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Filling Riemann surfaces by hyperbolic Schottky manifolds of negative volume

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ABSTRACT

We provide conditions under which a Riemann surface X is the asymptotic boundary of a convex co-compact hyperbolic manifold, homeomorphic to a handlebody, of negative renormalized volume. We prove that this is the case when there are on X enough closed curves of short enough hyperbolic length.

1. Introduction and results

1.1. Hyperbolic manifolds of smallest volume

The volume of a closed hyperbolic 3-manifold can be considered as a measure of its "complexity", and it is natural to ask what is the closed, orientable hyperbolic manifold of smallest volume. The answer is the Weeks manifold [14].

Consider now a compact Riemann surface X. We can extend the previous question in the following manner – the case of closed hyperbolic manifolds corresponds to $X = \emptyset$.

Question 1.1. Given X, what is the convex co-compact hyperbolic manifold M of smallest volume, with asymptotic boundary X?

Convex co-compact hyperbolic manifolds have infinite volume, but they have a well-defined *renormalized volume* (see Section 2.2.4 below) which we consider here. The notion of renormalized volume was introduced first in the physics literature by Skenderis and Solodukhin [9], and then quickly introduced in the mathematics study of conformally compact Einstein manifolds [16]. For 3-dimensional hyperbolic manifolds, it is closely connected [21] to the Liouville functional studied e.g. in [32,33]. More recently, an explicit upper bound on the renormalized volume of quasifuchsian manifolds in terms of the Weil-Petersson distance between the conformal metrics at infinity, as well as a bound on the difference between the renormalized volume and the volume of the convex core [29], led to bounds on the hyperbolic volume of mapping tori [6,20]. Moreover, the study of the gradient flow of the renormalized volume has brought a number of new results, see e.g. [1–3].

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Beyond those mathematical motivations, Question 1.1 also occurs naturally from a physical perspective, and specifically from the AdS/CFT correspondence. Very briefly, the AdS/CFT correspondence asserts the equality between the partition function of a conformal field theory (CFT) on a d-dimensional manifold X and a sum, over all $d+1$-dimensional manifolds M_i with boundary X, of a function of the action of a certain (super-)string theory on M_i. In a certain "gravity" limit, where many features disappear, it reduces to a very special and simplified statement: given a Riemann surface X, the partition function of a certain CFT on X should be recovered as a sum of exponential of minus a constant times the renormalized volumes of all convex co-compact hyperbolic manifolds M_i having X as asymptotic boundary:

$$\mathcal{A}(X) = a_0 \sum_{\partial M_i = X} e^{-cV_R(M_i)} ,$$

where a_0 and c are constants. In this simplified view, the main term on the $d+1$-dimensional "bulk" side corresponds to the convex co-compact manifold M_i with the smallest renormalized volume.

This AdS/CFT correspondence leads to some conjectural statements. For instance, if X is disconnected, the CFT should behave independently on the two connected component, and it might therefore be expected that the convex co-compact manifold of smallest volume "filling" X should also be disconnected (see [31] for a more elaborate analysis).

For instance, if $X = X_+ \cup X_-$ is the disjoint union of two connected Riemann surfaces of genus at least 2, with X_- equal to X_+ with opposite orientation, we can compare:

- the Fuchsian manifold M_F with ideal boundary $X_+ \cup X_-$, which has (with the normalization used here) renormalized volume zero,
- any possible filling of $X_+ \cup X_-$ by the disjoint union of two handlebodies M_+ and M_-, with $\partial_\infty M_+ = X_+$ and $\partial_\infty M_- = X_-$.

The heuristics above suggests that one of the disconnected fillings might have negative renormalized volume. This might be a motivation for the following conjecture, attributed to Maldacena (see [35]).

Conjecture 1.2. *Any connected Riemann surface of genus at least 2 is the asymptotic boundary of a Schottky manifold of negative renormalized volume.*

By "Schottky manifold" here we mean a convex co-compact hyperbolic manifold homeomorphic to a handlebody.

1.2. Results

In what follows S will always denote a closed orientable surface of genus at least 2.

1.2.1. Existence of fillings of minimal renormalized volume

Before we consider the questions above, it is useful to know that, given a Riemann surface X of finite type, there is at least one convex co-compact filling of X of minimum renormalized volume, and that the set of those minimum volume fillings is finite. Precisely, let $\mathcal{M}(X)$ be the set of convex co-compact hyperbolic manifolds with ideal conformal boundary X, then we think the following question should have a positive answer.

Question 1.3. Let $V := \inf_{M\in\mathcal{M}(X)} V_R(M, X)$. There exists $M_V \in \mathcal{M}(X)$ such that $V_R(M_V) = V$.

This extends to the case of closed hyperbolic manifold, when X is empty, in which the Weeks manifold is the unique smallest volume closed hyperbolic 3-manifold [14].

1.2.2. An upper bound on the renormalized volume

The main result here is an upper bound on the renormalized volume of a Schottky manifold, when it is obtained from a pants decomposition for which some of the curves are short. We denote by ε_0 the 2-dimensional Margulis constant, equal to $\varepsilon_0 = 2\text{arsinh}(1)$. Given a pants decomposition P of S, we denote by M_P the handlebody with boundary S in which all curves of P are null-homotopic in M_P (see §2.2.2), and by $M_P(X)$ the convex co-compact hyperbolic manifold homeomorphic to M_P with complex structure at infinity X. The complex structure has a unique hyperbolic metric in its conformal class and we will take lengths with respect to that. Thus, by $\ell_X(\gamma)$ we mean the length of γ with respect to the hyperbolic structure induced by X. In the case in which there is no ambiguity we will often use $\ell(\gamma)$.

Theorem 1.4. *Let X be a closed Riemann surface of genus $g \geq 2$. Assume that there are k disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ such that $\ell(\gamma_i) \leq 1, 1 \leq i \leq k$, and there are no other geodesic loops of length less or equal than 1 in X. Then there exists a pants decomposition P containing the γ_i's such that*

$$V_R(M_P(X)) \le -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \frac{1}{\ell(\gamma_i)} + \left(9 + \frac{3}{4}\coth^2\left(\frac{1}{4}\right)\right) k + 81\coth^2\left(\frac{1}{4}\right)\pi(3g-3-k)(g-1)^2 .$$

By imposing the right hand side of the estimate in Theorem 1.4 to be negative we obtain for instance the following corollary.

Corollary 1.5. *For all $g \in \mathbb{N}$ s.t. $g \ge 2$, $0 < k \le 3g-3$ and $0 < k_1 \le k$ there exists an explicit constant $A = A(g, k_1, k-k_1) > 0$ such that if X is a Riemann surface with k_1 geodesic loops of length less than A and k geodesic loops of length at most 1, then X admits a Schottky filling with negative renormalized volume.*

Remark 1.6. Let us see a couple of examples for Corollary 1.5 in the two limit cases.

- **Case $k = k_1 = 1$.** By the inequality of Theorem 1.4, we have

$$A(g,1) < \frac{\pi^3}{\sqrt{e}(9 + \frac{3}{4}\coth^2(1/4) + 81\coth^2(1/4)\pi(3g-4)(g-1)^2)} ,$$

  which is largest for genus $g = 2$, in which case the bound is

$$A(2,1) \le 0.00221.$$

  Note that, for large genus g, we obtain the asymptotic $A(g,1) \sim cg^{-3}$, for a $c > 0$ that can be read from the expression of $A(g,1)$.
- **Case $k = k_1 = 3g-3$.** By Theorem 1.4, since $k - k_1 = 0$, we are looking for an $A := A(g, 3g-3)$ such that

$$A < \frac{\pi^3}{\sqrt{e}(9 + \frac{3}{4}\coth^2(1/4))} ,$$

  we can then take

$$A(g, 3g-3) = 0.87458 .$$

This statement can be compared to [36, Corollary 5.6], also see [35, Theorem 2.1], which states that: if a Riemann surface X of finite type and genus $g \ge 2$ has $g-1$ closed curves $\gamma_1, \cdots, \gamma_{g-1}$ such that the complement of their union is a disjoint union of k-holed tori, and if

$$\frac{1}{\pi-2}\left(\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)}\right)^2 \le \pi(g-1) ,$$

then

$$V_R(X, \mathcal{P}) \le \pi(g-1)\left(3 - \frac{\pi(\pi-2)(g-1)}{\left(\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)}\right)^2}\right)$$

which is negative if

$$\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)} \le \left(\frac{\pi(\pi-2)(g-1)}{3}\right)^{\frac{1}{2}} ,$$

and, in the case $g = 2$, leads to a better $A(2,1) = \frac{\pi(\pi-2)}{3}$.

1.2.3. Outline of the proof

The proof of Theorem 1.4 follows several steps. First, we introduce in Section 3 a notion of "symmetric" Riemann surfaces – those which admit an orientation-reversing involution with quotient a surface with boundary. We prove that given any Riemann surface X of finite-type and any pants decomposition P of X, there is a symmetric surface X_s (for which the involution leaves P invariant component-wise) obtained from X by earthquakes along the curves of P (see Lemma 3.7).

Then, in Section 4, we estimate the renormalized volume of "symmetric" Schottky fillings of symmetric surfaces. In Section 5, we provide a formula for the difference of the renormalized volume of a filling under an earthquake path of the boundary surface (see Theorem 5.4). The result expresses the estimates in terms of the Schwarzian derivative at infinity (see Section 2.2.3) at the core of tubes associated to the pants curves. Finally, Section 6 contains the proofs of the main results.

