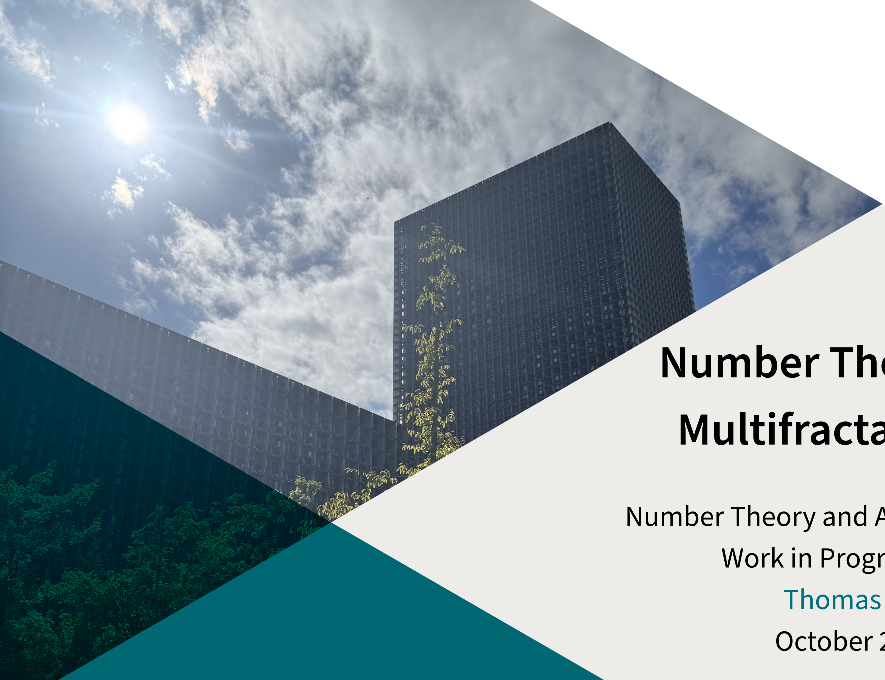


UNIVERSITÉ DU LUXEMBOURG

Number Theory Meets Multifractal Analysis

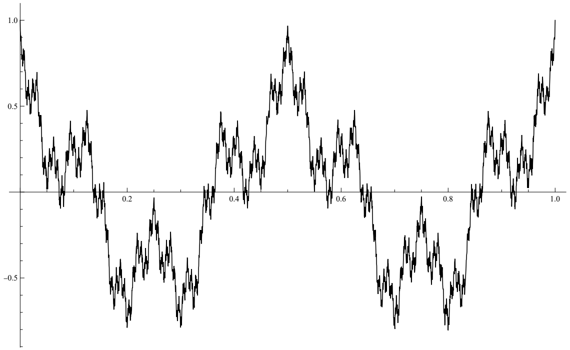
Number Theory and Arithmetic Geometry Work in Progress Seminar

Thomas Lamby

October 23, 2025

Notions of Regularity

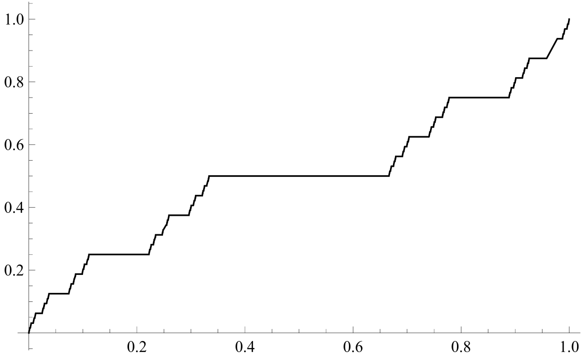
Weierstrass function



For all $t \in \mathbb{R}$,

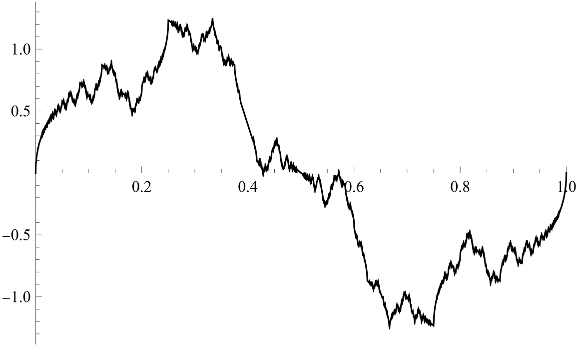
$$h(t) = -\frac{\log a}{\log b}.$$

Cantor staircase function



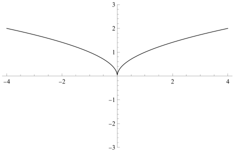
$$d(h) = \begin{cases} \frac{\log 2}{\log 3} & \text{if } h = \frac{\log 2}{\log 3}, \\ 1 & \text{if } h = \infty, \\ -\infty & \text{otherwise}. \end{cases}$$

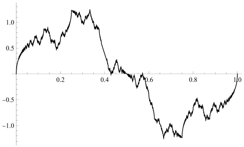
Riemann function



$$d(h) = \begin{cases} 4h - 2 & \text{if } h \in [1/2, 3/4], \\ 0 & \text{if } h = 3/2, \\ -\infty & \text{otherwise}. \end{cases}$$

