IRREDUCIBLE POLYADIC SEMIGROUPS ADMITTING THE ADJUNCTION OF A NEUTRAL ELEMENT

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Abstract. It was claimed in [4] that for any integer $n \geqslant 2$, a neutral element can be adjoined to an n-ary semigroup if and only if the n-ary semigroup is reducible to a binary semigroup. We show that the 'only if' direction of this statement is incorrect when n is odd. Moreover, we offer a comprehensive characterization of the class of irreducible n-ary semigroups, for odd n, that admit the adjunction of a neutral element.

1. Introduction

Let X be a nonempty set and let $n \geqslant 2$ be an integer. Recall that an n-ary operation $F\colon X^n \to X$ is said to be *associative* if it satisfies the following system of functional equations:

$$\begin{aligned} &F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\ &\quad = \quad F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}) \end{aligned}$$

for all $x_1,\ldots,x_{2n-1} \in X$ and all $1 \leqslant i \leqslant n-1$. The pair (X,F) is commonly known as an *n-ary semigroup*, or alternatively, a *polyadic semigroup* when the arity n is not specified. This concept was introduced by Dörnte [3] and has subsequently led to the development of the notion of an n-ary group, which was first studied by Post [6].

An associative n-ary operation $F\colon X^n \to X$ is said to be *reducible to* (or *derived from*) an associative binary operation $\circ\colon X^2 \to X$ if it satisfies the following equation (where $\circ$ is used in its standard infix notation):

$$\begin{aligned} F(x_1,x_2,x_3\ldots,x_n) \quad &= \quad (\cdots((x_1\circ x_2)\circ x_3)\circ\cdots)\circ x_n \\ &= \quad x_1\circ x_2\circ x_3\circ\cdots\circ x_n \qquad (x_1,x_2,x_3,\ldots,x_n \in X). \end{aligned}$$

In this case, the n-ary semigroup (X,F) is also said to be *reducible to* (or *derived from*) the semigroup $(X,\circ)$.

Let us provide a straightforward example of an irreducible ternary semigroup.

Example 1.1. The real associative operation $F\colon \mathbb{R}^3 \to \mathbb{R}$, defined by

$$F(x_1,x_2,x_3) \;=\; x_1 - x_2 + x_3 \qquad \text{for } x_1,x_2,x_3 \in \mathbb{R},$$

is irreducible. Indeed, suppose that this operation is reducible to an associative binary operation $\circ\colon \mathbb{R}^2 \to \mathbb{R}$ and let $c = 0\circ 0$. Then, for any $x \in \mathbb{R}\setminus\{c\}$, we have

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$c \circ x = x = x \circ c$ and hence also $x = c \circ x \circ c = 2c - x$, which implies $x = c$, a contradiction. $\lozenge$

Recall also that an element $e \in X$ is said to be *neutral* for an n-ary operation $F \colon X^n \to X$ (or an n-ary semigroup (X, F)) if, for any $x \in X$, the following condition is satisfied:

$$F(e^{k-1}, x, e^{n-k}) = x \qquad (k = 1, \dots, n).$$

Here and throughout, for any $k \in \{0, \dots, n\}$ and any $x \in X$, the symbol x^k represents the list $x, \dots, x$, with k copies of x. For instance, for any $x, y, z \in X$, we have

$$F(x^3, y^0, z^2) = F(x, x, x, z, z).$$

Recall that, when $n = 2$, the neutral element is unique and often also called an *identity* element.

Dudek and Mukhin [4, Proposition 2] made the following appealing and intriguing claim: "*one can adjoin a neutral element to an n-ary semigroup if and only if the n-ary semigroup is reducible to a binary semigroup*" (see Definition 2.2 below for the formal definition of adjoining a neutral element).

The main objective of this paper is to present a counter-example that unfortunately disproves the 'only if' direction of their statement when n is an odd integer. This means that *there exists at least one irreducible n-ary semigroup, where n is odd, that admits the adjunction of a neutral element.*

The structure of the paper is as follows. In Section 2, we revisit the 'if' direction of Dudek and Mukhin's statement and establish that the 'only if' direction holds when n is even. In Section 3, we provide a comprehensive characterization of the class of irreducible n-ary semigroups, for odd n, that admit the adjunction of a neutral element. Section 4 presents a method for constructing such n-ary semigroups, along with explicit examples that serve to refute the 'only if' direction when n is odd.

