ such that $\mathcal{V} = \mathbb{R} \cup \{\infty\}$, and for $\top, \bot, a, b \in \mathcal{B}$:*

- $R(\top) = 0$, $R(\bot) = \infty$,
- $R(a \cup b) = \min_{<}\{R(a), R(b)\}$,

How ranking measures specify the inference process is tied to the particular case of the conditional ranking measure:

Definition 28 (Conditional ranking)**.** *Let $R : \mathcal{B} \longrightarrow V$ be a ranking measure. Then, the conditional ranking measure $R(\cdot\,|\cdot\,) : \mathcal{B} \times \mathcal{B} \longrightarrow V$ is defined such that $R(a \mid b) = R(b \cap a) - R(b)$ if $R(b) \neq \infty$ and $R(a \mid b) = \infty$ if $R(b) = \infty$.*

A distinguished ranking measure that forms the basis of constructible inference techniques is the uniform measure:

Definition 29 (Uniform 0-measure)**.** *The distinguished ranking measure R_0 is the ranking measure such that for every $A \in \mathcal{B}$, $R_0(A) = 0$.*

Simply put, R_0 is the ranking measure that assigns 0 to every element of the domain.

For first-order default reasoning, product ranking measures are an important construct, definable as:

Definition 30 (Product ranking measures)**.** *For any ranking measures $R_1, \ldots, R_n$ over finite boolean algebras $\mathcal{B}_1, \ldots, \mathcal{B}_n$, the product ranking measure $R_1 \otimes \ldots \otimes R_n$ is defined to be the unique ranking measure R on $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$ such that for each $(A_1, \ldots, A_n)$ with $A_i \in \mathcal{B}_i$:*

$$R(A_1, \ldots, A_n) = R(A_1) + \ldots + R(A_n).$$

Product measures establish the relationship between the rank of an element in a boolean algebra and the resultant ranks of the elements in the Cartesian products in which that element appears. If there is a A in a boolean algebra $\mathcal{B}_1$, then the rank of A can reasonably be expected to influence the rank of the element $(A, C) \in \mathcal{B}_1 \times \mathcal{B}_2$. It is possible that there are circumstances in which this may not be desired to hold, but this thesis will deal with a context in which it is desirable, and therefore these descriptions of product ranking measures are crucial.

Conjecture 1 (Product ranking measures)**.** *For any finite tuple of ranking measures R_i over $\mathcal{B}_i$, the corresponding product ranking measure exists and is canonical over $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$*

The following property ensures that ranking measures assign the infinite element of the ranking algebra to all elements of the boolean algebra that evaluate to falsum:

Definition 31 (Possibility conservation)**.** *A ranking measure R is said to be possibility-conserving if for each $A \in \mathcal{B}$, $R(A) = \infty$ implies $A = \bot$.*

In general, an equivalence between false formulae, or empty predicates, and a ranking measure valuation of infinity is maintained. This maintains a link between a classical notion of impossibility and a defeasible one.

2.2.2 Propositional default reasoning

Let $\mathcal{L}_{PROP}$ be the standard propositional language over some arbitrary set of atomic symbols PS, the set of connectives $\{\neg, \vee, \wedge, \rightarrow, \top, \bot\}$ and the auxiliary symbols $\{(,)\}$. The set of all propositional $\mathcal{L}_{PROP}$-valuations is denoted W.

The general belief measures defined in the previous section may be used to extend the semantics of propositional logic for the task of interpreting default conditionals: syntactic constructions for the purpose of representing defeasible knowledge. That is, statements such " if x then typically y", which express information regarding a conditional degree of belief.

Definition 32 (Propositional default conditionals)**.** *If $\phi, \psi \in \mathcal{L}_{PROP}$ and $s \in [0, \infty]$ then a conditional is an expression $\phi \Rightarrow_s \psi$. Sets of conditionals are denoted with Δ.*

Definition 33 (Ranking measure model)**.** *A ranking measure model is a function* $R : \mathcal{B}_{\mathcal{L}_{PROP}} \longrightarrow [0, \infty]_{\mathbb{R}}$ *such that* $\mathcal{B}_{\mathcal{L}_{PROP}} = \{[\phi] \mid \phi \in \mathcal{L}_{PROP}\}$ *mapping each element of the propositional Boolean algebra to a non-negative real number, or infinity.*

In general, $R([\phi])$ will be abbreviated by $R(\phi)$ for any formula ϕ. Then, satisfaction between the propositional language and these ranking measure semantics are defined such that:

Definition 34 (Ranking measure satisfaction)**.** *A ranking measure model satisfies a conditional* $R \models \phi \Rightarrow_s \psi$ *if and only if:*

- $R(\phi \wedge \psi) + s \leq R(\phi \wedge \neg\psi)$ *(Strong semantics)*
- $R(\phi \wedge \psi) < R(\phi \wedge \neg\psi)$ *(Weak semantics)*

This notation is extended for R *satisfying sets of conditionals* Δ *in the natural way.*

The set of ranking measures satisfying a defeasible knowledge base Δ is denoted $Mod(\Delta)$. For singleton sets of defaults $\{\delta\}$, $Mod(\{\delta\})$ is abbreviated $Mod(\delta)$. With the model set of a default base defined, a monotonic Tarskian inference relation is straightforwardly defined:

Definition 35 (Monotonic ranked model inference)**.** *The monotonic inference relation* $\vdash$ *over the language of defaults is defined such that* $\Delta \vdash \delta$ *if and only if* $Mod(\Delta) \subseteq Mod(\delta)$ *between default statements, and for any* $\Gamma \subseteq \mathcal{L}_{PROP}$*,* $\phi \in \mathcal{L}_{PROP}$*,* $\Gamma \vdash_\Delta \phi$ *if and only if for every* $R \in Mod(\Delta)$*,* $R(\neg\phi \mid \bigwedge \Gamma) = \infty$*.*

However, monotonic inference relations are, after all, not defeasible (Shoham, 1988). What is sought after in this work is to have an inference relation $\mid\!\sim$ such that even if $A \mid\!\sim B$ for arbitrary formulae A and B, it is still possible for the presence of a new formula C that $A \wedge C \not\mid\!\sim B$ — that the conclusion B may be no longer inferred. This pattern of reasoning is at odds with a monotonicity as a property of the inference relation.

So, in order to define such a nonmonotonic inference relation, a summary of the ranking choice approach to defining defeasible entailment:

Definition 36 (Ranking choice functions (Weydert, 2003))**.** $\mathcal{F}$ *is a ranking choice function if and only if for each finite default knowledge base* Δ*:*

1. $\mathcal{F}(\Delta) \subseteq Mod(\Delta)$ *(Choice)*
2. $\{Mod(\delta) \mid \delta \in \Delta\} = \{Mod(\delta') \mid \delta' \in \Delta'\}$ *implies* $\mathcal{F}(\Delta) = \mathcal{F}(\Delta')$ *(Local semanticality)*
3. $Mod(\Delta) \neq \emptyset$ *implies* $\mathcal{F}(\Delta) \neq \emptyset$ *(Nonmonotonic consistency)*

Then, using ranking choice, general defeasible inference may be specified:

Definition 37 (Defeasible inference (Weydert, 2003))**.** *If* $\mathcal{F}$ *is a ranking choice function and* Δ *a satisfiable defeasible knowledge base, then for some finite set of propositional facts* Γ *and an arbitrary propositional formula* α*,* $\Gamma \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha$ *if and only if for every* $R \in \mathcal{F}(\Delta)$*,* $R(\neg\alpha \mid \bigwedge\Gamma) > 0$*.*

Defeasible inference relations defined using ranking choice satisfy certain properties from the general literature on defeasible reasoning:

Definition 38 (Properties for $\mathrel{|\!\sim}$)**.** *Given any satisfiable default knowledge base* Δ *and arbitrary formulae* ϕ*,* ψ*,* α*, then the following postulates from (Weydert, 2003) characterize different defeasible inference relations defined with respect to a ranking choice function* $\mathcal{F}$*, providing points of comparison with postulates and default entailment schemes in the literature:*

- $\dfrac{\phi \vdash \psi}{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi}$ *(Supraclassicality)*
- $\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \phi$ *(Reflexivity (REF))*
- $\dfrac{\phi \dashv\vdash \psi,\ \phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}{\psi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}$ *(Left logical equivalence (LLE))*
- $\dfrac{\psi \vdash \alpha,\ \phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi}{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}$ *(Right weakening (RW))*
- $\dfrac{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi,\ \phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi \wedge \alpha}$ *(AND)*
- $\dfrac{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi,\ \alpha \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi}{\phi \vee \alpha \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi}$ *(OR)*
- $\dfrac{\phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \psi,\ \phi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}{\phi \wedge \psi \mathrel{|\!\sim}^{\mathcal{F}}_{\Delta} \alpha}$ *(Cautious monotonicity (CM))*

The above postulates form the meta-level default inference version of preferentiality, as defined by (Kraus et al., 1990). Any ranking choice function $\mathcal{F}$ that induces a default inference relation $\mid\!\sim^{\mathcal{F}}$ is said to be preferential if it satisfies all of the above. That is:

- $\mid\!\sim^{\mathcal{F}}$ *satisfies Supraclassicality, LLE, RW, and AND* *(Logicality (LOG))*
- $\mid\!\sim^{\mathcal{F}}$ *satisfies LOG, CM, and OR and is referred to as preferential (Preferentiality (PREF))*

It is further said to be rational if it, in addition to being preferential, satisfies the following, from (Lehmann and Magidor, 1992):

- $\frac{\phi\mid\!\sim^{\mathcal{F}}_{\Delta}\psi,\ \phi\not\mid\!\sim^{\mathcal{F}}_{\Delta}\neg\alpha}{\phi\wedge\alpha\mid\!\sim^{\mathcal{F}}_{\Delta}\psi}$ *(Rational monotonicity (RM))*

(Weydert, 2003) further enumerates alternative, and additional, properties that a ranking choice function may be desired to fulfil:

- $\frac{\Delta\vdash\phi\Rightarrow_s\psi}{\phi\mid\!\sim^{\mathcal{F}}_{\Delta}\psi}$ *(Strong defeasible modus ponens (DMP))*
- $\frac{\phi\Rightarrow_s\psi\dashv\vdash\phi'\Rightarrow\psi'}{\mid\!\sim^{\mathcal{F}}_{\Delta\cup\{\phi\Rightarrow_s\psi\}}=\mid\!\sim^{\mathcal{F}}_{\Delta\cup\{\phi'\Rightarrow_s\psi'\}}}$ *(Local logical invariance (LLI))*
- $\frac{\Delta\dashv\vdash\Delta'}{\mid\!\sim^{\mathcal{F}}_{\Delta}=\mid\!\sim^{\mathcal{F}}_{\Delta'}}$ *(Global logical invariance (GLI))*
- $\Delta \nvdash \phi \Rightarrow_s \bot$ *implies* $\phi \not\mid\!\sim^{\mathcal{F}}_{\Delta} \bot$ *(Consistency preservation (CP))*
- *if* $\mathcal{L}_1$, $\mathcal{L}_2$ *are two propositional sublanguages over disjoint variable sets, then for any finite* $\Gamma_i \subseteq \mathcal{L}_i$, $\Delta_i \subseteq \mathcal{L}_i(\Rightarrow)$, $\Gamma_1 \cup \Gamma_2 \mid\!\sim^{\mathcal{F}}_{\Delta_1\cup\Delta_2} \phi$ *iff* $\Gamma_1 \mid\!\sim^{\mathcal{F}}_{\Delta} \phi$ *for any satisfiable* $\Gamma_2 \cup \Delta_2$ *(Strong irrelevance (IRR))*
- *Let* $\pi : B_1 \longrightarrow B_2$ *be an isomorphism between two boolean sub-algebras* $B_1, B_2 \subseteq B$ *and* $\pi' : \mathcal{L} \longrightarrow \mathcal{L}$ *be an associated syntactic map over the propositional syntax, such that* $[\pi'(\phi)] = \pi([\phi])$. *Then, if* $\Gamma^{\pi'} = \{\pi'(\phi) \mid \phi \in \Gamma\}$ *and* $\Delta^{\pi'} = \{\pi'(\phi) \Rightarrow_s \pi'(\psi) \mid \phi \Rightarrow_s \psi \in \Delta\}$ *then* $\Gamma \mid\!\sim^{\mathcal{F}}_{\Delta} \psi$ *if and only if* $\Gamma^{\pi'} \mid\!\sim^{\mathcal{F}}_{\Delta^{\pi'}} \pi'(\psi)$ *(Representation independence (RI))*

The question is then how to reasonably define these ranking choice functions such that the above properties may be satisfied, or ignored, in service of specifying particular defeasible inference relations. Here, the answer is the methodology of constructible inference relations: a ranking choice function defined by building a set of ranking measures according to the default knowledge base provided.

The essential device used in constructible inference techniques is shifting, an operation on ranking measures that increases the rank of the models of a proposition A:

Definition 39 (Shifting (Weydert, 2003)). *Shifting is a function mapping a ranking measure R to a ranking measure R^*, such that if $\mathcal{B}$ is a boolean algebra, $A \in \mathcal{B}$, $a \in [0, \infty]$, then $R^* = R + aA$ is a shifting operation such that for all $B \in \mathcal{B}$:*

- $R^*(B) = \texttt{min}\{R(B \wedge A) + a, R(B \wedge \neg A)\}$

The result of a shifting operation, such as $R + aA$, is that the rank of the formula to be shifted, A, is at least a.

Shifting is the primary operation that leads to the specific subset of defeasible inference — that of constructible inference relations, which may be seen as specific ranking choice functions satisfying certain desiderata:

Definition 40 (Constructibility (Weydert, 1998)). *If Δ is a defeasible knowledge base $\{\phi_i \Rightarrow \psi_i \mid i \leq n\}$, and $a_i \in [0, \infty]$ then a ranking measure R is constructible over Δ if and only if $R = R^0 + a_0[\phi_0 \wedge \neg\psi_0] + ... + a_n[\phi_n \wedge \neg\psi_n]$. $Constr(\Delta)$ is the set of all ranking measures constructible over Δ.*

The intuition with constructible defeasible inference is that holding a defeasible knowledge base as a collection of default statements containing information regarding what should be conditionally expected to be true, then a strategy of iterating through these statements and algorithmically generating a semantic interpretation that guides the defeasible inference process should be an optimal method of generating new beliefs. Constructible inference can be defined as:

Definition 41 (Constructible inference). *A defeasible inference relation $\mid\!\sim^{\mathcal{F}}$ is constructible if and only if $\mathcal{F}(\Delta) \subseteq Mod(\Delta) \cap Constr(\Delta)$*

That is, a defeasible inference relation is constructible if it is specified by a set of ranking measures, each of which is constructible over the default knowledge base.

One defeasible inference relation that permeates the literature is that of rational closure (Lehmann and Magidor, 1992) also referred to as System Z (Pearl, 1990). This ranking measure semantics allows for a specification of System Z much as was done by (Pearl, 1990) and (Goldszmidt and Pearl, 1992b). A semantic construction of this defeasible inference may be done as follows:

Definition 42 ($\preceq$-order over measures). *If R_1, R_2 are ranking measures, then $R_1 \preceq R_2$ if and only if $R_1(\phi) \leq R_2(\phi)$ for all $\phi \in \mathcal{L}$.*

With this ordering, the following result has been proved in the literature from, for example, (Giordano et al., 2015) and (Freund, 1998):

Proposition 1 (Z-rank existence). *If Δ is a satisfiable default knowledge base, then the minimal ranking measure model with respect to $\prec$ is unique and exists, and is denoted R^Z_Δ.*

Then, rational closure or System Z, is simply the defeasible inference relation such that:

Definition 43 (System Z). *If Δ is a default knowledge base, Γ a set of facts, and ϕ an arbitrary formula, then $\Gamma \mathrel{|\!\!\sim}^Z_\Delta \phi$ if and only if $R^Z_\Delta(\neg\phi \mid \bigwedge \Gamma) > 0$.*

An example of System Z in action:

Example 3 (System Z). *Let the following be a default knowledge base:*

- $B \Rightarrow F$
- $B \Rightarrow W$
- $P \Rightarrow \neg F$
- $P \wedge \neg B \Rightarrow \bot$

These statements induce the following strong ranking measure constraints:

- $R^Z(B \wedge \neg F) = 1$
- $R^Z(B \wedge \neg W) = 1$

- $R^Z(P \wedge F) = 2$
- $R^Z(P \wedge \neg B) = \infty$

Which leads to the following examples of defeasible inference relations:

- $P \mid\!\sim^Z_\Delta \neg F$
- $B \mid\!\sim^Z_\Delta \neg P$

Notably, in the above example 3, $P \not\mid\!\sim^Z_\Delta W$, due to System Z not verifying exceptional inheritance:

Definition 44 (Exceptional inheritance). *For the formulae ψ, ψ' that are logically independent given ϕ, and the default knowledge base $\Delta \coloneqq \{\phi \Rightarrow \psi, \phi \Rightarrow \psi'\}$:*

- $\{\phi, \neg\psi\} \mid\!\sim_\Delta \psi'$

Turning attention back to constructibility, the foundational constructible inference relation is System J (Weydert, 1996) defined as:

Definition 45 (System J). *If Δ is a default knowledge base, Γ a set of facts, and ϕ an arbitrary formula, then $\Gamma \mid\!\sim^J_\Delta \phi$ if and only if for every $R \in Mod(\Delta) \cap Constr(\Delta)$, $R(\neg\phi \mid \bigwedge \Gamma) > 0$.*

Example 4 (System J). *Let the following be a default knowledge base:*

- $B \Rightarrow F$
- $B \Rightarrow W$
- $P \Rightarrow \neg F$
- $P \wedge \neg B \Rightarrow \bot$

These statements induce the following strong ranking measure constraints, for $\alpha, \beta, \gamma \in (0, \infty)_{\mathbb{R}}$, such that every $R \in Mod(\Delta) \cap Constr(\Delta)$ needs to satisfy:

- $R(B \wedge F) + \alpha \leq R(B \wedge \neg F)$
- $R(B \wedge W) + \beta \leq R(B \wedge \neg W)$

- $R(P \wedge \neg F) + \gamma \leq R(P \wedge F)$
- $R(P \wedge B) + \infty \leq R(P \wedge \neg B)$

Which leads to the following examples of defeasible consequences:

- $P \mid\!\sim^{J}_{\Delta} \neg F$
- $P \mid\!\sim^{J}_{\Delta} W$
- $B \mid\!\sim^{J}_{\Delta} \neg P$

This concludes the summary of the various concepts from the literature upon which the rest of this thesis will be building. In the next chapter, extending classical first-order logic with these default reasoning concepts will be defined, as the first primary contribution of this work.

Chapter 3

Default quantifier first order logic

As discussed in section 1.2, there are numerous representational and reasoning challenges in defeasible first-order logic. Formalising statements such as "elements in X are typically also included in Y" comprises one of the primary goals of this thesis of extending classical logics with common-sense representations and reasoning. For the propositional case, conditional operators have been investigated extensively (Lehmann and Magidor, 1992; Boutilier, 1994; Gabbay, 1985; Goldszmidt and Pearl, 1992a; Weydert, 2003) as the primary vehicle for extending these logics with plausibility notions.

3.1 Default quantifiers

The primary syntactic focus of this work are default quantifiers (Schlechta, 1995; Weydert, 1997): variable binding binary connectives that convey the intuition of expressing what is plausibly the case. These variable binding conditional operators can be viewed as generalizations of propositional default conditionals that are appropriate in the first-order context, where this syntax more naturally captures which variable names are tied to the default information expressed by the formula.

Definition 46 (Default quantifier statements)**.** *A default quantifier formula is an expression $\alpha \Rightarrow_{\vec{x}} \beta$ such that $\alpha, \beta \in \mathcal{L}$ and $\vec{x}$ is a vector of variables.*

Default quantifiers are an object-level extension of first-order logic. The intention is that these default quantifiers extend the classical language with explicitly defeasible statements.

This represents a generalisation of propositional conditionals — propositional conditionals may be embedded using single variable default quantifier statements. Being able to bind various numbers of variables to the conditional operator allows for a great deal of control in expressing different typicality statements.

In particular, this allows for the necessary nuance when representing first-order defeasible information. Consider the difference between the following statements:

1. $Adult(x) \Rightarrow_x \exists y(Child(y) \wedge TakesCareOf(x, y))$
2. $Adult(x) \Rightarrow_{xy} Child(y) \wedge TakesCareOf(x, y)$

The first one operates over the set of individuals, and says simply that if an individual is an adult, then it should be expected that there exists a child that they take care of. The second, since it binds a pair of variables, is a statement expressing a conditional degree of belief about what is true for a pair of elements. That is, if there is a pair of elements, such that the left hand element is an adult then the expectation is that the other is a child, and it should be expected that the adult takes care of the child. This is contrasted with the former, on the other hand, which expresses information about individuals, specifically about what are normal adults.

An appropriate defeasible first order language is therefore just this language of default quantifiers. It should be noted that here nested default statements are not considered, and so therefore the language is straightforwardly defined:

Definition 47 (Default quantifier first-order language)**.** *The language $\mathcal{L}(\Rightarrow)$ is the set of all formulae $\alpha \Rightarrow_{\vec{x}} \beta$ such that $\alpha, \beta \in \mathcal{L}$ and $\vec{x}$ is an arbitrary vector of variables in* $\mathtt{V}$.

Since these default quantifiers are intended to be a lifting, or a generalisation, of propositional default conditional logic, appropriate properties from the literature on default conditionals should sensibly also be lifted to this context. A semantics to interpret $\mathcal{L}(\Rightarrow)$ intending to truly generalise existing approaches in default reasoning should at least verify well known properties on the propositional level (Lehmann and Magidor, 1992; Gabbay, 1985). These properties may be lifted to the language of $\mathcal{L}(\Rightarrow)$ as follows:

Definition 48 (Default conditional properties)**.** *The following properties represent default quantifier versions of the preferentiality postulates:*

1. $\frac{\phi(x) \Rightarrow_x \bot(x)}{\forall x \neg\phi(x)}$ *(Impossibility)*
2. $\frac{\phi \Rightarrow_{\vec{x}} \psi}{\phi[\vec{x}/\vec{y}] \Rightarrow_{\vec{y}} \psi[\vec{x}/\vec{y}]}$ *for $\vec{y}$ not in ϕ, ψ and length$(\vec{x})$ = length$(\vec{y})$ (Renaming)*
3. $\phi(\vec{x}) \Rightarrow_{\vec{x}} \phi(\vec{x})$ *(Reflexivity)*
4. $\frac{\forall x(\phi(x) \rightarrow \psi(x))}{\phi(x) \Rightarrow_x \psi(x)}$ *(Supraclassicality)*
5. $\frac{\phi(x) \Rightarrow_x \psi(x), \forall x(\psi(x) \rightarrow \alpha(x))}{\phi(x) \Rightarrow_x \alpha(x)}$ *(Right weakening (RW))*
6. $\frac{\phi(x) \Rightarrow_x \psi(x), \forall x(\phi(x) \leftrightarrow \alpha(x))}{\alpha(x) \Rightarrow_x \psi(x)}$ *(Left logical equivalence (LLE))*
7. $\frac{\phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x}), \phi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})}{\phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x}) \wedge \alpha(\vec{x})}$ *(AND)*
8. $\frac{\phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x}), \alpha(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x})}{\phi(\vec{x}) \vee \alpha(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x})}$ *(OR)*
9. $\frac{\phi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x}), \phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x})}{\phi(\vec{x}) \wedge \psi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})}$ *(Cautious monotonicity (CM))*

Default quantifiers satisfying the above properties are referred to as preferential.