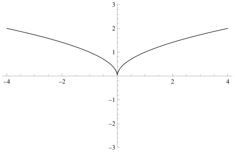
Hölder

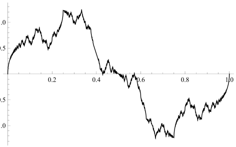




$\|f - P\|_{L^\infty(B(x_0, r))} \leq C r^\alpha$

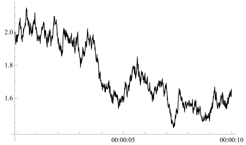
Hölder

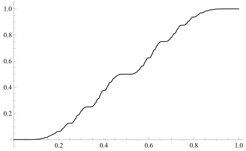




$\|f-P\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha$

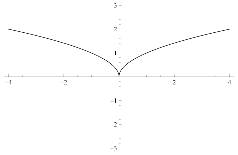
Weighted Hölder

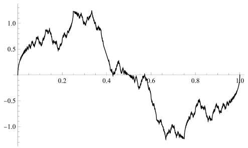




$\|f-P\|_{L^\infty(B(x_0,r))} \leq C\phi(r)$

Hölder





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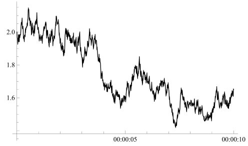
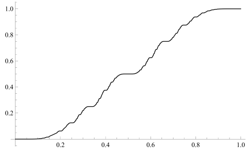
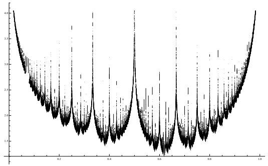
Weighted Hölder

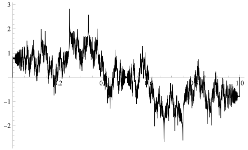




$\|f-P\|_{L^\infty(B(x_0,r))} \leq C\phi(r)$

Calderon-Zygmund





$r^{-d/p}\|f-P\|_{L^p(B(x_0,r))} \leq Cr^\alpha$

Let $x_0 \in \mathbb{R}^d$, $p \in [1, \infty]$, $\alpha > -d/p$, a function $f \in L^p_{\text{loc}}$ is in $T^p_\alpha(x_0)$ if there exist a constant $C > 0$ and a polynomial P of degree strictly smaller than α such that

$$r^{-d/p}\|f - P\|_{L^p(B(x_0,r))} \leq Cr^\alpha$$

for sufficiently small r.

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p-exponent

$$h_p(x_0) := \sup\{\alpha > -d/p : f \in T^p_\alpha(x_0)\}.$$

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p-exponent

$$h_p(x_0) := \sup\{\alpha > -d/p : f \in T^p_\alpha(x_0)\}.$$

p-spectrum

$$d_p(h) = \dim_{\mathcal{H}}(\{x \in \mathbb{R}^d : h_p(x) = h\}).$$

"Nowhere Regularity"

- Nowhere Continuous Functions :

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- Nowhere Differentiable Functions :

$$T(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-1} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Thomae's-type functions

Thomae's-type functions

Let $\theta > 0$,

$$T_\theta(x) = \begin{cases} 1 & \text{if } x = 0, \\ q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

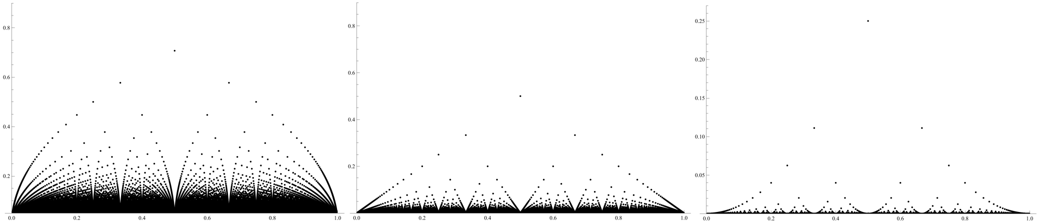

Figure 1: Representation of the function T_θ on $(0,1)$ for $\theta = 1/2$, 1 and 2.

Periodicity

Proposition

The Thomae function is periodic with period 1.

Rational-Irrational Dichotomy

Proposition

The function T_θ is discontinuous at rational points and continuous at irrational points.

Differentiability

Proposition

Let f be a function on $\mathbb{R}$ that is positive on the rationals and 0 on the irrationals. Then, there is an uncountable dense set of irrationals on which f is not differentiable.

Proposition

Let $(a_j)_j$ be a sequence of $\mathbb{R} \setminus \mathbb{Q}$. Then there exists a function that is positive on the rationals, zero on the irrationals, and differentiable at each point a_j.

Rational Approximations

$\tau(x) = \sup \left\{ u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N} : \left| x - \frac{p}{q} \right| < \frac{1}{q^u} \right\}.$

Dirichlet's Theorem

Let x be a real number and n a positive integer. Then there is a rational number p/q with $0 < q \leq n$, satisfying

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{(n+1)q}.$$

Corollary

Given any real number x, there exists a rational number p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Rational Approximations

Theorem

Let $x \in \mathbb{R}\backslash\mathbb{Q}$, then there are infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Hurwitz's Theorem

(i) Let $x \in \mathbb{R}\backslash\mathbb{Q}$, there are infinitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}.$$

(ii) If $\sqrt{5}$ is replaced by $C > \sqrt{5}$, then there are irrational numbers x for which statement (i) does not hold.

