Stochastic anisotropic aquifer characterization: a poroelastic finite element model and the potential of using InSAR data

Introduction

Groundwater extracted from aquifers supports critical needs for drinking water, agriculture, and industry. However, unsustainable pumping practices threaten long-term water security, contributing to land subsidence, saltwater intrusion, and declining water availability. To mitigate these risks, robust, data-informed models of aquifer behavior are essential for sustainable groundwater management. Among the many data sources now used to constrain such models, Interferometric Synthetic Aperture Radar (InSAR) has become an increasingly popular tool for measuring surface deformation, offering cost-saving, wide-area insight into subsurface processes.

Recent studies suggest that surface displacements observed by InSAR and Global Positioning System (GPS) contain signatures of aquifer hydraulic properties—particularly anisotropic hydraulic conductivity (AHC)—due to the coupling between fluid flow and geomechanical deformation. Anisotropy in AHC arises from geological features such as fractures and faults, which influence directional fluid flow at the aquifer scale. Despite this fact, previous Bayesian inversion frameworks based on InSAR data have typically assumed isotropic conductivity priors (Alghamdi et al., 2020, 2021), limiting their ability to capture the true structural complexity of many aquifers.

In this work, we focus on the Anderson Junction site—an aquifer known to exhibit strong directional anisotropy, as identified through pumping tests (Heilweil and Hsieh, 2006) and fracture mapping. We introduce a Bayesian modeling framework to estimate uncertainty in the AHC tensor. A key feature of our approach is the use of circular statistical models to represent uncertainty in the principal flow direction, calibrated against fracture outcrop data. Specifically, we construct a mixture of von Mises distributions to capture multi-modal angular structure in a probabilistic way, extending previous frequentist analyses (Lark et al., 2014) into the Bayesian domain.

We then embed this directional model into a stochastic framework for generating symmetric positive definite (SPD) tensors, leveraging a spectral decomposition approach (Shivanand et al., 2024). This construction allows us to separately model the magnitude and direction of AHC, ensuring mathematical consistency while incorporating geostatistical information. We propagate the resulting uncertainty through our poroelastic Partial Differential Equation (PDE)-based model of the Anderson Junction site (Salehian Ghamsari et al., 2025) to assess the impact of AHC uncertainty on predicted InSAR Line of Sight (LOS) surface displacements.

In summary, our contributions are: (1) A Bayesian mixture model for geological directionality, extending prior work on circular statistics (Taghia et al., 2014); (2) An extension of SPD tensor modeling to incorporate complex anisotropy (Shivanand et al., 2024); (3) A forward uncertainty analysis demonstrating how directional uncertainty affects predicted InSAR surface displacements. While our study is applied to hydrogeology, the methodology is general and applicable to other fields involving anisotropic material properties and coupled physical processes.

Case study

The Anderson Junction aquifer in Utah, USA, serves as the basis for our case study, originally investigated through a controlled pumping test involving one pumping well and two observation wells aligned with the principal directions of AHC (Heilweil and Hsieh, 2006). The confined Navajo Sandstone aquifer exhibited directional conductivity, with major and minor hydraulic conductivity values estimated at $1.1 \times 10^{-8} \pm 21\%$ and $4.7 \times 10^{-10} \pm 19\%$, respectively. Fracture orientations observed in the field, concentrated along north-south and east-west trends, support these anisotropic flow properties. Since previous modeling showed that vertical surface deformation from the original four-day test at $0.07\,\mathrm{m^3\,s^{-1}}$ was likely below the detection threshold of Sentinel-1 InSAR (Salehian Ghamsari et al., 2025), we simulate a hypothetical scenario with higher pumping ($0.28\,\mathrm{m^3\,s^{-1}}$ for 8 days) using the same intrinsic AHC values to better explore the effects of directional conductivity on surface displacement.