1.2.4. Convex co-compact fillings

The result in bounding the difference of renormalized volume under earthquake, see Theorem 5.4, can also be applied in the more general setting of convex co-compact manifolds. Specifically it makes sense in the setting where $N(X_0) \in CC(M)$ is a convex co-compact hyperbolic 3-manifold, homeomorphic to M, with conformal boundary $X_0 \in \mathcal{T}(\partial M)$. In this more general setting the boundary of M can be disconnected and can be decomposed as $\partial M = F_c \cup F_i$ where F_i does not compress in M and each component of F_c compresses (i.e. it has at least a loop bounding a disk in M).

Let $c_{\mathbf{t}}^{\mathfrak{m}} : [0,1] \to CC(M)$ be an earthquake path (we earthquake by a parameter t_i, with $\mathbf{t} = (t_1, \dots, t_n)$, along the curve γ_i) along a multi-curve $\mathfrak{m} = \{\gamma_i\}_{i=1}^n \subseteq S$ which, with respect to the reference hyperbolic metric X_0, can be subdivided into:

- $\mathfrak{m}_1^c$: the set of compressible geodesic loops γ of $\mathfrak{m}$ with length at most 1;
- $\mathfrak{m}_1$: the set of geodesic loops γ in F_c and not in $\mathfrak{m}_1^c$ such that any compressible geodesic loop α intersecting γ essentially has length at least 1;
- $\mathfrak{m}_\infty$: the set of geodesic loops γ of $\mathfrak{m}$ that are contained in F_i and so incompressible.

Note that not every $\mathfrak{m}$ admits such a decomposition with respect to the given X_0, as there could be a $\gamma_i \in \mathfrak{m}$ in a compressible component, of length more than 1 and intersecting a short compressible loop.

Theorem 1.7. *Let $X_0 \in \mathcal{T}(\partial M)$ and $\mathfrak{m} = \mathfrak{m}_\infty \cup \mathfrak{m}_1^c \cup \mathfrak{m}_1$ be a multi-curve and $c_{\mathbf{t}}^{\mathfrak{m}}$ be an earthquake path terminating at X_1. Then*

$$|V_R(X_1) - V_R(X_0)| \le \sum_{\gamma_i \in \pi_0(\mathfrak{m}_1^c)} (3\ell_i \coth^2(\ell_i/4)) t_i + C \sum_{\alpha_j \in \mathfrak{m}_1} t_j \ell_j + 3 \sum_{\beta_k \in \mathfrak{m}_\infty} t_k \ell_k \,,$$

for $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

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2. Notation and background

In this section we recall the main objects and tools that we will use in this work.

2.1. Hyperbolic surfaces

2.1.1. Teichmüller space

Good references for Teichmüller space are [18, Chapter 6-7] and [13], we now recall what we will need. Any closed, oriented, surface of genus ≥ 2 is hyperbolic, i.e. is homeomorphic to a quotient $\mathbb{H}^2/\Gamma$ of the hyperbolic space by a discrete, torsion-free subgroup of the orientation-preserving isometries of $\mathbb{H}^2$. The Teichmüller space of S can be defined in the following various ways, depending on the set-up, we will use the most suitable definition:

$$\mathcal{T}(S) = \{\text{h hyperbolic metric on } S\}/\mathrm{Diffeo}_0(S)\,,$$

$$\mathcal{T}(S) = \{c \text{ complex structure on } S\}/\mathrm{Diffeo}_0(S)\,,$$

$$\mathcal{T}(S) = \{[g] \text{ s.t. g is a Riemannian metric on S}\}/\mathrm{Diffeo}_0(S)\,.$$

Here $\mathrm{Diffeo}_0(S)$ is the group of diffeomorphisms of S isotopic to the identity, and it acts by pull-back, moreover $g_1 \in [g_2]$ if and only if there exists a smooth function $u_1 : S \to \mathbb{R}$ such that $g_2 = e^{u_1} g_1$, i.e. $[g]$ represents the class of Riemannian metrics conformal to g. In particular, to any complex structure on S corresponds a conformal class of metrics $[g]$, in which, by the Riemann uniformization Theorem [18, Theorem 1.1.1], there exists a unique hyperbolic metric $h \in [g]$.

2.1.2. Margulis tubes

Hyperbolic surfaces, and in general hyperbolic n-manifolds, have the important property that "short geodesics" have particularly nice neighbourhoods. We will mostly deal with the surface case and so we restrict ourselves to that setting.

Definition 2.1. By a *thin tube*, for a hyperbolic surface, we mean the set of points $\mathbb{T}(\ell)$ around a geodesic γ of length $\ell \leq \varepsilon_0$, with $\varepsilon_0 = 2\mathrm{arsinh}(1)$ the 2-dimensional Margulis constant, that are at a distance at most $L := \mathrm{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right)$. In $\mathbb{T}(\ell)$ the injectivity radius is bounded as

$$\frac{\ell}{2} \le \mathrm{inj}(p) = \operatorname{arsinh}\left(\sinh(\ell/2)\cosh(L-d)\right), \qquad d = d(p, \partial\mathbb{T}),$$

and its maximum is achieved on $\partial\mathbb{T}(\ell)$, see [7, Thm 4.1.6]. The hyperbolic metric on $\mathbb{T}(\ell)$ can be written as $d\rho^2 + \left(\frac{\ell}{2\pi}\right)^2\cosh^2(\rho)d\theta^2$, $\theta \in [0, 2\pi]$ and $\rho \in [-L, L]$. Moreover, any multi-curve P such that each component is simple and has length at most ε_0 can be completed to a pants decomposition of S. For details see [7, Thm 4.1.1].

2.1.3. Earthquakes along simple closed geodesics

We recall here some basic facts on earthquakes along closed geodesics, which will be needed. For more background see [13, Sec 10.7.3] and [8, Part III].

Given a simple closed geodesic γ on a hyperbolic surface (S, h) a (left) t-earthquake is a map $\varphi_{\gamma,t}$ from S to itself, discontinuous along γ, defined by cutting S along γ, twisting the left-hand side of γ by a fixed length t in the positive direction, and gluing back isometrically the two sides.[1]

Taking the push-forward of the hyperbolic metric by $\varphi_{\gamma,t}$ defines a new hyperbolic metric on S, and in this manner γ and t define a homeomorphism of $\mathcal{T}(S)$, which is also called the left earthquake of length t along γ, and denoted by $E_\gamma(t)$.

By continuously varying the twisting length t, one gets a path of diffeomorphisms of $S \setminus \gamma$, and by pulling back h through such a path, we get a path in $\mathcal{T}(S)$.

Let us now define earthquakes more carefully. Having fixed a simple closed curve γ in S, we consider the unique geodesic on (S, h) in the same isotopy class again by γ. In this way, the operation only depends on the isotopy class of γ. Let ℓ be the length of γ with respect to h, and $N_r \cong S^1 \times [-r, r] \cong \mathbb{R}/\ell\mathbb{Z} \times [-r, r]$ be the tubular r-neighbourhood of γ parameterized in such a way that $S^1 \times \{0\}$ isometrically identifies with γ and $\{e^{i\theta}\} \times [-r, r] \in S^1 \times [-r, r]$ with a geodesic segment of length $2r$ orthogonal to γ parameterized in unit velocity by the coordinate in $[-r, r]$. We now choose an arbitrary function $f : [-r, r] \to \mathbb{R}$ such that f is smooth on $[-r, r] \setminus \{0\}$, increasing on $[-r, 0]$, constantly equal to 0 in a neighbourhood of $-r$, constantly equal to 1 in a left neighbourhood of 0, and equal to 0 on $(0, r]$. We then define $\varphi_{\gamma,t} : S \to S$, with $t \in [0, \infty)$, as the diffeomorphism of $S \setminus \gamma$ such that $\varphi_{\gamma,t}$ is the identity outside N_r, and

$$\varphi_{\gamma,t}(e^{i\theta}, r) = \left(e^{i(\theta + \frac{2\pi}{\ell} t f(r))}, r\right)$$

for any $(e^{i\theta}, r) \in N_r$. Note that $\varphi_{\gamma,0}$ is the identity, and that $\varphi_{\gamma,\ell}$ extends to a diffeomorphism of S, which is called a *Dehn twist*.

As $\varphi_{\gamma,t}$ acts by isometry on $N_\varepsilon(\gamma) \setminus \gamma$ and fixes the metric on γ, the push-forward $(\varphi_{\gamma,t})_*(h)$ is a new well defined hyperbolic Riemannian metric on S. We say that $(\varphi_{\gamma,t})_*(h)$ is obtained by a *(left) earthquake of parameter t* along γ.

We define $\varphi_\gamma : [0, a] \to \mathcal{T}(S)$, $a > 0$, to be a *earthquake path along* γ by $\varphi_\gamma(t) = (\varphi_{\gamma,t})_*(h)$. The *infinitesimal earthquake along* γ is the derivative of φ_γ in t at $t = 0$, this can also be seen as a vector field v on S by differentiating the path of diffeomorphisms $(\varphi_{\gamma,t})_{t\in[0,\varepsilon]}$ with respect to t and evaluating it at $t = 0$. For more background see [13, Sec 10.7.3] and [8, Part III].

2.2. Hyperbolic 3-manifolds

Some references on hyperbolic 3-manifolds are [17,25,26], we now recall what we will need in this work. A 3-manifold M is hyperbolic if it is homeomorphic to $\mathbb{H}^3/\Gamma$ for Γ a discrete, torsion free subgroup of $\mathbb{P}SL(2, \mathbb{C})$, the positive isometry group of $\mathbb{H}^3$.