Rational Approximations

Theorem

Let $\varepsilon > 0$. For almost every $x \in [0,1]$, there exist only finitely many rational numbers p/q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}.$$

Differentiability

Proposition

For $\theta \in (0, 2]$, T_θ is not differentiable at any point.

Differentiability at 0

We put $T_\theta(0) = 1$ in order to have the periodicity. Consider

$$\widetilde{T}_\theta(x) = \begin{cases} q^{-\theta} & \text{if } x \text{ is rational with } x = p/q, \gcd(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational or } x = 0. \end{cases}$$

As one might expect, $\widetilde{T}_\theta$ becomes continuous at 0 and the dichotomy no longer holds. A more interesting fact is that $\widetilde{T}_\theta$ becomes differentiable at 0 for $\theta > 1$.

Differentiability at 0

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As one might expect, $\widetilde{T}_\theta$ becomes continuous at 0 and the dichotomy no longer holds. A more interesting fact is that $\widetilde{T}_\theta$ becomes differentiable at 0 for $\theta > 1$. Indeed, if the derivative at 0 exists, it must be equal to 0. Thus, we must show that if $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in (-\delta, \delta) \implies \left| \frac{\widetilde{T}_\theta(x) - \widetilde{T}_\theta(0)}{x - 0} \right| = \left| \frac{\widetilde{T}_\theta(x)}{x} \right| < \varepsilon.$$

If x is irrational, then this difference quotient is equal to $0 < \varepsilon$. Suppose x is a nonzero rational number. There exists a positive integer n such that $\frac{1}{n^{\theta-1}} < \varepsilon$. There exists a $\delta > 0$ such that every nonzero rational number in the interval $(-\delta, \delta)$ has denominator $q > n$. Thus, if $x = \frac{p}{q}$ with $\gcd(p,q) = 1$, then for $|x| < \delta$ we have $q > n$, and hence:

$$\left| \frac{\widetilde{T}_\theta(x)}{x} \right| = \left| \frac{q^{-\theta}}{p/q} \right| = \left| \frac{1}{pq^{\theta-1}} \right| < \varepsilon.$$

Therefore, the difference quotient is less than ε for all $x \in (-\delta, \delta)$, and the derivative of $\widetilde{T}_\theta$ at 0 exists and equals 0.

Regularity of T_θ

- T_θ is discontinuous at rational points and continuous at irrational points.

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- For $\theta \in (0, 2]$, T_θ is not differentiable at any point.
- Exact regularity of T_θ at each of its points ?

Pointwise Regularity of T_θ

Lemma

Let $\theta, \alpha > 0$ and $x \in (0,1) \setminus \mathbb{Q}$. If $T_\theta \in \Lambda^\alpha(x)$, then the polynomial P of degree less than α appearing in Definition of $\Lambda^\alpha(x)$ must necessarily be the zero polynomial.

Theorem

Let $\theta > 0$, then

$$h_{T_\theta}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ \theta/\tau(x) & \text{otherwise,} \end{cases}$$

where

$$\tau(x) = \sup\left\{u : \exists \text{ an infinity of coprime pairs } (p,q) \in \mathbb{Z} \times \mathbb{N} : \left|x - \frac{p}{q}\right| < \frac{1}{q^u}\right\}.$$

- If $\theta < 2$, T_θ is nowhere differentiable.
- T_2 is nowhere differentiable and $h_{T_2} = 1$ almost everywhere !
- When $\theta > 2$, T_θ is differentiable at x_0 when $\tau(x_0) < \theta$. For example, T_9 is differentiable at algebraic irrationals numbers, $e, \pi, \pi^2, \ln(2)$.

Spectrum of T_θ

Jarnik's Theorem

Let $a, b \in \mathbb{R}$ with $a < b$, $\tau \geq 2$, then

$$\dim_{\mathcal{H}}(\{x \in [a,b] : \tau(x) = t\}) = \frac{2}{t} \quad \forall t \in [2, \infty] .$$

Spectrum of T_θ

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Theorem

The Hölder-spectrum is given by

$$d_{T_\theta}(h) = \begin{cases} \frac{2h}{\theta} & \text{if } h \in [0, \theta/2], \\ -\infty & \text{otherwise.} \end{cases}$$

Bigger class of Thomae's type functions

We consider continuous functions $\phi : (0,1) \to (0,\infty)$ such that

$$0 < \underline{\phi}(t) := \inf_{s<1} \frac{\phi(ts)}{\phi(s)} \leq \overline{\phi}(t) := \sup_{s<1} \frac{\phi(ts)}{\phi(s)} < \infty,$$

for any $t < 1$. The *lower* and *upper indices* of ϕ are defined by

$$\underline{s}(\phi) = \lim_{t\to 0} \frac{\log \underline{\phi}(t)}{\log t} \quad \text{and} \quad \bar{s}(\phi) = \lim_{t\to 0} \frac{\log \overline{\phi}(t)}{\log t}.$$

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We define

$$T_\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \phi(1/q) & \text{if } x = p/q, \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

where $\underline{s}(\phi) = \overline{s}(\phi) = \theta \in (0,2]$.