Additionally, the following two rationality properties:

- $\frac{\phi(\vec{x}) \vee \psi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})}{\phi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x}) \text{ or } \psi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})}$ *(Disjunctive rationality (DR))*
- $\frac{\phi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x}), \phi(\vec{x}) \not\Rightarrow_{\vec{x}} \neg\psi(\vec{x})}{\phi(\vec{x}) \wedge \psi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})}$ *(Rational monotonicity (RM))*

Represent extensions of preferentiality referred to as disjunctive rationality, if DR is satisfied but not RM, or rationality if RM is satisfied.

Most of the properties above are adapted from the literature (Lehmann and Magidor, 1992; Weydert, 2003), where they are well established postulates for defeasible connectives. Of note is the renaming postulate, which requires a notion of *permutation invariance*: that the specific variables in the vector are irrelevant to the semantic interpretation of the default. That is given the two defaults:

- $\phi \Rightarrow_{xy} \psi$
- $\phi \Rightarrow_{yx} \psi$

Then the semantic interpretation should be the same, and the specific variable names do not cause them to be evaluated differently. Indeed, the two defaults above should be interchangeable and completely equivalent. In general, the precise variable names chosen in each default should not affect the outcome of the semantic interpretation of a knowledge base.

For default quantifiers to reasonably be considered to correspond to a first-order counterpart to propositional defaults, it would be expected that they satisfy at least the preferential properties in the definition above.

For ease of representing information regarding variables and constant names, a named predicate symbol is introduced as the singleton set representing a constant:

Definition 49 (Constant predicates). *For any constant name c, the unary predicate name P_c is a distinguished predicate name representing a variable being equal to a constant. That is, $P_c(x)$ if and only if $x = c$.*

Default quantifiers require an extension to the notion of free variables:

Definition 50 (Free variables). *Given a default quantifier statement $\phi \Rightarrow_{\vec{x}} \psi$, then $FV(\phi \Rightarrow_{\vec{x}} \psi) = FV(\phi) \cup FV(\psi) - \vec{x}$.*

It should be noted that while the definition of closed formulae is the same in $\mathcal{L}(\Rightarrow)$ as in classical logic, since the definition of a free variable is adjusted, this affects which formulae are closed. The sentence $\phi(x, y) \Rightarrow_x \psi(x, y)$ is not closed, while $\phi(x, y) \Rightarrow_{xy} \psi(x, y)$ is closed.

A finite collection of closed default quantifier formulae together constitutes a default quantifier knowledge base, which will also be referred to as a defeasible knowledge base:

Definition 51 (Default quantifier knowledge base). *A default quantifier knowledge base $\Delta \subset \mathcal{L}(\Rightarrow)$ is a finite set of default quantifier statements $\alpha_i \Rightarrow_{\vec{x}_i} \beta_i$.*

An example from the literature (Delgrande, 1998) that formulates the relationships between elephants and their keepers is given below:

Example 5 (Elephant zoo). *Given the following natural language statements:*

- *Elephants usually like their keepers.*
- *Elephants usually do not like the keeper Fred.*
- *Clyde the elephant does usually like the keeper Fred.*

Using the signature $Elephant(x)$ = *"x is an elephant",* $Keeper(x)$ = *"x is an elephant keeper", and* $Likes(x, y)$ = *"x likes y"*
(Delgrande, 1998) formulates these statements as:

- $Elephant(x) \wedge Keeper(y) \Rightarrow_{xy} Likes(x, y)$
- $Elephant(x) \wedge Keeper(Fred) \Rightarrow_{x} \neg Likes(x, Fred)$
- $Elephant(Clyde) \wedge Keeper(Fred) \Rightarrow Likes(Clyde, Fred)$

Using default quantifiers and constant predicates, it may be formalised as:

- $Elephant(x) \wedge Keeper(y) \Rightarrow_{xy} Likes(x, y)$
- $Elephant(x) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} \neg Likes(x, y)$
- $(Elephant(x) \wedge P_{Clyde}(x)) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} Likes(x, y)$

One aspect of this example that is noteworthy in this framework is the care in using predicates formed from constants. Contrasting with the formulation from (Delgrande, 1998), where each successive statement drops a variable from the conditional connective, it may be observed that like this the statement loses the two dimensional semantics regarding the most normal tuples in the domain, becoming a one dimensional statement that regards only the most normal elephants, in the second statement, and in the final statement, it seems indistinguishable from a classical implication. This is not to say that in this framework the latter meaning cannot be captured, but rather that this syntax is flexible enough to accommodate various nuances when defeasibly reasoning about singletons or n-tuples.

3.2 Ranking first order interpretations

The fundamental semantic extension to define a model theory for default quantifiers studied here are ranking measures Weydert (1994), which are defined here with respect to the powerset of the variable substitution functions of a first-order interpretation.

Definition 52 (Ranking measures over 2^S). *A ranking measure* $R : 2^S \longrightarrow ([0, \infty]_{\mathbb{R}}, 0, \infty, +, <)$ *having as the domain the powerset of all variable assignments* S *with respect to a first order structure* $\mathcal{I}$*, such that:*

- $R(S) = 0$
- $R(\emptyset) = \infty$
- *for any* $S', S'' \in 2^S$*,* $R(S' \cup S'') = \min(R(S'), R(S''))$

Defining ranking measures for first-order default quantifier semantics with reference to variable assignment functions is motivated by the fact that default quantifiers are a type of arbitrary quantifier that binds finite sequences of variables, and therefore should be semantically linked to sets of variable substitutions.

These ranking measures are defined over the powerset of the set of variable substitutions, rather than just the set itself, as it provides a straightforward algebra with boolean operations over which to define the measure, and will also prove useful in the case of extensions and generalisations to this framework (e.g. nested default quantifier statements.)

The ranking algebra chosen for this framework is the non-negative half of the real number set extended by an infinite element. The necessary components are a distinguished zero and an infinite value, an additive structure, and density. The first is necessary as those distinguished elements are the measures for those sets that do not have any disbelief associated with them, and the second for those that are impossible. Additivity is necessary for the definition of product measures and inference relations based on constructibility to follow below, and density allows for defining various sophisticated defeasible inference relations.

These ranking measures over the variable substitutions are paired with a first-order structure to form the elementary semantic structures for interpreting default expressions.

Definition 53 (Ranked structure). *A ranked structure over* $\mathcal{L}$ *is a pair* $\langle \mathcal{I}, R \rangle$ *consisting of a first-order interpretation* $\mathcal{I} = \langle D, I \rangle$ *over* $\mathcal{L}$ *and a ranking measure* R *over the powerset of all variable assignments* 2^S*.*

Classical first-order structures are extended with a ranking measure over the powerset variable substitutions of that interpretation. $\mathbb{I}$ denotes the set of all ranked structures.

Ranked structures also need to satisfy constant predicates, introduced in the previous section for the purposes of incorporating constants into defaults while preserving the cardinality of the bound vector of variables, and so the criterion for satisfaction of these atomic constant predicates is presented in the following definition:

Definition 54 (Satisfaction of constant predicates)**.** *For a constant* c*, a ranked structure* $\langle \mathcal{I}, R \rangle$ *and a variable assignment* s *satisfies the associated constant predicate* $P_c(x)$*,* $\mathcal{I}, s \models P_c(x)$ *if and only if* $s(x) = c^{\mathcal{I}}$*.*

A ranked structure satisfies a default quantifier sentence, with respect to an arbitrary interpretation, if the set of variable substitutions for the variable names bound to the default quantifier, with respect to that interpretation, that verify the satisfaction set of the default statement is ranked lower than the set of variable substitutions that satisfy the violation set:

Definition 55 (Satisfaction of default quantifiers)**.** *Let* $\phi \Rightarrow_{\vec{x}} \psi$ *be a sentence of* $\mathcal{L}(\Rightarrow)$*,* $\langle \mathcal{I}, R \rangle$ *a ranked structure, and* S *a set of variable assignments for* $\mathcal{I}$*, then* $\langle \mathcal{I}, R \rangle \models \phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x})$ *if and only if for every* $S' := \{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \psi(\vec{x})\}$ *and* $S'' := \{s \mid \mathcal{I}, s \not\models \phi(\vec{x}) \wedge \neg\psi(\vec{x})\}$*, for some* $a \in [1, \infty]_{\mathbb{R}}$*:*

- *then* $R(S') < R(S'')$ *(Weak semantics)*
- *then* $R(S') + a \leq R(S'')$ *(Strong semantics)*

The parameter a *for the strong semantics satisfaction requirement is taken by default to be* 1*, if not otherwise specified.*

Satisfaction of defeasible knowledge bases is straightforwardly defined with respect to ranked structures as above:

Definition 56 (Satisfaction of defeasible knowledge bases)**.** *Let* Δ *be a set of sentences of* $\mathcal{L}(\Rightarrow)$*, then a ranked structure* $\langle \mathcal{I}, R \rangle$ *satisfies* Δ *if and only if it satisfies every sentence of* Δ*.*

The following definition is an important convenience:

Definition 57. *Let* $\phi \in \mathcal{L}$ *be an arbitrary formula, and* $\langle \mathcal{I}, R \rangle$ *a ranked structure, then* $\widehat{\phi^{\mathcal{I}}} := \{s \mid \mathcal{I}, s \models \phi\}$*.*

Firstly, logical equivalence clearly results in equivalent ranks:

Lemma 1 (Equivalence)**.** *For any $\phi, \psi \in \mathcal{L}$, if $\phi \dashv\vdash \psi$, then for any $\langle \mathcal{I}, R \rangle$, $R(\widehat{\phi^{\mathcal{I}}}) = R(\widehat{\psi^{\mathcal{I}}})$.*

Proof. Since ϕ and ψ are logically equivalent, then the same set S of variable assignments will satisfy both formulae in any structure $\mathcal{I}$, and so $R(\{s \mid \mathcal{I}, s \models \phi\}) = R(\{s \mid \mathcal{I}, s \models \psi\})$. □

Then, the following lemma is clear.

Lemma 2. *If $\phi, \psi \in \mathcal{L}$, then for any ranked structure $\langle \mathcal{I}, R \rangle$, $\widehat{(\phi \wedge \psi)}^{\mathcal{I}} = \widehat{\phi^{\mathcal{I}}} \cap \widehat{\psi^{\mathcal{I}}}$.*

Proof. The set $\widehat{(\phi \wedge \psi)}^{\mathcal{I}}$ is comprised of, for some $\mathcal{I}$, every s such that $\mathcal{I}, s \models \phi \wedge \psi$, whereas $\widehat{\phi^{\mathcal{I}}} \cap \widehat{\psi^{\mathcal{I}}} := \{s \mid \mathcal{I}, s \models \phi\} \cap \{s \mid \mathcal{I}, s \models \psi\}$. Clearly, given any $\mathcal{I}$, then $s \in \{s \mid \mathcal{I}, s \models \phi\} \cap \{s \mid \mathcal{I}, s \models \psi\}$ if and only if $\{s \mid \mathcal{I}, s \models \phi \wedge \psi\}$. □

Along with the disjunctive version:

Lemma 3. *If $\phi, \psi \in \mathcal{L}$, then for any ranked structure $\langle \mathcal{I}, R \rangle$, $\widehat{(\phi \vee \psi)}^{\mathcal{I}} = \widehat{\phi^{\mathcal{I}}} \cup \widehat{\psi^{\mathcal{I}}}$.*

Proof. For any $\mathcal{I}$, the set $\widehat{(\phi \vee \psi)}^{\mathcal{I}}$ is comprised of every s such that $\mathcal{I}, s \models \phi \vee \psi$, whereas $\widehat{\phi^{\mathcal{I}}} \cup \widehat{\psi^{\mathcal{I}}} := \{s \mid \mathcal{I}, s \models \phi\} \cup \{s \mid \mathcal{I}, s \models \psi\}$. Clearly, $\{s \mid \mathcal{I}, s \models \phi\} \cup \{s \mid \mathcal{I}, s \models \psi\} = \{s \mid \mathcal{I}, s \models \phi \vee \psi\}$ □

The ranked structure semantics defined for interpreting default quantifiers validate the postulates in definition 48:

Theorem 1 (Preferential postulates)**.** *Default quantifier logic interpreted with ranked structures under the weak semantics verifies all the preferentiality postulates in definition 48, and the rationality postulates under the strong semantics.*

Proof. 1. (Impossibility) $\phi(x) \Rightarrow_x \bot$ induces the constraint, for every $\mathcal{I}$, $R(\{s \mid \mathcal{I}, s \models \phi(x) \wedge \bot\}) < R(\{s \mid \mathcal{I}, s \models \phi(x) \wedge \top)$ which is only satisfiable if $R(s \mid \mathcal{I}, s \models \phi(x)) = \infty$, and so therefore $\neg\phi(x)$ is true for all x, since $R(s \mid \mathcal{I}, s \models \phi(x)) = \infty$ implies that there are no possible substitutions satisfying $\phi(x)$.

2. (Renaming) Standard substitution principle. For any $d \in \Delta$, $R(\{s[x \mapsto d]\}) = R(\{s[y \mapsto d]\})$.

3. (Reflexivity) Clearly follows, since $R(\widehat{\phi(x) \land \neg\phi(x)}^{\mathcal{I}}) = \infty$ is necessary.

4. (Supraclassicality) Since the premise requires $R(\{s \mid \mathcal{I}, s \models \exists x(\phi(x) \land \neg\psi(x))\}) = \infty$ for any $\mathcal{I}$, then the constraint induced by $\phi(x) \Rightarrow_x \psi(x)$, $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\psi(x)\})$ is satisfied since the right hand side is already established to be ∞.

5. (Right weakening) for any interpretation $\mathcal{I}$, $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\psi(x)\})$ and $\mathcal{I} \models \forall x(\psi(x) \to \alpha(x))$ implies that $R(\{s \mid \mathcal{I}, s \models \psi(x) \land \neg\alpha(x)\}) = \infty$. $\phi(x) \Rightarrow_x \alpha(x)$ requires that, for any $\mathcal{I}$, $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \alpha(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\alpha(x)\})$. Since $\psi(x) \land \neg\alpha(x)$ is ranked ∞ in every $\mathcal{I}, s$ pair under consideration, and therefore impossible, $\psi(x) \land \alpha(x)$ is necessary, and so in every $\langle \mathcal{I}, R \rangle$ under these assumptions, $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\psi(x)\})$ implies $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x) \land \alpha(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg(\psi(x) \land \alpha(x))\})$ implies $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \alpha(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\alpha(x)\})$ which implies that $\phi(x) \Rightarrow_x \alpha(x)$.

6. (Left logical equivalence) the premises induce the constraints that for any interpretation $\langle \mathcal{I}, R \rangle$, $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\psi(x)\})$ and $R(\{s \mid \mathcal{I}, s \models (\phi(x) \land \neg\alpha(x)) \lor (\alpha(x) \land \neg\phi(x))\}) = \infty$. Therefore, all possible substitutions for x satisfying $\phi(x)$ also satisfy $\alpha(x)$ and so for all interpretations $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \neg\psi(x)\})$ implies $R(\{s \mid \mathcal{I}, s \models \phi(x) \land \alpha(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(x) \land \alpha(x) \land \neg\psi(x)\})$ which implies $R(\{s \mid \mathcal{I}, s \models \alpha(x) \land \psi(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \alpha(x) \land \neg\psi(x)\})$ which means that $\alpha(x) \Rightarrow_x \psi(x)$ is satisfied.

7. (Right conjunction) the premises result in the constraints $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \land \psi(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \land \neg\psi(\vec{x})\})$ and $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \land \alpha(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \land \neg\alpha(\vec{x})\})$ for any $\langle \mathcal{I}, R \rangle$. Using lemma 2, the left hand sides may be combined into $\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \land \psi(\vec{x})) \land (\phi(\vec{x}) \land \alpha(\vec{x}))\}$, which after applying distributivity results in $\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \land (\psi(\vec{x}) \land \alpha(\vec{x}))\}$ and the right hand sides may be combined with lemma 3 into $\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \land \neg\psi(\vec{x})) \lor (\phi(\vec{x}) \land \neg\alpha(\vec{x}))\}$

and likewise by distributivity obtain $\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\psi(\vec{x}) \vee \neg\alpha(\vec{x})\}$. Therefore, the ranking measure constraint $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge (\psi(\vec{x}) \wedge \alpha(\vec{x}))\})+1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge (\neg\psi(\vec{x}) \vee \neg\alpha(\vec{x}))\})$ which corresponds to the conclusion $\phi(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x}) \wedge \alpha(\vec{x})$.

8. (Left disjunction) the premises result in the constraints $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \psi(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\psi(\vec{x})\})$, and $R(\{s \mid \mathcal{I}, s \models \alpha(\vec{x}) \wedge \psi(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \alpha(\vec{x}) \wedge \neg\psi(\vec{x})\})$. The left hand sides may be combined with lemma 3 into $\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \wedge \psi(\vec{x})) \vee (\alpha(\vec{x}) \wedge \psi(\vec{x}))\}$ which is equivalent via distributivity to $\{s \mid \mathcal{I}, s \models \psi(\vec{x}) \wedge (\phi(\vec{x}) \vee \alpha(\vec{x}))\}$, and similarly the right hand sides may be combined with lemma 3 to $\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \wedge \neg\psi(\vec{x})) \vee (\alpha(\vec{x}) \wedge \neg\psi(\vec{x}))\}$ which reduces to $\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \vee \alpha(\vec{x})) \wedge \neg\psi(\vec{x})\}$ and so the constraint $R(\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \vee \alpha(\vec{x})) \wedge \psi(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \vee \alpha(\vec{x})) \wedge \neg\psi(\vec{x})\})$ is obtained which corresponds to the default $\phi(\vec{x}) \vee \alpha(\vec{x}) \Rightarrow_{\vec{x}} \psi(\vec{x})$.

9. (Cautious monotonicity) the premises correspond to the constraints $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \alpha(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\alpha(\vec{x})\})$ and $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \psi(\vec{x})\})+1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\psi(\vec{x})\})$. Assume that the conclusion is false, and therefore it holds that $R(\widehat{(\phi \wedge \psi) \wedge \neg\alpha}) < R(\widehat{(\phi \wedge \psi) \wedge \alpha}) + 1$. However, this constraint, using distributivity and lemma 2 is equivalent to $R(\widehat{\phi \wedge \neg\alpha}) < R(\widehat{\phi \wedge \alpha}) + 1$ and $R(\widehat{\psi \wedge \neg\alpha}) < R(\widehat{\psi \wedge \alpha}) + 1$ which amount to a negation of the premises, and so a contradiction is derived, and the conclusion holds.

10. (Disjunctive rationality) The premise corresponds to the constraint $R(\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \vee \psi(\vec{x})) \wedge \alpha(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models (\phi(\vec{x}) \vee \psi(\vec{x})) \wedge \neg\alpha(\vec{x})\})$ over the ranking measures of any given interpretation $\mathcal{I}$. This constraint may be expressed with adjusted notation as $R(\widehat{(\phi \vee \psi) \wedge \alpha}) + 1 \leq R(\widehat{(\phi \vee \psi) \wedge \neg\alpha})$, using distributivity and lemma 3 the constraints $R(\widehat{\phi \wedge \alpha} \cup \widehat{\psi \wedge \alpha}) + 1 \leq R(\widehat{\phi \wedge \neg\alpha} \cup \widehat{\psi \wedge \neg\alpha})$, which may be expressed as the constraints: $R(\widehat{\phi \wedge \alpha}) + 1 \leq R(\widehat{\phi \wedge \neg\alpha})$ or $R(\widehat{\psi \wedge \alpha}) + 1 \leq R(\widehat{\psi \wedge \neg\alpha})$ which correspond to the conclusions: $\phi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})$ or $\psi(\vec{x}) \Rightarrow_{\vec{x}} \alpha(\vec{x})$.

11. (Rational monotonicity) The premises correspond to the constraints $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \alpha(\vec{x})\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\alpha(\vec{x})\})$ and $R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \neg\psi(\vec{x})\}) + 1 > R(\{s \mid \mathcal{I}, s \models \phi(\vec{x}) \wedge \psi(\vec{x})\})$.

There must exist a minimal subset $S \subseteq \widehat{\phi}$ that satisfies ψ, as the second constraint necessitates, and this S must be a minimal subset of $\widehat{\phi \wedge \psi}$, since for any minimal $S' \subset \widehat{\phi \wedge \psi}$, it cannot be the case that $R(S') < R(S)$. This means that S' is also a minimal subset of $\widehat{\phi}$, and so by the first constraint, S' must also satisfy $\alpha(\vec{x})$, and so the conclusion is validated.

□

The preferentiality postulates tie these default quantifiers into the broader literature of conditional and preferential logics.

It can also be remarked that if $\langle \mathcal{I}, R \rangle$ is a model of a defeasible knowledge base Δ, then R should be defined for, at least, n dimensions, that is the number of variables to be considered, where n is the largest cardinality for a vector of bound variables in Δ: $n = \max(\{|\vec{x}| \mid \phi \Rightarrow_{\vec{x}} \psi \in \Delta\})$.

The above obviously is a result of the fact that a ranking measure defined for two variables cannot reasonably model a default statement defined for three variables, for example. In order to propagate information from lower dimensions to higher dimensions, the machinery of product measures are exploited. Straightforwardly, product measures are built from compositing two ranking measures defined over, for example, n and m variable names, into one ranking measure defined over $n + m$ variable names, if necessary.