The action of Γ on $\mathbb{H}^3$ can be naturally extended to $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$, with $\hat{\mathbb{C}}$ the Riemann sphere, but it does not remain properly discontinuous, that is, the closure of the orbit of a point $x \in \mathbb{H}^3$ has non-empty set of accumulation points $\Lambda_x(\Gamma)$ in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. One can show that actually $\Lambda_x(\Gamma)$ does not depend on x. We then denote it simply by $\Lambda(\Gamma)$ and we call the complement $\Omega(\Gamma) = \partial\mathbb{H}^3 \setminus \Lambda(\Gamma) = \hat{\mathbb{C}} \setminus \Lambda(\Gamma)$ the *domain of discontinuity of* Γ. We observe that $\Lambda(\Gamma)$ is closed, and that both $\Lambda(\Gamma)$ and $\Omega(\Gamma)$ are Γ-invariant. The action of Γ on $\Omega(\Gamma)$ is properly discontinuous, we can then define the *boundary at infinity of* $M = \mathbb{H}^3/\Gamma$ as the surface

$$\partial_\infty M = \Omega(\Gamma)/\Gamma.$$

Since $\Omega(\Gamma)$ is an open subset of $\hat{\mathbb{C}}$, and the elements of $\mathbb{P}SL(2, \mathbb{C})$ are in particular bi-holomorphism of $\hat{\mathbb{C}}$, the boundary at infinity $\partial_\infty M$ of M is naturally equipped with a *complex projective structure*, and thus also a complex structure. A complex projective structure on a surface S is an atlas of charts to $\hat{\mathbb{C}}$ whose transition maps are restriction of Möbius transformations. Equivalently, a complex projective structure is the datum of a holonomy representation of $\pi_1(S)$ in $\mathbb{P}SL(2, \mathbb{C})$, and an equivariant *developing map*, that is, an immersion of the universal cover $\widetilde{S}$ equipped with the lifted complex projective structure in $\hat{\mathbb{C}}$, which locally restricts to projective charts. The developing map is unique up to composition with Möbius transformations. The deformation space of complex projective structures forms a holomorphic vector boundle π on

[1] Note that this definition requires the choice of an orientation of γ, but the result does not depend on which orientation is chosen.

the Teichmüller space of S, of dimension $12g-12$ (see [10]). Given $X \in \mathcal{T}(S)$, the fiber $\pi^{-1}(X)$ is parameterized by the Schwarzian derivative of the developing map of each point in the fiber (see Section 2.2.3).

2.2.1. The convex core

We define the *convex core* of $M = \mathbb{H}^3/\Gamma$ as

$$C(M) = \mathrm{Hull}(\Lambda(\Gamma))/\Gamma\ ,$$

where $\mathrm{Hull}(\Lambda(\Gamma))$ is the convex envelope of the points of $\Lambda(\Gamma)$ in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. The convex core of M is also characterized as the smallest non-empty *strongly geodesically convex*[2] subset of M, that is, the smallest convex subset of M which is also homotopically equivalent to M. It is also not difficult to prove that if M has finite volume, then the limit set $\Lambda(\Gamma)$ coincides with $\hat{\mathbb{C}}$, and so $C(M) = M$. Here, we will be interested in the case of M having infinite volume. The convex core $C(M)$ is generically a 3-dimensional domain, but in some cases, it can be a totally geodesic surface in M, possibly with geodesic boundary.

Definition 2.2. A hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is *convex co-compact* if its convex core $C(M)$ is compact.

When M is convex co-compact then $\overline{M} = M \cup (\partial_\infty M)$ is its manifold compactification and $\partial_\infty M$ is homeomorphic to the closed surface $S = \partial\overline{M}$, and so

$$[\partial_\infty M] \in \mathcal{T}(S)\ .$$

We call *end* a connected component of $M \setminus C(M)$, or, more generally, of the complement of a strongly geodesically convex compact subset of M. An end is homeomorphic to $S^i \times [0, +\infty)$, with S^i a connected component of the boundary $S = \partial_\infty M$, and it has infinite hyperbolic volume.

We denote by $CC(M)$ the space of convex co-compact hyperbolic structures on M considered up to homotopy equivalence. The deformation space $CC(M)$ is parameterized by the one of conformal structures on the boundary at infinity, see [24, Thm 5.1.3.] and [26, Thm 5.27]:

$$CC(M) = \mathcal{T}(\partial\overline{M})\Big/T_0(D)\,,$$

where $T_0(D) \subseteq MCG(\partial\overline{M})$ is the subgroup generated by Dehn twists along compressible curves[3] of $\partial\overline{M}$ and $\mathcal{T}(\partial\overline{M})$ is the product of the Teichmüller spaces of the connected components of $\partial\overline{M}$.

2.2.2. Handlebodies

We will think of an handlebody H_g of genus $g \geq 1$ as the following data. Given a surface $S = S_g$ and a pants decomposition P on S we can form the 3-manifold H_0 by attaching $3g-3$ thickened disks $\mathbb{D}^2 \times I$ to $S \times I$ by gluing each $\partial\mathbb{D}^2 \times I$ to $N_\varepsilon(\gamma) \times \{0\}$ for $\gamma \in P$. The manifold H_0 has then a genus g boundary component and $2g-2$ sphere boundary components. After filling each sphere component with a 3-ball we obtain a handlebody $H_P \cong H_g$, this is unique up to isotopy. We will think of this as the handlebody induced by P. Note that H_P is well defined up to isotopy.

We now define what it means to *fill* a given conformal structure $X \in \mathcal{T}(S)$ via a (complete) hyperbolic 3-manifold M so that M is homeomorphic to a handlebody and its conformal boundary is X.

Definition 2.3. Given a conformal structure $X \in \mathcal{T}(S_g)$ and a pants decomposition P on S_g we say that $M_P(X)$ is the *Schottky filling* of X with pants curve P if it is the (complete) hyperbolic 3-manifold obtained by uniformising H_P so that its conformal boundary is X. By $CC_P(S_g)$ we denote the deformation space of a hyperbolic genus g handlebody obtained by gluing disks along P.

Remark 2.4. More generally, a handlebody H_g, topologically, is any irreducible compact 3-manifold M with a unique boundary component such that the map induced by the inclusion $\partial M \hookrightarrow M$ on the fundamental groups is surjective, [17]. Thus, the manifold $M := F \times I$, for F a compact orientable surface with non-empty boundary, is also a handlebody with boundary given by the double of F along ∂F. In the case that F is not-orientable then we can consider the twisted I-bundle[4] $N = F \tilde{\times} I$ in which ∂N is given by the orientation double cover of F.

2 Here we say that a subset $K \subseteq M$ is "strongly geodesically convex" if any geodesic segment in M with endpoints in K is entirely contained in K.

3 An essential loop γ in $\partial\overline{M}$ is compressible if it is null-homotopic in M, i.e. it bounds a compressing disk in M.

4 Recall that a twisted I-bundle is a non-trivial I-bundle, i.e. $N \ncong F \times I$.

2.2.3. The Schwarzian derivative at infinity

Given a Riemann surface, a holomorphic quadratic differential is a holomorphic section of the symmetric square of its holomorphic cotangent bundle, and in holomorphic coordinate it can be expressed as $\varphi(z)dz \otimes dz = \varphi(z)dz^2$. Let $D \subseteq \mathbb{C}$ be a connected open set, and $f : D \to \hat{\mathbb{C}}$ a locally injective holomorphic map. The *Schwarzian derivative* of f is the holomorphic quadratic differential

$$\mathcal{S}(f) = \left(\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right) dz^2 .$$

The Schwarzian derivative has the following properties:

(1) Let f and g be locally injective holomorphic maps such that the composition is well defined, then

$$\mathcal{S}(f \circ g) = g^* \mathcal{S}(f) + \mathcal{S}(g) .$$

(2) For any holomorphic map $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is an open subset, $\mathcal{S}(f) = 0$ if and only if $f \in \mathbb{P}SL(2, \mathbb{C})$, that is, if and only if f is the restriction to U of a Möbius transformation.

We will be interested in considering the Schwarzian derivative of the developing map $f : \mathbb{H}^2 \to \Omega(\Gamma)$ of a non simply connected domain of discontinuity $\Omega(\Gamma)$, whose quotient by Γ gives the boundary at infinity $\partial_\infty M = \mathbb{H}^2/\Gamma'$ of $M = \mathbb{H}^3/\Gamma$. The map f is a covering of $\Omega(\Gamma)$, hence locally univalent, and it is (Γ', Γ)-equivariant, thus, thanks also to property (2), the Schwarzian $\mathcal{S}(f)$ of f descends to a holomorphic quadratic differential on $\partial_\infty M$.

2.2.4. The renormalized volume

If one is willing to talk about volumes for convex co-compact hyperbolic structures on M, being this infinite, some kind of renormalization will be needed. A possibility is to consider the function

$$V_C : CC(M) \longrightarrow \mathbb{R}_{\geq 0} ,$$

which associates to any convex co-compact structure M the volume of its convex core $\mathrm{Vol}(C(M))$. The renormalized volume is some kind of relative of the function V_C, which presents much better analytic properties.

The idea is to consider an exhaustion of M by strongly geodesically convex compact subsets $\{C_r\}_r$ coming together with an equidistant foliation of the ends, and to renormalize the associated volumes $\mathrm{Vol}(C_r)$ in order to get a finite number which does not depend on r.

Before giving the definition of renormalized volume, we need to introduce some preliminary notions.