Bigger class of Thomae's type functions

For example, one can consider

$$T_{\log}(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{\log(q)}{q} & \text{if } x = p/q, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

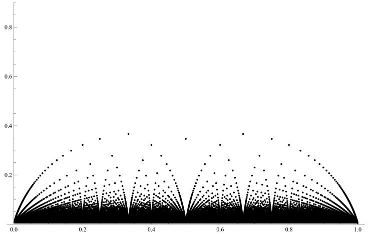

Figure 2: Representation of the function $T_{\log}$ on $(0, 1)$.

Bigger class of Thomae's type functions

- What about $\underline{s}(\phi) < \overline{s}(\phi)$?
- Negatives indices ?
- Interchange the dichotomy ?

Different indices

For example, define

$$\phi(t) = \begin{cases} t^{\alpha} & \text{if } t \in (0, s], \\ t^{\beta} & \text{if } t \in (s, 1). \end{cases}$$

$\rightsquigarrow$ Only few particular points. A more complex example : consider the increasing sequence $(j_n)_n$ defined by

$$\begin{cases} j_0 = 0, \\ j_1 = 1, \\ j_{2n} = 2j_{2n-1} - j_{2n-2}, \\ j_{2n+1} = 2^{j_{2n}}. \end{cases}$$

Then, define

$$\sigma_j := \begin{cases} 2^{j_{2n}} & \text{if } j_{2n} \leq j \leq j_{2n+1}, \\ 2^{j_{2n}} 4^{j - j_{2n+1}} & \text{if } j_{2n+1} \leq j < j_{2n+2}. \end{cases}$$

The sequence σ oscillates between $(j)_j$ and $(2^j)_j$. By setting

$$\phi(t) = \frac{1/\sigma_j - 1/\sigma_{j+1}}{2^j}(t - 2^{-j-1}) + 1/\sigma_{j+1} \quad \text{if} \quad t \in (2^{-j-1}, 2^{-j}],$$

we have $\underline{s}(\phi) = 0$ and $\overline{s}(\phi) = 1$. $\rightsquigarrow$ Partial Results : $h_{T_\phi}(x) \in [\underline{s}(\phi)/\tau(x), \overline{s}(\phi)/\tau(x)]$ if $x \in \mathbb{R} \backslash \mathbb{Q}$.

$\theta < 0$

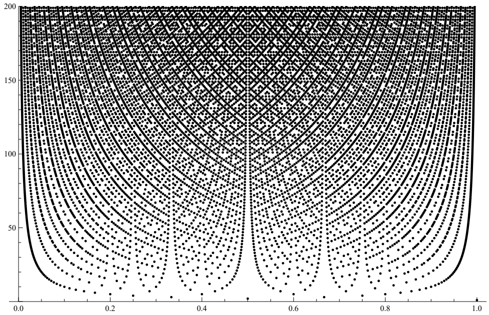

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.

$\theta < 0$

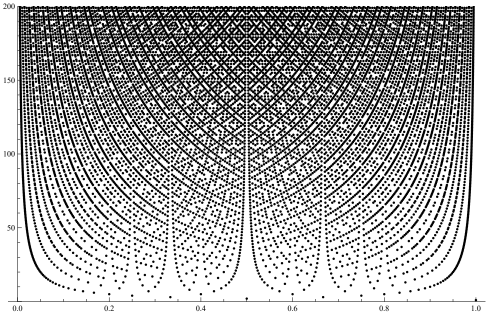

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.
- $\int_{\mathbb{R}} T_\theta(x)dx = \int_{\mathbb{R}\setminus\mathbb{Q}} T_\theta(x)dx = 0.$

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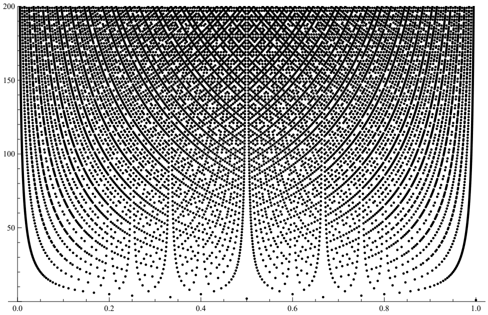

Figure 3: Representation of the function T_{-1} on $(0, 1)$.

- Easy construction of a nowhere locally bounded function.
- $\int_{\mathbb{R}} T_\theta(x)dx = \int_{\mathbb{R}\setminus\mathbb{Q}} T_\theta(x)dx = 0$. $\rightsquigarrow$ Notion of p-exponents not adapted.

Interchange the dichotomy?

Is there a function that is continuous on the rational numbers and discontinuous on the irrational numbers ?

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$\rightsquigarrow$ No, since the set of discontinuities of a function $\mathbb{R} \to \mathbb{R}$ is always a F_σ-set.

Interchange the dichotomy?

