Geometrical cake cutting

We are going to cut some cakes into two equal parts, with just one linear cut.
In fact, we will cut *mathematical cakes* whose shape is perfectly accurate, and that have a perfectly homogeneous pastry. Moreover, our cakes have the same height everywhere, so we are left with a problem of plane geometry. For example, to cut a round cake we simply need to cut a disk.

How can we cut a disk in two? We cut it through the midpoint. What about a square (for cutting a square cake)? By symmetry, every cut along a line through the square center will divide the square in two. The same holds for parallelograms. As the lines through the center cover all directions, these are the only possible cutting lines (if we move a line through the center with a translation, we are increasing one of the two slices and decreasing the other, loosing the property that the two slices are the same).

Let's now handle triangular cakes. We can cut the triangle through the vertex along a median, and indeed the intersection of the medians is the centroid (which is the center of gravity of the triangle if we assign to it a homogeneous weight). We may also cut the triangle parallel to one side, generating a smaller copy of the triangle and a trapezoid. The small triangle should then have area that is half of the area of the large triangle, hence its height should be $1/\sqrt{2}$ the height of the large triangle (the ratio of the area for similar figures uses the square of the scaling factor for the lengths). What about cutting the triangle along another line of which we fix a direction? Mathematicians would say that by continuity and monotonicity (of the function measuring the ratio of the two slices by translating the line) there is precisely one suitable line in the given family of parallel lines. Supposing that the cutting line is vertical, a cutting line too much on the left (respectively, right) will make the left (respectively, right) slice too small and we find the right balance somewhere in the middle.

Let us also cut isosceles trapezoids (supposing they are not parallelograms). We can cut them though one vertex, producing a triangle and a convex quadrilateral. Such lines all go through the center of gravity of the trapezoid, which by symmetry is located on the symmetry axis. To locate the center of gravity, and also to investigate a different kind of cut, we now cut the trapezoid into two parts with a line parallel to the bases (then the center of gravity is the intersection of this line with the symmetry axis of the trapezoid).

Suppose to cut the isosceles trapezoid with a line parallel to the bases into two equal parts (such line exists unique, again by continuity and monotonicity). The cut produces two trapezoids that have one basis in common, which we call "upper trapezoid" and "lower trapezoid", assuming that the small basis is horizontal and at the top as it usually is. With this orientation, all the heights that we consider are vertical.
The cutting problem is already solved for parallelograms, so we may exclude this case. By taking heights at the vertices of the small basis we decompose the

isosceles trapezoid into a rectangle and two triangles. We call b the small basis and ℓ the sum of the triangle bases.

Suppose up to a rescaling that the trapezoid has height 1, and call p the distance from the cutting line to the small basis (which in this case is also the ratio between the height of the upper trapezoid and the height of the original trapezoid).
The upper trapezoid is also divided into a rectangle and two triangles.
The area of the triangles in the upper trapezoid is $\frac{1}{2}\ell p^2$, while the area of the rectangle in the upper trapezoid is bp. What we want, comparing the area of the upper trapezoid to the one of the original trapezoid, is

$$\frac{1}{2}\ell p^2 + bp = \frac{1}{2} \cdot (\frac{1}{2}(2b + \ell))$$

Assuming ℓ, b to be known, the quadratic equation allows to determine p and there is precisely one solution strictly between 0 and 1. A challenge for teachers is inventing good numbers, such that all ℓ, b, p are rational numbers. An equivalent quadratic equation has discriminant

$$\left((r + \frac{1}{2})^2 + \frac{1}{4}\right)$$

where r denotes the ratio b/ℓ. So if we start with ℓ, b that are rational numbers, we need to ensure that the above expression in r is a square. Building on Pythagorean triples, we see that we may take $r = 1/6$ and obtain the nice value $p = 2/3$. By rescaling the original trapezoid to have height $H = 6$, the bases are $b = 1$ and $B = 7$ and the height of the upper trapezoid is $h = 4$ (see the picture).

The above reasoning works well for any acute trapezoid, and also for right trapezoids (in this case, the decomposition consists in one rectangle and only one triangle). We leave it as an exercise to the reader to consider obtuse trapezoids, for which a different decomposition leads to different computations.