Definition 2.5. Let M be convex co-compact and let $C \subseteq M$ be a compact, convex subset with smooth boundary. We define the *W-Volume of C* as

$$W(C) = \mathrm{Vol}(C) - \frac{1}{2} \int_{\partial C} H dA_{\partial C} ,$$

where $\mathrm{Vol}(C)$ is the hyperbolic volume of C with respect to the metric of M, H is the mean curvature of ∂C, and $dA_{\partial C}$ is the area form of the induced metric on the boundary ∂C.

The *mean curvature* is half the trace of the *shape operator* $B(X) = \nabla_X(N)$ with N the outer unit normal to ∂C and ∇ the Levi-Civita connection, and any vector field $X \in T(\partial C)$.

In what follows, we assume the compact subset C to be strongly geodesically convex, so that it is homotopically equivalent to M and can be used to decompose the manifold M in neighbourhoods of its convex co-compact ends and a compact piece containing the convex core $C(M)$. The additional term with the mean curvature in the definition just above is the right one to get a good renormalization. In [22], it is indeed proven that, denoting by C_r the r-neighbourhood of C in M, for any $r \geq 0$

$$W(C_r) + r\pi\, \chi(\partial_\infty M) = W(C) \tag{1}$$

where $\chi(\cdot)$ is the Euler characteristic.

Definition 2.6. Let E be an end of M. An *equidistant foliation* is a foliation $\{S_r\}_{r \geq r_0}$ of a neighbourhood of $\partial_\infty M$ in $\overline{M}$ in convex surfaces, such that for any $r' > r > r_0 \geq 0$ the surface $S_{r'}$ lives between S_r and $\partial_\infty M$, and its points stay at constant distance $r' - r$ from S_r.

By definition, the boundaries of the C_r's form an equidistant foliation $\{\partial C_r\}_r$ of the ends in $M \setminus C$.

Given any $C \subseteq M$ as above, and any end $E_i = S^i \times [0, +\infty)$ in $M \setminus C$, we can consider the associated equidistant foliation $\{\partial^i C_r\}_r$, for $r \geq 0$, where with ∂^i we mean the connected component of ∂C_r facing S^i in the boundary at infinity $S = \partial_\infty M$. Let us call g_r the induced metric on ∂C_r. Then we can define a metric on the boundary at infinity as

$$g := \lim_{r\to+\infty} 4e^{-2r} g_r \in [\partial_\infty M] , \tag{2}$$

see [29, Def. 3.2], or [30, Def. 3.2] for a slightly different point of view. A key property of this metric g is that it is in the conformal class at infinity of M, that is, it is compatible with the complex structure at infinity of M (note that this remains true if we change the factor 4 in (2)). Vice-versa, up to scaling by a big enough positive constant, any representative in $[\partial_\infty M]$ can be realized in this way, [11]. This leads to the following bijective correspondence:

$$\left\{ \begin{array}{c} \text{Riemannian metrics g on } S \\ \text{such that g} \in [\partial_\infty M] \\ \text{up to multiplication by } s \in \mathbb{R}^+ \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Equidistant convex foliations} \\ \text{of a neighbourhood of } \partial_\infty M \\ \text{up to } \sim_{\mathcal{F}} \end{array} \right\}$$

where two such foliations are $\sim_{\mathcal{F}}$-equivalent if and only if they are equal outside a compact set; rescaling by a positive constant a Riemannian metric in $[\partial_\infty M]$ corresponds to a reindexing of the associated foliation.

Definition 2.7. We define the W-Volume of M with respect to $g \in [\partial_\infty M]$ as

$$W(M, g) = W(C_r(g)) + \pi r \chi(\partial_\infty M) ,$$

where $\{C_r(g)\}_{r\geq r_0}$, with r_0 big enough, is the exhaustion in compact strongly geodesically convex subsets defined by the equidistant foliation associated to g, indexed in such a way that the sequence of induced metrics g_r on $\partial C_r(g)$ satisfies (2).

Thanks to equation (1) above, the W-volume $W(M, g)$ is well defined. We also remark that choosing the factor 4 for the corresponding metric at infinity as in (2), is the scaling which makes the geodesically convex subset C_0 associated to the hyperbolic metric in the conformal boundary at infinity of a Fuchsian manifold to be isometric to the induced metric on its 2-dimensional convex core.

We can finally define the renormalized volume of M.

Definition 2.8. Given a convex co-compact hyperbolic 3-manifold $M \in CC(N)$, its *renormalized volume* is defined as

$$V_R(M) = W(M, h) ,$$

with $h \in [\partial_\infty M]$ the hyperbolic representative.

Thanks to the parametrization of the space of convex co-compact structures $CC(M)$, we can think about the renormalized volume as a function from the Teichmüller space:

$$V_R : \mathcal{T}(\partial\overline{M}) \longrightarrow \mathbb{R} .$$

Remark 2.9. It is possible to define the W-volume also for the convex core $C(M)$ of M. In this case the boundary $\partial C(M)$ is not smooth, and the integral mean curvature of the boundary is replaced by the length of the *measured pleating lamination* (see [12,34]):

$$W(C(M)) = \mathrm{Vol}(C(M)) - \frac{1}{4} L(\beta_M) .$$

The renormalized volume satisfies the following differential formula, see [29, Corollary 3.11].

Theorem 2.10. *Let M be a convex co-compact hyperbolic 3-manifold, φ_M the holomorphic quadratic differential given by the Schwarzian derivative of the developing map of the projective structure of $\partial_\infty M$, and $\mu \in T_{[\partial_\infty M]}\mathcal{T}(\partial\overline{M})$. Then, the differential of the renormalized volume at $[\partial_\infty M]$ satisfies*

$$\mathrm{d}V_R(\mu) = Re(\langle \mu, \varphi_M \rangle).$$

Here the space of holomorphic quadratic differentials on $\partial_\infty M$ is identified with the cotangent bundle $T^*_{[\partial_\infty M]}\mathcal{T}(\partial\overline{M})$ through the Bers embedding, and the pairing $\langle\, ,\, \rangle$ is the duality one with $T_{[\partial_\infty M]}\mathcal{T}(\partial\overline{M})$, which is the space of harmonic Beltrami differentials [18], [15].

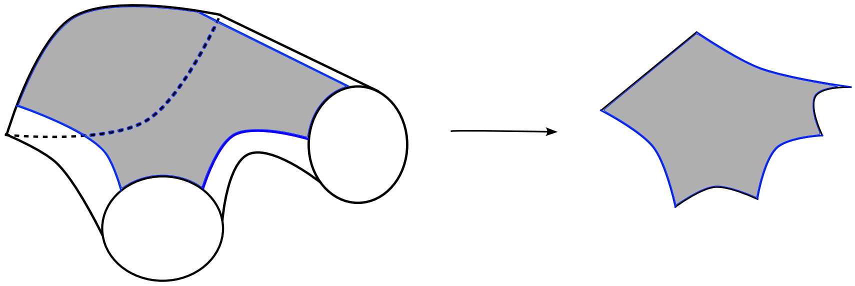

Fig. 1. The seams (in blue) in a pair of pants with the two hexagons H_1 (shaded), H_2 and the π_Q map. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

3. Earthquakes to symmetric surfaces

In this section we study conformal structures X on a surface S that admit an orientation-reversing involution $\sigma : X \to X$ such that, if X is equipped with its unique compatible hyperbolic metric, $X_\sigma := X/\sigma$ is a hyperbolic surface with totally geodesic boundary. The main result of this section is Lemma 3.7, which states that given $X \in \mathcal{T}(S)$ and $P \subseteq S$ a pants decomposition there exists a symmetric conformal structure $X' \in \mathcal{T}(S)$ and a path in $CC_P(S)$ from $M_P(X)$ to $M_P(X')$ which is obtained by doing earthquakes of bounded length along the curves of P.

Definition 3.1. Let $X \in \mathcal{T}(S)$, then X is a *symmetric surface* if S admits an orientation reversing involution $\sigma : S \to S$ that is a local isometry for (the hyperbolic metric on) X and such that $X_\sigma := X/\sigma$ is a surface with non-empty boundary. The subset of Teichmüller space of surfaces for which σ is a local isometry will be denoted by $\mathcal{T}_\sigma(S)$ and the subspace of surface admitting an involution σ by $\mathcal{T}_s(S) = \cup_\sigma \mathcal{T}_\sigma(S)$.

Remark 3.2. The surface X_σ does not have to be orientable.

Lemma 3.3. *Let X be a hyperbolic surface with an orientation reversing involution $\sigma : X \to X$ that is a local isometry. Then, $\partial X_\sigma = Fix(\sigma)$ is given by a multi-curve $\mathfrak{m} \subseteq X$ such that for each $\gamma \in \pi_0(\mathfrak{m})$ we have $\sigma|_\gamma = \mathrm{id}_\gamma$.*

Proof. By [19, Theorem 1.10.15] the set of fixed points is a closed totally geodesic sub-manifold, thus it is the union of a closed multi-curve $\mathfrak{m}$ and possibly a finite collection of points. By looking at the action on a small enough ball around an isolated fixed point (so that the centre is the unique fixed point) one can see that, being σ orientation reversing, isolated fixed points are not possible and so the fixed set has to be a geodesic multi-curve.