2. Preliminaries

Let us review two important results given by Dudek and Mukhin. We begin by examining the following straightforward lemma, which complements [4, Lemma 1].

Lemma 2.1. *If an n-ary semigroup (X, F) has a neutral element $e \in X$, then the structure $(X, \circ)$ defined by the equation*

$$x \circ y = F(x, e^{n-2}, y) \qquad \text{for } x, y \in X,$$

is a semigroup and, for any $k \in \{1, \dots, n\}$, we have

$$x_1 \circ \cdots \circ x_k = F(x_1, \dots, x_k, e^{n-k}) \qquad (x_1, \dots, x_k \in X).$$

In particular, (X, F) is reducible to the semigroup $(X, \circ)$; this semigroup also has e as its neutral element and is the unique reduction of (X, F) having this property.

Proof. We first observe that the operation $\circ$ defined in the statement is associative. In fact, for any $x, y, z \in X$, we have

$$\begin{aligned} (x \circ y) \circ z &= F(F(x, e^{n-2}, y), e^{n-2}, z) \\ &= F(x, e^{n-2}, F(y, e^{n-2}, z)) = x \circ (y \circ z). \end{aligned}$$

Moreover, it is clear that e is also the neutral element for $\circ$.

Let us now prove the stated formula by induction on k. The identity holds trivially for $k = 1$ since e is a neutral element for F. Assume now that it holds for some $1 \leqslant k \leqslant n-1$, and let us show that it also holds for $k+1$. By using associativity and the fact that e is a neutral element for F, we obtain

$$\begin{aligned}
(x_1 \circ \cdots \circ x_k) \circ x_{k+1} &= F(x_1 \circ \cdots \circ x_k, e^{n-2}, x_{k+1}) \\
&= F(F(x_1 \circ \cdots \circ x_k, e^{n-2}, x_{k+1}), e^{n-1}) \\
&= F(x_1 \circ \cdots \circ x_k, F(e^{n-2}, x_{k+1}, e), e^{n-2}) \\
&= F(x_1 \circ \cdots \circ x_k, x_{k+1}, e^{n-2}) \\
&= F(F(x_1, \ldots, x_k, e^{n-k}), x_{k+1}, e^{n-2}) \\
&= F(x_1, \ldots, x_k, F(e^{n-k}, x_{k+1}, e^{k-1}), e^{n-k-1}) \\
&= F(x_1, \ldots, x_k, x_{k+1}, e^{n-k-1}),
\end{aligned}$$

which shows that the identity still holds for $k+1$. Taking $k = n$ immediately establishes the reducibility of (X, F).

Now, suppose that $(X, \diamond)$ is a reduction of (X, F) that has e as the neutral element. Then, for any $x, y \in X$, we have

$$x \diamond y = \underbrace{x \diamond e \diamond \cdots \diamond e \diamond y}_{n} = F(x, e^{n-2}, y) = x \circ y.$$

Therefore, the semigroup $(X, \circ)$ is the unique reduction of (X, F) that has e as the neutral element. □

It is well known and easy to verify (see, e.g., Clifford and Preston [2, p. 4]) that a neutral element can always be adjoined to a semigroup. In the next proposition, we restate [4, Proposition 1] and its proof, using our notation; it asserts that this property also holds for any reducible n-ary semigroup. We begin with a formal definition.

Definition 2.2. We say that *a neutral element can be adjoined to an n-ary semigroup* (X, F) if there exists an n-ary semigroup (X^*, F^*), where $X^* = X \cup \{e\}$ for some $e \notin X$, such that the operation $F^* \colon (X^*)^n \to X^*$ satisfies the following two properties:

- $F^*|_{X^n} = F$;
- e is neutral for F^*.