In general, the idea is to preserve information true for an arbitrary tuple $\langle a_1, ..., a_n \rangle$ to a tuple in which it is subsumed, e.g. $\langle a_1, ..., a_n, ..., a_p \rangle$, via the additive property of the ranking algebra. Therefore, product measures, that previously have been defined for abstract Cartesian products of boolean algebras, is here specified for the case of ranked structures, where it is a property that is imposed on the ranking measures of the ranked structures:

Definition 58 (Product ranking measures)**.** *Let $\langle \mathcal{I}, R \rangle$ be a ranking measure structure, S the set of all variable substitutions to the domain $D \in \mathcal{I}$, and $\vec{x}$ and $\vec{y}$ disjoint vectors of variables. Let $\vec{A} = A_1, ..., A_n \in D \times ... \times D$, and $\vec{B} = B_1, ..., B_n \in D \times ... \times D$. Then, for any $S_{\vec{x}} := \{s \mid s(x_1) = A_1, ..., s(x_n) = A_n\}$, $S_{\vec{y}} := \{s \mid s(y_1) = B_1, ..., s(y_n) = B_n\}$, if $S_{\vec{x}\vec{y}} := \{s \mid s(x_1) = A_1, ..., s(x_n) = A_n, s(y_1) = B_1, ..., s(y_n) = B_n\}$ it holds that $R(S_{\vec{x}\vec{y}}) = R(S_{\vec{x}}) + R(S_{\vec{y}})$.*

This definition effectively allows information to propagate "upwards" into higher cardinality of tuples of individuals. Product measures are a crucial aspect of these semantics, and which allows for reasoning about defeasible knowledge bases containing statements over various sizes of variable vectors.

This machinery is what is used to unify these differently dimensioned sentences into a single multi-dimensional ranking measure that can be used to reason about the knowledge base as a whole.

A simple example illustrating the idea behind product measures:

Example 6 (Product measures)**.** *Let the following be a defeasible knowledge base:*

1. $\top(x) \Rightarrow_x \phi(x)$

Let $\mathcal{I} \coloneqq \langle D, I\rangle$ *be a first order structure such that* $D = \{0, 1, 2\}$ *and* $I = \phi^{\mathcal{I}} = \{0\}$*, let* $S = \{s_0, s_1, s_2\}$ *such that* $s_0 = (x \mapsto 0)$*,* $s_1 = (x \mapsto 1)$*,* $s_2 = (x \mapsto 2)$*, and* R^1 *be a ranking measure defined over just one variable that may straightforwardly assign any substitution for* x *satisfying* ϕ *to* 0*, and any substitution satisfying* $\neg\phi$ *to* > 0*, which here may be assigned, without loss of generality, to* 1*. Therefore, for example* $R(\{s_0\}) = 0$ *and* $R(\{s_1, s_2\}) = 1$*. By slightly abusing notation,* $R_x(\phi(x)) = 0$*, and* $R_x(\neg\phi(x)) = 1$ *will be used as shorthand for the preceding statements. A ranking measure then defined for two variables,* x *and* y*, and denoted* R_{xy} *then should composite ranking measures defined over either of them, into one ranking measure defined over both of them. Considering a two variable sentence such as* $\neg\phi(x) \wedge \neg\phi(y)$*, the variable assignments over tuples need to be considered for evaluation:* $s_{00} = (x \mapsto 0, y \mapsto 0)...s_{12} = (x \mapsto 1, y \mapsto 2)...s_{22} = (x \mapsto 2, y \mapsto 2)$ *and so forth (each variable substitution being subscripted with the ordered tuple of domain elements). Then, the product ranking measure results in the ranking value of, for example,* $R(\{s_{12}\}) = R(\{s_1\}) + R(\{s_2\}) = 1 + 1 = 2$*, and* $R(\{s_{01}\}) = R(\{s_0\}) + R(\{s_1\}) = 0 + 1 = 1$*.*

It is already established that $R_x(\neg\phi(x)) = 1$*, and similarly* $R_y(\neg\phi(y)) = 1$*. Product measures then dictate the result that* $R_{xy}(\neg\phi(x) \wedge \neg\phi(y)) = R_x(\neg\phi(x)) + R_y(\neg\phi(y)) = 2$*.*

The intuition of the above example is that since x and y are independent variables, the degree of surprise of both of them violating the default is greater than the degree of surprise of just one of them violating it, which is naturally greater than the surprise of them both conforming to the default. If there is the statement that by default, things are normally in ϕ, then the observation that x and y are not in ϕ is more surprising than finding out just one of them is not in ϕ, which is again of course more surprising than finding out they are both in ϕ.

Example 7 (Product measures II). *Let the following be a defeasible knowledge base:*

1. $\top(x) \Rightarrow_x \phi(x)$
2. $\neg\phi(x) \wedge \psi(y) \Rightarrow_{xy} \alpha(x, y)$

The ranking measure constraints may be represented, by using formulae as shorthand for satisfying sets of variable assignments, such that:

1. $R_x(\phi(x)) < R_x(\neg\phi(x))$
2. $R_{xy}(\neg\phi(x) \wedge \psi(y) \wedge \alpha(x, y)) < R_{xy}(\neg\phi(x) \wedge \psi(y) \wedge \neg\alpha(x, y))$

Constraint 1 is defined over assignments for a single variable name, whereas constraint 2 is defined over assignments for two variable names. The first constraint requires shifting of all variable assignments satisfying $\neg\phi(x)$*, which here can be taken such that* $R(\neg\phi(x)) = 1$ *without any loss of generality, which propagates, via the machinery of product measures, into the two variable case:*

$$R_{xy}(\neg\phi(x)\wedge\psi(y)\wedge\alpha(x, y)) = R_x(\neg\phi(x)\wedge\psi(y)\wedge\alpha(x, y))+R_{xy}(\neg\phi(x)\wedge\psi(y)\wedge\alpha(x, y)) = 1+0$$

Therefore, the second constraint needs to be greater than 1, since there is an exceptional formula in the antecedent which was incorporated from the single variable default.

Below, the elephant zoo example will be continued by defining a particular ranking measure model of the knowledge base:

Example 8 (Elephant zoo semantics). *Given the knowledge base given in example 5, a possible model could be:* $D = \{\texttt{Hyde}, \texttt{Clyde}, \texttt{Ed}, \texttt{Fred}\}$*, with* I *given by:*

- $Clyde^{\mathcal{I}} = \texttt{Clyde}$, $Fred^{\mathcal{I}} = \texttt{Fred}$
- $Elephant^{\mathcal{I}} = \{\texttt{Hyde}, \texttt{Clyde}\}$
- $Keeper^{\mathcal{I}} = \{\texttt{Ed}, \texttt{Fred}\}$
- $Likes^{\mathcal{I}} = \{(\texttt{Hyde}, \texttt{Ed}), (\texttt{Clyde}, \texttt{Fred}), (\texttt{Clyde}, \texttt{Ed})\}$

Lastly, with the ranking measure $R(\{S_2 \coloneqq [x = \texttt{Hyde}, y = \texttt{Fred}]\}) = 1$, $R(\{S_4 \coloneqq [x = \texttt{Clyde}, y = \texttt{Fred}]\}) = 2$, $R(S \setminus \{s_1, s_2\}) = 0$.

Satisfaction under the weak semantics is obtained since:

- $Elephant(x) \wedge Keeper(y) \Rightarrow_{xy} Likes(x, y)$ *requires that* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge Likes(x, y)\}) < R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x, y)\})$. *Observe that since the only pairs of elements in* D *that satisfy* $Elephant(x) \wedge Keeper(y)$ *are* $\{(\texttt{Hyde}, \texttt{Ed}), (\texttt{Hyde}, \texttt{Fred}), (\texttt{Clyde}, \texttt{Ed}), (\texttt{Clyde}, \texttt{Fred})\}$, *the relevant variable substitutions to analyse are*

  - $S_1 = (x \mapsto \texttt{Hyde}, y \mapsto \texttt{Ed})$
  - $S_2 = (x \mapsto \texttt{Hyde}, y \mapsto \texttt{Fred})$
  - $S_3 = (x \mapsto \texttt{Clyde}, y \mapsto \texttt{Ed})$
  - $S_4 = (x \mapsto \texttt{Clyde}, y \mapsto \texttt{Fred})$

  Since $R^2(\{S_1\}) = 0$, $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge Likes(x, y)\}) = R^2(\{S_1, S_3, S_4\}) = 0$, *and* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x, y)\}) = R^2(\{S_2\}) = 1$, *therefore the first default is satisfied.*

- $Elephant(x) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} \neg Likes(x, y)$ *requires that* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge \neg Likes(x, y)\}) < R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x, y)\})$. *The relevant elements of* $D \times D$ *are:* $\{(\texttt{Hyde}, \texttt{Fred}), (\texttt{Clyde}, \texttt{Fred})$ *Since* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge \neg Likes(x, y)\}) = R^2(\{S_2\}) = 1$ *and* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x, y)\}) = R^2(\{S_4\}) = 2$, *the second default is also satisfied.*

- $(Elephant(x) \wedge P_{Clyde}(x)) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} Likes(x, y)$ *requires that* $R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models (Elephant(x) \wedge P_{Clyde}(x)) \wedge (Keeper(y) \wedge P_{Fred}(y)) \wedge Likes(x, y)\}) < R^2(\{s[(x, y)] \mid \mathcal{I}, s[(x, y)] \models (Elephant(x) \wedge P_{Clyde}(x)) \wedge (Keeper(y) \wedge P_{Fred}(y)) \wedge \neg Likes(x, y)\})$. *The relevant element of* $D \times D$ *is:* $(\texttt{Clyde}, \texttt{Fred})$, *and since* $R^2(\{S_4\}) = 2$, *and* $R^2(\emptyset) = \infty$, $R^2(\{S_4\}) < R^2(\emptyset)$, *the final default is satisfied.*

A significant feature of these semantics is how information represented over different sizes of vectors of variables are combined to form a cohesive single model of the knowledge base. This is achieved through product measures, which requires a ranking algebra with an additive structure. Adding ranking values propagates as dimensions increase, allowing for both representing information over any number of variables, and for querying formulae of arbitrary variable count.

Consider the knowledge base:

- $\mathit{Adult}(x) \Rightarrow_x \mathit{Employed}(x)$
- $\mathit{Adult}(x) \wedge \mathit{Child}(y) \Rightarrow_{xy} \mathit{TakesCareOf}(x, y)$

For any fixed ranked structure $\langle \mathcal{I}, R \rangle$, these induce the ranking measure constraints:

- $R_1(S_1) < R_1(S_2)$ such that S_1 is the set of all variable assignments s, such that $\mathit{Adult}(s(x))^{\mathcal{I}} \wedge \mathit{Employed}(s(x))^{\mathcal{I}}$ is satisfied and S_2 is the set of all s such that $\mathit{Adult}(s(x))^{\mathcal{I}} \wedge \neg \mathit{Employed}(s(x))^{\mathcal{I}}$ is satisfied.
- $R_2(S_1) < R_2(S_2)$ with S_1 the set of variable assignments s such that $\mathit{Adult}(s(x))^{\mathcal{I}} \wedge \mathit{Child}(s(y))^{\mathcal{I}} \wedge \mathit{TakesCareOf}(s(x), s(y))^{\mathcal{I}}$ is satisfied and S_2 the set such that $\mathit{Adult}(s(x))^{\mathcal{I}} \wedge \mathit{Child}(s(y))^{\mathcal{I}} \wedge \neg \mathit{TakesCareOf}(s(x), s(y))^{\mathcal{I}}$ is satisfied.

Combining both constraints into a single ranking measure satisfying them requires at least the max number of dimensions expressed in the knowledge base, which in this case is just two dimensions.

3.3 Monotonic inference

Having defined the kind of interpretations used here to give our default quantifiers a model theory, this naturally leads to a monotonic entailment relation. Firstly, some notational niceties in order to refer to the model sets needed for future definitions:

Definition 59. *The set of all ranked structures satisfying a defeasible knowledge base is denoted* $\mathit{Mod}(\Delta) \coloneqq \{\langle \mathcal{I}, R \rangle \mid \langle \mathcal{I}, R \rangle \models \Delta\}$.

It will also be convenient to refer to the model set of a classical formula, or a set of classical formulae:

Definition 60. *The set of all ranked structures satisfying some set of $\mathcal{L}$-formulae Γ with respect to a defeasible knowledge base is denoted $Mod_\Delta(\Gamma) \coloneqq \{\langle \mathcal{I}, R\rangle \mid \langle \mathcal{I}, R\rangle \models \Delta, \langle \mathcal{I}, R\rangle \models \Gamma\}$.*

For convenience, $Mod_\Delta(\{\phi\})$ will be abbreviated $Mod_\Delta(\phi)$ for individual formulae.

Then, a monotonic inference relation between default bases and individual default quantifier statements may be defined:

Definition 61 (Monotonic inference of defaults)**.** *Given a satisfiable knowledge base Δ and a default quantifier sentence $\phi \Rightarrow_{\vec{x}} \psi$, then $\Delta \vdash \phi \Rightarrow_{\vec{x}} \psi$, that is Δ monotonically infers $\phi \Rightarrow_{\vec{x}} \psi$ if and only if $Mod(\Delta) \subseteq Mod(\{\phi \Rightarrow_{\vec{x}} \psi\})$.*

These output default conditionals are the defeasible relationships between classical formulae that hold in all of the ranked interpretations satisfying a default base. This is, of course, a different inferential relation than one among classical formulae, which is next to be defined.

Such an inference relation, between classical formulae, is defined by straightforward model-set inclusion:

Definition 62 (Monotonic entailment)**.** *A monotonic inference relation defined with respect to a satisfiable finite defeasible knowledge base $\Delta \subseteq \mathcal{L}(\Rightarrow)$, denoted $\vdash_\Delta$, is defined such that $\Gamma \vdash_\Delta \alpha$, for any $\Gamma \subseteq \mathcal{L}$, and $\alpha \in \mathcal{L}$ if and only if every ranked structure in $Mod(\Delta)$ satisfying Γ also satisfies α. If $\Delta = \emptyset$, then $\vdash_\Delta$ is denoted $\vdash$.*

By slightly abusing notation, this relation is also extended over a single formula, that is $\phi \vdash_\Delta \psi$ will be used as an abbreviation for $\{\phi\} \vdash_\Delta \psi$. This inference relation is Tarskian, and therefore monotonic.

For deriving monotonic entailment relations, the following proposition is useful:

Proposition 2 (Monotonic entailment)**.** *If Δ is a satisfiable finite default base, Γ a finite set of formulae, and α a formula, then $\Gamma \vdash_\Delta \alpha$ if and only if for every $\langle \mathcal{I}, R\rangle \in Mod(\Delta)$, if $\langle \mathcal{I}, R\rangle \models \Gamma$, then $R(\{s \mid \mathcal{I}, s \models \neg\alpha\}) = \infty$.*

Proof. For the only if part: $\Gamma \vdash_\Delta \alpha$ means that for every $\langle \mathcal{I}, R\rangle$ in $Mod(\Delta)$ such that $\langle \mathcal{I}, R\rangle \models \Gamma$, $\langle \mathcal{I}, R\rangle \models \alpha$. This implies that $\neg\alpha$ is false in all relevant ranked models and therefore that $R(\widehat{\neg\alpha}) = \infty$ for every ranked structure.

For the if part: if $R(\widehat{\neg\alpha^{\mathcal{I}}}) = \infty$ for every ranked structure $\langle \mathcal{I}, R\rangle \in Mod(\Delta)$ such that $\langle \mathcal{I}, R\rangle \models \Gamma$, then this means that $\neg\alpha$ is impossible, and therefore $\neg\alpha^{\mathcal{I}} = \emptyset$, and therefore that $\alpha^{\mathcal{I}} \neq \emptyset$ in every ranked structure, and therefore that $\langle \mathcal{I}, R \rangle \models \alpha$ is true in each ranked structure, and therefore $\Gamma \vdash_{\Delta} \alpha$. □

Theorem 2 (Preferentiality of $\vdash$)**.** *The inference relation $\vdash_{\Delta}$ over $\mathcal{L}$ is preferential, that is it satisfies the following, for any finite set of default quantifier statements Δ, a finite knowledge base $\Gamma \subseteq \mathcal{L}$, and arbitrary formulae $\phi, \psi \in \mathcal{L}$:*

- $\phi \vdash_{\Delta} \phi$ *(Reflexivity)*
- $\frac{\phi \vdash_{\Delta} \psi, \psi \vdash \alpha}{\phi \vdash_{\Delta} \alpha}$ *(Right weakening)*
- $\frac{\phi \vdash_{\Delta} \psi, \phi \dashv\vdash \alpha}{\alpha \vdash_{\Delta} \psi}$ *(Left logical equivalence)*
- $\frac{\phi \vdash_{\Delta} \psi, \phi \vdash_{\Delta} \alpha}{\phi \vdash_{\Delta} \psi \wedge \alpha}$ *(Right conjunction)*
- $\frac{\phi \vdash_{\Delta} \psi, \alpha \vdash_{\Delta} \psi}{\phi \vee \alpha \vdash_{\Delta} \psi}$ *(Left disjunction)*
- $\frac{\phi \vdash_{\Delta} \alpha, \phi \vdash_{\Delta} \psi}{\phi \wedge \psi \vdash_{\Delta} \alpha}$ *(Cautious monotonicity)*

Proof. The preferentiality of $\vdash_{\Delta}$ can be verified by observing that it preserves the preferentiality results of $\Rightarrow$.

- (Reflexivity) Given any $\phi \in \mathcal{L}$, $Mod_{\Delta}(\phi)$ is clearly a subset of itself, and so $\phi \vdash_{\Delta} \phi$.

- (Right weakening) if $\psi \vdash \alpha$, then every $\mathcal{I}$ satisfying ψ also satisfies α. This means that, for any Δ, every $\langle \mathcal{I}, R \rangle \in Mod_{\Delta}(\psi)$ also satisfies α. Then, since $\phi \vdash_{\Delta} \psi$, it holds that $Mod_{\Delta}(\phi) \subseteq Mod_{\Delta}(\psi)$, which means that every $\langle \mathcal{I}, R \rangle \in Mod_{\Delta}(\phi)$ also must satisfy α, and so therefore $\phi \vdash_{\Delta} \alpha$.

- (Left logical equivalence) $\phi \dashv\vdash \alpha$ implies that any $\mathcal{I}$ that satisfies one must satisfy the other, i.e. they have the same model set. This also implies that for any Δ, $Mod_{\Delta}(\phi) = Mod_{\Delta}(\alpha)$. So, if $\phi \vdash_{\Delta} \psi$, then $Mod_{\Delta}(\phi) \subseteq Mod_{\Delta}(\psi)$, and since $Mod_{\Delta}(\phi) = Mod_{\Delta}(\alpha)$, then $Mod_{\Delta}(\alpha) \subseteq Mod_{\Delta}(\psi)$, and so $\alpha \vdash_{\Delta} \psi$.

- (Right conjunction) $\phi \vdash_\Delta \psi$ implies $Mod_\Delta(\phi) \subseteq Mod_\Delta(\psi)$ and $\phi \vdash_\Delta \alpha$ implies $Mod_\Delta(\phi) \subseteq Mod_\Delta(\alpha)$. Since $Mod_\Delta(\phi)$ is contained in both $Mod_\Delta(\psi)$ and $Mod_\Delta(\alpha)$, it must be contained in their intersection, i.e. $Mod_\Delta(\psi) \cap Mod_\Delta(\alpha)$. This intersection is precisely the model set of the formula $\psi \wedge \alpha$, and so $Mod_\Delta(\psi) \cap Mod_\Delta(\alpha)$ must be equal to $Mod_\Delta(\psi \wedge \alpha)$, and so $Mod_\Delta(\phi) \subseteq Mod_\Delta(\psi \wedge \alpha)$, which implies $\phi \vdash_\Delta \psi \wedge \alpha$.

- (Left disjunction) $\phi \vdash_\Delta \psi$ implies $Mod_\Delta(\phi) \subseteq Mod_\Delta(\psi)$ and $\alpha \vdash_\Delta \psi$ implies $Mod_\Delta(\alpha) \subseteq Mod_\Delta(\psi)$. This means that every $\langle \mathcal{I}, R \rangle \in Mod_\Delta(\phi) \cup Mod_\Delta(\alpha)$ is also an element in $Mod_\Delta(\psi)$. $Mod_\Delta(\phi) \cup Mod_\Delta(\alpha)$ is the exact model set of $Mod_\Delta(\phi \vee \alpha)$, i.e. $Mod_\Delta(\phi) \cup Mod_\Delta(\alpha) = Mod_\Delta(\phi \vee \alpha)$, and so it holds that $Mod_\Delta(\phi \vee \alpha) \subseteq Mod_\Delta(\psi)$ which implies $\phi \vee \alpha \vdash_\Delta \psi$

- (Cautious monotonicity) $\phi \vdash_\Delta \alpha$ implies $Mod_\Delta(\phi) \subseteq Mod_\Delta(\alpha)$, and $\phi \vdash_\Delta \psi$ implies $Mod_\Delta(\phi) \subseteq Mod_\Delta(\psi)$. Since $Mod_\Delta(\phi) \subseteq Mod_\Delta(\psi)$, it holds that every $\langle \mathcal{I}, R \rangle \in Mod_\Delta(\phi)$ also satisfies ψ, which implies that $Mod_\Delta(\phi) = Mod_\Delta(\phi \wedge \psi)$. Since $Mod_\Delta(\phi) \subseteq Mod_\Delta(\alpha)$, and $Mod_\Delta(\phi) = Mod_\Delta(\phi \wedge \psi)$, it holds that $Mod_\Delta(\phi \wedge \psi) \subseteq Mod_\Delta(\alpha)$ which implies $\phi \wedge \psi \vdash_\Delta \alpha$.

□

Example 9. *Consider the following knowledge base* Δ*:*

- $\top \Rightarrow_x A(x)$
- $\top \Rightarrow_{xy} \Phi(x, y) \rightarrow \neg A(y)$

These defaults are satisfied by all $\langle \mathcal{I}, R \rangle$ *such that:*

- $R(\{s \mid \mathcal{I}, s \models A(x)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \neg A(x)\})$
- $R(\{s \mid \mathcal{I}, s \models \neg\Phi(x, y) \vee \neg A(y)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models \Phi(x, y) \wedge A(y)\})$

A selection of output conditional statements that follow from these constraints are:

- $\Delta \vdash \top \Rightarrow_{xy} \neg\Phi(x, y)$
- $\Delta \vdash \neg A(x) \Rightarrow_{xy} A(y)$

- $\Delta \vdash \Phi(x,y) \Rightarrow_{xy} A(x) \wedge \neg A(y)$

However, the inferences that may be derived with this inference relation are not defeasible.