We now want to show that $\mathfrak{m}$ is the boundary of X_σ. Let $B \subseteq X$ be a small enough ball such that $\mathfrak{m} \cap B$ separates B in two balls and $B = \sigma(B)$. Then, B/σ is homeomorphic to a half disk with boundary in $\mathfrak{m}$. By connectedness and continuity this shows that $\mathfrak{m} \subseteq \partial X_\sigma$. The reverse containment follows from the fact that $\sigma : X \setminus \mathfrak{m} \to X \setminus \mathfrak{m}$ is a 2 to 1 cover and so $(X \setminus \mathfrak{m})/\sigma$ is a surface without boundary. □

In each pair of pants, a seam is the orthogeodesic connecting two distinct boundary components, so each pair of pants has 3 such arcs, see Fig. 1. For every pair of pants Q we have on each boundary component γ_i two marked points x_i^1, x_i^2, endpoints of the seams of Q. We define a marked pants decomposition P^m to be P together with a choice of either x_i^1 or x_i^2 for each pair of pants Q and each boundary curve of P.

Let $X \in \mathcal{T}(S)$ be a hyperbolic surface, P be a pants decomposition of X, and $\mathcal{S}$ be the set of the induced seams with marked endpoints, i.e. a marked pants decomposition P^m. Then, we define the Fenchel-Nielsen coordinates for X as follows: $FN(X) = (\ell_i, t_i)_{i=1}^{3g-3}$ where the ℓ_i are the hyperbolic lengths of the pants curve in the hyperbolic structure on X and the t_i are the twist parameters with respect to the two marked points on the curve γ_i. The twist parameters are computed by fixing lifts in the universal cover of γ_i and then taking their signed euclidean distance. For details on Fenchel-Nielsen coordinates see [7, Sec 6.2] or [13, Sec 10.6].

Thus, if $t_i = 0$, the seams match up and the two marked points are identified. If $t_i = \ell_i/2$, the seams match up but the marked points are opposite to each other.

Moreover, the seams cut each pair of pants Q into two isometric right-angled hexagons H_1 and H_2. We can then define an orientation-reversing involution $\sigma_Q : Q \to Q$ which maps H_1 to H_2 and H_2 to H_1 isometrically, and is the identity on the seams, see Fig. 1. The quotient of Q by σ_Q is then a right-angled hexagon E_Q, on which Q projects by a map $\pi_Q : Q \to E_Q$ which is a local isometry outside of the seams.

Remark 3.4. The maps $\{\pi_Q\}_{Q\in P}$ glue together to a map $\pi : X \to X$ that is an orientation reversing local isometry (outside of the seams) if all seams match up. Moreover, if that is the case then X_π is a surface, not necessarily orientable, whose boundary is given, by Lemma 3.3, by the union of the seams.

Lemma 3.5. *Let $X \in \mathcal{T}(S)$, and let $P = \{c_1, \cdots, c_{3g-3}\}$ be a marked pants decomposition of X and let (ℓ_i, t_i) be the corresponding Fenchel-Nielsen coordinates. Then, the Riemann surface X_0 with Fenchel-Nielsen coordinates (ℓ_i, t_i'), $t_i' = 0, \ell_i/2$, admits an orientation-reversing isometry which leaves invariant each curve of P.*

Proof. We want to show that the surface X_0 defined by (ℓ_i, t_i') admits an orientation-reversing isometry mapping each geodesic loop in P to itself.

The surface X_0 is obtained by gluing $3g-3$ pairs of pants with boundary lengths given by the ℓ_i's and in the pattern given by P such that if two pairs of pants Q_1 and Q_2 (Q_1 could be equal to Q_2) are glued along a geodesic loop $c_i \in \pi_0(P)$ then the endpoint of the seam $y_1 \in Q_1 \cap c_i$ is glued to $y_2 \in Q_2 \cap c_i$ without any twist. The pairs of pants Q_i, $i = 1, 2$, are obtained by doubling regular hexagons E_i along the seams, and each P_i is equipped with an orientation-reversing isometry map $\pi_i : Q_i \to Q_i$ exchanging the two hexagons. The fixed point set of this map is exactly the seams of Q_i.

Since the seams on $c_i \subseteq Q_1 \cap Q_2$ have endpoints that are $\ell_i/2$ apart and by our glueing condition one of them matches up we know that they both do. Therefore, all the seams with endpoints on c_i match-up and we can glue the maps π_1 and π_2 to obtain an orientation-reversing isometry from $Q_1 \cup Q_2$ to $Q_1 \cup Q_2$. By doing this for all pants we obtain the required statement. □

Remark 3.6. Given X and $\sigma : X \to X$ then, for specific markings in the FN-coordinates the quotient surface is orientable and equal to a thickening of the glueing graph of the pants decomposition in which if a curve c_i has twist parameter equal to $\ell_i/2$ then the quotient edge is glued with an half-twist.

Lemma 3.7. *Given a pants decomposition P on S and $X \in \mathcal{T}(S)$, there exists $X', X_s \in \mathcal{T}(S)$ such that X_s is symmetric, $M_P(X') \overset{isom}{\cong} M_P(X)$, and X_s is obtained from X', in FN_P coordinates, by twisting at most $\ell(c_i)/4$ (in the positive or negative direction) over each curve in P.*

Proof. First note that in $CC_P(S)$ we can do full twists along curves of P and get isometric structures, see [24, Thm 5.1.3.]. Recall that we denote by $M_P(X) \in CC_P(S)$ the structure corresponding to $X \in \mathcal{T}(S)$ with compressible curves given by P.

We use P to define the Fenchel-Nielsen coordinates by choosing seams $y \in \{x_1^i, x_2^i\} \subseteq c_i$, see Lemma 3.5. Also note that a full twist along c_i has length ℓ_i. Let X be the given structure, then $FN_P(X) = (\ell_i(X), t_i(X))_{i=1}^{3g-3}$. By doing full twists along the c_i's we can find a hyperbolic structure X' with the same length parameters, while the twists parameters are between 0 and $\ell_i(X') = \ell_i(X)$, and $M_P(X) \overset{isom}{\cong} M_P(X')$.

By doing twists of length at most $\ell_i(X)/4$ we get a surface X_s with the same length parameters and all seams of pair of pants matching up. Then, the twist parameters are equal to either zero or $\ell_i(X)/2$. □

4. The renormalized volume of symmetric surfaces

In this section we estimate the renormalized volume of a Schottky filling of a surface $X \in \mathcal{T}_s$ corresponding to a "symmetric" pants decomposition. This will be used in the proof of Theorem 1.4. In the next two sections, we will bound the variation of the renormalized volume under a variation of the twist parameters in the Fenchel-Nielsen coordinates, and as a consequence we will be able to obtain an upper bound on the renormalized volume of Schottky fillings which are non-symmetric by comparing their renormalized volume to that of a symmetric surface obtained by changing the twist parameters.

In the following lemma we will deal with manifold whose convex core is 2-dimensional. Thus, it will be useful to use a slightly modified definition of convex core boundary which is more compatible with the corresponding conformal boundary. In what follows we denote by $\partial\widehat{C}(M)$ the "boundary" of $C(M)$ for M any convex co-compact hyperbolic manifold and we define:

- $\partial\widehat{C}(M)$ is the boundary of $C(M)$ in the usual sense if $C(M)$ has non-empty interior,
- if $C(M)$ is a totally geodesic orientable surface $\Sigma \subseteq M$, then $\partial\widehat{C}(M)$ is the union of two copies of Σ with opposite orientation, if $\partial\Sigma \neq \varnothing$ then the two copies of Σ are glued along their common totally geodesic boundary.
- if $C(M)$ is a totally geodesic non-orientable surface $\Sigma \subseteq M$, then $\partial\widehat{C}(M)$ is the orientation double-cover of Σ.

In all cases, $\partial\widehat{C}(M)$ is homeomorphic to $\partial_\infty M$. Specifically, the hyperbolic Gauss map, which sends a unit vector normal to a support plane of $C(M)$ to the endpoint at infinity of the geodesic ray it defines, is a homeomorphism from the unit normal bundle of $C(M)$ – which is itself homeomorphic to $\partial\widehat{C}(M)$ – to $\partial_\infty M$.

The "boundary" $\partial\widehat{C}(M)$ is equipped with an induced metric m, which is hyperbolic. However, it is *pleated* along a *measured lamination* β which is geodesic for m, with the transverse measure recording the amount of pleating along the leaves, see [12,34]. When $C(M)$ is a totally geodesic surface Σ, the support of β corresponds to the boundary of Σ, with each leave equipped with a weight π.

Let X be the conformal structure at infinity of M. Then X is obtained from m and β by a geometric construction called *grafting*, see e.g. [10]. Given a closed surface S of genus at least 2, this grafting operation defines a map

$$\mathrm{gr} : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S) ,$$

where $\mathcal{ML}(S)$ denotes the space of measured laminations on S. The key property that is important to us here is a result of Scannell and Wolf [28]: if $\lambda \in \mathcal{ML}(S)$ is fixed, the map $\mathrm{gr}(\cdot, \lambda) : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism.

Now, if M is a convex co-compact hyperbolic manifold with convex core $C(M)$ a totally geodesic surface Σ with boundary, then the lamination β is the fixed-point set of an orientation-reversing involution $\sigma : \partial\widehat{C}(M) \to \partial\widehat{C}(M)$ such that $\Sigma = \partial\widehat{C}(M)/\sigma$, and the induced metric m on $\partial\widehat{C}(M)$ is invariant under σ. Since m and β are both invariant under σ, so is the conformal structure at infinity $X = \mathrm{gr}(m, \beta)$.