Is there a function that is continuous on the rational numbers and discontinuous on the irrational numbers ?
$\rightsquigarrow$ No, since the set of discontinuities of a function $\mathbb{R} \to \mathbb{R}$ is always a F_σ-set.
Starting from a F_σ subset of $\mathbb{R}$, $A := \bigcup_n F_n$, we define

$$T_A(x) = \begin{cases} 1/n & \text{if } x \text{ rational and } n \text{ is minimal s.t. } x \in F_n, \\ -1/n & \text{if } x \text{ irrational and } n \text{ is minimal s.t. } x \in F_n, \\ 0 & \text{if } x \notin A. \end{cases}$$

The set of discontinuities of T_A is given by A.

Brjuno functions

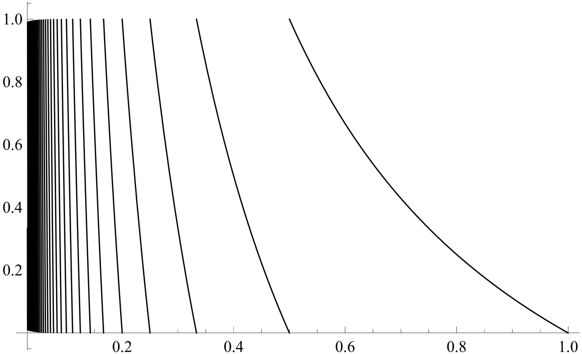
Brjuno number

Let $x \in \mathbb{R} \backslash \mathbb{Q}$ and let $(p_n/q_n)_{n \geq 0}$ be the sequence of the convergents of its continued fraction expansion. A Brjuno number is an irrational number x such that

$$\sum_{n \geq 0} \frac{\log q_{n+1}}{q_n} < \infty.$$

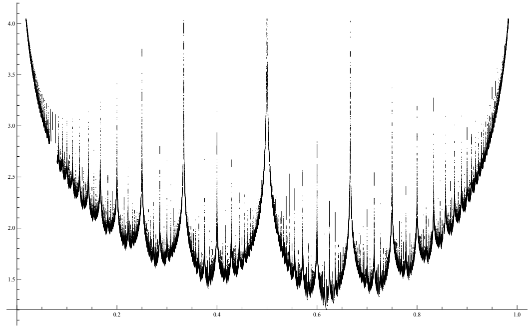
The importance of Brjuno numbers comes from the study of one–dimensional analytic small divisors problems. In the case of germs of holomorphic diffeomorphisms of one complex variable with an indifferent fixed point, extending a previous result of Siegel, Brjuno proved that all germs with linear part $e^{2\pi i x}$ are linearizable if x is a Brjuno number.

Gauss map



$$A : (0,1) \to [0,1] \quad x \mapsto |\frac{1}{x} - \lfloor \frac{1}{x} \rfloor|.$$

Brjuno function



$$B : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} x_0 x_1 \ldots x_{n-1} \log x_n,$$

where $x_0 = |x - \lfloor x \rfloor|$ and $x_{n+1} = A(x_n)$.

Regularity of B

S. Jaffard, B. Martin

Let $p \in [1, \infty)$; the p-exponents of B are given by

$$h_p^{(B)}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$$

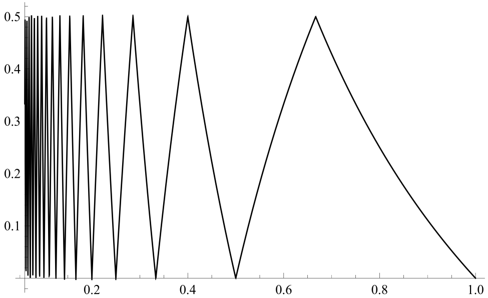
Moreover, the p-spectrum is given by

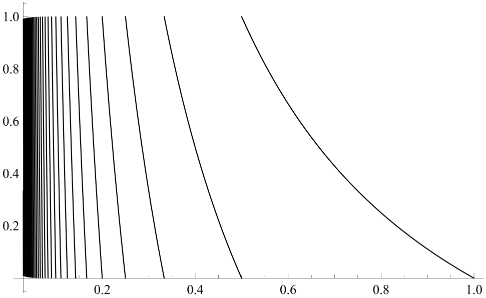
$$d_p(h) = \begin{cases} 2h & \text{si } h \in [0, 1/2], \\ -\infty & \text{sinon.} \end{cases}$$

Modified Gauss map associated to the NCFE

$$A_{1/2} : (0, 1/2) \to [0, 1/2] \quad x \mapsto |\frac{1}{x} - [\frac{1}{x}]_{1/2}|,$$

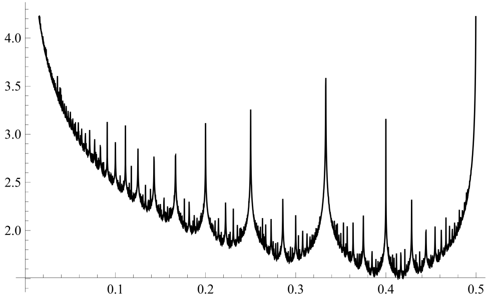
where $[y]_{1/2} = \lfloor y + 1/2 \rfloor$.





$A_{1/2}$ and A.