To revisit the elephants and keepers example, popular in this literature, in this monotonic defeasible logic:

Example 10 (Elephant zoo monotonic inference)**.** *Consider the same knowledge base from example 5:*

- $Elephant(x) \wedge Keeper(y) \Rightarrow_{xy} Likes(x,y)$
- $Elephant(x) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} \neg Likes(x,y)$
- $(Elephant(x) \wedge P_{Clyde}(x)) \wedge (Keeper(y) \wedge P_{Fred}(y)) \Rightarrow_{xy} Likes(x,y)$

Then, all ranked structures $\langle \mathcal{I}, R \rangle$ that satisfy the following constraints are models of the knowledge base:

- $R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x,y)\})$
- $R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge \neg Likes(x,y)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x,y)\})$
- $R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x,y)\}) + 1 \leq R(\{s \mid \mathcal{I}, s \models Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge \neg Likes(x,y)\})$

Then, the following default statements may be monotonically inferred:

- $\Delta \vdash (Elephant(x) \wedge P_{Clyde}(x)) \wedge Keeper(y) \Rightarrow_{xy} \neg P_{Fred}(y) \wedge Likes(x,y)$
- $\Delta \vdash (Elephant(x) \wedge P_{Clyde}(x)) \wedge Keeper(y) \Rightarrow_{xy} \neg P_{Fred}(y) \wedge Likes(x,y)$

There are many other default statements that may be inferred. However, of interest is the ability to defeasibly infer facts, or sentences, however, here again there are no interesting inferences regarding which facts typically may be concluded from a set of given facts.

This inference relation is, naturally, monotonic, and therefore is more of a necessary formality rather than a central interest of this work. This is since monotonic inference is too conservative to entail defeasible consequences. One major goal of common-sense reasoning, or uncertain reasoning, is to define ampliative reasoning processes Booth et al. (2019): entailment relations capable of conditionally inferring adventurous defeasible facts. This requires a nonmonotonic inference relation — a form of entailment where adding conditional facts may result in a retraction. Next chapter, these nonmonotonic, or defeasible, inference relations will be lifted from the propositional case to this logic.

Chapter 4

Defeasible inference for default quantifier logic

In chapter 3, a first order language extended with default quantifier statements was defined, along with the associated extended model theory with which to interpret those statements. Finally, a Tarskian style monotonic inference defined by model set inclusion was described. These three levels, syntax, semantics, and inference, provide a full logic for defeasible reasoning.

How this defeasible inference relation is defined is not straightforward, even just conceptually. Default quantifiers contain information regarding the conditional degree of belief regarding the category to which an individual belongs, and in the local semantics — that is, ranked structures — it is conceptually reasonably clear what needs to be obtained: a representation of conditional belief over the individuals in the domain. When considering inference, however, there are different strategies one can use to define what is conditionally entailed by manipulating the set of ranked structures considered for any given default base.

The general strategy employed here is as follows: select a distinguished subset of ranked structures that satisfy certain desiderata in line with propositional default reasoning. In aid of this, and to make the default inference itself easier to follow, ranked structures are mapped to corresponding epistemic ranking measures: rankings over the associated Lindenbaum algebra over the first-order language along with the classical semantic notions of inference.

These epistemic ranking measures allow to lift the various definitions and associated notions of default reasoning from the propositional case to this novel forum of default quantifier first-order logic.

4.1 Epistemic ranking measures

Epistemic ranking measures are essentially a method to unify the defeasible information imparted in a framework that combines the disparate domains of each ranked structure into a shared Lindenbaum algebra, defined as follows:

Definition 63 (Epistemic ranking measure)**.** *A epistemic ranking measure $\mathcal{R}$ over the Lindenbaum algebra $\mathcal{LB}_{(\mathcal{L},\vdash)}$ maps elements of the Lindenbaum algebra constructed from the language $\mathcal{L}$ with the Tarskian classical inference relation $\vdash$ to a ranking algebra $\mathcal{V}$. The set of epistemic ranking measures is denoted ER.*

Note that the Lindenbaum algebra forming the domain of these epistemic measures is constructed with the classical monotonic inference, rather than the monotonic inference of the default quantifier logic defined in the previous chapter.

The mapping from ranked structures to epistemic measures is described as such:

Definition 64 (Ranked structures to epistemic ranking measures)**.** *The function $\mathcal{E} : \mathbb{I} \longrightarrow ER$ maps a ranking structure $\langle \mathcal{I}, R \rangle$ to a epistemic ranking measure $\mathcal{E}(\langle \mathcal{I}, R \rangle) = \mathcal{R}^{\langle \mathcal{I}, R \rangle}$ such that for every $\phi(\vec{x}) \in (\mathcal{L}, \vdash)$ it holds that $\mathcal{R}^{\langle \mathcal{I}, R \rangle}(\phi(\vec{x})) = R(\{s \mid \mathcal{I}, s \models \phi(\vec{x})\})$. For a fixed default base Δ the set of $\mathcal{R}$ corresponding to the model set of Δ, $Mod(\Delta)$, is denoted $ER(\Delta) \coloneqq \{\mathcal{E}(\langle \mathcal{I}, R \rangle) \mid \langle \mathcal{I}, R \rangle \models \Delta\}$.*

For ease of notation, for any $\delta \in \mathcal{L}(\Rightarrow)$, $ER(\{\delta\})$ will be abbreviated by $ER(\delta)$. One remark is that if $\Delta = \emptyset$, then for every $\phi \in \mathcal{LB}$, $\mathcal{R}(\phi) = 0$ or $\mathcal{R}(\phi) = \infty$ depending solely on the strict knowledge Γ that may be present. When restricted to the set of epistemic measures satisfying Δ under the strong semantics, then it will be denoted $ER_S(\Delta)$.

It may also be noted that the relationship between epistemic measures and ranked structures is one-to-many: many ranked structures may be mapped to any particular epistemic ranking measure.

Defeasible inference relations in this framework may then be defined by reconstructing the same methods of defining entailment relations in defeasible extensions to propositional logic. The fundamental construction in this method is ranking choice. First, for any default knowledge base Δ, let $ER^{sg}(\Delta) = \{ER(\delta) \mid \delta \in \Delta\}$.

Definition 65 (Ranking choice)**.** *If Δ is a finite default knowledge base, then a ranking choice function $\mathcal{F} : 2^{\mathcal{L}(\Rightarrow)} \longrightarrow 2^{ER}$ maps any Δ to a set of epistemic ranking measures such that:*

- $\mathcal{F}(\Delta) \subseteq ER(\Delta)$ *(choice)*
- $ER^{sg}(\Delta) = ER^{sg}(\Delta')$ *implies* $\mathcal{F}(\Delta) = \mathcal{F}(\Delta')$ *(local semanticality)*
- $ER(\Delta) \neq \emptyset$ *implies* $\mathcal{F}(\Delta) \neq \emptyset$ *(nonmonotonic consistency)*

Every set of epistemic measures $\mathcal{F}(\Delta)$ implicitly refers to a set of ranked structures that are a subset of $Mod(\Delta)$. This grounding of such sets of inference driving ranking measures within the local semantics is an important point to keep in mind, since these ranked structures are the core semantics that determine the truth of a default quantifier.

Ranking choice functions, just as in propositional default reasoning, specify a defeasible inference relation:

Definition 66 (General default inference)**.** *Given a default knowledge base Δ, then a ranking choice function $\mathcal{F}$ defines a defeasible entailment relation such that for a finite set of facts $\Gamma \subset \mathcal{L}$, and an arbitrary formula $\alpha \in \mathcal{L}$, $\Gamma \mid\!\sim^{\mathcal{F}}_{\Delta} \alpha$ if and only if for every $\mathcal{R} \in \mathcal{F}(\Delta)$, $\mathcal{R}(\neg\alpha \mid \bigwedge \Gamma) > 0$.*

This general scheme to define defeasible inference is enough to guarantee some basic desiderata for default inference:

Theorem 3 (Default inference basic principles)**.** *$\mid\!\sim^{\mathcal{F}}$ satisfies DMP, LLI, LOG, PREF.*

Proof.

- For DMP: $\Delta \vdash \phi \Rightarrow_{\vec{x}} \psi$ means that for every $\mathcal{R} \in ER(\Delta)$, $\mathcal{R}(\phi \wedge \psi) + 1 \leq \mathcal{R}(\phi \wedge \neg\psi)$ which necessitates that $\mathcal{R}(\neg\psi \mid \phi) > 0$ for all $\mathcal{R} \in ER(\Delta)$, which means that $\phi \mid\!\sim_{\Delta} \psi$.
- For LLI: this is clear from observing that $\phi \Rightarrow_{\vec{x}} \psi \dashv\vdash \alpha \Rightarrow_{\vec{x}} \beta$ implies that $Mod(\{\phi \Rightarrow_{\vec{x}} \psi\}) = Mod(\{\alpha \Rightarrow_{\vec{x}} \beta\})$ and therefore $Mod(\Delta) \cup Mod(\{\phi \Rightarrow_{\vec{x}} \psi\}) = Mod(\Delta) \cup Mod(\{\alpha \Rightarrow_{\vec{x}} \beta\})$ implies that $\mid\!\sim_{\Delta \cup \{\phi \Rightarrow_{\vec{x}} \psi\}} = \mid\!\sim_{\Delta \cup \{\alpha \Rightarrow_{\vec{x}} \beta\}}$.

- For PREF, each property follows directly from the definitions and the preferentiality of the default quantifier language.

□

The objective is to define suitable ranking choice functions that specify desirable default inference patterns that, for example, follow a plausibility maximisation strategy, or that verify exceptional inheritance. Reconstructing some examples of these inference strategies will be the focus of the following sections.

4.2 Reconstructing Z-inference

Similarly to propositional default reasoning, the most straightforward defeasible inference scheme is that which follows plausibility maximization (Pearl, 1990; Goldszmidt and Pearl, 1992a; Lehmann and Magidor, 1992), referred to as System Z or rational closure. This amounts, in ranking measure based frameworks, to selecting the epistemic ranking measure that minimizes the ranking values while still satisfying each default statement. In the propositional case, the inference scheme defined using the ranking measure corresponding to this strategy of plausibility maximization is referred to as System Z, and so here the same nomenclature is followed.

In order to reconstruct System Z such that it corresponds to a lifting over default quantifier logic, a partial ordering over global ranking measures is necessary:

Definition 67 (Order over ER). *For any two epistemic ranking measures $\mathcal{R}_1$, $\mathcal{R}_2$, then $\mathcal{R}_1 \preceq \mathcal{R}_2$ if and only if for every $\phi \in \mathcal{LB}_{(\mathcal{L},\vdash)}$, $\mathcal{R}_1(\phi) \leq \mathcal{R}_2(\phi)$.*

This partial order over the ranking measures has been shown, for instance by (Freund, 1998; Giordano et al., 2015) to have a distinguished minimum.

Proposition 3 ($\preceq$-min element in ER). *For a finite default knowledge base Δ, under the strong semantics, there is a minimal element $\mathcal{R}^Z_\Delta \in ER_S(\Delta)$ with respect to the $\preceq$ ordering. $\mathcal{R}^Z_\Delta$ is referred to as the Z-ranking measure of the default knowledge base Δ.*

The Z-epistemic ranking measure is the distinguished canonical epistemic measure that is the minimal epistemic measure using the strong semantics for the satisfaction relation.

This distinguished epistemic measure is used to calculate defeasible inference. In essence, System Z is specified by the ranking choice function $\mathcal{F}^Z(\Delta) = \{\mathcal{R}^Z_\Delta\}$.

Definition 68 (System Z inference)**.** *Let Δ be a defeasible knowledge base. Then for a finite $\Gamma \subset \mathcal{L}$ and arbitrary $\phi \in \mathcal{L}$, $\Gamma \mathrel{|\!\sim}^Z_\Delta \phi$ if and only if $\mathcal{R}^Z_\Delta(\neg\phi \mid \bigwedge \Gamma) > 0$.*

This lifting of a Z-inference relation over the first-order language, specified by a default quantifier base, can be illustrated with the following example:

Example 11 (Elephant zoo Z inference)**.** *Taking the defeasible knowledge base, Δ, regarding the elephant zoo previously covered in examples 5 and 8, then a canonical epistemic ranking measure may be constructed in order to calculate a defeasible inference relation.*

The epistemic ranking measure over the Lindenbaum-algebra has to satisfy the following constraints under the strong semantics:

1. $\mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)) + 1 \leq \mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x,y))$

2. $\mathcal{R}(Elephant(x) \wedge (Keeper(y) \wedge P_{Fred}(y)) \wedge \neg Likes(x,y)) + 1 \leq \mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge Likes(x,y))$

3. $\mathcal{R}(Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x,y)) + 1 \leq \mathcal{R}((Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y)) \wedge \neg Likes(x,y))$

These constraints may be visualised as in the below table, where the horizontal constraints over x are seen in conjunction with the vertical constraints over y, and the abbreviations $E = Elephant$ and $K = Keeper$ are used:

$E(x) \wedge P_{Clyde}(x)$	¬ Likes(x,y) 1 Likes(x,y) 0	¬ Likes(x,y) 3 Likes(x,y) 2
$E(x)$	¬ Likes(x,y) 1 Likes(x,y) 0	¬ Likes(x,y) 1 Likes(x,y) 2
	$K(y)$	$K(y) \wedge P_{Fred}(y)$

Table 4.1: Z-model representation

The above representation should be read such that, for example, the far bottom left triangle encodes the formula $Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)$

and the associated epistemic ranking measure $\mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)) = 0$*. Similarly, the top right triangle represents the conjunction of* $Elephant(x) \wedge P_{Clyde}(x)$ *and* $Keeper(y) \wedge P_{Fred}(y)$ *and the associated ranking measure value located inside the triangle is* 3*. The table is employed to attempt to visualise the two dimensional nature of the constraints — which formulae are true of each variable must be considered in conjunction.*

Using Z-inference, the following inferential consequences are obtained:

- $\{Elephant(a), Keeper(b)\} \mid\!\sim^{Z}_{\Delta} Likes(a,b)$
- $\{Elephant(Clyde), Keeper(b)\} \mid\!\sim^{Z}_{\Delta} Likes(Clyde,b)$
- $\{Elephant(Clyde), Keeper(Fred)\} \mid\!\sim^{Z}_{\Delta} Likes(Clyde,Fred)$
- $\{Elephant(a), Keeper(Fred)\} \mid\!\sim^{Z}_{\Delta} \neg Likes(a,Fred)$

Examining the table, it may be verified that the rank of $Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)$ *is* 0*, whereas the rank of* $Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x,y)$ *is* 1*, which is the explanation behind the first statement regarding some generic elephant and keeper pair. When the pair of Clyde and Fred is considered, the top right quadrant must be considered, where it can be verified that the rank of Clyde liking Fred is lower than the rank of Clyde not liking Fred,* $2 < 3$*, and so System Z infers the third statement above.*

Unsurprisingly, plausibility maximisation will assume that Clyde is a typical elephant in any other respect, other than what is explicitly noted to be atypical, i.e. that Clyde likes Fred, therefore Z-inference concludes that Clyde likes any other Keeper in general.

One thing that can be noted is the inability to conclude universally quantified formulae. Consider $Elephant(Clyde) \mid\!\sim_{\Delta} \forall x(Keeper(x) \rightarrow Likes(Clyde,x))$, i.e. does $Clyde$ like all keepers? The answer is that this cannot be concluded, at least not with the Z-ranking, because the rank of $Elephant(Clyde) \wedge (Keeper(x) \wedge \neg Likes(Clyde,x))$ is less than infinite. This means there are local ranked structures where there exist regular keepers $Clyde$ does not like. For example, consider the ranked structure such that $\langle \mathcal{I}, R\rangle := D^{\mathcal{I}} = \{c,f,a,b,d\}$, $Clyde^{\mathcal{I}} = c^{\mathcal{I}}$, $Fred^{\mathcal{I}} = f^{\mathcal{I}}$, $Elephant^{\mathcal{I}} = \{c^{\mathcal{I}}, a^{\mathcal{I}}\}$, $Keeper^{\mathcal{I}} = \{f,b,c\}$, $Likes^{\mathcal{I}} = \{(c,f),(c,b),(a,b),(a,d)\}$ with the associated ranking measure R that maps the assignments to the relevant tuples such that $R(s) = 2$ if $s(x) = c$ and $s(y) = f$ and $R(s) = 1$ if $s(x) = c$ and $s(y) = d$

and $R(s) = 0$ otherwise. This ranked structure will map to the epistemic measure $\mathcal{R}^Z$, but does not verify $\forall x Likes(Keeper(x) \rightarrow Clyde(x))$.

The straightforward plausibility maximisation that is at the core of Z-inference is insufficient as the ultimate goal of defeasible reasoning. It fails to verify exceptional inheritance in general, for example, and depending on specific application, more ampliative inference relations are desirable. Z-inference is, however, an important default inference to realise in any framework, as it is recognised in the literature as the "nonmonotonic core" of any defeasible reasoning framework (Giordano et al., 2012), being sound for the KLM postulates, specifically including rational monotonicity.

4.3 Epistemic ranking construction

In order to realise the final goal of defining sophisticated default inference schemes in this logic, the general strategy regarding constructible inference relations needs to be lifted from propositional default reasoning. The idea behind constructibility is to build up a distinguished set of ranking measures, in this case epistemic ranking measures, that constitute the ranking choice function, that in turn specifies the default inference relation.

The key operation necessary for constructible inference is that of shifting:

Definition 69 (Shifting)**.** *Shifting is a function that associates an epistemic ranking measure $\mathcal{R}$ to another epistemic ranking measure $\mathcal{R} + a\phi$ with parameters $\phi \in \mathcal{L}$ and $a \in [0, \infty]$ such that for all $\psi \in \mathcal{L}$:*

- $(\mathcal{R} + a\phi)(\psi) = \texttt{min}(\mathcal{R}(\psi \wedge \phi) + a, \mathcal{R}(\psi \wedge \neg\phi))$

Shifting is a transformation over an epistemic measure in order to tailor an epistemic measure to a particular default base to realise the expressed constraints.

Constructibility over a default knowledge base is the general property satisfied by ranking measures that may be defined by iteratively revising the rankings over default statements, that convey information regarding comparative rankings over the ranking domain:

Definition 70 (Constructibility)**.** *An epistemic ranking measure $\mathcal{R}$ is constructible over a satisfiable default knowledge base $\Delta = \{\phi_i \Rightarrow_{\vec{x}} \psi_i \mid i \leq n\}$ if and only if*

- $\mathcal{R} = R_0 + a_0[\phi_0 \wedge \neg\psi_0] + ... + a_n[\phi_n \wedge \neg\psi_n]$ *for* $a_i \in [0, \infty]$

The set of all constructible epistemic ranking measures over Δ *is denoted* $Constr(\Delta)$*.*

Constructibility defines a general restriction over the total set of epistemic ranking measures to narrow down the set of ranking measures to just those that may be built by iteratively shifting ranks according to each default conditional.

A ranking choice function is then said to be constructible, if all epistemic measures selected by it are as well:

Definition 71 (Constructible inference)**.** *A defeasible inference relation* $\mid\!\sim^{\mathcal{F}}$ *is constructible if and only if* $\mathcal{F}(\Delta) \subseteq Constr(\Delta)$*.*

The most straightforward constructible inference relation is the one that defined from the full set of constructible epistemic measures (Weydert, 2003):

Definition 72 (System J)**.** *The System J inference relation* $\mid\!\sim^{J}_{\Delta}$ *is specified by the ranking choice function* $\mathcal{F}^J(\Delta) = Constr(\Delta)$*.*

System J satisfies DMP, LLI, PREF, CP, RI, and IRR. Additionally, it satisfies exceptional inheritance, which is the primary improvement over Z-inference. However, it may still be considered to be too conservative in the inferences it is willing to conclude. Ultimately, the goal is to define a ranking choice function that selects a single distinguished epistemic ranking measure which will solely determine all inferences.

Example 12 (Elephant zoo J inference)**.** *Taking the defeasible knowledge base,* Δ*, regarding the elephant zoo previously covered in examples 5 and 8, then a canonical epistemic ranking measure may be constructed in order to calculate a defeasible inference relation.*

Then the epistemic ranking measure over the Lindenbaum-algebra has to satisfy the following constraints using the strong semantics:

1. $\mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge Likes(x,y)) + 1 \leq \mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge \neg Likes(x,y))$

2. $\mathcal{R}(Elephant(x) \wedge (Keeper(y) \wedge P_{Fred}(y)) \wedge \neg Likes(x,y)) + 1 \leq \mathcal{R}(Elephant(x) \wedge Keeper(y) \wedge Likes(x,y))$

3. $\mathcal{R}(Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y) \wedge Likes(x, y)) + 1 \leq \mathcal{R}((Elephant(x) \wedge P_{Clyde}(x) \wedge Keeper(y) \wedge P_{Fred}(y)) \wedge \neg Likes(x, y))$

Let $\alpha, \beta, \gamma \in (0, \infty]_{\mathbb{R}}$. The resultant rankings of the relevant formulae are listed below, using the abbreviations $E = Elephant$ and $K = Keeper$, such that for every $\mathcal{R} \in \mathcal{F}^J(\Delta)$:

1. $\mathcal{R}(E(x) \wedge K(y) \wedge Likes(x, y)) = 0$
2. $\mathcal{R}(E(x) \wedge K(y) \wedge \neg Likes(x, y)) = \alpha$
3. $\mathcal{R}(E(x) \wedge P_{Fred}(y) \wedge K(y) \wedge Likes(x, y)) = \alpha + \beta$
4. $\mathcal{R}(E(x) \wedge P_{Fred}(y) \wedge K(y) \wedge \neg Likes(x, y)) = \alpha$
5. $\mathcal{R}(E(x) \wedge P_{Clyde}(x) \wedge K(y) \wedge Likes(x, y)) = 0$
6. $\mathcal{R}(E(x) \wedge P_{Clyde}(x) \wedge K(y) \wedge \neg Likes(x, y)) = \alpha$
7. $\mathcal{R}(E(x) \wedge P_{Clyde}(x) \wedge P_{Fred}(y) \wedge K(y) \wedge Likes(x, y)) = \alpha + \beta$
8. $\mathcal{R}(E(x) \wedge P_{Clyde}(x) \wedge P_{Fred}(y) \wedge K(y) \wedge \neg Likes(x, y)) = \alpha + \beta + \gamma$

Using the above rankings, defeasible consequences may be calculated using the conditional ranking measures that can be deduced to be common to all of the above.

Therefore, the following inferential consequences are obtained:

- $\{Elephant(Clyde), Keeper(a)\} \mid\!\sim^J_\Delta Likes(Clyde, a)$
- $\{Elephant(Clyde), Keeper(Fred)\} \mid\!\sim^J_\Delta Likes(Clyde, Fred)$
- $\{Elephant(a), Keeper(Fred)\} \mid\!\sim^J_\Delta \neg Likes(a, Fred)$

In search of this ranking choice function, a further restriction on the set of constructible epistemic measures is that of justifiable constructibility (Weydert, 1998):

Definition 73 (Justifiable constructibility)**.** *For any default knowledge base $\Delta = \{\phi_i \Rightarrow_{\vec{x}_i} \psi_i \mid i \leq n\}$ a ranking construction scheme is justifiably constructible if and only if $\mathcal{R} = \mathcal{R}_0 + a_1[\phi_1 \wedge \neg\psi_1] + ... + a_n[\phi_n \wedge \psi_n]$ such that for every $0 < a_i$, $\mathcal{R}(\neg\psi_i \mid \phi_i) = 1$. Let the set of all justifiably constructed global ranking measures over Δ be denoted $Constr_{JJ}(\Delta)$.*

Justifiable constructibility requires that no formula in the Lindenbaum algebra is shifted any more than is necessary to ensure the satisfaction of each default statement. This excludes the many epistemic measures that, for example, satisfy the defeasible statements by arbitrarily large numbers.