Lemma 4.1. *Let $\sigma : S \to S$ be an orientation-reversing involution with $\mathrm{Fix}(\sigma) \neq \emptyset$ and quotient surface $\Sigma = S/\sigma$. Then, for any invariant conformal structure $X \in \mathcal{T}_\sigma(S)$ there exists a handlebody H with a convex co-compact hyperbolic structure such that the convex core of H is homeomorphic to Σ and the conformal boundary of H is X.*

Proof. Let $\beta = \partial\Sigma$. We claim that the restriction map

$$\mathrm{gr}(\cdot, \beta)_{|\mathcal{T}_\sigma(S)} : \mathcal{T}_\sigma(S) \to \mathcal{T}_\sigma(S)$$

is onto. Indeed, let $X \in \mathcal{T}_\sigma(S)$. Since $\mathrm{gr}(\cdot, \beta) : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism, there exists a unique $Y \in \mathcal{T}(S)$ such that $\mathrm{gr}(Y, \beta) = X$. But then

$$\mathrm{gr}(\sigma^*Y, \beta) = \mathrm{gr}(\sigma^*Y, \sigma^*\beta) = \sigma^*\mathrm{gr}(Y, \beta) = \sigma^*X = X = \mathrm{gr}(Y, \beta) ,$$

and since Y is unique, $\sigma^*Y = Y$, so that $Y \in \mathcal{T}_\sigma(S)$.

Let $Y_\sigma = Y/\sigma$, homeomorphic to Σ, be the quotient surface of the hyperbolic surface Y by the locally isometric involution σ. Then, Y_σ has a uniformization $\Gamma < \mathrm{Isom}^\pm(\mathbb{H}^2)$. By considering Γ inside $\mathrm{Isom}^\pm(\mathbb{H}^3)$, by the natural inclusion, and the corresponding quotient $\mathbb{H}^3/\Gamma$, we obtain a hyperbolic 3-manifold whose convex core is Y_σ. By the above discussion, we also know that the conformal boundary is X. The fact that $H = \mathbb{H}^3/\Gamma$ is a handlebody follows from the fact that H is homeomorphic to either $Y_\sigma \times I$, if Y_σ has non-empty boundary or is orientable, or to the twisted bundle $Y_\sigma \tilde{\times} I$, if Y_σ is non-orientable with empty boundary. As $\Sigma \cong Y_\sigma$ has boundary by Remark 2.4 this yields a handlebody. □

Remark 4.2. The following remark is not needed in the rest of this paper, however, one should note that Lemma 4.1 also works for fixed point free involution if one allows for the topological condition to be that of a twisted I-bundle $K \tilde{\times} I$ over a closed non-orientable surface K.

Remark 4.3. In the case we have a pants decomposition P such that, for each $\gamma \in P$, $\sigma(\gamma) = \gamma$, we can also infer from Lemma 4.1 and Lemma 3.3 that $H \in CC_P(S)$, i.e. P compresses in H and the seams of P form ∂X_σ.

For a convex co-compact hyperbolic 3-manifold M,

$$V_R(M) \leq V_C(M) - \frac{1}{4}L(\beta_M) , \tag{3}$$

see [29, Lemma 4.1] (and also [1, Theorem 3.7]). In the case considered here, the bending lamination is given by a multicurve with bending measure given by assigning the weight π to each curve, see Lemma 4.1. Then, its length is given by:

$$L(\beta_M) = \pi \sum_{\gamma \in \pi_0} \ell_Y(\gamma) , \tag{4}$$

for Y the hyperbolic structure on the convex-core boundary and π_0 the set of the simple closed curves composing the multicurve. Thus, one has $L(\beta_M) > 0$ and so, by Lemmas 3.3 and 4.1 we obtain the following statement.

Theorem 4.4. *Let $X \in \mathcal{T}_s(S)$, and let $\sigma : S \to S$ be such that $X \in \mathcal{T}_\sigma(S)$. Then there exists a handlebody filling H_X such that*

$$V_R(H_X) \leq -\frac{\pi}{4}\ell_{X_\sigma}(\partial X_\sigma) < 0 .$$

Proof. By equation (3) and (4), as the convex core volume of H_X is zero, we have:

$$V_R(H_X) \leq -\frac{\pi}{4}\ell_{Y_\sigma}(\partial\Sigma)\,,$$

for Y_σ the hyperbolic structure on the convex core induced by Lemma 4.1. Recall that by Lemma 3.3 and Remark 4.3, as isotopy classes of loops in S, we have that $\partial\Sigma$ and ∂X_σ are the same. We remark that, with respect to the metric obtained by grafting Y_σ along $\partial\Sigma$ (without reuniformizing), i.e., along the bending lamination of H_X, the boundary $\partial\Sigma$ has the same length as in Y_σ (see [10, Section 4.1] for the geometric definition of grafting). Moreover, the *grafting metric* coincides with the *Thurston metric*, which is conformal to X, and defined as

$$h_{Th}(z) = \inf_{\Omega(\Gamma)} h_D(z)\,,$$

where $H_X = \mathbb{H}^3/\Gamma$, and the infimum is taken on the round disks D immersed in $\Omega(\Gamma)$, with h_D the hyperbolic metric on D. By the Schwarz Lemma, the Thurston metric is bigger then the hyperbolic metric at infinity. Therefore

$$V_R(H_X) \leq -\frac{\pi}{4}\ell_{Y_\sigma}(\partial\Sigma) \leq -\frac{\pi}{4}\ell_{X_\sigma}(\partial X_\sigma). \quad \square$$

If we know some curves are short in X and the pants decomposition is fixed component by component we obtain the following estimate, Lemma 4.5. This is the main such estimate we will use in this work. For completeness we also prove the other option in Lemma 4.6.

Lemma 4.5. *There exist universal constants $S, Q \geq 0$ as follows. Let $X \in \mathcal{T}_s(S)$, σ so that $X \in \mathcal{T}_\sigma(S)$, $M = M_P(X)$ be the Schottky manifold corresponding to any pants decomposition for which each curve is invariant under $\sigma : S \to S$, and such that there are $0 \leq k \leq 3g-3$ geodesic loops of P of length $\ell_X(\gamma_i) \leq 1$. Then,*

$$V_R(M_P(X)) \leq -\frac{S}{4}\sum_{i=1}^{k}\frac{1}{\ell_X(\gamma_i)} + \frac{Q}{4}k \leq \frac{k}{4}(-S+Q) < 0\,.$$

Specifically, one can take $S = \frac{4\pi^3}{\sqrt{e}} \sim 75.225$ and $Q = 4\pi\log\left(\frac{\pi e^{0.502\pi}}{\operatorname{arsinh}(1)}\right) \sim 35.7901 \leq 36$.

Proof. By Lemma 4.1 the convex-core of M is a totally geodesic surface and so $V_C(M) = 0$. However, by Theorem 2' of [4], we have

$$\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right) \leq L(\beta_M).$$

Applying it to equation (3), one gets:

$$V_R(M_P(X)) \leq -\frac{1}{4}\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right) = -\frac{S}{4}\sum_{i=1}^{k}\frac{1}{\ell_X(\gamma_i)} + \frac{Q}{4}k \leq \frac{k}{4}(-S+Q)\,,$$

concluding the proof. $\square$

The case in which the pants curves are not fixed component-wise requires introducing some auxiliary functions from [5, Corollary 1], these functions will only be needed here. For $m = \cosh^{-1}(e^2)$ we define

$$g(x) = e^{-m}\frac{e^{-\pi^2/2x}}{2}$$

and

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh(x/2)}{\sqrt{1-\sinh^2(x/2)}}\right)\,.$$

Since F is invertible we let $K(x) = \frac{2\pi}{F^{-1}(x)}$ and then define $L(x) = 1 + K(g(x))$.

Lemma 4.6. *Let $X \in \mathcal{T}_s(S)$, and $M(X)$ be the Schottky manifold with flat convex core and conformal boundary X. Let $\mathfrak{m} = \{\gamma_1, \dots, \gamma_k\}$ be the collection of geodesic loops point-wise invariant by σ and let ρ_X be half of the length of the shortest simple closed compressible geodesic in X. Then,*

$$V_R(M(X)) \leq -\frac{\pi}{4L(\rho_X)} \sum_{i=1}^{k} \ell_X(\gamma_i) \ .$$

Proof. By Lemma 4.1 the convex-core of M is a totally geodesic surface Y and so $V_C(M) = 0$. Moreover, by Lemma 3.3 ∂X_σ is given by the multi-curve $\mathfrak{m}$ of geodesic loops that are point-wise fixed by σ. Thus, by Corollary 1 of [5] we have:

$$\ell_Y(\gamma) \geq \frac{1}{L(\rho_X)} \ell_X(\gamma) \ .$$

Then, by applying it to equation (3) we obtain the required result. □

5. Variation of the renormalized volume under an earthquake

In this section we compute how the renormalized volume changes under earthquake paths in the deformation space.

5.1. First-order variation of the renormalized volume

We start the section with a formula for $\mathrm{d}V_R$ at $M_P(X) = \mathbb{H}^3/\Gamma$. Recall that by $\mathcal{S}(f)$ we are denoting the Schwarzian derivative of the developing map of the domain of discontinuity $\Omega(\Gamma)$ of the Schottky hyperbolic 3-manifold $M_P(X)$, and by S the boundary $\partial\overline{M}$. We will sometimes refer to $\mathcal{S}(f)$ just as the Schwarzian of $M_P(X)$.