Proposition 2.3. *A neutral element can be adjoined to any reducible n-ary semigroup.*

Proof. Let (X, F) be an n-ary semigroup that is reducible to a semigroup $(X, \circ)$ and let $(X^*, *)$ be the semigroup obtained from $(X, \circ)$ by adjoining a neutral element $e \notin X$. One can readily see that the n-ary structure (X^*, F^*), where the n-ary operation $F^* \colon (X^*)^n \to X^*$ is defined by the equation

$$F^*(x_1, \ldots, x_n) = x_1 * \cdots * x_n \qquad \text{for } x_1, \ldots, x_n \in X^*,$$

is a (reducible) n-ary semigroup with e as a neutral element, whose restriction to X is the n-ary semigroup (X, F). □

Remark 2.4. If an n-ary semigroup has a neutral element e, then it is reducible by Lemma 2.1. Moreover, by Proposition 2.3, we can always adjoin a neutral element to it. However, the element e need not remain a neutral element for the resulting semigroup. ◊

The following example demonstrates that it is not possible to adjoin a neutral element to the real ternary semigroup mentioned in Example 1.1.

Example 2.5. The irreducible ternary semigroup $(\mathbb{R}, F)$, defined by

$$F(x_1, x_2, x_3) = x_1 - x_2 + x_3 \qquad \text{for } x_1, x_2, x_3 \in \mathbb{R},$$

does not admit the adjunction of a neutral element. Indeed, suppose on the contrary that we can adjoin a neutral element $e \notin \mathbb{R}$ to it. By Lemma 2.1, the resulting semigroup is then the ternary extension of a unique monoid $(X^*, *)$, with $X^* = \mathbb{R} \cup \{e\}$ and neutral element e. Now, let $u \in \mathbb{R}$ and define $v = u * u$. For any $x \in \mathbb{R} \smallsetminus \{v\}$, we then have

$$x * v = x * u * u = F(x, u, u) = x = F(u, u, x) = u * u * x = v * x,$$

and hence

$$x = v * x * v.$$

If $v \in \mathbb{R}$, then this implies $x = F(v, x, v) = 2v - x$, which is impossible. This proves that $u * u = e$, i.e., the square of every element of X^* is the neutral element e, thus $(X^*, *)$ is a group of exponent 2. Such groups are commutative, hence

$$y = F(x, x, y) = x * x * y = x * y * x = F(x, y, x) = 2x - y$$

for all $x, y \in \mathbb{R}$, a contradiction. $\Diamond$

In the following proposition, we show that, when n is even, any n-ary semigroup that admits the adjunction of a neutral element is reducible. We first consider a definition and a lemma.

Definition 2.6. We say that an n-ary semigroup (X, F) is an n-ary *IN-semigroup* if it is irreducible and admits the adjunction of a neutral element.

According to Examples 1.1 and 2.5, the real ternary semigroup $(\mathbb{R}, F)$ with $F(x_1, x_2, x_3) = x_1 - x_2 + x_3$ is irreducible, but not an IN-semigroup, thereby showing that not all irreducible n-ary semigroups are n-ary IN-semigroups.

Lemma 2.7. *Let (X, F) be an n-ary IN-semigroup, let (X^*, F^*) be the n-ary semigroup obtained from (X, F) by adjoining a neutral element $e \notin X$, and let $(X^*, *)$ be the reduction of (X^*, F^*) whose operation $*$ is defined by*

$$x * y = F^*(x, e^{n-2}, y) \qquad (x, y \in X).$$

Then, there exist $a, b \in X$ such that $a * b = e$.

Proof. We only need to prove that X cannot be closed under the operation $*$. Suppose, on the contrary, that X is closed under $*$. Then $(X, *)$ is a semigroup and, for any $x_1, \ldots, x_n \in X$, we have

$$F(x_1, \ldots, x_n) = F^*(x_1, \ldots, x_n) = x_1 * \cdots * x_n \in X,$$

and hence (X, F) is reducible to the semigroup $(X, *)$, a contradiction. It follows that there must exist $a, b \in X$ such that $a * b \notin X$, i.e., $a * b = e$. $\square$

Proposition 2.8. *Let (X, F) be an n-ary semigroup, where n is even. If we can adjoin a neutral element to (X, F), then it is reducible.*

Proof. Suppose on the contrary that (X,F) is an n-ary IN-semigroup and let (X^*,F^*) and $(X^*,*)$ be the semigroups defined in Lemma 2.7. Then, there exist $a,b\in X$ such that $a*b=e$. We then have

$$\begin{aligned} e &= \overbrace{e*\cdots*e}^{n/2} = (a*b)*\cdots*(a*b)\\ &= F^*(\underbrace{a,b,\ldots,a,b}_{n}) = F(\underbrace{a,b,\ldots,a,b}_{n}) \in X\,, \end{aligned}$$

a contradiction (since $e\notin X$). □

Corollary 2.9. *If n is even, then a neutral element can be adjoined to an n-ary semigroup if and only if it is reducible.*

3. A Characterization of all the n-ary IN-Semigroups

In the following theorem, we provide a characterization of the family of all n-ary IN-semigroups, i.e., all the n-ary semigroups that, while irreducible, admit the adjunction of a neutral element. We first introduce a special class of monoids.