This set of justifiably constructible epistemic measures naturally is itself a ranking choice function that therefore specifies a defeasible inference relation:

Definition 74 (System JJ)**.** *The System JJ inference relation* $\mid\!\sim^{JJ}$ *is specified by the ranking choice function* $\mathcal{F}^{JJ}(\Delta) = Constr_{JJ}(\Delta)$*.*

One straightforward ranking choice function that fulfils the goal of specifying a single canonical epistemic measure that determines the defeasible inferences is the result of taking the greatest lower bound of all ranking values for each formula in the domain, which corresponds to the minimal epistemic measure in the set of all justifiably constructible epistemic measures:

Definition 75 (System JJR)**.** *The System JJR inference relation* $\mid\!\sim^{JJR}$ *is specified by the ranking choice function* $\mathcal{F}^{JJR}(\Delta) = \mathtt{inf}_{\preceq} Constr_{JJ}(\Delta)$*.*

Both JJ-inference and JJR-inference satisfy many of the significant properties that System J satisfies.

Lifting the construction procedure detailed in (Weydert, 1998) over the Lindebaum algebra $LB_{\mathcal{L},\vdash}$, it is also possible to specify the JZ epistemic measure $\mathcal{R}^{JZ}$ over a default knowledge base.

Conjecture 2 (JZ ranking)**.** *For any default quantifier knowledge base* Δ*, the JZ epistemic measure* $\mathcal{R}^{JZ}_{\Delta} \in Constr_{JJ}(\Delta)$ *exists and is canonical.*

This canonical JZ epistemic measure informs a defeasible inference relation in the usual way:

Definition 76 (System JZ)**.** *System JZ is the inference relation informed by the ranking choice function* $\mathcal{F}^{JZ}(\Delta) = \{\mathcal{R}^{JZ}_{\Delta}\}$ *such that for any finite* $\Gamma \subset \mathcal{L}$ *and* $\phi \in \mathcal{L}$*,* $\Gamma \mid\!\sim^{JZ}_{\Delta} \phi$ *if and only if* $\mathcal{R}^{JZ}_{\Delta}(\neg\phi \mid \bigwedge \Gamma) > 0$

System JZ verifies all the properties of System J, and additionally satisfies RM, and so is a rational consequence relation.

A simple example of JZ-inference, lifted from propositional default reasoning (Weydert, 2003):

Example 13 (Simple ranking construction)**.** *Let the following be a defeasible knowledge base* Δ*:*

- $\top \Rightarrow_x A(x)$
- $\top \Rightarrow_x B(x)$
- $\top \Rightarrow_x A(x) \wedge B(x)$

These defaults result in the corresponding epistemic ranking measure constraints:

- $\mathcal{R}(A(x)) + 1 \leq \mathcal{R}(\neg A(x))$
- $\mathcal{R}(B(x)) + 1 \leq \mathcal{R}(\neg B(x))$
- $\mathcal{R}(A(x) \wedge B(x)) + 1 \leq \mathcal{R}(\neg A(x) \vee \neg B(x))$

Since these constraints are unary — only a single variable is bound to the default quantifiers — then this knowledge base is effectively identical to a propositional default knowledge base.

The resultant constraints combined into a single-variable ranking measure may be visualised as follows, just as in the propositional case:

B	0	1
$\neg B$	1	1.5
	A	$\neg A$

Table 4.2: JZ-model representation

The inferential relations that hold in the propositional case also hold in the single variable case, i.e.:

- $\{A(a)\} \mid\!\sim_\Delta B(a)$
- $\{\neg B(a)\} \mid\!\sim_\Delta A(a)$
- $\{\neg A(a)\} \mid\!\sim_\Delta B(a)$

Since the defaults are single-dimensional, multi-variable inferential relationships are semantically determined purely by exploiting product measure machinery in order to evaluate the conditional ranking values, providing expected inferences, such as:

- $\{\neg A(a), \neg B(b)\} \mid\!\sim_\Delta B(a) \wedge A(b)$

Single variable default bases naturally are not the most compelling examples, as a core aspect of first-order reasoning is handling multi-variable sentences, and default bases. An example of a multi-variable default base is:

Example 14 (Multidimensional ranking construction)**.** *Let the following be a variation on the previous default knowledge base:*

- $\top \Rightarrow_x A(x)$
- $\top \Rightarrow_x B(x)$
- $\top \Rightarrow_{xy} A(x) \wedge B(y)$

Under System J, these defaults result in the corresponding epistemic ranking measure constraints:

- $\mathcal{R}(A(x)) + \alpha \leq \mathcal{R}(\neg A(x))$
- $\mathcal{R}(B(x)) + \beta \leq \mathcal{R}(\neg B(x))$
- $\mathcal{R}(A(x) \wedge B(y)) + \gamma \leq \mathcal{R}(\neg A(x) \vee \neg B(y))$

Greek letters are used for these strong semantic constraints, in order to abstract across all possible J-models. Since the third default, and corresponding ranking measure constraint, is defined over two variables, the resulting constructed ranking measure must be over at least two variables in order to incorporate all of the information in the defaults. The following table is a model representing the class of epistemic ranking measures selected by the System J ranking choice function.

$\neg A(x) \wedge \neg B(x)$	$\alpha + \beta + \gamma$	$\alpha + 2 * \beta + \gamma$	$2 * \alpha + \beta + \gamma$	$2 * \alpha + 2 * \beta + \gamma$
$\neg A(x) \wedge B(x)$	$\alpha + \gamma$	$\alpha + \beta + \gamma$	$2 * \alpha + \gamma$	$2 * \alpha + \beta + \gamma$
$A(x) \wedge \neg B(x)$	β	$2 * \beta + \gamma$	$\alpha + \beta$	$2 * \beta + \gamma$
$A(x) \wedge B(x)$	0	$\beta + \gamma$	α	$\alpha + \beta + \gamma$
	$A(y) \wedge B(y)$	$A(y) \wedge \neg B(y)$	$\neg A(y) \wedge B(y)$	$\neg A(y) \wedge \neg B(y)$

Table 4.3: J-ranking

A selection of inference relations under System J are:

- $\{A(a), \neg B(b)\} \mathrel{|\!\sim}_\Delta B(a) \wedge A(b)$

The above example demonstrates product ranking measures, and how it preserves the patterns of reasoning existing in the propositional case to the multi-variable first-order case. However, since the predicates in question are still unary, the complexity is limited. The primary reason first-order reasoning is multi-variable is to define relational information, as in the following example:

Example 15 (Taking care)**.** *An example from defeasible description logics (Casini et al., 2021), originally formulated as* $\mathcal{D} := \{\top \sqsubseteq\!\!\!\sim A \sqcap \forall r.\neg A\}$*,* $\mathcal{A} := \{r : (a,b)\}$*. Here, it may formulated as the following defeasible knowledge base:*

- $\top \Rightarrow_x A(x) \wedge \forall y(\Phi(x,y) \rightarrow \neg A(y))$

The default statement above results in the following ranking measure constraint over the Lindenbaum algebra:

- $\mathcal{R}_x(A(x) \wedge \forall y(\Phi(x,y) \rightarrow \neg A(y))) + \alpha \leq R_x(\neg A(x) \vee \exists y(\Phi(x,y) \wedge A(y)))$

Since the default statement here effectively merges what could be considered two properties, that of being in A*, and that of being in* $\forall y(\Phi(x,y) \rightarrow \neg A(y))$ *into one statement, or source of information. This effectively removes the language independence between these properties. Since the default quantifier binds a single variable name, and therefore the independence between the elements, or variable names, remains. All epistemic ranking measures* $\mathcal{R} \in Constr(\Delta) \cap ER(\Delta)$ *over two variable names then have to conform to the scheme below:*

$A(x)$	¬ Φ(x,y) α Φ(x,y) α	¬ Φ(x,y) 0 Φ(x,y) α
$\neg A(x)$	¬ Φ(x,y) 2α Φ(x,y) 2α	¬ Φ(x,y) α Φ(x,y) α
	$\neg A(y)$	$A(y)$

Table 4.4: J-model representation

Conditioning with the strict information $\Phi(a,b)$ *constrains the first-order interpretations accordingly. Under the J-ranking illustrated above, the only inference obtainable is:* $\Phi(a,b) \mid\!\sim^J_\Delta A(a) \vee A(b)$*.*

The previous example can be viewed as inadequate modelling: compressing the information into a single statement quantified over single individuals is not exploiting the full expressivity this logic affords. Expressing the same statement but defined over pairs of individuals results in the following case:

Example 16 (Taking care II)**.** *Taking the previous example 15, and modifying the knowledge base such that the default quantifier is two-dimensional, shown as such:*

- $\top \Rightarrow_{xy} A(x) \wedge (\Phi(x,y) \rightarrow \neg A(y))$

This results in the two dimensional constraints over the ranking measures:

- $R_{xy}(A(x) \wedge (\Phi(x,y) \rightarrow \neg A(y)) + \alpha \leq R_{xy}(\neg A(x) \vee (\Phi(x,y) \wedge A(y))$

The compliant class of epistemic measures conform to the following schema:

$A(x)$	$\neg$ Φ(x,y) 0 Φ(x,y) α	$\neg$ Φ(x,y) 0 Φ(x,y) α
$\neg A(x)$	$\neg$ Φ(x,y) α Φ(x,y) α	$\neg$ Φ(x,y) α Φ(x,y) α
	$\neg A(y)$	$A(y)$

Table 4.5: J-model representation

The following consequences are then:

- $A(a) \mid\!\sim^{J}_{\Delta} \neg\Phi(a,b)$

Combining the information regarding individuals and pairs in the same default clearly causes a loss of information: in example 15, the unary default was able to propagate into the two dimensional ranking measures the distinction between one abnormal individual and two abnormal individuals, however in the previous example this information is rolled into the single default that is providing information about (ab)normal binary relations. Therefore, default information needs to be expressed with respect to the correct arity of tuples. Splitting up this information into two defaults of different arity results in the following case:

Example 17 (Taking care III). *If the same example was reformulated across two statements - breaking up the conjunction into two default statements - then both System J-like constructible inference methods may be demonstrated, as well as ranking measures utilising product measures in the construction. Splitting the statement up as the following:*

- $\top \Rightarrow_x A(x)$
- $\top \Rightarrow_{xy} (\Phi(x,y) \rightarrow \neg A(y))$

Results in the following strong semantic constraints over the Lindenbaum algebra:

- $\mathcal{R}(A(x)) + \alpha \leq \mathcal{R}(\neg A(x))$
- $\mathcal{R}(\neg\Phi(x,y) \vee \neg A(y)) + \beta \leq \mathcal{R}(\Phi(x,y) \wedge A(y))$

A representation of the schema by which any $\mathcal{R} \in Constr(\Delta) \cap ER(\Delta)$:

$A(x)$	$\neg$ Φ(x,y) α Φ(x,y) α	$\neg$ Φ(x,y) 0 Φ(x,y) β
$\neg A(x)$	$\neg$ Φ(x,y) 2α Φ(x,y) 2α	$\neg$ Φ(x,y) α Φ(x,y) $\alpha + \beta$
	$\neg A(y)$	$A(y)$

Table 4.6: J-model representation

Therefore, System J is only able to infer $\Phi(a,b) \mid\!\sim^{J}_{\Delta} A(a)$ due to the only partial comparability of the ranking measure values α, β, $\alpha + \beta$ and 2α, that is just $\alpha < 2\alpha$, $\alpha < \alpha + \beta$, and $\beta < \alpha + \beta$. What System J does clarify, is that whether or not b is in A or $\neg A$ depends on which default is preferred, which arises in the ranking measure values via the comparability of α and β.

Using an approach by which a canonical ranking measure is constructed, it is possible to have specific ranking measure values. The System JZ ranking, for instance, can be visualised as follows:

$A(x)$	$\neg$ Φ(x,y) 1 Φ(x,y) 1	$\neg$ Φ(x,y) 0 Φ(x,y) 1
$\neg A(x)$	$\neg$ Φ(x,y) 2 Φ(x,y) 2	$\neg$ Φ(x,y) 1 Φ(x,y) 2
	$\neg A(y)$	$A(y)$

Table 4.7: JZ-model representation

The JZ ranking, however, does not differentiate between the defaults either, and so the inferential benefits over System J are marginal, in this case. In this framework, specifying ranking constraints such as:

- $\mathcal{R}(A(x)) + 1 \leq \mathcal{R}(\neg A(x))$
- $\mathcal{R}(\neg\Phi(x,y) \vee \neg A(y)) + 0.5 \leq \mathcal{R}(\Phi(x,y) \wedge A(y))$

Will result in the custom constructible ranking measure, called $R^{e.g.}$ for convenience:

$A(x)$	¬ Φ(x,y) 1 Φ(x,y) 1	¬ Φ(x,y) 0 Φ(x,y) 0.5
$\neg A(x)$	¬ Φ(x,y) 2 Φ(x,y) 2	¬ Φ(x,y) 1 Φ(x,y) 1.5
	$\neg A(y)$	$A(y)$

Table 4.8: Custom ranking measure $R^{e.g.}$

Under this ranking measure, the inferential relation $\{\Phi(a,b)\} \mid\!\sim^{e.g.}_{\Delta} A(b)$ is verified by this ranking, and so, in fact, the inference $\{\Phi(a,b)\} \mid\!\sim^{e.g.}_{\Delta} A(a) \wedge A(b)$ are obtained.

The above examples are intended to provide an insight into the inferential machinery at work in this framework, and to help convey the intuitions behind these definitions.

A question might occur when modelling information, are the statements $\phi \Rightarrow_{\vec{x}} \psi$ and $\top \Rightarrow_{\vec{x}} \phi \rightarrow \psi$ equivalent? Although it might seem, at first glance, like these statements could semantically be equivalent, they result in very different constraints, that when integrated in a default base result in different epistemic measures. This is easily illustrated with an example:

Example 18. *Let the following be a defeasible knowledge base:*

- $\top \Rightarrow_x A(x)$
- $\top \Rightarrow_{xy} (\phi(x) \wedge \psi(y)) \rightarrow \neg A(y)$

These induce the associated epistemic ranking measure constraints are:

- $\mathcal{R}(A(x)) + \alpha \leq \mathcal{R}(\neg A(x))$

- $\mathcal{R}(\neg(\phi(x) \wedge \psi(y)) \vee \neg A(y)) + \beta \leq \mathcal{R}((\phi(x) \wedge \psi(y)) \wedge A(y))$

These constraints inform that every $\mathcal{R} \in Constr(\Delta) \cap ER(\Delta)$ conforms to the following schema:

$A(x) \wedge \phi(x)$	1	1	0	1
$A(x) \wedge \neg\phi(x)$	1	1	0	0
$\neg A(x) \wedge \phi(x)$	2	2	1	1
$\neg A(x) \wedge \neg\phi(x)$	2	2	1	1
	$\neg A(y) \wedge \neg\psi(y)$	$\neg A(y) \wedge \psi(y)$	$A(y) \wedge \neg\psi(y)$	$A(y) \wedge \psi(y)$

Table 4.9: Z-model representation over $\langle x, y \rangle$ tuples

These result in the following selected inferences:

- $\{\phi(a)\} \mid\!\sim^Z_\Delta A(a) \wedge A(b) \wedge \neg\psi(b)$
- $\{\psi(b)\} \mid\!\sim^Z_\Delta A(a) \wedge \neg\phi(a) \wedge A(b)$

Additionally, the following are not inferred:

- $\{\neg A(a), \psi(b)\} \not\mid\!\sim^Z_\Delta \phi(a)$ *or* $\neg\phi(a)$
- $\{\neg A(a), \phi(a)\} \not\mid\!\sim^Z_\Delta \psi(b)$ *or* $\neg\psi(b)$

Now, compare the aforementioned with the following defeasible knowledge base:

- $\top \Rightarrow_x A(x)$
- $\phi(x) \wedge \psi(y) \Rightarrow_{xy} \neg A(y)$

Notice the changed second default statement distinguishing this from the previous default base. These induce the associated epistemic ranking measure constraints:

- $\mathcal{R}(A(x)) + \alpha \leq \mathcal{R}(\neg A(x))$
- $\mathcal{R}((\phi(x) \wedge \psi(y)) \wedge \neg A(y)) + \beta \leq \mathcal{R}((\phi(x) \wedge \psi(y)) \wedge A(y))$

$A(x) \wedge \phi(x)$	1	1	0	2
$A(x) \wedge \neg\phi(x)$	1	1	0	0
$\neg A(x) \wedge \phi(x)$	2	2	1	2
$\neg A(x) \wedge \neg\phi(x)$	2	2	1	1
	$\neg A(y) \wedge \neg\psi(y)$	$\neg A(y) \wedge \psi(y)$	$A(y) \wedge \neg\psi(y)$	$A(y) \wedge \psi(y)$

Table 4.10: Z-model representation over $\langle x, y \rangle$ tuples

These result in the following selected inferences:

- $\{\phi(a)\} \mid\!\sim^Z_\Delta A(a) \wedge A(b) \wedge \neg\psi(b)$
- $\{\psi(b)\} \mid\!\sim^Z_\Delta A(a) \wedge \neg\phi(a) \wedge A(b)$

Additionally, the following are inferred:

- $\{\neg A(a), \psi(b)\} \mid\!\sim^Z_\Delta \neg\phi(a)$
- $\{\neg A(a), \phi(a)\} \mid\!\sim^Z_\Delta \neg\psi(b)$

The second default base distinguishes the satisfying and violating sets more finely than the first, where in order to satisfy the second default a much larger set of tuples are taken as reference, whereas in the second case the second default admits far fewer tuples as the reference for shifting.

This chapter defined a coherent approach to defining defeasible inference over a first-order logic extended with default quantifiers. The default reasoning framework used was that of ranking measure theory, well defined in propositional default reasoning, where specifically the notions of ranking choice functions and constructible inference are developed in the literature. These notions were lifted and redefined here for the first-order default quantifier case into a complete first-order default reasoning framework. However, there are many aspects and concepts in this framework worthy of a more discursive treatment. Certain decisions have to always be made in formalising such a framework, which will be elaborated in the final chapter.

Chapter 5

Defeasible description logics

5.1 Defeasible $\mathcal{ALC}$

A straightforward method to express defeasible information in the context of Description Logic is to extend the $\mathcal{ALC}$-language with defeasible concept inclusion statements of the form $C \precsim D$ (Britz et al., 2020). $C \precsim D$ is here assumed to mean that if all is known is that some individual is an instance of the concept C, then it may be expected by default that it is also an instance of D. It may be read as "instances of C typically/usually are also instances of D".

Definition 77 ($\mathcal{DALC}$ Language)**.** *A $\mathcal{DALC}$-language $\mathcal{L}_{\mathcal{D}} = \mathcal{L}(\precsim)$ is obtained by enriching the logical vocabulary of an $\mathcal{ALC}$-language $\mathcal{L}$ with the default subsumption symbol $\precsim$, i.e. $V^{log}_{dalc} = V^{log}_{alc} \cup \{\precsim\}$, and adding to $\mathcal{L}_{\mathcal{D}}$ all the all the defeasible subsumption statements, i.e. $\mathbb{S}_{\mathcal{D}} = \mathbb{S} \cup \{C \precsim D \mid C, D \in \mathbb{C}\}$.*

A $\mathcal{DALC}$-knowledge base is an $\mathcal{ALC}$-knowledge base $\mathcal{T} \cup \mathcal{A}$ extended by a finite collection $\mathcal{D}$ of $\mathbb{S}_{\mathcal{D}}$-statements called a DBox.

A standard requirement for defeasible subsumption $\precsim$ is that it is preferential (Britz et al., 2019). It is imposed in addition that it is possibility preserving, that is non-empty concepts are not defeasibly subsumed by $\bot$.

To impose more linearity one may also ask for disjunctive rationality or rational monotony.

Definition 78 (Preferential and rational defeasible subsumption)**.** *The defeasible subsumption relation $\precsim$ is called preferential if and only if it satisfies for all $C, D, E \in \mathbb{C}$:*

- $C \precsim \bot$ *implies* $C \sqsubseteq \bot$ *(Consistency preservation)*

- $C \precsim C$ *(Reflexivity)*

- $C \equiv D$, $C \precsim E$ *implies* $D \precsim E$ *(Left logical equivalence)*

- $C \precsim D$, $D \sqsubseteq E$ *implies* $C \precsim E$ *(Right weakening)*

- $C \precsim D$, $C \precsim E$ *implies* $C \precsim D \sqcap E$ *(Right conjunction)*

- $C \precsim E$, $D \precsim E$ *implies* $C \sqcup D \precsim E$ *(Left disjunction)*

- $C \precsim E$, $C \precsim D$ *implies* $C \sqcap D \precsim E$ *(Cautious monotonicity)*

It is called disjunctively rational if it validates in addition

- $C \sqcup D \precsim E$ *implies* $C \precsim E$ *or* $D \precsim E$ *(Disjunctive rationality)*

and respectively rational if it satisfies in addition:

- $C \precsim D$ *and not* $C \precsim \neg E$ *implies* $C \sqcap E \precsim D$ *(Rational monotonicity)*

The failure of $\top \sqsubseteq \bot$ in $\mathcal{ALC}$ obviously implies the failure of $\top \precsim \bot$ in $\mathcal{DALC}$, referred to as a nontriviality postulate. The TBox and the DBox then specify our initial knowledge about strict and defeasible subsumption links between various concepts. Formally speaking, these rules correspond to those known from standard conditional logics for strict and defeasible implication. This relationship can be exploited fruitfully to guide the specification of new monotonic and nonmonotonic semantics for DDLs.

5.2 Monotonic reasoning over defeasible description logics

With the syntax defined for a defeasible $\mathcal{ALC}$, a semantics capable of giving these default conditionals a reasonable interpretation is naturally necessary. Now, a ranking measure semantics for defeasible description logics is defined, along with the associated monotonic inference.