Lemma 5.1. *Let μ be an infinitesimal earthquake (at unit velocity) along a simple closed geodesic on X, parameterised at unit velocity by $\gamma : \mathbb{R}/\ell\,\mathbb{Z} \to X$. Then, for $q = \mathcal{S}(f)$:*

$$\mathrm{d}V_R(\mu) = -\frac{1}{2} \int_{\mathbb{R}/\ell\,\mathbb{Z}} \mathrm{Re}(q(i\dot{\gamma}(t), \dot{\gamma}(t)))dt = \mathrm{Im}\left(\frac{1}{2} \int_{\mathbb{R}/\ell\,\mathbb{Z}} q(\dot{\gamma}(t), \dot{\gamma}(t)) \right) \ .$$

Proof. Let v be a vector field realizing the infinitesimal earthquake along the image of γ. That is, v is the vector field obtained by differentiating at zero, with respect to the time parameter t, the family of diffeomorphisms $\varphi_\gamma(t)$ corresponding to a length t earthquake along γ. We assume that v is discontinuous along $\gamma(\mathbb{R}/\ell\,\mathbb{Z})$, that is, it has limit zero on the right side and equal to $\dot{\gamma}(t)$ along $\gamma(\mathbb{R}/\ell\,\mathbb{Z})$ and is continuous on the left.

The first-order variation of the complex structure associated to v is then determined by the Beltrami differential $\mu = \overline{\partial}v$. Where here by $\overline{\partial}v$ we mean the L^∞ weak* limit of $\overline{\partial}v_n$ for v_n smooth compactly supported vector fields that are C^∞ approximations of v, converging in the uniform topology on compact sets of $S \setminus \gamma$. Specifically, choosing a complex coordinate z, we can write:

$$v = 2\omega\,(\partial_z + \partial_{\overline{z}}) \ ,$$

and note that ω vanishes on the right half-neighbourhood of γ.

Consider the area form $dx \wedge dy$ associated to $z = x + iy$, and note that $d\overline{z} \wedge dz = 2i(dx \wedge dy)$. We have

$$\overline{\partial}v = 2(\overline{\partial}\omega)(d\overline{z} \otimes \partial_z + d\overline{z} \otimes \partial_{\overline{z}}) \ ,$$

and so if $q = g(z)dz^2$,

$$\begin{aligned} \langle q, \overline{\partial}v\rangle &= \frac{1}{2i} \int_S 2g(z)(\overline{\partial}\omega(z))(d\overline{z} \wedge (dz^2(\partial_z)) + d\overline{z} \wedge (dz^2(\partial_{\overline{z}}))) \\ &= \int_S 2g(z)(\overline{\partial}w(z))dx \wedge dy \end{aligned}$$

by definition of the duality product commonly used between Beltrami differentials and holomorphic quadratic differentials, see [18].

Consider now the one-form defined by $\alpha = q(v,\cdot) = 2\omega(z)g(z)dz$. Then

$$\begin{aligned}\overline{\partial}\alpha &= \overline{\partial}(2\omega(z)g(z)dz)\\ &= 2(\overline{\partial}\omega(z))g(z)d\overline{z}\wedge dz + 2\omega(z)\overline{\partial}g(z)d\overline{z}\wedge dz\\ &= 4i(\overline{\partial}\omega(z))g(z)dx\wedge dy\ ,\end{aligned}$$

because g is holomorphic, $\overline{\partial}g = 0$, and $d\overline{z}\wedge dz = 2i(dx\wedge dy)$.

The outcome of this discussion is that

$$\int_S \overline{\partial}\alpha = \int_S 4ig(z)(\overline{\partial}\omega(z))dx\wedge dy = 2i\int_S 2g(z)(\overline{\partial}\omega(z))dx\wedge dy = 2i\langle q,\overline{\partial}v\rangle\ .$$

Therefore, we get

$$\langle q,\mu\rangle = -\frac{i}{2}\int_S \overline{\partial}\alpha\ .$$

However, α is a complex 1-form, so that $\partial\alpha = \partial(2gw)dz\wedge dz = 0$, and as a consequence

$$d\alpha = (\partial+\overline{\partial})\alpha = \overline{\partial}\alpha\ .$$

Using Stokes on $S' = S\setminus\gamma(\mathbb{R}/\ell\mathbb{Z})$, we obtain that, since α vanishes on one component of $\partial S'$:

$$\langle q,\overline{\partial}v\rangle = -\frac{i}{2}\int_S d\alpha = -\frac{i}{2}\int_{\partial S'}\alpha(\dot\gamma(t))dt = -\frac{i}{2}\int_0^\ell \alpha(\dot\gamma(t))dt\ .$$

However, by definition of α we obtain that

$$\langle q,\overline{\partial}v\rangle = -\frac{i}{2}\int_0^\ell q(v|_{\gamma(t)},\dot\gamma(t))dt = -\frac{1}{2}\int_0^\ell q(i\dot\gamma(t),\dot\gamma(t))dt\ .$$

The first order variation of the renormalized volume, thanks to Theorem 2.10, is equal to:

$$\begin{aligned}\mathrm{d}V_R(\mu) &= \mathrm{Re}\left(\langle q,\overline{\partial}v\rangle\right) = \mathrm{Re}\left(-\frac{1}{2}\int_0^\ell q(i\dot\gamma(t),\dot\gamma(t))dt\right)\\ &= -\frac{1}{2}\int_0^\ell \mathrm{Re}\left(q(i\dot\gamma(t),\dot\gamma(t))\right)dt\ ,\end{aligned}$$

completing the proof. □

Definition 5.2. An *earthquake path* $c_{\mathbf{t}}:[0,1]\to CC_P(S)$, with $\mathbf{t} = (t_1,\dots,t_{3g-3})$, is a path which at time $s\in[0,1]$ twists $st_i\in\mathbb{R}$ along each pants curve $\gamma_i\in P$ of $c_{\mathbf{t}}(0)$.

For a loop γ we use $\mathrm{inj}|_\gamma$ to denote half of the length of the shortest loop δ such that δ is compressible and δ is either γ or δ intersects γ essentially in M. Note that if γ is a compressible geodesic loop of length $\le\varepsilon_0$ in X, then $\mathrm{inj}|_\gamma = \frac{\ell_X(\gamma)}{2}$.

Lemma 5.3. *Let $c_t(s)$, for $s\in[0,1]$ and a fixed $t\in\mathbb{R}$, be an earthquake path along a simple geodesic loop γ starting at the Riemann surface X_0. Then, the following bound for $|\mathrm{d}(V_R\circ c_t)|$ holds at any $s\in[0,1]$:*

$$|\mathrm{d}(V_R\circ c_t)| \le 3\ell_{X_0}(\gamma)\coth^2\left(\frac{\mathrm{inj}|_\gamma}{2}\right)t\ .$$

In particular, if $\mathrm{inj}|_\gamma \ge 1/2$ we have

$$|\mathrm{d}(V_R\circ c_t)| \le 3\ell_{X_0}(\gamma)\coth^2\left(\frac{1}{4}\right)t = C\ell_{X_0}(\gamma)t\ ,\quad C = 3\coth^2\left(\frac{1}{4}\right) < 50.013\ .$$

Proof. First, observe that the length of γ remains constantly equal to $\ell_{X_0}(\gamma)$ along the earthquake path $c_t(s)$. Moreover, since earthquaking forms a flow (i.e. $c_t(s_1+s_2) = c_t(s_1) \circ c_t(s_2)$), the scaling by t of the infinitesimal earthquake μ_s along γ at $X_s = c_t(s)$ coincides with the derivative of $c_t(s)$ at s. Then, at any $s \in [0,1]$, we can use the integration by part of Lemma 5.1. Denoting by $\mathcal{S}(f_s)(z) = q_s(z)dz^2$ the Schwarzian associated through uniformization to $c_t(s)$, we can estimate $|q_s(z)| \le 6\coth^2\left(\frac{\operatorname{inj}|_\gamma}{2}\right)$ (see [1, Corollary 2.12], and note that the factor 4 comes from the hyperbolic metric), yielding the first bound. The second estimate follows by direct computation. □

5.2. Earthquake paths and V_R estimates

In this section we compute the change of renormalized volume under a path $c_{\mathbf{t}} : [0,1] \to CC_P(S)$ obtained by doing earthquakes along geodesic loops in the pants decomposition P.

Theorem 5.4. *Let $c_{\mathbf{t}} : [0,1] \to CC_P(S)$ be an earthquake path, and let $\ell_i = \ell_{X_0}(\gamma_i)$. Then*

$$|V_R(X_1) - V_R(X_0)| \le \sum_{i=1}^{k} (3\ell_i \coth^2(\ell_i/4)) t_i + C \sum_{i=k+1}^{3g-3} t_i \ell_i ,$$

where $\gamma_1, \dots, \gamma_k$ are the geodesic loops of P with $\ell_i < 1$ and for all $j > k$ we have $2\operatorname{inj}|_{\gamma_j} \ge 1$, and $C = 3\coth^2(1/4)$.

Proof. Pick a 1-thick/thin pants decomposition with k geodesic loops less than 1 and integrate Lemma 5.3. □

Since, by Lemma 3.7, to reach a symmetric surface we need to twist at most $\ell_X(\gamma_i)/4$, we can take $t_i \le \ell_X(\gamma_i)/4$ in the above expression and obtain the following statement.