Definition 3.1. We say that a semigroup $(M,*)$, endowed with a neutral element e, is a *W-monoid* if there exists an element $a\in M$ such that the following three conditions are satisfied:

(W1) for any $x,y\in M$, we have
$$x*y=e \quad\Longleftrightarrow\quad (x=a \text{ and } y=a) \text{ or } (x=e \text{ and } y=e);$$

(W2) for any $x,y\in M$, we have
$$x*y=a \quad\Longleftrightarrow\quad (x=a \text{ and } y=e) \text{ or } (x=e \text{ and } y=a);$$

(W3) a is noncentral for $*$.

Note that condition (W1) ensures the uniqueness of a, while condition (W3) implies that $a\neq e$.

Theorem 3.2. *If (X,F) is an n-ary IN-semigroup, then n is odd. Adjoining a neutral element $e\notin X$ to (X,F), we obtain an n-ary semigroup (X^*,F^*) that is the n-ary extension of the W-monoid $(X^*,*)$ defined by the equation*
$$x*y = F^*(x,e^{n-2},y) \qquad \text{for } x,y\in X^*.$$
Moreover, $(X^*,*)$ is the only such monoid with neutral element e.

Conversely, let $(M,*)$ be any W-monoid with neutral element e, and let us consider its n-ary extension $F^*\colon M^n\to M$ defined by
$$F^*(x_1,\ldots,x_n) = x_1*\cdots*x_n \qquad (x_1,\ldots,x_n\in M).$$
If n is odd, then the n-ary semigroup $(X,F^*|_{X^n})$ with $X=M\smallsetminus\{e\}$ is an n-ary IN-semigroup.

Proof. Let us prove the first part of the theorem. Let (X,F) be an n-ary IN-semigroup, i.e., (X,F) is irreducible and admits the adjunction of a neutral element $e\notin X$. By Proposition 2.8, n must be odd, i.e., $n=2k+1$ for some integer $k\geqslant 1$. By Lemma 2.1, the n-ary semigroup (X^*,F^*) obtained from (X,F) by adjoining a neutral element $e\notin X$ is reducible to the binary semigroup $(X^*,*)$ defined by
$$x*y = F^*(x,e^{n-2},y),$$

and $(X^*, *)$ is the unique reduction that has e as its neutral element.

Now, we only need to prove that $(X^*, *)$ is a W-monoid. By Lemma 2.7, there exist $a, b \in X$ such that $a * b = e$. The following claim states that e cannot be expressed as a composition of three elements of X.

Claim 1. For any $x, y, z \in X^*$ such that $x * y * z = e$, we have $e \in \{x, y, z\}$.

Proof of Claim 1. Suppose on the contrary that $x * y * z = e$ for some $x, y, z \in X$. Since e is a neutral element for $*$, we have

$$\begin{aligned} F(x,y,z,a,b,\ldots,a,b) &= F^*(x,y,z,a,b,\ldots,a,b) \\ &= (x*y*z)*\underbrace{(a*b)*\cdots*(a*b)}_{k-1\text{ pairs}} = e. \end{aligned}$$

This is a contradiction since F ranges in X and $e \notin X$. ◊

Let us now show that condition (W1) of Definition 3.1 holds. By Claim 1, we see that

$$e = (a*b)*(a*b) = a*(b*a)*b$$

implies that $b * a = e$, since $a \neq e \neq b$. From this, it follows that

$$a*(b*b)*a = (a*b)*(b*a) = e = (b*a)*(a*b) = b*(a*a)*b.$$

Applying Claim 1 again, we obtain $a * a = b * b = e$. We then see that $b = a$, since

$$b = e*b = (a*b)*b = a*(b*b) = a*e = a.$$

Now, if e has another factorization $e = c * d$ with $c, d \in X$, then we can derive similarly that $c = d$. Moreover, we have

$$e = (a*a)*(c*c) = a*(a*c)*c,$$

and then $a * c = e$ by Claim 1, and hence $a = c$ by the preceding argument. This shows that the only nontrivial factorization of e is $e = a * a$, thus proving condition (W1).