Methodology

We begin by applying a poroelastic finite element formulation based on Biot's theory of poroelasticity (see eq. (1)). This formulation involves solving a system of PDE to model the interaction between fluid pressure, deformation, and fluid flux within a porous medium. The Biot equations are discretized using finite elements in the DOLFINx environment, with appropriate boundary conditions applied to the solid and fluid domains. The poroelastic finite element model is detailed in Salehian Ghamsari et al. (2025).

$$(S_\varepsilon p + \alpha \nabla \cdot u)_t + \nabla \cdot q = f_p \text{ on } \Omega \times (0,T], \tag{1a}$$

$$-\nabla \cdot \bar{\sigma}(u,p) = f_u \text{ on } \Omega \times (0,T], \tag{1b}$$

$$q + \widetilde{k} \nabla p = 0 \text{ on } \Omega \times (0,T]. \tag{1c}$$

The domain of the aquifer system is represented in three-dimensional space, where the primary challenge lies in determining the hydraulic conductivity tensor, which is assumed to be anisotropic, with direction-dependent properties as follows

$$\widetilde{k} = \left[\begin{array}{cc|c} k_{xx} & k_{xy} & 0 \\ k_{yx} & k_{yy} & 0 \\ \hline 0 & 0 & k_{zz} \end{array}\right]. \tag{2}$$

Using spectral decomposition, the SPD AHC tensor is decomposed into two components: a diagonal matrix Λ, whose entries are the eigenvalues representing the magnitudes of hydraulic conductivity along the principal directions, and an orthogonal matrix Q, whose columns are the eigenvectors defining the orientation of these principal directions.

$$k = Q\Lambda Q^T. \tag{3}$$

Calibration of the AHC tensor is performed using Bayesian inference. Specifically, we employ a mixture of von Mises distributions to model the rotation angles associated with the anisotropic tensor's principal directions. The von Mises distribution is well-suited for circular directional data and is used to account for uncertainties in the principal orientations of the hydraulic conductivity. A hierarchical Bayesian model is constructed, where the parameters of the von Mises distribution, including the mean and concentration parameters, are estimated from the fracture outcrop data.

We write the joint random model for $n \geq 2$ mixtures on the latent parameters $\theta^n = (\mu_1, \kappa_1, c_1, w_1, m_1 \ldots, \mu_n, \kappa_n, c_n, w_n, m_n)$ and p independent and identically distributed (iid) rotation angles $\phi_1, \phi_2, \ldots, \phi_p$

$$\mu_1, \ldots, \mu_n \overset{\text{iid}}{\sim} \text{VonMises}(\hat{\mu}, \hat{\kappa}), \tag{4a}$$

$$\kappa_1, \ldots, \kappa_n \overset{\text{iid}}{\sim} \text{Gamma}(\hat{\alpha}, \hat{\beta}), \tag{4b}$$

$$c_1, \ldots, c_n \overset{\text{iid}}{\sim} \text{VonMises}(\mu_i, \kappa_i), \tag{4c}$$

$$w_1, \ldots, w_n \sim \text{Dirichlet}(\hat{a}_1, \ldots, \hat{a}_n), \quad n \geq 2, \tag{4d}$$

$$w_1 = 1, \quad n = 1, \tag{4e}$$

$$m_1, \ldots, m_n \sim \text{Categorical}(w_1, \ldots, w_n), \tag{4f}$$

$$\phi_1, \phi_2, ..., \phi_p \overset{\text{iid}}{\sim} \text{Mixture}((m_1, \ldots, m_n), (c_1, \ldots, c_n)), \tag{4g}$$

where the VonMises distribution, characterized by a mean direction μ and a concentration parameter κ, serves as the circular analogue of the normal distribution. In constructing the mixture model, each component c_i is assigned a VonMises prior over its mean μ_i and a Gamma prior over its concentration κ_i, with the Gamma distribution defined by hyperparameters $\hat{\alpha}$ (shape) and $\hat{\beta}$ (rate). The weights $w_1, \ldots, w_n$ that govern the mixture proportions are drawn from a Dirichlet prior with concentration parameters $\hat{a}_1, \ldots, \hat{a}_n$. These weights are then used within a Categorical distribution to generate random component selectors $m_1, \ldots, m_n$. Ultimately, the rotation angles are modeled as iid samples from the resulting mixture of von Mises components $c_1, \ldots, c_n$, with selection governed by $m_1, \ldots, m_n$.

To determine the optimal number of mixture components n for modeling the observed fracture outcrop data, we address the model selection problem by balancing fit and parsimony. We use Leave-one-out cross-validation (LOOCV), which, despite being more computationally intensive than alternatives, provides more reliable performance estimates for complex models such as the mixtures used in this study.

We model the eigenvalues of the tensor as log-normal random variables to ensure they remain positive. This approach aligns with prior work (Shivanand et al., 2024) that models their logarithms as normal variables.