5.2.1 Ranking measures for DDL

Most semantics for DDLs are based on preference orders over the interpretation domain Δ. The simplicity and robustness of this approach are appealing, but it seems the lack of expressiveness is less well suited for representing, relating, and aggregating defeasible ontologies. In particular it is insufficient to handle independence in a general, theoretically and intuitively appropriate way (Weydert, 2003). This motivates developing a semantics using a more fine-grained, quantitative but still robust plausibilistic semantics based on ranking measures. These are rooted in Spohn's integer-valued ranking functions (Spohn, 1988, 2012), also known as κ-functions, which are the simplest instance of the general mathematical notion of a ranking measure introduced in (Weydert, 1995). They play an important role in graded belief modelling, iterated belief revision, and default reasoning. Ranking measures are quasi-probabilistic implausibility valuations defined over a boolean algebra, and whose value domain carries a commutative additive structure. This allows to obtain a reasonable independence notion and to define (Jeffrey-)conditionalization. For our purposes, general default reasoning over finite knowledge bases, it is necessary and in general sufficient that the value range encompasses the (countably many) positive rationals with infinity: $[0, \infty]_{\mathbb{Q}}$. For the sake of generality, we will however include all the positive reals (with infinity) as possible values: $[0, \infty]_{\mathbb{R}}$.

Definition 79 (Real-valued ranking measures)**.** *A real-valued ranking measure is a function* $R : (\mathbb{B}, \sqcap, \sqcup, \neg) \longrightarrow ([0, \infty]_{\mathbb{R}}, 0, \infty, +, <)$ *such that:*

- $R(\top) = 0$, $R(\bot) = \infty$,
- $R(A \sqcup B) = min_{<}\{R(A), R(B)\}$,

They are possibility-conserving if for each $A \in \mathbb{B}$, $R(A) = \infty$ *implies* $A = \bot$. *The associated conditional ranking measure* $R(.|.)$ *is defined by*
$R(A \mid B) = R(A \sqcap B) - R(B)$ *for* $R(B) \neq \infty$, *and* $R(A \mid B) = 0$ *otherwise.*

For example, to interpret defeasible subsumption, consider the boolean algebra consisting of the congruence classes of concept terms modulo logical equivalence. Each class is represented by any of its members. Hence, for all $C, D \in \mathbb{C}$, $\vdash_{\mathcal{ALC}} C \equiv D$ if and only if $R(C) = R(D)$.

For evaluating defeasible inferential relationships involving role assertions, one needs in addition ranking measure products, in analogy to products of probability measures. In a general scenario, one may want to define the product of a tuple of ranking measures $R_1, \dots, R_n$, respectively defined over the boolean algebras $\mathbb{B}_1, \dots, \mathbb{B}_n$. W.l.o.g. it is assumed that each $\mathbb{B}_i$ is a set algebra over a universe W_i. The product will be denoted $R_1 \otimes \dots \otimes R_n$.

The first step is to specify the product boolean algebra associated with $\mathbb{B}_1, \dots, \mathbb{B}_n$. This is by definition the boolean closure of the product base consisting of all the cross-products $A_1 \times \dots A_n$ for $A_i \in \mathbb{B}_i$. Note that the product base is closed under intersections. Let $\mathbb{B}_1 \otimes \dots \otimes \mathbb{B}_n$ denote the resulting product boolean algebra. It is not difficult to see that each of its elements can be written as a finite union of product base elements. Therefore, to define a product ranking measure over $\mathbb{B}_1 \otimes \dots \otimes \mathbb{B}_n$, because of the min-rule for finite unions, it is enough to do two things:

1. Identify the ranking measure values for the elements of the product base.

2. Prove that for each $B \in \mathbb{B}_1 \otimes \dots \otimes \mathbb{B}_n$, any reconstruction as a finite union of product base elements produces the same ranking measure value.

The idea is that $R_1 \otimes \dots \otimes R_n$ constitutes the independent aggregate of the individual ranking measures R_i so that the ranking measure value of a cross-product, which represents an independent aggregate of sets, is just the sum of the ranking measure values of the individual component sets. In particular, for each $A \in \mathbb{B}_i$,

$$(R_1 \otimes \dots \otimes R_n)(W_1 \times \dots \times W_{i-1} \times A \times W_{i+1} \times \dots W_n) = R_i(A).$$

This gives us the following definition.

Definition 80 (Product ranking measures)**.** *For any ranking measures $R_1, \dots, R_n$ over boolean algebras $\mathbb{B}_1, \dots, \mathbb{B}_n$, the product ranking measure $R_1 \otimes \dots \otimes R_n$ is defined to be the unique ranking measure R on $\mathbb{B}_1 \otimes \dots \otimes \mathbb{B}_n$ such that for each $A_1 \times \dots A_n$ with $A_i \in \mathbb{B}_i$,*

$$R(A_1 \times \dots \times A_n) = R(A_1) + \dots + R(A_n).$$

Proposition 4 (Product ranking measures)**.** *For any finite tuple of ranking measures R_i over $\mathbb{B}_i$, the corresponding product ranking measure exists and is canonical over $\mathbb{B}_1 \otimes \dots \otimes \mathbb{B}_n$*

However, one further step is needed. In fact, arbitrary role relations may not be reconstructible as finite unions of cross-products of concepts. Even if a canonical boolean algebra $\mathbb{C} = \mathbb{B}_{\beth}$ (for all $0 < i \leq n$) is assumed, there is no guarantee that a given role relation r is representable in $\mathbb{C}^{\nvDash} = \mathbb{C} \otimes \mathbb{C} = \otimes^2\mathbb{C}$. Therefore, to express and evaluate the plausibility of arbitrary role relationships, $\otimes^n\mathbb{R}$ is lifted to some boolean algebra $\mathbb{C}^*$ extending $\otimes^n\mathbb{C}$ and representing all the definable relations resulting from admissible combinations of roles and concepts.

Fortunately, it is not necessary to discuss in detail what constitutes a definable relation. What can be done is to lift $R_1 \otimes \ldots \otimes R_n$ to the boolean subset algebra $\mathbb{S}(W^n)$ based on 2^{W^n}, the full power set of W^n, which is stipulated to constitute a — in some specific way — maximal instantiation for the relevant signature.

Definition 81 (Completion)**.** *Let $R = R_1 \otimes \ldots \otimes R_n$ be a product ranking measure over the boolean algebra $\otimes^n\mathbb{C}$, which can be interpreted as a subalgebra of a set algebra $\mathbb{S}(W^n)$. The completion R^* of R is defined for each $A \subseteq W^n$ by*

$$R^*(A) = sup\{R(B) \mid A \subseteq B, B \in \otimes^n\mathbb{C}\}.$$

One can now show that the completion process gives us a functions with the appropriate features, namely possibility-conserving real-valued product ranking measures which are coherent across dimensions.

Conjecture 3. *Let R^* be the completion of $R = R_1 \otimes \ldots \otimes R_n$ over $\otimes^n\mathbb{C}$. Then:*

1. *R^* is a real-valued ranking measure over 2^{W^n}.*

2. *R^* is possibility-conserving if this holds for R and the image set of each factor measure R_i is finite.*

3. *If R^{**} is the completion of $R = R_1 \otimes \ldots \otimes R_{n+m}$ with domain $2^{W^{n+m}}$, then the projection of R^{**} to 2^{W^n} is R^*.*

The standard scenario in defeasible description logic is that a basic ranking measure R over $\mathbb{C}$ has to be extended to the product measures $\otimes^n R$ to handle all the relational expressions definable through roles and concept assertions in the given language context. The chosen semantic framework, the inferential needs, and possible representational purposes point here to a natural extension of $\mathcal{ALC}$ where the roles are closed under boolean operations and the inverse operator.

This would be a relatively moderate generalization which ensures that, for a finite signature, the number of explicit role names stays finite. However, because there may be countably infinitely many distinct combined concepts, the combination of role and concept assertions in ABoxes still allows countably many relational descriptions in each dimension. Note that the minimal dimension is determined by the number of individual constants appearing in the ABox. The important point is that it is enough to specify for each dimension n a countable subalgebra of the full set-algebras over 2^{W^n}, ensuring cross-dimensional compatibility. It is obtained by restricting the full completion of each product measure $\otimes^n R$ to the relations definable over the relevant sub-signature. Now there is a ranking measure value for each role induced from the original ranking measure over concepts. In fact, this value can be determined without specifying the full completion, but the definition above may be slightly more convenient.

Definition 82 (Ranking roles)**.** *Given a role $r \in \mathsf{N}_{\mathscr{R}}$, its ranking measure value, sloppily written $R(r)$, is defined to be $R^*(r)$ where R^* is the completion of the product measure $R \otimes R$.*

If in an $\mathcal{ALC}$-context the domain and the range of r are subsumed by the concepts A_{dom} respectively A_{rg}, which can be expressed in $\mathcal{ALC}$, then it holds that $R(r) \geq R(A_{dom}) + R(A_{rg})$. If moreover there are no less plausible subsuming subconcepts A'_{dom} or $A'_{rg} \sqsubseteq A_{RG}$, i.e. verifying $R(A'_{dom}) > R(A_{dom})$ or $R(A'_{rg}) > R(A_{rg})$, then the definitions ensure that $R(r) = R(A_{dom}) + R(A_{rg})$. Note that this observation relies on the absence of direct defeasible information about roles, e.g. as provided by defeasible role subsumptions, not available in $\mathcal{ALC}$.

Now the basic machinery for interpreting and reasoning with defeasible concept subsumptions is defined.

5.2.2 Defeasible description logic: Semantics

To interpret the defeasible subsumption statements in $\mathcal{DALC}$ the $\mathcal{ALC}$-interpretations are expanded, which are just a special instance of classical first-order structures, with ranking measures.

Definition 83 ($\mathcal{DALC}$-interpretation). *A $\mathcal{DALC}$-interpretation over $\mathcal{L}$ is a pair (I, R) consisting of an $\mathcal{ALC}$-interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ over $\mathcal{L}$ and a ranking measure R over the boolean algebra of the concept interpretations induced by $\mathcal{I}$ such that for any concept $C \in \mathbb{C}$, $R(C^{\mathcal{I}}) = \infty$ iff $C^{\mathcal{I}} = \emptyset$. If $\mathcal{I}$ is fixed, $R(C^{\mathcal{I}})$ is abbreviated by $R(C)$. $\mathbb{I}$ denotes the set of all $\mathcal{DALC}$-interpretations for a given relational signature $(\mathsf{N}_{\mathscr{C}}, \mathsf{N}_{\mathscr{R}})$. $\mathcal{DALC}$-interpretations are also referred to as ranking interpretations.*

A monotonic $\mathcal{DALC}$-semantics for a given $\mathcal{DALC}$-language $\mathcal{L}(\sqsubsim)$ is characterized by a $\mathcal{DALC}$-satisfaction relation $\models_d$ which determines for each $\mathcal{DALC}$-interpretation $(\mathcal{I}, R)$ the $\mathcal{DALC}$-formulas φ it verifies. Of course, $\models_d$ is expected to be a conservative extension of $\models_{alc}$ over the $\mathcal{ALC}$-language $\mathcal{L}$. That is, $(\mathcal{I}, R)$ should $\models_d$-satisfy exactly those $\varphi \in \mathcal{L}$ which are $\models_{alc}$-satisfied by $\mathcal{I}$. What remains to be done is to specify under which conditions a $\mathcal{DALC}$-interpretation $(\mathcal{I}, R)$ $\models_d$-satisfies a given defeasible subsumption statement.

The basic idea, well-known from the logic of default conditionals, is to say that $C \sqsubsim D$ holds if and only if $C \sqcap D$ is sufficiently more plausible than $C \sqcap \neg D$. The question is then what one may consider sufficient. Here there are essentially two alternatives: on the one hand, simple dominance without any further conditions, which is the weakest and most common approach, and on the other hand, threshold dominance, which is stronger and offers more conceptual and technical flexibility, e.g. to define advanced inference methods based on minimization. Because all the real-valued ranking measure values $\neq 0, \infty$ are structurally equivalent[1], it is fortunately enough to consider a single threshold, by convention $\tau = 1$. In what follows the stronger truth condition will be exploited. Note that for the traditional integer-valued ranking functions, both variants are equivalent. Another, maybe more productive way to experiment with strength would be to encode the different possibilities at the object-level, e.g. by introducing graded defeasible subsumption. But here the focus on the single-strength scenario.

Definition 84 ($\mathcal{DALC}$ satisfaction relation). *Let $(\mathcal{I}, R)$ be a ranking measure interpretation over $\mathcal{L}$, $C, D \in \mathbb{C}$, and $\varphi \in \mathcal{L}$.*

- $(\mathcal{I}, R) \models_d \varphi$ *if and only if* $\mathcal{I} \models_{alc} \varphi$,
- $(\mathcal{I}, R) \models_d C \sqsubsim D$ *if and only if* $R(C \sqcap D) + 1 \leq R(C \sqcap \neg D)$.

[1] They can be exchanged by automorphisms on the value structure

For $\Gamma \subseteq \mathcal{L}(\, \underset{\sim}{\sqsubset} \,)$*, let* $Mod_d(\Gamma) = \{(\mathcal{I}, R) \in \mathbb{I} \mid (\mathcal{I}, R) \models_d \varphi$ *for each* $\varphi \in \Gamma\}$ *be the corresponding model set. For* $\varphi \in \mathcal{L}(\, \underset{\sim}{\sqsubset} \,)$*, is written* $Mod(\varphi) = Mod(\{\varphi\})$*.*

The $\mathcal{DALC}$-semantics now determines in the usual way a corresponding (monotonic) Tarskian entailment relation.

Definition 85 ($\mathcal{DALC}$-entailment)**.** *A knowledge base* $\Gamma \subseteq \mathcal{L}(\, \underset{\sim}{\sqsubset} \,)$ *is said to monotonically entail a formula* $\varphi \in \mathcal{L}(\, \underset{\sim}{\sqsubset} \,)$*, written* $\Gamma \vdash_d \sigma$*, if and only if* $Mod_d(\Gamma) \subseteq Mod_d(\sigma)$*.*

Under this interpretation, $\underset{\sim}{\sqsubset}$ shows the expected properties.

Conjecture 4. $\vdash_d$ *validates:*

- *the preferentiality of* $\underset{\sim}{\sqsubset}$ *over* $\sqsubseteq$*,*
- *disjunctive rationality for* $\underset{\sim}{\sqsubset}$*, but*
- *does not validate rational monotonicity for* $\underset{\sim}{\sqsubset}$*.*

One feature of the description language is that it allows to some extent to translate logical connectives into boolean operations over concept terms.

Conjecture 5. *Let* $a \in \mathsf{N}_{\mathscr{I}}$ *and* $C, D \in \mathbb{C}$*. Then:*

- $\vdash_d \; a\!:\!C \wedge a\!:\!D \;\leftrightarrow\; a\!:\!C \sqcap D$*,*
- $\vdash_d \; a\!:\!C \vee a\!:\!D \;\leftrightarrow\; a\!:\!C \sqcup D$*,*
- $\vdash_d \; \neg(a\!:\!C) \;\leftrightarrow\; a\!:\!\neg C$*.*

This will be useful to induce ranking measures over assertions from ranking measures over concepts.

5.2.3 Ranking measure semantics for assertions

Up to this point, ranking measure semantics for concept subsumption statements has been described, but without explicitly considering assertions. So, now the focus will be shifted to considering the primary challenge considered here, that of defeasible reasoning about assertions.

Now, ranking measures need to be extended to assertions. As a baseline, observe that the concept assertions restricted to one individual constitute a Lindenbaum algebra:

Definition 86 (Assertion algebra)**.** *The algebra of assertions* $(\mathbb{A}, \vee, \wedge, \sim)$ *is such that, given any individual name a:*

- $a : C \wedge a : D \iff a : C \sqcap D$
- $a : C \vee a : D \iff a : C \sqcup D$
- $\sim (a : C) \iff a : \neg C$

The assertion algebra provides a domain for induced ranking measures, defined relative to ranking measures over the concept algebra. These structures allow:

Definition 87 (Induced ranking measures)**.** *An induced ranking measure* $\hat{R}$*, is derived from a ranking measure function* R *such that* $\hat{R} : \mathbb{A} \longrightarrow [0, \infty]_{\mathbb{R}}$ *such that* $\hat{R}(a : C) \coloneqq R(C)$*.*

Induced ranking measures by themselves only represent the perspective of a single individual. Different individuals introduce interacting ranking measures which require aggregating in order to combine different individuals into a canonical ranking function. This multidimensional aspect to the reasoning process is handled by product measures:

Definition 88 (Product measures)**.** *The rank of the conjunction of any two concept assertions in an induced ranking measure is:* $\hat{R}(a : C \wedge b : D) = \hat{R}(a : C) + \hat{R}(b : D)$*.*

It is more tricky to define the rank of a role assertion, $(a, b) : r$, seeing as ranks on roles are not explicitly defined. In $\mathcal{DALC}$ there is no way to explicitly reason about roles, and so for every role $r \in \mathsf{N}_{\mathscr{R}}$, and the ranking measure R^2 over the cross-product of concepts, for now $R^2(r) = 0$ is stipulated.

However, in general:

Definition 89 (Role assertion ranks)**.** *Given any role assertion, the rank is the same as the rank of the underlying role* $\hat{R}((a, b) : r) = R^2(r)$*.*

This will be useful for extensions to $\mathcal{DALC}$ where explicit statements about role typicality are explored.

The next definition is crucial for nonmonotonic entailment on the level of assertions. Ranking conditionals over assertion statements is necessary for specifying a default inference relation on the level of the assertion language.

In order to evaluate whether an assertion statement follows from a knowledge base including an Abox, the rank of an assertion needs to be well-defined, and then the conditional ranking over assertions allows for specifying default inference of assertions:

Definition 90 (Conditional induced ranking measure)**.** *A conditional ranking measure statement over assertions $\alpha, \beta \in \mathbb{A}$, $\hat{R}(\alpha \mid \beta)$, is defined such that $\hat{R}(\alpha \mid \beta) = \hat{R}(\alpha \wedge \beta) - R(\beta)$*

5.3 Semantics for nonmonotonic reasoning

5.3.1 General defeasible inference

Defeasible reasoning over $\mathcal{DALC}$ has focused primarily on the task of inferring defeasible subsumption statements from a default subsumption base, leaving aside the assertion language and the ABox. Less studied is the task of defeasibly inferring assertions: that is defining a nonmonotonic inference relation to defeasibly infer concept assertions or role assertions from a default subsumption base and assertion box $\mathcal{T} \cup \mathcal{D} \cup \mathcal{A} \mid\!\sim a : C$ or $\mathcal{T} \cup \mathcal{D} \cup \mathcal{A} \mid\!\sim (a, b) : r$.

The usual semantics for defeasible reasoning in the literature revolve around preferential and ranked interpretations (Britz et al., 2019), which are adapted from the KLM framework defined for propositional default conditional logics (Kraus et al., 1990).

The main focus at the core of this chapter is the defeasible inference of assertions. The following example will illustrate much of what is tackled in this chapter with respect to role assertions (Britz et al., 2018):

Example 19 (Abnormality choice)**.** *Consider the knowledge base $\mathcal{T} = \emptyset$, $\mathcal{D} = \{\top \mathrel{\underset{\sim}{\sqsubset}} A \sqcap \forall r.\neg A\}$, and $\mathcal{A} = \{(a, b) : r\}$. Intuitively either a or b should be normal, but this requires the other to be abnormal with respect to the default statement. Britz et al (Britz et al., 2018) introduces a procedure based on System Z that calculates two extensions, as they are referred to in their work, of the ABox: $\{a : A, a : \exists r.\neg A, b : \neg A\}$ and $\{b : A, a : \exists r.A\}$, and proposes the intersection of the consequences of these choices, $\{a : \exists r.\top\}$ as a canonical solution, and Bonatti (Bonatti, 2019) considers each extension as equally valid.*

However, there are some preliminary considerations before true nonmonotonic inference may be introduced.

5.3.2 Ranking choice

To specify proper nonmonotonic reasoning using our ranking measure semantics, the idea is to choose a preferred subset of $Mod(\mathcal{T} \cup \mathcal{D} \cup \mathcal{A})$ to realise a justifiable defeasible inference relation. This is done via ranking choice functions (Weydert, 2003):

Definition 91 (Ranking choice functions)**.** *A ranking choice function* $\mathcal{F} : 2^{\mathbb{S}} \mapsto 2^{\mathbb{I}}$ *associates to a finite default knowledge base a set of ranking measure interpretations such that:*

1. $\mathcal{F}(\mathcal{T} \cup \mathcal{D}) \subseteq Mod(\mathcal{T} \cup \mathcal{D})$ *(Choice)*
2. $\{Mod(\delta) \mid \delta \in \mathcal{D}\} = \{Mod(\delta') \mid \delta' \in \mathcal{D}'\}$ *implies* $\mathcal{F}(\mathcal{T} \cup \mathcal{D}) = \mathcal{F}(\mathcal{T} \cup \mathcal{D}')$ *(local semanticality)*
3. $Mod(\mathcal{T} \cup \mathcal{D}) \neq \emptyset$ *implies* $\mathcal{F}(\mathcal{T} \cup \mathcal{D}) \neq \emptyset$ *(nonmonotonic consistency)*

Ranking choice functions are used to define a defeasible inference relation by selecting specific subsets of ranking interpretations to specify entailment (Weydert, 2003):

Definition 92 (Ranking choice inference)**.** *Let* $\mathcal{F}$ *be ranking choice function,* $\mathcal{T}$ *a TBox,* $\mathcal{D}$ *a DBox,* $\mathcal{A}$ *an ABox, and* α *an assertion, then* $\mathcal{T} \cup \mathcal{D} \cup \mathcal{A} \mid\!\sim_{\mathcal{F}} \alpha$ *if and only if for every* $\langle \mathcal{I}, R \rangle \in \mathcal{F}(\mathcal{T} \cup \mathcal{D})$ *then the resulting product measure* $\hat{R}$ *over all induced ranking measures* $\hat{R}_a$ *for each named individual* a *in* $\mathcal{A}$*,* $\hat{R}(\neg\alpha \mid \mathcal{A}) > 0$*.* $\mathcal{A} \mid\!\sim^{\mathcal{F}}_{\mathcal{T} \cup \mathcal{D}} \alpha$ *may be used as an abbreviation.*

Now, a defeasible inference relation for $\mathcal{DALC}$ has been properly defined. The next question is how to define specific ranking choice functions in order to obtain the inferences that best align with intuitions regarding commonsense reasoning, and that satisfy desired principles in the literature.

5.3.3 Reconstructing System Z

Among the most popular default inference relations is System Z, or rational closure (Lehmann and Magidor, 1992; Pearl, 1990). In the context of our approach, System Z corresponds to assigning the minimal ranking measure values that satisfies our subsumption statements (Giordano et al., 2015). This amounts to a ranking measure choice function that seeks to minimally satisfy, with respect to the ranking algebra values, the default subsumption statements, which is comparable to inference schemes based on plausibility maximization. For further explanation it is covered in detail by Britz et al (Britz et al., 2018).