Let us now show that condition (W2) holds. If $a = x * y$ for some $x, y \in X^*$, then $e = a * a = (x * y) * a$, and hence $e \in \{x, y\}$ by Claim 1, which also implies that $\{x, y\} = \{a, e\}$, as e is the neutral element for $*$.

To verify that condition (W3) also holds, suppose for contradiction that a is a central element for $*$. Then, for any $x \in X$ and any $k \in \{1, \ldots, n\}$, we have

$$F(a^{k-1},x,a^{n-k}) = F^*(a^{k-1},x,a^{n-k}) = a^{k-1}*x*a^{n-k} = x*\underbrace{a*\cdots*a}_{n-1\text{ (even)}} = x,$$

which shows that a is a neutral element for F. This is a contradiction, since F is irreducible.

Let us now prove the second part of the theorem. Let $(M, *)$ be a W-monoid with neutral element $e \in M$, and let $a \in M \setminus \{e\}$ be as defined in Definition 3.1. Let also n be an odd integer, and let (M, F^*) be the n-ary extension of $(M, *)$.

Claim 2. For any $x_1, \ldots, x_n \in M$, we have $x_1 * \cdots * x_n \in \{a, e\}$ if and only if $x_1, \ldots, x_n \in \{a, e\}$. Moreover, in this case we have

$$x_1*\cdots*x_n = \begin{cases} a, & \text{if } |\{i : x_i = a\}| \text{ is odd,} \\ e, & \text{if } |\{i : x_i = a\}| \text{ is even.} \end{cases}$$

Proof of Claim 2. By conditions (W1) and (W2), we have $x_1 * x_2 \in \{a,e\}$ if and only if $x_1, x_2 \in \{a,e\}$, and then a routine induction argument shows that

$$x_1 * \cdots * x_n \in \{a,e\} \quad\iff\quad x_1,\ldots,x_n \in \{a,e\}.$$

The second statement of the claim follows from the fact that the subsemigroup $(\{a,e\},*)$ is isomorphic to $(\mathbb{Z}_2,+)$ under the isomorphism $e \mapsto 0$, $a \mapsto 1$. $\lozenge$

Set $X = M \smallsetminus \{e\}$, and let us show that X is closed under F^*.

Claim 3. For any $x_1,\ldots,x_n \in X$, we have $F^*(x_1,\ldots,x_n) \in X$.

Proof of Claim 3. Suppose on the contrary that there exist $x_1,\ldots,x_n \in X$ such that $F^*(x_1,\ldots,x_n) = e$. By Claim 2, we have $x_1,\ldots,x_n \in \{a,e\}$ and $|\{i : x_i = a\}|$ is even. Since n is odd, this implies that some x_i equals e, contrary to our assumption. $\lozenge$

Now we can define the restriction $F = F^*|_{X^n}$, and we will show that the n-ary semigroup (X,F) is an n-ary IN-semigroup. It is clear by definition that it admits the adjunction of the neutral element e. To see that it is irreducible, we first establish a claim.

Claim 4. The n-ary semigroup (X,F) has no neutral element.

Proof of Claim 4. Suppose on the contrary that (X,F) has a neutral element $u \in X$. We then have $F(u^{n-1},a) = a$, that is,

$$\underbrace{u * \cdots * u}_{n-1} * a \;=\; a, \qquad \text{and hence} \qquad \underbrace{u * \cdots * u}_{n-1} \;=\; e.$$

It follows by Claim 2 that $u = a$ (since $u \neq e$), which means that a is a neutral element for F. But then, for any $x \in X$, we have (by Claim 2, using that $n-2$ is odd)

$$x \;=\; F(a,x,a^{n-2}) \;=\; F^*(a,x,a^{n-2}) \;=\; a * x * \underbrace{a * \cdots * a}_{n-2} \;=\; a * x * a,$$

which, upon multiplying by a on the left, implies that a is a central element for $*$, a contradiction. $\lozenge$