Corollary 5.5. *Let $X \in \mathcal{T}(S)$ and $P = \{\gamma_i\}_{i=1}^{3g-3}$ be a pants decomposition in which the first k curves have length less than 1 and the others have injectivity radius at least 1. Then, there exists a symmetric surface X_0 such that*

$$|V_R(X) - V_R(X_0)| \le \frac{3}{4}\sum_{i=1}^{k} \coth^2(\ell_i/4)\, \ell_i^2 + \frac{C}{4}\sum_{i=k+1}^{3g-3} \ell_i^2,$$

with $\ell_i = \ell_{X_0}(\gamma_i)$ and $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

The above estimates also work in the setting of general convex co-compact manifolds. Let $CC(M)$ be the deformation space which is also parameterised by the quotient of $\mathcal{T}(\partial M)$ by Dehn twists along disks. Let $c_{\mathbf{t}}^{\mathfrak{m}} : [0,1] \to CC(M)$ be an earthquake path along a multi-curve $\mathfrak{m} \subseteq S$. Assume that the multi-curve $\mathfrak{m}$ can be subdivided, according to the reference metric X_0, in the following way:

- $\mathfrak{m}_1^c$ is the set of geodesic loops γ of $\mathfrak{m}$ that are compressible and have length at most 1;
- $\mathfrak{m}_1$ is the set of geodesic loops γ contained in compressible components of ∂M and not in $\mathfrak{m}_1^c$, and such that any compressible loop intersecting γ essentially has length at least 1;
- $\mathfrak{m}_\infty$ is the set of geodesic loops γ of $\mathfrak{m}$ that are contained in components of ∂M that are incompressible.

Note that not every $\mathfrak{m}$ admits such a decomposition with respect to the given X_0.

Theorem 5.6. *Let $X_0 \in \mathcal{T}(\partial M)$ and $\mathfrak{m} = \mathfrak{m}_1^c \cup \mathfrak{m}_1 \cup \mathfrak{m}_\infty$ be a multi-curve and $c_{\mathbf{t}}^{\mathfrak{m}}$ be an earthquake path terminating at X_1. Then*

$$|V_R(X_1) - V_R(X_0)| \le \sum_{\gamma_i \in \pi_0(\mathfrak{m}_1^c)} (3\ell_i \coth^2(\ell_i/4)) t_i + C \sum_{\alpha_j \in \pi_0(\mathfrak{m}_1)} t_j \ell_j + 3 \sum_{\beta_k \in \pi_0(\mathfrak{m}_\infty)} t_k \ell_k ,$$

for $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

Proof. The first two cases follow by the previous computations and integrating Lemma 5.3. For the last case we bound the norm of the Schwarzian on the geodesic loops in $\mathfrak{m}_\infty$ by the Kraus-Nehari estimate [23,27] and then integrating gives the result. □

6. Main results

We now put together the results from the previous sections to prove the main Theorem 1.4 and Corollary 1.5.

Theorem 1.4. *Let X be a closed Riemann surface of genus $g \geq 2$. Assume that there are k disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ such that $\ell(\gamma_i) \leq 1$, $1 \leq i \leq k$, and there are no other geodesic loops of length less or equal to 1 in X. Then there exists a pants decomposition P containing the γ_i's such that*

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}}\sum_{i=1}^{k}\frac{1}{\ell(\gamma_i)} + \left(9 + \frac{3}{4}\coth^2\left(\frac{1}{4}\right)\right)k + 81\coth^2\left(\frac{1}{4}\right)\pi(3g-3-k)(g-1)^2 .$$

Proof. Let P be a pants decomposition containing the k geodesic loops $\gamma_1, \ldots, \gamma_k$ shorter than 1 and the α_i, $i = k+1, \ldots, 3g-3$, being *Bers pants curves* (see [13, Theorem 12.8]).

That is, we have:

- $\ell_X(\gamma_i) \leq 1$ for $i \leq k$;
- $1 < \ell_X(\alpha_i) \leq B_g \leq 6\sqrt{3\pi}(g-1)$, see [7, Theorem 5.1.4], and $\mathrm{inj}\,|_{\alpha_i} \geq 1$ for $k < i \leq 3g-3$;
- P has seams such that in the FN coordinates induced by P, $FN(X)$ has no twists bigger than $\ell_X(\gamma_i)/4$ or $\ell_X(\alpha_i)/4$ (see Lemma 3.5 and Lemma 3.7).

Let $c_{\mathbf{t}}$ be the path in FN coordinates from X to X_s, the symmetric surface. Then, $c_{\mathbf{t}}$ can be thought of doing $3g-3$ twists along each pants curve, each of length at most $\ell_X(\gamma_i)/4$ or $\ell_X(\alpha_i)/4$, see Lemma 3.7. Then, for $C = 3\coth^2\left(\frac{1}{4}\right)$, by Corollary 5.5 we get:

$$\begin{aligned}
|V_R(X) - V_R(X_s)| &\leq \frac{3}{4}\sum_{i=1}^{k}\coth^2(\ell_i/4)\,\ell_i^2 + \frac{C}{4}\sum_{i=k+1}^{3g-3}\ell_i^2 \\
&\leq \frac{C}{4}k + \frac{C}{4}\sum_{i=k+1}^{3g-3}B_g^2 \\
&\leq \frac{C}{4}k + \frac{C}{4}(3g-3-k)B_g^2 \\
&\leq \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2,
\end{aligned}$$

where we used the fact that $B_g \leq 6\sqrt{3\pi}(g-1)$ and $\coth^2(x/4)\,x^2$ is an increasing function. Thus, we get that:

$$V_R(X) \leq V_R(X_s) + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2.$$

Since $\ell_i \leq 1$ for $i \leq k$ by using Lemma 4.5 to estimate $V_R(X_s)$ we have:

$$V_R(X_s) \leq -\frac{1}{4}\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right),$$

for $S = \frac{4\pi^3}{\sqrt{e}}$ and $Q = 4\pi\log\left(\frac{\pi e^{0.502\pi}}{\operatorname{arsinh}(1)}\right) \sim 35.7901 \leq 36$. Then, we obtain the following bound:

$$\begin{aligned}
V_R(X) &\leq \sum_{i=1}^{k}\left(-\frac{S}{4\ell_X(\gamma_i)} + \frac{Q}{4}\right) + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2 \\
&\leq \sum_{i=1}^{k}\left(-\frac{\pi^3}{\sqrt{e}\ell_X(\gamma_i)}\right) + 9k + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2 \\
&\leq -\frac{\pi^3}{\sqrt{e}}\sum_{i=1}^{k}\left(\frac{1}{\ell_X(\gamma_i)}\right) + \left(9 + \frac{C}{4}\right)k + 27C\pi(3g-3-k)(g-1)^2 .
\end{aligned}$$

Substituting for $C = 3\coth^2\left(\frac{1}{4}\right)$ concludes the proof. □

Corollary 1.5. *For all $g \in \mathbb{N}$ s.t. $g \geq 2$, $0 < k \leq 3g-3$ and $0 < k_1 \leq k$ there exists a constant $A = A(g, k_1, k-k_1) > 0$ such, that if X is a Riemann surface with k_1 geodesic loops of length less than A and k geodesic loops of length at most 1, then X admits a Schottky filling with negative renormalized volume.*

Proof. Let P be a pants decomposition containing the k_1 geodesic loops, $\gamma_1, \ldots, \gamma_k$ shorter than A and $k - k_1$ geodesic loops $\gamma_{k_1+1}, \ldots, \gamma_k$ of length at most 1 and the α_i, $i = k+1, \ldots, 3g-3$ are Bers pants curves.

That is, we have:

- $\ell_X(\gamma_i) < A$ for $1 \leq i \leq k_1$;
- $\ell_X(\gamma_i) \leq 1$ for $k_1 < i \leq k$;
- $1 < \ell_X(\alpha_i) \leq B_g \leq 6\sqrt{3\pi}(g-1)$ and $\mathrm{inj}|_{\alpha_i} \geq 1$ for $k < i \leq 3g-3$.

Then, by Theorem 1.4 we get:

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \frac{1}{\ell(\gamma_i)} + \left(9 + \frac{C}{4}\right) k + 27C\pi(3g-3-k)(g-1)^2 ,$$

which can be further decomposed in:

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}} \left(\sum_{i=1}^{k_1} \frac{1}{\ell(\gamma_i)} + \sum_{i=k_1+1}^{k} \frac{1}{\ell(\gamma_i)} \right) + \left(9 + \frac{C}{4}\right) k + 27C\pi(3g-3-k)(g-1)^2 .$$

Since for $i \leq k_1$ we have that $\frac{1}{\ell_X(\gamma_i)} \geq \frac{1}{A}$ and, similarly, for $k_1 + 1 \leq i \leq k$ we have that $\frac{1}{\ell_X(\gamma_i)} \geq 1$ we get:

$$V_R(X) \leq -\frac{\pi^3}{\sqrt{e}} \left(\frac{k_1}{A} + k - k_1 \right) + \left(9 + \frac{C}{4}\right) k + 27C\pi(3g-3-k)(g-1)^2 .$$

We want to find an upper bound on A that makes the above expression negative. Note that

$$B := -\frac{\pi^3}{\sqrt{e}}(k-k_1) + \left(9 + \frac{C}{4}\right) k + 27C\pi(3g-3-k)(g-1)^2 > 2k > 0 ,$$

as the smallest case for B is for $k = 3g-3$ and $k_1 = 0$. Then, to have

$$-\frac{\pi^3}{\sqrt{e}} \frac{k_1}{A} + B < 0 ,$$

it suffices to take:

$$A < \frac{\pi^3}{\sqrt{e}} \frac{k_1}{B},$$

concluding the proof. □

Declaration of competing interest

No conflicts of interests to declare.

Data availability

No data was used for the research described in the article.

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