For the construction of System Z in this framework, a useful definition in the context of ranking choice regards how to compare two ranking functions:

Definition 93 (Comparison of ranking measures). *Given two ranking functions, R_1 and R_2, we set $R_1 \preceq R_2$ if and only if for every $C \in \mathbb{C}$, $R_1(C) \leq R_2(C)$.*

In natural language, this definition states that a ranking measure that assigns every concept to the same rank or lower than another ranking function is at least as preferred. Two ranking functions are incomparable if there exists $C, D \in \mathbb{C}$ such that $R_1(C) < R_2(C)$ and $R_2(D) < R_1(D)$.

Conjecture 6 (Existence and uniqueness of Z ranking). *Given the preference ordering over a set of ranking measures in definition 93, then there always exists a unique ranking measure R^Z such that $R^Z \preceq R$ for all other ranking measures R, i.e. R^Z is a global minimum with respect to $\preceq$.*

Then, this global minimum can be used to define the set of System Z models as such:

Definition 94 (Z model). *Given a defeasible knowledge base, the Z-ranking model $\langle \mathcal{I}, R^Z_{\mathcal{T}\cup\mathcal{D}} \rangle$ is such that for all ranking measure functions defined using strong, i.e. threshold, semantics, $R_{\mathcal{T}\cup\mathcal{D}}$, $R^Z_{\mathcal{T}\cup\mathcal{D}} \preceq R_{\mathcal{T}\cup\mathcal{D}}$.*

Having the Z-ranking measure for the concept language, allows the specification of the System Z inference relation for the assertion language via the usage of induced ranking measures:

Definition 95 (System Z). *For a knowledge base, $\mathcal{T}$, $\mathcal{D}$, $\mathcal{A}$, and an assertion $\alpha \in \mathbb{A}$, then $\mathcal{T} \cup \mathcal{D} \cup \mathcal{A} \mathrel{|\!\!\sim}^Z \alpha$ if and only if for the product measure $\hat{R}^Z$ obtained over all induced ranking measures defined from the individuals in $\mathcal{A}$, it is such that $\hat{R}^Z(\neg\alpha \mid \mathcal{A}) > 0$. $\mathcal{A} \mathrel{|\!\!\sim}^Z_{\mathcal{T}\cup\mathcal{D}} \alpha$ is used as shorthand.*

Example 20 (Abnormality choice cont.). *Let us take the knowledge base from example 19: $\mathcal{T} = \emptyset$, $\mathcal{D} = \{\top \sqsubseteq\!\!\!\!\sim A \sqcap \forall r.\neg A\}$, $\mathcal{A} = \{(a, b) : r\}$*

The default provides the (strong semantics using the default parameter 1) constraint over any satisfying ranking measure:

- $R(A \sqcap \forall r.\neg A) + 1 \leq R(\neg A \sqcup \exists r.A)$

Under System Z, the distinguished Z-ranking measure $R^{Z}_{\mathcal{T}\cup\mathcal{D}}$ obtained by minimally satisfying the above constraint, then is represented in the following table, where the integers are to be read as the ranking value of the intersection of the concepts on the left hand side and the top row, such that each cell in the central area shows the rank in $R^{Z}_{\mathcal{T}\cup\mathcal{D}}$ of the intersection of the concepts that coincide at that cell:

	A	$\neg A$
$\forall r.A \sqcap \forall r.\neg A$	0	1
$\exists r.\neg A \sqcap \forall r.\neg A$	0	1
$\forall r.A \sqcap \exists r.A$	1	1
$\exists r.\neg A \sqcap \exists r.A$	1	1

Table 5.1: Single dimensional induced ranking Z-values

So, $R(\forall r.A \sqcap \forall r.\neg A \sqcap \neg A) = 1$ which corresponds to the top right octant of the table of integers, and similarly $R(\exists r.\neg A \sqcap \forall r.\neg A \sqcap A) = 0$ which may be read off the second row from the top and left hand side octant.

The above ranking measure is naturally defined over concepts, however since default inference among assertions is of interest here, it is necessary to define ranking values over the assertions themselves. Therefore, the above ranking measure will be translated into ranking measures over the assertion language.

Since there are two individual names here, a and b, the induced ranking measures, following definition 88, may be represented as follows:

	$b : A$	$b : \neg A$
$a : \neg A \sqcap (\forall r.A \sqcap \forall r.\neg A)$	1	2
$a : \neg A \sqcap (\exists r.\neg A \sqcap \forall r.\neg A)$	1	2
$a : \neg A \sqcap (\forall r.A \sqcap \exists r.A)$	1	2
$a : \neg A \sqcap (\exists r.\neg A \sqcap \exists r.A)$	1	2
$a : A \sqcap (\forall r.A \sqcap \forall r.\neg A)$	0	1
$a : A \sqcap (\exists r.\neg A \sqcap \forall r.\neg A)$	0	1
$a : A \sqcap (\forall r.A \sqcap \exists r.A)$	1	2
$a : A \sqcap (\exists r.\neg A \sqcap \exists r.A)$	1	2

Table 5.2: Two dimensional induced ranking Z-values

Similarly as table 5.1, the integers in the central area represent the ranking value by combining the assertions on the left column and top row. So, $\hat{R}(\{a : \neg A \sqcap (\forall r.A \sqcap \forall r.\neg A), b : A\}) = 1$, *corresponding to the top left cell, and so on.*

The ABox $\mathcal{A} = \{(a, b) : r\}$ *provides a role assertion. This role assertion imposes constraints based on the concepts* a *can be: first* a, *by monotonic entailment, is in either* $\exists r.A$ *or* $\exists r.\neg A$, *i.e.* $a : \exists r.A \sqcup \exists r.\neg A$, *which is equivalent to* $a : \neg(\forall r.\neg A \sqcap \forall r.A)$, *i.e.* $\hat{R}(a : \forall r.\neg A \sqcap \forall r.A \mid (a, b) : r) = \infty$. *Taking this further, conditioning additionally with* $b : A$ *gives* $a : \neg\forall r.\neg A$, *and* $b : \neg A$ *gives* $a : \neg\forall r.A$, *and so* $\hat{R}(a : \neg\forall r.\neg A \mid b : A, (a, b) : r) = 0$ *and* $\hat{R}(a : \neg\forall r.A \mid b : \neg A, (a, b) : r) = 0$.

Thus, a visual representation of conditioning the induced product measure with the ABox produces:

	$b : A$	$b : \neg A$
$a : \neg A \sqcap (\forall r.A \sqcap \forall r.\neg A)$	∞	∞
$a : \neg A \sqcap (\exists r.\neg A \sqcap \forall r.\neg A)$	∞	1
$a : \neg A \sqcap (\forall r.A \sqcap \exists r.A)$	0	∞
$a : \neg A \sqcap (\exists r.\neg A \sqcap \exists r.A)$	0	1
$a : A \sqcap (\forall r.A \sqcap \forall r.\neg A)$	∞	∞
$a : A \sqcap (\exists r.\neg A \sqcap \forall r.\neg A)$	∞	0
$a : A \sqcap (\forall r.A \sqcap \exists r.A)$	0	∞
$a : A \sqcap (\exists r.\neg A \sqcap \exists r.A)$	0	1

Table 5.3: Two dimensional induced ranking Z-values

Reading the ranking values conditioned with the given ABox information, now observe that the ranking values have changed. To evaluate inference, the same process of checking conditional ranking measure values is followed: to evaluate if $b : A$ *is inferred, for example, the ranking value of* $b : \neg A$ *needs to be checked, and specifically* $b : A$ *may be inferred if* $\hat{R}(b : \neg A) > 0$ *in the above representation of the conditional ranking measure. Since the table contains a cell where* $b : \neg A$ *is* 0, *this is the value of* $\hat{R}(b : \neg A)$ *and so* $b : A$ *may not be inferred. The opposite may also be checked — if* $\hat{R}(b : A) > 0$ *then* $b : \neg A$ *may be inferred. Again, there are (many) instances of* $\hat{R}(b : A) = 0$ *as well in the representation above. Therefore, there is no way to infer whether* b *is in* A *or* $\neg A$. *Is there something else that may be inferred? Not at the moment, but observe that there are some more conditional inferences — if* $a : \neg A$ *were* *also to be added to the ABox, then the equivalent of the bottom half of the representation above is no longer considered, and then* $\hat{R}(b : \neg A) = 1$ *which satisfies the inference requirement, and so then* $b : A$ *would be inferred.*

5.4 Hypothesis-driven inference

One possible conclusion that may drawn so far is that having a classical language that allows for expressing strict information about binary relations, without also having the capability to express defeasible relational information results in an over-restriction of the modelling process that can adversely affect the defeasible inference processes.

5.4.1 Hypotheses

Consider the example from before:

Example 21. *Given the knowledge base* $\mathcal{T} = \emptyset$, $\mathcal{D} = \{\top \precsim A \sqcap \forall r.\neg A\}$, *and* $\mathcal{A} = \{(a, b) : r\}$.

The Z-ranking on the concepts is then $R^Z(\neg A \sqcup \exists r.A) = 1$ *and for all other concepts* C, $R^Z(C) = 0$. *All roles have rank* 0, *and so* $R((a, b) : r) = 0$. *So, the conditional rankings are therefore:*

1. $R^Z(a : \neg A \mid \mathcal{A}) = 1$
2. $R^Z(a : A \mid \mathcal{A}) = 0$
3. $R^Z(a : \exists r.A \mid \mathcal{A}) = 1$
4. $R^Z(a : \exists r.\neg A \mid \mathcal{A}) = 0$
5. $R^Z(a : \forall r.\neg A \mid \mathcal{A}) = R^Z(a : \forall r.A \mid \mathcal{A}) = 0$
6. $R^Z(b : A \mid \mathcal{A}) = 0$
7. $R^Z(b : \neg A \mid \mathcal{A}) = 1$

Therefore the inconsistent set of inferences: $\{(a, b) : r, a : A, a : \exists r.\neg A, a : \forall r.\neg A, b : A\}$ *can be naïvely concluded.*

The primary issue targeted here is that simply lifting System Z (Pearl, 1990) also known as rational closure (Lehmann and Magidor, 1992) to defeasible description logics in our approach appears to lead to some inconsistent defeasible inferences as demonstrated above.

One reason for System Z being insufficient is the presence of role assertions from which a naïve lifting of default reasoning methods from propositional logic cannot extract useful information regarding role restrictions and instantiations. The projection principle applied to System Z, referred to as System PZ, is one attempt to try to instantiate and formalize this extraction.

Here, projection is conceptualised as a "hypothesis", or a speculative, meta-defeasible lemma, that works as a pre-processing step before the inference scheme is applied.

To formalise this notion of a hypothesis, first define a hypothesis box, or HBox, $\mathcal{H}$, as a collection of hypothetical, speculative statements $C \leq D$ where C and D are either atomic concepts, or complex concepts depending on the scheme chosen. These defeasible constraints are comparable to negated conditionals (Booth and Paris, 1998) as in both cases semantic constraints are introduced that are also satisfied by the relevant formulae having the same ranking value. This introduces new challenges in incorporating a new collection of weak, comparative constraints and resolving possible inconsistencies.

Definition 96. *A hypothesis box $\mathcal{H}$ describes a collection of speculative defeasible constraints on ranking measure functions. For any two concepts C, D, here constraints of the form $C \leq D$ are considered.*

In general, an HBox may be induced from the terminological information, or prescribed by a knowledge engineer. Here, one scheme for the former is of interest. The scheme here is as follows: given a TBox and DBox, a set of speculative statements are induced, and collected in an HBox. Then, these speculative statements provide further constraints on the set of models that may satisfy the knowledge base.

The methods for inducing these speculative statements from the regular terminology are theoretically many. Here, one general type of scheme is presented in focus: projection.

5.4.2 Projection principle

The projection principle is a family of speculative methods to extend the strict and defeasible terminological information, in order to extract more information from role assertions. With the current logic so far defined, there is no clear way to incorporate implicit knowledge from role assertions into defeasibly inferring concept assertions.

In order to combine information from defaults and both classical and default statements containing concepts constructed from role relations, consider a hypothesis box containing defeasible assumptions about concepts related through a role.

A collection of concepts are selected from the terminology, and projection constraints are generated using those concepts as the basis. These constraints are represented in the HBox and used to constrain the set of ranking measure interpretations.

5.5 System PZ

The first kind of hypotheses considered are constraints generated by the projection principle:

Definition 97 (Projection HBox induction)**.** *For a defeasible terminology $\mathcal{T} \cup \mathcal{D}$, then the function $\mathcal{H}(\mathcal{T} \cup \mathcal{D}) = \mathcal{H}_{\mathcal{T} \cup \mathcal{D}}$ generates speculative constraints, in the case of the projection principle: for a set of complex concepts $\mathcal{C} \subseteq \mathbb{C}$, add a set of speculative, defeasible ranking measure constraints of the form $\{\forall r.C \sqcap \exists r.C \geq C \mid \text{ for every } C \in \mathcal{C}\}$ to the hypothesis box, denoted $\mathcal{H}$.*

One way of conceptualising projection is it prevents individuals from "forcing" other individuals into exceptionality. The scenario where a being normal causes b to be exceptional should be somehow less preferred to the scenario where b being normal simply causes us to be ignorant regarding a.

Ranking measure semantics need to be extended to consider hypothesis boxes:

Definition 98 (HBox satisfaction)**.** *A ranking measure interpretation $\langle\langle\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle, R\rangle$ satisfies all constraints in the hypothesis box if and only if $R(C) \leq R(D)$ for all $C \leq D \in \mathcal{H}$.*

The results of these projection constraints can be various, and investigations into schemes for how to choose the set $\mathcal{C}$ is an interesting topic. One possible scheme is to define $\mathcal{C}$ as the set of concepts C such that $R(C) > 0$ after only the DBox is considered.

Considering the structure of projection constraints, one question is whether it is possible to express and reason about these constraints using the syntax of $\mathcal{DALC}$. It seems convincing that they should not be.

Conjecture 7. *$\mathcal{H}$ constraints are not expressible in the regular language of $\mathcal{DALC}$.*

The above appears to be true due to, firstly, the simple fact that the default constraints induced by defeasible statements are always of the form $C \sqcap D < C \sqcap \neg D$ whereas hypothetical constraints may take the form of $D \leq C$ where C and D may be logically independent concepts. Secondly, the comparison between the concepts in defaults are, with the exception of defaults parametrized with 0, strictly less-than. This seems incompatible with the possible equality allowed in hypothesis box statements.

Once the HBox is processed, then the nonmonotonic inference scheme is applied. One main inference schemes are considered now, although in theory many may be defined.

Definition 99 (Hypothesis preprocessing)**.** *Given a default base $\mathcal{T}$, $\mathcal{D}$, $\mathcal{H}$, then $\mathcal{S}(\mathcal{T} \cup \mathcal{D}, \mathcal{H}) \subseteq \mathcal{H}$ is a selection function that chooses what subset of the speculative statements to use to constrain the set of models available to the ranking choice function.*

5.5.1 PZ closure

The general task to be addressed now is how to identify reasonable defeasible inference notions in the context of a hypothesis generation mechanism $\mathcal{H}$ and a DDL-inference notion $\mid\!\sim^{\mathcal{F}}_{dalc}$, both defined over the extended $\mathcal{ALC}$-language. Within our ranking measure based framework for DDL this amounts to determine for any TD-Box $\mathcal{T} \cup \mathcal{D}$ a suitable preferred collection of plausible ranking measure models $\mathcal{F}_{\mathcal{H}}(\mathcal{T} \cup \mathcal{D}) \subseteq Mod_{rk}(\mathcal{T} \cup \mathcal{D})$. As the parameter $\mathcal{H}$ indicates, this choice function will exploit hypothesis generation.

Suppose there is a finite subsumption base $\mathcal{T} \cup \mathcal{D}$. $\mathcal{H}$ then generates an associated set of justifiable speculative hypotheses $\mathcal{H}(\mathcal{T} \cup \mathcal{D})$. We recall that these hypotheses may be formulated over an extended language (including concept relations beyond $\sqsubseteq$ and $\underset{\sim}{\sqsubset}$) as long as they are interpretable by $\mathcal{ALC}$ ranking measure models. This defines a bi-prioritized base pair $(\mathcal{T} \cup \mathcal{D}, \mathcal{H}(\mathcal{T} \cup \mathcal{D}))$, where the first component describes hard ranking measure constraints over $\mathcal{DALC}$-interpretations, and the second component possibly conflicting soft hypothetical ranking measure constraints. Its evaluation will fix the choice set $\mathcal{F}_{\mathcal{H}}(\mathcal{T} \cup \mathcal{D})$, also taking into account the origin and nature of the hypotheses.

The idea is to develop an hypothesis-driven approach which still relies on the defeasible background semantics characterized by $\mathcal{F}$, but also on a well-informed selection of hypotheses among those provided by $\mathcal{H}$. That is, a hypothesis selection function $\mathcal{H}^*_{\mathcal{F}}$ is specified, refining $\mathcal{H}$ for filtering hypotheses in the context of $\mathcal{F}$ and $\mathcal{H}$. This step is necessary because $\mathcal{H}$ will be typically inconsistent with $\mathcal{T} \cup \mathcal{D}$. At this level of generality only three conditions on $\mathcal{H}^*_{\mathcal{F}}$ are imposed:

- **Selection** $\mathcal{H}^*_{\mathcal{F}}(X) \subseteq \mathcal{H}(X)$
- **Consistency** $Mod_{rk}(\mathcal{H}^*_{\mathcal{F}}(X)) \neq \emptyset$ if $Mod_{rk}(X) \neq \emptyset$
- **Local semanticality** $\mathcal{H}^*_{\mathcal{F}}(X) = \mathcal{H}^*_{\mathcal{F}}(X')$ if $\{Mod_{rk}(\eta) \mid \eta \in X\} = \{Mod_{rk}(\eta) \mid \eta \in X'\}$

Based on $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{H}^*_{\mathcal{F}}$, then the corresponding hypothesis-driven inferential semantics $\mathcal{F}_{\mathcal{H},\mathcal{H}^*}$ can be defined as follows:

$$\mathcal{F}_{\mathcal{H},\mathcal{H}^*}(\mathcal{T} \cup \mathcal{D}) = \mathcal{F}(\mathcal{T} \cup \mathcal{D} \cup \mathcal{H}^*_{\mathcal{F}}(\mathcal{T} \cup \mathcal{D})).$$

In other words, in the context of $\mathcal{F}$ and $\mathcal{H}$ we identify and add preferred hypotheses specified by $\mathcal{H}^*$ to the input base $\mathcal{T} \cup \mathcal{D}$ and use $\mathcal{F}$ to evaluate the resulting extended base.

This general approach may be illustrated by taking a closer look at the family of those instances where (1) the inferential semantics is based on plausibility maximization (generalizing System Z), i.e. $\mathcal{F} = \mathcal{F}^Z$, and (2) the hypothesis generation $\mathcal{H}$ only relies on the Projection Principle. In fact, most existing proposals in DDL profit in one way or another from System Z, exploiting its simplicity and its computational features. Presupposing (1) ad (2) has several interesting consequences.

1. Global semanticality of $\mathcal{F}$: The ranking measure choice function $\mathcal{F}^Z$ is invariant under logical equivalence w.r.t. $\mathcal{DALC}$-entailment. That is, it is fully determined by the model sets of the input bases $\mathcal{T} \cup \mathcal{D}$. Note that this condition is violated by well-behaved inference notions supporting exceptional inheritance.

2. Ranking measure monotony: The canonical Z-ranking measure R^Z_Γ associated with a subsumption base Γ pointwisely increases (weakly) with Γ, i.e., $\Gamma \subseteq \Gamma'$ implies $R^Z_\Gamma(A) \leq R^Z_{\Gamma'}(A)$. This is useful for the stability inductive ranking measure constructions, like those exploited for computing $\mathcal{F}_\mathcal{H}$.

3. Choice cautiousness/skepticality: The speculative character of the projection principle suggests to drop the resulting hypotheses in the presence of any noticeable conflicts with other hypotheses of similar quality. This has quite an impact on the specification of $\mathcal{F}_\mathcal{H}$.

The central task is now to specify in this context a suitable hypothesis selection method $\mathcal{H}^*_\mathcal{F}$. Because this work restricts itself to $\mathcal{F} = \mathcal{F}^Z$, the sub/superscript $\mathcal{F}$ is dropped.

A standard strategy would be to introduce an extension concept $Ext_\mathcal{H}$ over the induced HBoxes $\mathcal{H}(\mathcal{T} \cup \mathcal{D})$:

$$Ext_\mathcal{H}(\mathcal{T} \cup \mathcal{D}) \subseteq Pow(\mathcal{H}).$$

The selected hypothesis set $\mathcal{H}^*(\mathcal{T} \cup \mathcal{D})$ could then be identified with the intersection of all the extensions $\cap Ext_\mathcal{H}(\mathcal{T} \cup \mathcal{D})$. Focusing on this robust collection of unquestioned hypotheses, and their conclusions, as opposed to a more adventurous approach including the joint consequences of all the extensions, seems more in line with speculative hypotheses. What would be reasonable demands for the extensions?

- **1. Relative Consistency:** The extensions should be consistent with the TDBox $\mathcal{T} \cup \mathcal{D}$.
- **2. Relative Maximality:** They should be maximal relative to the TDBox.
- **3. Possibility conservation:** Adding the extension to the TDBox does not entail new strict subsumptions.

The first requirement is obvious. The second one reflects the desire not to spoil information without good reasons and therefore requires maximal admissible sets of hypotheses. The third one is another way to implement cautiousness w.r.t. speculative hypotheses. One should not assume emptiness or impossibility without a logical or theoretical necessity to do so. This may be seen as an anti-circumscriptive principle. This property, which strengthens (relative) consistency, is referred to as *(relative) coherence*. It has the advantage that individual observations compatible with the TDBox cannot invalidate preferred hypotheses.