Let us now show that the n-ary semigroup (X,F) is irreducible. Suppose, on the contrary, that (X,F) is reducible to a semigroup $(X,\circ)$. Then, for any $x \in X$, we have

$$\begin{aligned}\overbrace{a \circ \cdots \circ a}^{n-1} \circ x \;&=\; F(a^{n-1},x) \;=\; F^*(a^{n-1},x)\\ &=\; \underbrace{a * \cdots * a}_{n-1} * x \;=\; e * x \;=\; x,\end{aligned}$$

and similarly,

$$x \circ \underbrace{a \circ \cdots \circ a}_{n-1} \;=\; x * \underbrace{a * \cdots * a}_{n-1} \;=\; x.$$

Therefore, $\overbrace{a \circ \cdots \circ a}^{n-1}$ is a neutral element for $(X,\circ)$ and hence also for (X,F). This contradicts Claim 4 and therefore completes the proof of the theorem. $\square$

In the next section, we present examples of W-monoids which, in view of Theorem 3.2, establish the existence of n-ary IN-semigroups for odd n. This observation enables us to refute the 'only if' direction of Dudek and Mukhin's statement in [4, Proposition 2] for odd n. In fact, the flaw in their argument lies only in their proof of this implication. In our notation, they erroneously assert that the existence of elements $a, b \in X$ such that $a * b = e$ (see Lemma 2.7) contradicts the assumption that $e \notin X$.

4. Constructions of W-monoids

In this final section, we outline a procedure for constructing W-monoids via specific ideal extensions of semigroups. We begin by recalling some fundamental definitions. For background, the reader may consult, e.g., Clifford and Preston [2], Petrich [5], and Rees [7].

Given a semigroup $(S, *)$, a subset $I \subset S$ is called an *ideal* of $(S, *)$ if, for all $i \in I$ and $x \in S$, we have $i * x \in I$ and $x * i \in I$. The *Rees congruence* associated with I is the relation $\sim$ on S defined by $x \sim y$ if and only if $x, y \in I$ or $x = y$. The corresponding quotient S/I is referred to as the *Rees quotient*.

An *ideal extension* of a semigroup S by a semigroup T with a zero element is a semigroup Σ that contains S as an ideal and for which the Rees quotient Σ/S is isomorphic to T.

Using these concepts, we can easily derive the following proposition, which describes the monoids satisfying conditions (W1) and (W2) of Definition 3.1 in terms of Rees quotients and ideal extensions. The proof is straightforward and is therefore omitted.

Proposition 4.1. *Let $(M, *)$ be a monoid with neutral element e and let $a \in M \smallsetminus \{e\}$. Then the following assertions are equivalent.*

(i) *The monoid $(M, *)$ satisfies conditions (W1) and (W2) of Definition 3.1.*
(ii) *The subset $I = M \smallsetminus \{a, e\}$ is an ideal of $(M, *)$ and the Rees quotient M/I is isomorphic to the set $T = \{-1, 0, 1\}$ endowed with the usual multiplication, where the equivalence class $[a]$ corresponds to -1.*
(iii) *$(M, *)$ is an ideal extension of a semigroup S by the semigroup $T = \{-1, 0, 1\}$ endowed with the usual multiplication.*

According to Proposition 4.1, to establish the existence of W-monoids, it suffices to construct an ideal extension $(M, *)$ of a semigroup S by the semigroup $T = \{a, 0, e\} \cong \{-1, 0, 1\}$, in which the element a is noncentral for $*$. To this end, the following theorem adapts a result due to Clifford [1] (see also Clifford and Preston [2, Theorem 4.19]). For simplicity, we denote the operations in both T and S by concatenation.

Theorem 4.2. *Any semigroup homomorphism $h \colon T^* = \{-1, 1\} \cong \mathbb{Z}_2 \to S$ defines an ideal extension $(M, *)$ of S by $T = \{-1, 0, 1\}$, where $M = S \cup T^*$ and the operation $*$ is defined by*

$$x * y \;=\; \begin{cases} x\,y, & \text{if } x, y \in T^*, \\ h(x)\,y, & \text{if } x \in T^* \text{ and } y \in S, \\ x\,h(y), & \text{if } x \in S \text{ and } y \in T^*, \\ x\,y, & \text{if } x, y \in S. \end{cases}$$

Moreover, if S has a neutral element, then every ideal extension of S by T arises in this manner.

As stated, Theorem 4.2 provides a straightforward method for constructing a wide range of W-monoids. The following example illustrates a particularly simple instance of such monoids.