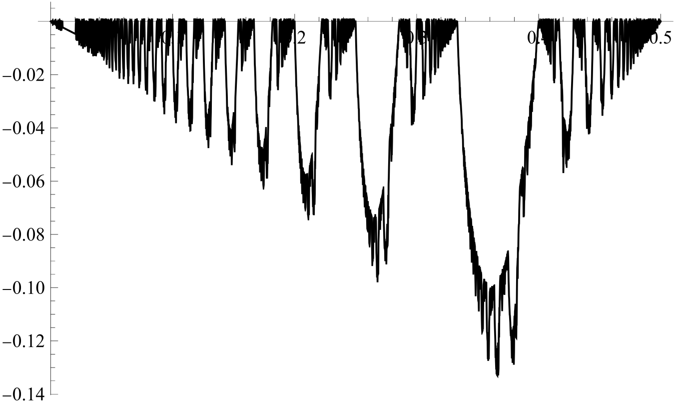
Brjuno function $B_{1/2}$



$$\mathfrak{B} : \mathbb{R} \setminus \mathbb{Q} \to \overline{\mathbb{R}} \quad x \mapsto -\sum_{n=0}^{\infty} x_0 x_1 \ldots x_{n-1} \log x_n,$$

where $x_0 = |x - [x]_{1/2}|$ and $x_{n+1} = A_{1/2}(x_n)$.

Difference $B - B_{1/2}$



NCFE

Set $x_0 = |x - [x]_{1/2}|$ and $a_0 = [x]_{1/2}$. Consequently, $x_0 = a_0 + \varepsilon_0 x_0$, where

$$\varepsilon_0 = \begin{cases} 1 & \text{if } x \geq a_0, \\ -1 & \text{otherwise.} \end{cases}$$

This initialization defines $x_{n+1} = A_{1/2}(x_n)$ and

$$a_{n+1} = [\frac{1}{x_n}]_{1/2} \geq 1,$$

for $n \in \mathbb{N}_0$ if it is meaningful. Subsequently, $x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1}$, where

$$\varepsilon_{n+1} = \begin{cases} 1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

NCFE

The n-th α-convergent of x is given by

$$\frac{p_n}{q_n} = [(a_0, \varepsilon_0), \ldots, (a_{n-1}, \varepsilon_{n-1}), a_n] = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{\ddots + a_{n-1} + \cfrac{\varepsilon_{n-1}}{a_n}}}$$

Set $\tau_n^{(1/2)}(x)$ as

$$|x - \frac{p_n}{q_n}| = \frac{1}{q_n^{\tau_n^{(1/2)}(x)}},$$

we introduce the $1/2$-irrationality exponent of x as

$$\tau^{(1/2)}(x) = \limsup_{n\to\infty} \tau_n^{(1/2)}(x).$$

Results

L., B. Martin, S. Nicolay

$$\tau^{(1/2)}(x) = \tau(x)$$

for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

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Let $p \in [1, \infty)$; the p-exponents of $\mathfrak{B}$ are given by

$$h_p(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1/\tau(x) & \text{otherwise.} \end{cases}$$

General α

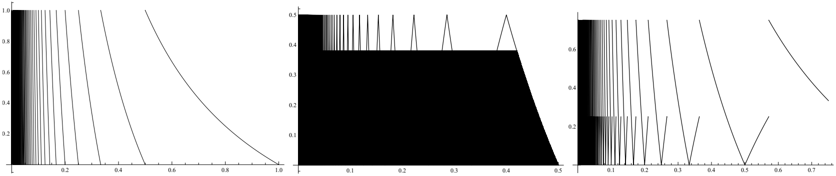


Figure 4: $A_\alpha : (0, \alpha) \to [0, \alpha]$ with resp. $\alpha = 1$, $\alpha = 1/2$ and $\alpha = 3/4$.

<u>Admissible α :</u>

$$\alpha \in \left\{1/2, \frac{\sqrt{5}-1}{2}, 1\right\} \cup \left\{1 - \frac{1}{k}, k \geq 3\right\} \cup \left\{\frac{-k + \sqrt{k^2 + 4k}}{2}, k \geq 2\right\}.$$

Minkowski question mark function

Construction of M

Set $M(0) = 0$ and $M(1) = 1$. Then, for the mediant $\frac{1}{2} = \frac{0+1}{1+1}$, set

$$M(\frac{1}{2}) = M(\frac{0+1}{1+1}) = \frac{M(0)+M(1)}{2} = \frac{1}{2}.$$

Similarly,

$$M(\frac{1}{3}) = M(\frac{0+1}{1+2}) = \frac{M(0)+M(1/2)}{2} = \frac{1}{4}$$

and

$$M(\frac{2}{3}) = M(\frac{1+1}{2+1}) = \frac{M(1/2)+M(1)}{2} = \frac{3}{4}.$$

In general, if two consecutive fractions p/q and $\tilde{p}/\tilde{q}$ are defined, we et

$$M(\frac{p+\tilde{p}}{q+\tilde{q}}) = \frac{M(p/q)+M(\tilde{p}/\tilde{q})}{2}.$$