However, such a straightforward extension-based approach has some drawbacks. One is that it makes no distinction between hypotheses, but this ignores the fact that in the context of $\mathcal{T} \cup \mathcal{D}$, adding different hypotheses may have different impacts. For instance, making changes affecting concepts more exceptional from the perspective of System Z is informationally less disruptive than doing so for less exceptional ones. This actually reminds the minimal change principle from belief revision. So one may want to give priority to those hypotheses whose addition to the TDBox has an impact on higher ranked (more implausible) levels when plausibility maximization is applied. For hypothesis generation based on the projection principle, the hypotheses have the form $A \leq B$. Only those hypotheses where $R^Z_{\mathcal{T}\cup\mathcal{D}}(A) > R^Z_{\mathcal{T}\cup\mathcal{D}}(B)$ will modify plausibility maximization, that is, $R^Z_{\mathcal{T}\cup\mathcal{D}\cup\{A\leq B\}} \neq R^Z_{\mathcal{T}\cup\mathcal{D}}$. But only those layers $\geq R^Z_{\mathcal{T}\cup\mathcal{D}}(B)$ will be affected. This suggests the following prioritization notion for $\mathcal{H}$ in the context of a TDBox and plausibility maximization:

Relative hypothesis prioritization: Let Γ be a TDBox and $(A \leq B), (A' \leq B') \in \mathcal{H}(\Gamma)$. Then

$$(A \leq B) \prec_\Gamma (A' \leq B') \text{ iff } R^Z_\Gamma(B) < R^Z_\Gamma(B')$$

The relevant structure to evaluate will therefore be $(\mathcal{H}(\Gamma), \prec_\Gamma)$. This immediately gives us a prioritized extension concept

$$Ext_{\mathcal{H}}^{\prec_{\mathcal{T}\cup\mathcal{D}}}(\mathcal{T}\cup\mathcal{D}).$$

The corresponding prioritized extensions are obtained in the usual way, namely through maximizing the hypothesis sets, starting at the highest priority levels and moving downwards, but ensuring coherence. This defines a simple priority-driven hypothesis selection function given by

$$\mathcal{H}^*_{hz}(\mathcal{T}\cup\mathcal{D}) = \cap Ext_{\mathcal{H}}^{\prec_{\mathcal{T}\cup\mathcal{D}}}(\mathcal{T}\cup\mathcal{D})$$

However, it is possible to do even better. In fact, the above handling of priorities may be too rigid and thereby unable to exploit all the information available in the context of a TDBox, $\mathcal{H}$ and $\mathcal{F}^Z$. The current prioritization approach relies on the Z-ranking induced by the original TDBox, but one could also imagine an iterated construction with evolving Z-rankings. The idea is to first consider the top priority hypotheses, locally identifying the maximal coherent subsets and taking their intersection, as was done before over the full hypothesis set. Next, on the one hand the preferred hypotheses obtained from the local evaluation to the TDBox may be added, forming a new TDHBox, and on the other hand subtract the top priority hypotheses from the hypothesis set, because they have already been taken into account. The preferred ones are part of the extended TDHBox, whereas speculative hypotheses involved in conflicts are automatically removed.

The procedure will then be restarted from the revised pair consisting of the extended TDHBox and the reduced hypothesis set. The new TDHBox also determines a revised Z-ranking with ranking values increasing or staying constants. It follows that the prioritization of the hypothesis base must be recomputed as well, defining a fresh set of top hypotheses. These steps are repeated until no hypotheses are left, but this means that it is enough to apply plausibility maximization on the completed TDHBox to determine $\mathcal{F}_{\mathcal{H}}(\mathcal{T} \cup \mathcal{D})$. Besides the conceptual appeal of this incremental procedure, one may also mention the possible computational advantages resulting from less costly local maximizations and intersections, as well as the anytime flavour of the algorithm. Formally, the iterated priority-driven hypothesis selection function $\mathcal{H}^*_{ihz}$ is specified as follows.

Let $\mathcal{H}$ determine for each finite TDBox $\mathcal{T} \cup \mathcal{D}$ a finite collection of $\leq$-statements figuring as hypotheses and backed by the projection principle. The idea is to specify by parallel induction two sequences of finite bases over the extended language, namely $(\Gamma_i)_{i<\infty}$ and $(H_i)_{i<\infty}$, with growing Γ_i and shrinking H_i.

The starting points of the induction are $\Gamma_0 = \mathcal{T} \cup \mathcal{D}$ and $H_0 = \mathcal{H}(\mathcal{T} \cup \mathcal{D})$. Suppose Γ_i and H_i are given. Set $M_i = Max_{\prec_{\Gamma_i}}(H_i)$. Let $Ext_X(KB)$ be our basic extension function, which collects the maximal subsets of X coherent with KB. Let $N_i = \cap Ext_{M_i}(\Gamma_i) \cap M_i$. We set $\Gamma_{i+1} = \Gamma_i \cup N_i$ and $H_{i+1} = H_i - M_i$. Because H_i is always finite, maxima exist and $M_i \neq \emptyset$. Hence $H_{i+1} \subset H_i$ and the H_i will become empty after finitely many steps. When $H_n = \emptyset$, Γ_n is maximal, and the process stops. $\mathcal{H}^*_{ihz}(\Gamma_0) = \cup N_i$.

System PZ is a family of inference notions for DDL which extend System Z to handle speculative hypotheses suggested by different restrictions or variations of the projection principle. For any specific finitary PP-hypothesis choice function $\mathcal{H}$ the associated characteristic ranking choice function for PZ is defined by

$$\mathcal{F}^{PZ}_{\mathcal{H}}(\mathcal{T} \cup \mathcal{D}) = \mathcal{F}^{Z}(\mathcal{T} \cup \mathcal{D} \cup \mathcal{H}^{*}_{ihz}(\mathcal{T} \cup \mathcal{D}))$$

Consistent HBox

The case of an HBox that is consistent with the existing knowledge base is relatively straightforward. Consider the following example:

Example 22. *Take the same knowledge base from example 19.*

Using System PZ allows to revise the rankings such that:

- $R^{Z}(\neg A \sqcup \exists r.A) = 1$
- *Hypothesis:* $R(\forall r.(\neg A \sqcup \exists r.A) \sqcap \exists r.(\neg A \sqcup \exists r.A)) = 1$

The second constraint is provided by the projection principle — it states that any element contained in the domain of a role, such that it is only related to one type of concept through that role, may be defeasibly assumed to be at least as exceptional as that concept. In this case, the result is as shown above.

The ranking measure values may visually represented as below, where the difference thanks to projection is highlighted in red, compared to the ranking measure in table 5.1.

A	$\neg A$	
0	*1*	$\forall r.A \sqcap \forall r.\neg A$
1	*1*	$\exists r.\neg A \sqcap \forall r.\neg A$
1	*1*	$\forall r.A \sqcap \exists r.A$
1	*1*	$\exists r.\neg A \sqcap \exists r.A$

The above representation of the PZ ranking measure over the concept language of this knowledge base is read as usual throughout this thesis.

These ranking measure constraints induce the following conditional ranking measure results:

- $R^{PZ}(a : \neg A \mid \mathcal{A}) = 0$
- $R^{PZ}(a : \exists r.A \mid \mathcal{A}) = 0$
- $R^{PZ}(b : \neg A \mid \mathcal{A}) = 1$

So, it holds that $\mathcal{A} \mid\!\sim^{\mathcal{T}\cup\mathcal{D}}_{PZ} b : A$. *This may be verified as* $b : \neg a$ *is ranked higher than* 0 *given the ABox as a condition. However, this is a consistent, stable entailment relation that is able to produce a distinguished set of inferences from this default base.*

This chapter presented an approach to formalising a defeasible extension to the description logic $\mathcal{ALC}$ using the syntactic extension of adding a set of default conditional statements between concepts, as is the standard in the literature. However, in this work the semantics are extended with ranking measures. Representational and reasoning shortcomings were identified and illustrated, and a general approach to address these was presented: that of speculative default reasoning with hypotheses. These hypotheses are a syntactic extension that specify certain defeasible relationships that are semantically interpreted as "pre-processing" constraints to which to attempt to conform. One full inference relation using hypothesis box reasoning was presented: that of System PZ — a modification of System Z, also referred to as rational closure, incorporating such speculative information. Nevertheless, HBox reasoning is highly flexible and there could be very many methods by which to define defeasible inference incorporating such hypotheses. Therefore, this work represents the first steps in this approach to default reasoning.

Chapter 6

Discussion and conclusion

The first-order logic of default quantifiers presented is the primary novel contribution of this project. The formulation of $\mathcal{ALC}$ extended by conditionals interpreted by ranking measure semantics is an interesting alternative semantics to the literature, and the proposition to extend defeasible $\mathcal{ALC}$ with speculative inference properties by extending the language with $\mathcal{H}$ statements and associated inference relations is another novel contribution.

To conclude this thesis, some final general remarks will be made about certain aspects of this work that were illustrated during the presentation of the techniques, and may benefit from further critical emphasis.

6.1 Discussion

Taking this work in the broader context of common-sense reasoning in first-order logics, certain issues warrant a discursive treatment. Particular choices in defining a nonmonotonic semantics bake in assumptions about the nature of the world and the forms of reasoning to be employed. These considerations should be explicated and therefore taken into account in future work in this area; defeasible first-order reasoning need not prioritise the same desiderata as this approach, and instead explicitly take into account which assumptions to import and which to eschew.

6.1.1 Neutrality

An important element when considering defeasible inference in the first-order context is the topic of neutrality: a general concept that here is defined to be the principle of not privileging any particular first-order structure for the purposes of default inference. Ranking choice effectively selects a distinguished set of ranked structures, as all epistemic measures are generated from an underlying, or a set of underlying, ranked first-order structures. There is a choice to be made, then, whether the resulting set of epistemic measures used to specify inference is aggregated from sets of ranked structures including those assigning the empty set to certain exceptional predicate names, which under these semantics would assign those predicates ∞, as opposed to some ranking value greater than 0. For certain sophisticated constructible inference relations, this is a non-trivial point, and can affect how one might decide is best to define the default inference process.

The decision made in the framework presented in this thesis was to keep the constructible inference processes coherent with the defeasible frameworks in propositional default reasoning. This amounts to defining constructibility notions over epistemic ranking measures, and then performing inference with the associated ranked structures underlying these epistemic measures. This has the advantage of easing the process of calculating the inferences, and reliably replicating the examples lifted from propositional default reasoning. However, this procedure may possibly neglect neutrality. When constructing a canonical epistemic measure, the underlying set of ranked structures may not contain all those that are models of the default knowledge base under the desired scheme. There are two perspectives one can take on this: on the one hand, defeasible inference is always a procedure of model selection — certain semantic elements are chosen, or structured, in order to gain a desired inference, and this may be seen as no different, as this general strategy is the same. On the other hand, for this particular case of first-order reasoning, certain structures having certain predicates map to the empty set while still satisfying the default base may be viewed as relevant information that should be incorporated.

An alternative procedure to the one taken here could be to generate an epistemic measure by aggregating a set of ranked structures. This amounts to defining a map from a set of ranked structures to a single epistemic measure that aggregates the values according to a particular strategy. In such a framework, where the model set of the default base is aggregated to a single canonical epistemic measure to determine the various inferences, constructibility would be defined on the level of the ranked structures, and then the aggregation scheme would have to ensure that the resulting epistemic measure was also constructible over the default knowledge base. In this case, neutrality is relevant as emptiness is one simple route towards specifying certain difficult examples for aggregating constructible epistemic measures from constructible ranked structures. Generating an epistemic measure from a set of ranked structures requires defining aggregation strategies that can reliably replicate desirable results, for example from known propositional default reasoning examples. The challenge is that given a set of ranked structures with respect to a set of default statements, it is almost certain that there will exist models of the knowledge base such that $R(\phi) = \infty$, where ϕ is an arbitrary formula representing some exceptional, or atypical, sentence. Formulating aggregation strategies that reasonably take this into account, which precludes straightforward strategies like taking the minimum ranking for each formula across the set of ranked structures, is a non-trivial task.

6.1.2 Independence

Product ranking measures, that facilitate defeasible reasoning from n-variable formulae using default bases perhaps only defined over $j < n$ variables, contain within their usage assumptions about the nature of the defeasible knowledge. Given a tuple $\langle a, b\rangle$, then defining the product measure of this tuple such that $R(\langle a, b\rangle) = R(a) + R(b)$ assumes that the (ab)normality of a and b are, essentially, independent events. This reading, that if a is abnormal, then still nothing is assumed of b (or rather, the degree of surprise of b being atypical is unchanged), is not necessarily the only way of modelling uncertain reasoning. In this work, it is motivated by inheriting the quasi-probabilistic nature of ranking measure theory, and that it is a natural assumption in the defeasible reasoning literature to define frameworks that maximise normality (as is the assumption in rational closure (Lehmann and Magidor, 1992) and approaches based on maximum entropy (Peñaloza and Potyka, 2016; Goldszmidt et al., 1993)). However, approaches that modify this independence property could prove effective, for example by investigating the loss of independence if individuals are connected via some type of relation.

This independence issue is distinct from independence issues already well understood in the propositional case. First-order reasoning provides another dimension, that among individuals, over the existing independence criteria with respect to the language, and formulae themselves. Naturally, independence among formulae is replicated in this framework, on the level of the ranking measures considering a single individual, and independence on the level of individuals, and tuples of individuals, is taken here to hold. The conjecture behind these comments is that just as language independence on the level of the syntax is studied in propositional default reasoning ((Heyninck et al., 2023) for instance), tuple independence (whether singletons or n-tuples) is an issue worthy of further examination for first-order default reasoning.

6.1.3 Speculative reasoning

In the context of defeasible description logics, this thesis defined a framework for expanding the defeasible reasoning capabilities via the syntactic extension of a hypothesis box, referred to as an HBox, and adjusting the associated ranking measure semantics in order to interpret them. The motivation for this extension is multifaceted. Compared to default quantifiers in defeasible first-order logic, the expressivity of a given default in defeasible description logic is far lower. Defeasible description logic defaults are essentially propositional default conditional statements. However, the structures also contain information regarding binary relations. Since these defaults in $\mathcal{DALC}$ all express a default conditional relationship between two unary predicates, i.e. subsets of individuals of the domain, it is *a priori* not possible to extract any information regarding the typical binary relations in a structure. This limitation especially affects the reasoning process as concepts make use of roles, i.e. binary relations, in their construction; therefore, in order to address the goal of an inference relation on the level of assertions, it is necessary to be able to reasonably condition with role assertions, which is not possible if there is no easy way for default information to inform what is a typical role relation and what is atypical. Given only default information regarding individuals, and so corresponding single-dimension ranking measure constraints, when presented with an assertion box with a single role relation with which to condition there is no way to infer the rank of the role assertion.

The goal of HBox reasoning is to enrich the inference process with defeasible assumptions about the knowledge base. These assumptions are intended to make up for this gap in default knowledge resulting from the syntax limitations described above. In this thesis, the form of speculative reasoning investigated was that of projection: a set of defeasible assumptions intended to overcome specifically this inability to infer the rank of role assertions from a set of $\mathcal{DALC}$ default conditionals. However, the idea behind speculative inference, or HBox reasoning, is broader than just the projection based inferences explored here. The kinds of statements possible to include in an HBox are theoretically very many, not just the projection based system presented here.

6.2 Future work

This thesis proposes a semantic framework for interpreting and defeasibly inferring a logic of default quantifiers, and this work has opened up many lines of inquiry, both theoretical and practical. It should be emphasised that while this work present a coherent semantic framework for first-order default reasoning, it exposes many lines of future research that are out of scope of this thesis. Before concluding, a selection of potential topics for future work will be discussed.

6.2.1 Proof theory

While postulates that are sound for the model theory were covered in this thesis, the semantic framework presented here lacks a completeness proof, and so finding a set of rules of inference which are sound and complete for the semantics presented in this thesis would be a major theoretical advance. A proof theory would provide an important step into defining algorithmic implementations of these inference relations. Of course, algorithmic implementations of these logics would assist in theoretical investigations as well.

6.2.2 Complexity analysis

A thorough complexity analysis of this logic, and specifically the question of the complexity of an algorithm that can decide if a particular formula is inferred by some set of formulae under a specific default base and specific inferential strategy, was out of scope for this thesis, and so remains as future work for this project.

Following from comparable defeasible logics, some conjectures may be made: first-order logic itself is semi-decidable, and so naturally the relevant complexity questions are primarily relevant for deciding tautologies. Relevant algorithms could be defined as a series of classical entailment checks, as they are in propositional default reasoning (Lehmann and Magidor, 1992; Pearl, 1990), where the complexity is normally limited to a polynomial number of classical entailment checks in the size of the knowledge base.

6.2.3 Epistemic measure spaces

In this thesis, inference was done via defining corresponding epistemic measures over the Lindenbaum algebra unifying all the ranked structures of a default base. The ultimate result of this approach is that the inference relation effectively concludes about any particular individual, or model of an individual, what is believed about the individual across all ranked structures selected by the ranking choice function.

This intersects with a general point regarding first-order default reasoning: whether and when to order over individuals, as opposed to, or in conjunction with, ordering over worlds (Delgrande, 1998). In this framework, the ranking measures provide a type of ordering over sets of variable assignments, which implies an ordering over the powerset of all sequences, of all sizes, of individuals in a ranked structure. Default inference as defined here is then, functionally, a matter of selecting a set of these models and tallying the default inferences regarding generic individuals on which they agree. The approach taken here for default inference is analogous to giving all structures that model the default base equal rank (rank 0), and assigning to structures that do not satisfy the default base rank ∞. The inferences are what all these models agree are defeasibly inferred from the default base. Note that the preference ordering that defines System Z in the minimal model semantic definition given in section 4.2 is not a ranking employed in the inference relation itself. It is used to select the epistemic measure, and corresponding set of ranked structures, that specifies the inference relation.

An alternative approach to defeasible inference is to define a global structure that provides a ranking over the ranked models themselves. Instead of assigning all models of the default base the same rank of 0, there would be an ordering over them as well. How this ordering would be determined would be the primary goal of such a framework, and a non-trivial consideration. One could define the ordering based on how the ranked structure satisfies the default base — assigning a normality metric that encodes a way to measure how exceptional a particular ranked structure is, for example. Otherwise, one could also bind a variable to the default quantifier itself that is interpreted as the "world" itself. These would amount to specific statements regarding what are the most normal, respectively abnormal, worlds, i.e. ranked structures.

This essentially amounts to having a two tiered semantics: local ranked structures as defined here, which provide a local ranking measure that defines degrees of belief over individuals, and then a global ranking measure semantics that ranks these structures, in what could be termed an epistemic measure space. These global ranking measures would encode a degree of belief in these worlds themselves. How this degree of belief would be determined would be the immediate question: one can imagine that this would intersect with the issue of neutrality, as whether emptiness in a local structure affects the credibility of a particular structure in the global ranking measure would be a relevant decision in defining relevant default inference relations in the context of these epistemic measure spaces.

Consider example 11: one question it may be relevant to ask is what could be defeasibly inferred about all variables. As was noted, defeasibly inferring such universally quantified formulae is not condoned by this framework. However, it is not conceptually unreasonable to desire the default inference "in the most typical worlds, *Clyde* likes *all* regular keepers". This would be reasonably achievable in a framework such as the one described, where, in theory, the most typical worlds could verify universally quantified conclusions.

In the context of defeasibility over individuals versus defeasibility over worlds, these two approaches to defining default inference may be contrasted by prioritising one over the other. The approach defined in this thesis prioritises default inference regarding what is (defeasibly) believed regarding an individual, whereas an epistemic measure space would prioritise default inference regarding what is (defeasibly) believed about the world in general.

6.2.4 Defeasible description logics

Here, a defeasible extension of $\mathcal{ALC}$ was defined, which extends the base classical logic with a default conditional relation over the concept language collected in a default box, much as has been done by (Britz et al., 2020; Bonatti, 2019; Giordano et al., 2010), which guides the defeasible inference procedure. The focus here was therefore on building a semantics for this defeasible logic based around ranking measures, and investigating the general problem of extending defeasible reasoning for the assertion language. The main approaches that attempt to characterize a defeasible inference for assertions as well (Britz et al., 2018; Bonatti, 2019) tend to avoid semantically investigating the defeasible inference relation specified by a default box between an assertion box and an individual assertion. Future projects could focus on lifting these defeasible extensions to more expressive description logics, such as $\mathcal{SROIQ}$ (Horrocks et al., 2006), building on work such as (Britz and Varzinczak, 2017).

In this thesis, a remedy to some semantic issues surrounding inconsistent inferences was proposed: that is the framework of speculative inference, a specific instantiation of which, the projection principle, was described here. One primary point of interest regarding this strategy is the novelty. While all defeasible extensions add both a syntactic and a semantic novelty, the specific syntactic quality of $\mathcal{H}$ statements separates them from syntactic notions such as default conditionals that try to specify defeasible relationships between elements of the language, which then have to be interpreted in the semantics in a particular way, whereas $\mathcal{H}$ statements essentially try to directly specify the relationships between semantic elements and extend the language in this way. What remains to be seen is what kind of various other issues in defeasible reasoning the general approach of HBox reasoning, or speculative inference ideas, may effectively address. Other possible approaches aside from projection are theoretically possible, and perhaps worthy of the experimentation.

Another way to specify defeasible reasoning in description logics, similar to a recent approach by (Hahn et al., 2024), is to specify the default conditionals by extending a more expressive base logic, and combining this with a classical knowledge base in description logic. This has the benefit of allowing one to express a lot more information about the unary and binary predicates that make up a description logic knowledge base, however does mean that you introduce syntax from outside the base classical language. One approach investigated briefly within this project but not developed properly was to define default bases in $\mathcal{ALC}$ by extending the assertion language with default quantifiers. This has the detriment of introducing variables into a logic that specifically aims to exclude them, for complexity reasons, but does allow for far more expressivity regarding default information, specifically with regard to default information regarding binary relations.

6.3 Conclusion

This thesis presented a novel logic for uncertain reasoning about first-order information, both in representing statements regarding typicality, and for defeasible inference over those statements. On the syntactic level, formulating statements that can express default information about individuals or n-ary relations using default quantifiers continues work first described in (Weydert, 1997; Delgrande, 1998), but has not been picked up since for thorough investigation.

On the semantic level, a novel method for interpreting default quantifier statements was defined, by lifting the ranking measure semantic framework (Weydert, 1998) used for interpreting propositional default conditional logics to first-order structures. These ranking measures were defined over the powerset of the variable assignment functions of a first-order structure, allowing them to assign a truth value to any dimension of default quantifier, i.e. any number of variables bound to the quantifier. On the inferential level, lifting various concepts for defining sophisticated defeasible inference relations is an important contribution for defeasible first-order logics, as even frameworks that attempted to formalise defeasible first-order information typically did not formalise associated default inference relations.

Finally, a defeasible extension of the specific case of the fragment of first-order logic, the description logic $\mathcal{ALC}$ (Baader et al., 2007), was investigated to address the open problem of default reasoning of assertions in this area (Bonatti, 2019). A possible framework using the same ranking measure semantics for default reasoning was proposed, and some limitations pointed out. To address this, a novel proposal for speculative default reasoning in description logics was proposed, where additional information regarding the world may be specified in a hypothesis box, a collection of default information expressed in an extended syntax, that guides the inference procedure in a preprocessing like stage.

This work should ideally stimulate new ideas and investigations into formalising uncertain reasoning approaches in first-order logic and fragments thereof. A proof theoretic characterization, algorithmic implementations, and alternative semantic inference formulations were all described as future works. Analysing the shortcomings of the approach made here in defining defeasible description logics represent problems to be kept in mind when defining defeasible extensions to description logics, particularly on the representation side. Ultimately, these many lines of future work point to the variety of projects that could be built on this framework.

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