Example 4.3. Let $(S,\circ)$ be a monoid with a neutral element id, and let $A\in S$ be an element satisfying $A\circ A=\mathrm{id}$. It is straightforward to verify that the map $h\colon T^*=\{-1,1\}\to S$ defined by $h(1)=\mathrm{id}$ and $h(-1)=A$ is a semigroup homomorphism. By Theorem 4.2, this induces an ideal extension $(M,*)$ of S by T, where $M=S\cup T^*$. Moreover, if A is noncentral in S, then it remains noncentral in M, and consequently, $(M,*)$ is a W-monoid. As a concrete example, one can consider the general linear group $S=\mathrm{GL}_2(\mathbb{R})$ of all invertible 2×2 matrices over $\mathbb{R}$ and the involutive matrix $A=\mathrm{diag}(1,-1)$. ◊

Remark 4.4. If the semigroup $(M,*)$ constructed by Theorem 4.2 is a monoid, then it is not difficult to see that $h(1)$ must be a neutral element for S. Conversely, according to the last statement of the theorem, all W-monoids $(M,*)$ such that $S=M\smallsetminus\{a,e\}$ has a neutral element are necessarily constructed along the lines of Example 4.3. ◊

We now observe that a result by Yoshida [8] (see also Petrich [5, Theorem III.2.2]) offers a description of all ideal extensions, based on the concept of the translational hull. To proceed, we first recall some fundamental concepts related to semigroups (see [5, p. 63]). Given a semigroup $(S,\circ)$,

- a *left translation* of S is a map $\lambda\colon S\to S$ satisfying
$$\lambda(x\circ y) ~=~ \lambda(x)\circ y \qquad (x,y\in S);$$
- a *right translation* of S is a map $\rho\colon S\to S$ satisfying
$$\rho(x\circ y) ~=~ x\circ\rho(y) \qquad (x,y\in S);$$
- a *bitranslation* of S is a pair (λ,ρ) such that
$$x\circ\lambda(y) ~=~ \rho(x)\circ y \qquad (x,y\in S),$$
where λ is a left translation and ρ is a right translation of S;
- the *translational hull* of S is the semigroup $\Omega(S)$ of all the bitranslations.

Associativity implies that the pair (λ,ρ) with $\lambda(y)=a\circ y$ and $\rho(x)=x\circ a$ is a bitranslation for any given $a\in S$.

Applying Yoshida's result, we obtain the following characterization of the class of W-monoids.

Theorem 4.5. *Let $(S,\circ)$ be a semigroup and consider a bitranslation (L,R) of S satisfying $L^2=R^2=\mathrm{id}$, $LR=RL$, and $L\neq R$. Then, the structure $(M,*)$, where $M=S\cup\{a,e\}$ for some $a,e\notin S$, with e neutral for $*$, and*
$$x*y ~=~ \begin{cases} x\circ y, & \text{if } x,y\in S,\\ L(y), & \text{if } x=a \text{ and } y\in S,\\ R(x), & \text{if } x\in S \text{ and } y=a,\\ e, & \text{if } x=a \text{ and } y=a.\end{cases}$$
is a W-monoid. Moreover, all W-monoids can be constructed in this manner.

Proof. By Proposition 4.1, a W-monoid is an ideal extension $(M,*)$ of a semigroup $(S,\circ)$ by $T=\{a,0,e\}$, with the additional properties that e is a neutral element for $*$ and a is noncentral for $*$. We first observe that $T^*=\{a,e\}$ forms a group, and hence the *ramification set* [5, p. 68] of T is empty. Consequently, such an extension is determined by a homomorphism $\theta\colon T^*\to\Omega(S)$, which maps T^* onto a set of permutable bitranslations. Since e is the neutral element for $*$, we must have $\theta(e)=(\mathrm{id},\mathrm{id})$. Moreover, setting $\theta(a)=(L,R)$, the map θ is an homomorphism if and only if $L^2=R^2=\mathrm{id}$. In addition, it has *permutable values* [5, p. 68] if and only if $LR=RL$. Note also that the condition $L\neq R$ is equivalent to a being noncentral for $*$. Finally, Yoshida's result (see [5, Theorem III.2.2]) guarantees that $(M,*)$ is a W-monoid and that all W-monoids can be constructed in this way. □

We can actually prove Theorem 4.5 without making use of Yoshida's theory or any semigroup theory beyond the basic definition of a bitranslation. We now present such an elementary proof.