Construction of M

If $x = [a_0, a_1, a_2, ...]$ is irrational, then we define

$$M(x) = a_0 + 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k}}.$$

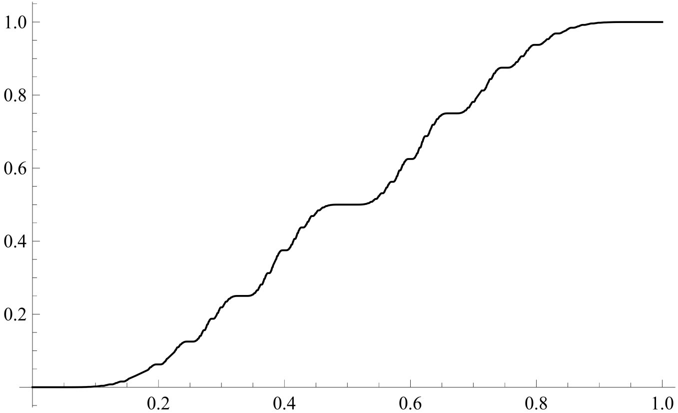
If $x = [a_0, a_1, a_2, ..., a_m]$ is rational, then we define

$$M(x) = a_0 + 2\sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{a_1+\ldots+a_k}}.$$

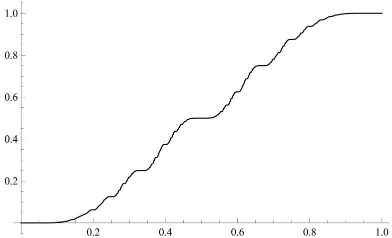
Example :

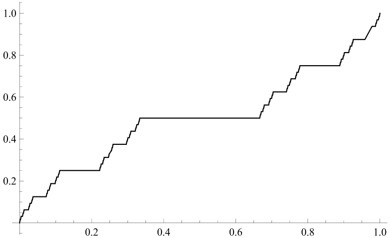
$$M(1/\varphi) = M([0, 1, 1, ...]) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{2^{k+1}} = \frac{2}{3}.$$

Graph of M

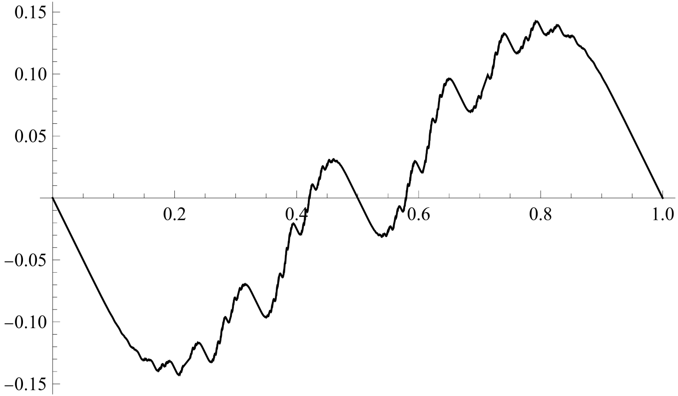


Comparaison with the Cantor staircase function





Graph of $M-\mathrm{id}$



Regularity of M

Global regularity

M belongs to the space $\Lambda^{\frac{\log 2}{2\log\varphi}}(\mathbb{R})$. Therefore, for all $x \in \mathbb{R}$,

$$h(x) \geq \frac{\log 2}{2\log\varphi}.$$

This lower bound cannot be improved since $h(1/\varphi) = \frac{\log 2}{2\log\varphi}$.

Regularity of M

Pointwise regularity

Let $x \in (0,1)\backslash\mathbb{Q}$, if

$$a_1(x) + \ldots + a_n(x) < \frac{n \log 2}{2 \log \varphi}$$

for sufficiently large n, then

$$h(x) \in [\frac{\log 2}{2 \log \varphi}, 1].$$

If

$$a_1(x) + \ldots + a_n(x) > \kappa_2 n$$

for sufficiently large n, then

$$h(x) \geq 1,$$

where $\kappa_2 \simeq 4.401$.

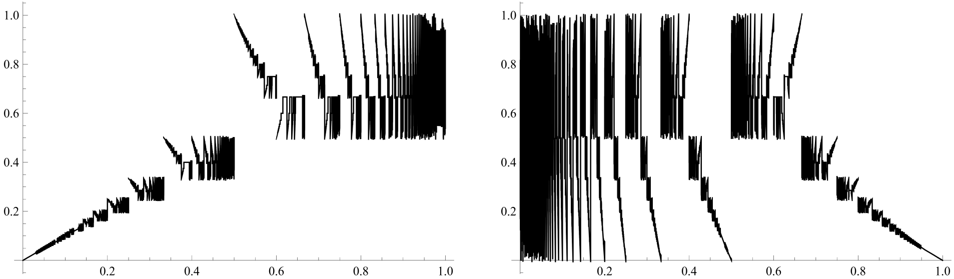
Cantor functions

Definitions and graphs of $\mathcal{C}$

Let $I = (0,1)\backslash\mathbb{Q}$.

$$\mathcal{C} : I \mapsto I^2 : x = [a_0, a_1, a_2, a_3, \ldots] \mapsto (\mathcal{C}_1(x), \mathcal{C}_2(x)),$$

where $\mathcal{C}_1(x) = [a_1, a_3, \ldots]$ and $\mathcal{C}_2(x) = [a_2, a_4, \ldots]$.