Then, we construct a stochastic model for the AHC tensor by extending the deterministic decomposition introduced in previous work Shivanand et al. (2024). Specifically, the AHC tensor k is defined as a function of a random variable $\omega := (\phi, \lambda_x, \lambda_y)$, where ϕ represents the rotation angle and λ_x, λ_y are the eigenvalues.

We propose three levels of stochastic modeling for k. The full model introduces randomness in both scaling and rotation through the formulation:

$$k(\omega) = R(\phi)\widehat{Q}\Lambda(\lambda_x, \lambda_y)\widehat{Q}^T R(\phi)^T, \tag{5}$$

where $R(\phi)$ is a rotation matrix, $\widehat{Q}$ is a fixed eigenvector, and $\Lambda(\lambda_x, \lambda_y)$ is the diagonal matrix of eigenvalues. This setup captures the coupled uncertainty inherent in heterogeneous subsurface systems.

Two simplified models are also considered. The randomness in the scaling model k_s assumes perfect knowledge of orientation (ϕ), resulting in variability solely from the eigenvalues. Conversely, the randomness in the rotation model k_r fixes λ_x, λ_y to their calibrated values, introducing randomness purely through orientation. These reduced models offer useful abstractions for isolating the impact of individual sources of uncertainty.

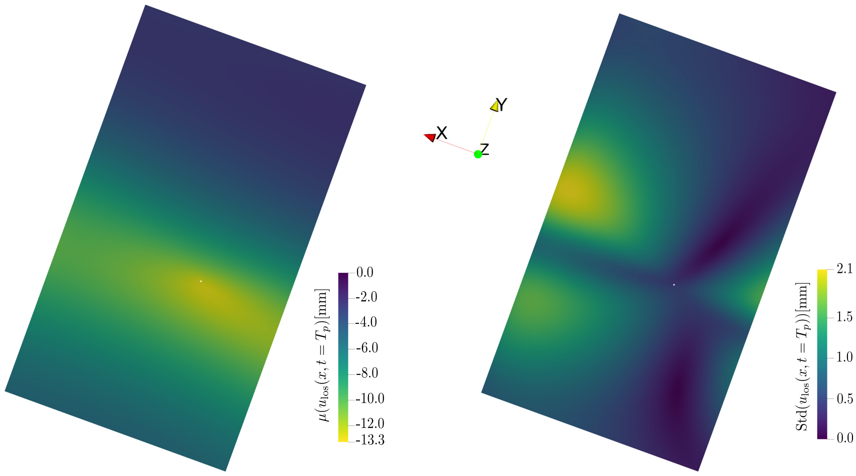
Once the AHC tensor is calibrated, we propagate the resulting randomness through the conceptual model of the Anderson Junction aquifer system. We compute summary statistics of the LOS surface displacement using Monte Carlo simulations to examine how uncertainty in the AHC tensor leads to the uncertainty in the LOS surface displacement.

Results

We apply the methodology to develop two distinct models of the Anderson Junction site, reflecting two different states of subjective Bayesian belief. The first modeling scenario assumes the availability of fracture outcrop data and estimates of the hydraulic conductivity (k) from the pump test by Heilweil and Hsieh (2006). Based on the pump test results, we construct a rotation angle model that captures the complex distribution of fracture orientations, particularly emphasizing that the major principal direction of the AHC aligns with the east-west axis. In fig. 1, we present the mean and standard deviation of LOS displacement calculated using AHC with random scaling and rotation.

Conclusions

This study introduces a random model for AHC, designed to incorporate uncertainty from structural directional data. A primary objective of this work was to develop a model that can serve as a prior in Bayesian inference, where InSAR-derived LOS data provides insights into AHC. By calibrating the model using fracture outcrop and pump test data from Anderson Junction, we were able to explore two distinct conceptual states of belief. Our findings demonstrate that the proposed methodology is flexible enough to model uncertainty effectively and that directional uncertainty plays a more significant role in influencing the deviance of the LOS displacement compared to the magnitudes of hydraulic conductivity in the principal directions.



(a) Mean of LOS displacement (random scaling and rotation AHC).

(b) Standard deviation of LOS displacement (random scaling and rotation AHC).

Figure 1: Mean and standard deviation of the LOS displacement outputs from executing the poroelastic model using AHC with randomness in both scaling and rotation. T_p is the time when the pumping finished after 8 d of pumping. The mean values are truncated at zero from above.

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