Self-contained proof of Theorem 4.5. Assume first that $(M,*)$ is a W-monoid with neutral element e, and let $a\in M$ be the element provided by Definition 3.1. Conditions (W1) and (W2) guarantee that $(S,\circ)$, with $S:=M\smallsetminus\{a,e\}$, is a subsemigroup of M (here $\circ$ denotes the restriction of $*$ to S in order to be consistent with the notation of the theorem). Let $L(y)=a*y$ and $R(x)=x*a$; then (L,R) is a bitranslation of M. It is clear from (W1) and (W2) that S is closed under L and R, hence their restriction to S constitutes a bitranslation of $(S,\circ)$. Associativity immediately implies $LR=RL$, while associativity together with $a*a=e$ shows that $L^2=R^2=\mathrm{id}$. Condition (W3) ensures that $L\neq R$, and the formula for $*$ stated in the theorem follows from the definition of $\circ$, L, and R.

Conversely, if $(M,*)$ is constructed as described in the theorem, then a simple verification yields that $*$ is associative (using the associativity of $\circ$, the definition of a bitranslation, and the assumptions $L^2=R^2=\mathrm{id}$ and $LR=RL$). The definition of the operation $*$ shows that $x*y\in\{a,e\}$ can happen only if $x,y\in\{a,e\}$, and then (W1) and (W2) follow from $a*a=e$ and from the fact that e is the neutral element for $(M,*)$. Finally, $L\neq R$ implies condition (W3). □

In the following example, we construct a W-monoid using Theorem 4.5 that cannot be obtained via Theorem 4.2.

Example 4.6. Let $(X,\diamond)$ be a monoid with a neutral element id and let $i,j\in X\smallsetminus\{\mathrm{id}\}$ be elements satisfying $i\diamond i=j\diamond j=\mathrm{id}$. Consider also the semigroup $(S,\circ)$, where $S=X\times X$ and $\circ\colon S^2\to S$ is the binary operation defined by

$$(x,y)\circ(x',y') ~=~ (x\diamond x',y').$$

It is then easy to verify that the maps $L\colon S\to S$ and $R\colon S\to S$ defined by

$$L((x,y)) ~=~ (i\diamond x,y)\quad\text{and}\quad R((x,y)) ~=~ (x\diamond i,y\diamond j)$$

satisfy the properties stated in Theorem 4.5. For instance, for any $x,y\in X$, we have

$$R^2((x,y)) ~=~ R(x\diamond i,y\diamond j) ~=~ (x\diamond i\diamond i,y\diamond j\diamond j) ~=~ (x,y).$$

Using Theorem 4.5, we immediately obtain a W-monoid. If $|X|\geqslant 2$, then S does not have a neutral element, hence the corresponding W-monoid cannot be obtained via Theorem 4.2 (see Remark 4.4). As a concrete example, we can take $X=\{0,1\}\cong\mathbb{Z}_2$ and $i=j=1$, and the binary operation $\diamond$ is the addition modulo 2. ◊

References

[1] A. H. Clifford. Extensions of semigroups. *Trans. Amer. Math. Soc.*, 68:165–173, 1950.

[2] A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups.* Vol. I. American Mathematical Society, Providence, RI, 1961.

[3] W. Dörnte. Untersuchungen über einen verallgemeinerten Gruppenbegriff. *Math. Z.*, 29:1–19, 1928.

[4] W. A. Dudek and V. V. Mukhin. On n-ary semigroups with adjoint neutral element. *Quasigroups and Related Systems*, 14(2):163–168, 2006.

[5] M. Petrich. *Introduction to Semigroups.* Merrill Research and Lecture Series. Charles E. Merrill Publishing Co., Colombus, OH, 1973.

[6] E. L. Post. Polyadic groups. *Trans. Amer. Math. Soc.*, 48:208–350, 1940.

[7] D. Rees. On semi-groups. *Proc. Cambridge Philos. Soc.*, 36:387–400, 1940.

[8] R. Yoshida. Ideal extensions of semigroups and compound semigroups. *Mem. Res. Inst. Sci. Eng. Ritumeikan Univ.*, 13:1–8, 1965.

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