$\mathcal{C}_1$ and $\mathcal{C}_2$.

Regularity of $\mathcal{C}$

Pointwise Regularity

For almost all $x \in I$,

$$h^{(\mathcal{C}_1)}(x), h^{(\mathcal{C}_2)}(x) \in [\frac{\log \kappa_0}{2 \log \kappa_1}, \frac{\log \kappa_1}{2 \log \kappa_0}].$$

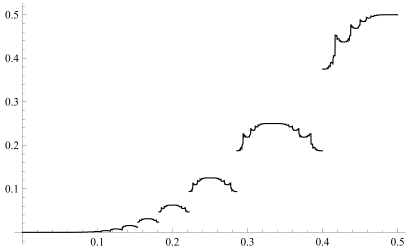
Let x_1, x_2, x_3 defined by

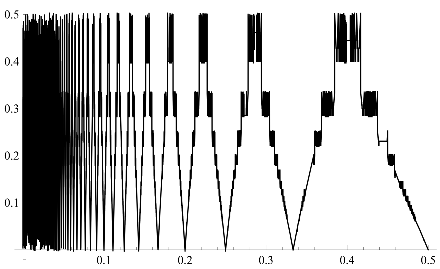
$$a_j(x_1) = \begin{cases} 2^j & j \text{ even}, \\ 1 & j \text{ odd}, \end{cases} \qquad a_j(x_2) = 2^j, \qquad a_j(x_3) = \begin{cases} 1 & j \text{ even}, \\ 2^j & j \text{ odd}, \end{cases}$$

then,

$$h^{(\mathcal{C}_1)}(x_1) = 0, \quad h_\infty^{(\mathcal{C}_1)}(x_2) = 1/2, \quad h^{(\mathcal{C}_1)}(x_3) = 1.$$

Minkowski and Cantor using NCFE





Perspectives

Pointwise regularity of

- Investigations of modified versions of Thomae's type functions
- Brjuno functions
- Minkowski function
- Cantor function
- General functions of the form

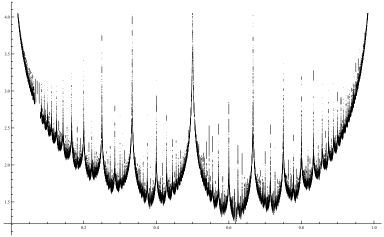
$$f(x) = \sum_{n \geq 0} f_1(x,n)\, f_2(x,n),$$

  where f_1 is linked to an iteration of a transformation that involves the position of real numbers relative to nearby integers, and f_2 plays the role of a singularity.

Perspectives

$$f(x) = \sum_{n \geq 0} f_1(x,n)\, f_2(x,n),$$

$f_1(x,n)$	$f_2(x,n)$	**Pointwise reg of f at x**
$A_1 \cdots A_1^{n-1}$	$\log(1/A_1^n)$	$1/\tau(x)$
$A_1 \cdots A_1^{n-1}$	1	$1/\tau(x)$
$(A_1 \cdots A_1^{n-1})^\theta$	$\log(1/A_1^n)$	$\theta/\tau(x)$
$A_\alpha \cdots A_\alpha^{n-1}$	$\log(1/A_\alpha^n)$	$1/\tau(x)$
$(-1)^n A_1 \cdots A_1^{n-1}$	$\log(1/A_1^n)$	$1/\tau(x)$
$S \cdots S^{n-1}$	$\log(1/S^n)$	?



Perspectives

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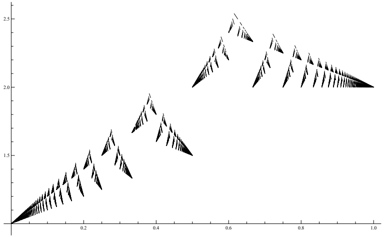

Perspectives

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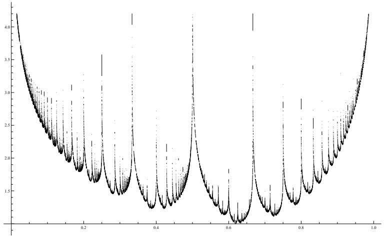

Perspectives

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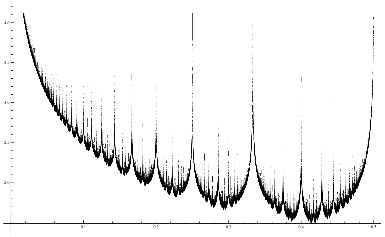

Perspectives

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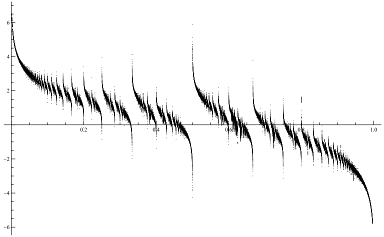

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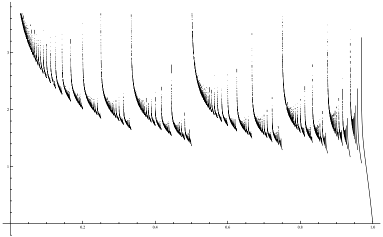

Thank you for your attention !