PhD-FSTM-2025-078
Faculty of Science, Technology and Medicine

Università degli studi di Pisa,
Dipartimento di Matematica

DISSERTATION

Defence held on 28 July 2025 in Esch-sur-Alzette

to obtain the degree of

DOCTEUR DE L'UNIVERSITÉ DU LUXEMBOURG

EN MATHÉMATIQUES

AND

DOTTORE DI RICERCA IN MATEMATICA

DELL'UNIVERSITÀ DI PISA

By

Viola GIOVANNINI

Born on 2 March 1996 in Pietrasanta (Italy)

ON THE RENORMALIZED VOLUME OF HYPERBOLIC 3-MANIFOLDS WITH COMPRESSIBLE BOUNDARY

DISSERTATION DEFENCE COMMITTEE

Prof. Dr Bruno MARTELLI, Supervisor
Full professor in Mathematics, UNIVERSITÀ DI PISA / Pisa / Italy

Prof. Dr Jean-Marc SCHLENKER, Supervisor
Full professor in Mathematics, UNIVERSITÉ DU LUXEMBOURG / Esch-sur-Alzette / Luxembourg

Prof. Dr Karin MELNICK, Chair
Full professor in Mathematics, UNIVERSITÉ DU LUXEMBOURG / Esch-sur-Alzette / Luxembourg

Prof. Dr Francesco BONSANTE, Member
Full Professor in Mathematics, UNIVERSITÀ DEGLI STUDI DI PAVIA

Prof. Dr Sara MALONI, Member
Professor in Mathematics, UNIVERSITY OF VIRGINIA / USA

Prof. Dr Yilin WANG, Member
Junior Professor in Mathematics, IHES

Ai miei nonni: Alberto, Fiorella, Giovanni e Sonia

Acknowledgements

I would like to sincerely thank my supervisors, Bruno Martelli and Jean-Marc Schlenker, for all the math they passionately taught me, and for always supporting and believing in me, much more than I do myself. A deep and due thank you to Jean-Marc for the invaluable time he dedicated me, and without whom this thesis would not exist.

Thank you, Tom, for all the math we have shared over these years, and for being such a precious and thoughtful friend.

I wish to thank Sara Maloni and Francesco Bonsante for their helpful participation as part of my CET, and for accepting to be members of the tribunal for the defense, together with Karin Melnick and Yilin Wang, whom I also warmly thank. I want to extend a special thanks to Francesco and Yilin for agreeing to report this thesis. No words would be enough to express my gratitude to Yilin for offering me a PostDoc position in her group.

I am extremely grateful for the life I shared with many awesome people of the math department in Luxembourg. First, I would like to thank Hugo Parlier for all the wise advice, whether in front of a beer or not. I cannot imagine my years here without the friends of the geometry group: Nathaniel, Mingkun, Chris, Abder, Marie, Bruno, Katie, Pallavi, Alexey, Wayne, Dídac, Danyu, Tien, Naomi. Thank you all for the care you show me everyday - for the hugs, the laughs, the chats and deep exchanges, the wooonderfuls, the concerts, the chocolates, the shrimpos, the teas, the beers, the nights, and everything else. Antigona, I can't even express how important you've been to me through all this time, thank you for being you and for making me someone important in your life. Thank you also to Bryan, Shiwi, Pierre, Thilo, Flavio, and Francesco, for the time spent together, I am glad I got to know you. Thank you, Franci, for keeping me alive by taking me running outside, and saving the treadmill I'd probably have worn out otherwise.

It would be impossible to mention all the people I was lucky to meet during workshops and conferences, but I especially would like to thank Filippo, Enrico and Andrea, for all the good conversations we had.

If I enjoyed the life in Belval, it's also thanks to my flatmates: Rabie, Baseer, Clemont and Michele. Fra, destiny made us live next to each other, thank you for always being there to listen to my complaints, for the dinners in our kitchens, and for becoming one of my closest friends.

I also want to thank my friends who made my first year of PhD in Pisa so beautiful: Beppe, Ludo, Alice, Matti, Matti, Diego, Dome, and Filippo. The time spent in and outside our office has been great.

Thank you Ele, Luca, Jaco, Chiara, Fra, Gaia, and Dario, for always being there, even from the various and furthest parts of the world, and for making me feel I can always count on you. I don't know how I'd hold it together without you. Grazie per farmi sentire vicina.

Un immenso grazie a mamma, a papá, e alla mia famiglia, per il supporto, la stima e l'amore che mi infondono sempre, anche da cosí lontano. Grazie per avermi dato tutto ció di cui si ha bisogno per arrivare fin qua(ssú).

Grazie David, per avermi reso questi anni tanto ricchi e felici. Gràcies per haver compartit i compartir amb mi realment tot d'aquesta vida, dalla matematica alle feste, dai viaggi alle giornate al parco di Belval, dalle sere in ufficio a quelle in palestra, del cafè del matí al vi del vespre, dels somriures al plor. Grazie, per ser casa.

Abstract

We investigate the renormalized volume of convex co-compact hyperbolic 3-manifolds in the setting where the boundary is compressible.

We make some steps forward in the study of Maldacena's question, which asks: given a Riemann surface, what is the most efficient way to fill it with a convex co-compact hyperbolic 3-manifold in order to minimize the renormalized volume? In particular, we prove that every closed Riemann surface of genus at least two, with enough curves of sufficiently short hyperbolic length, is the conformal boundary at infinity of a convex co-compact handlebody of negative renormalized volume.

We give an explicit description of the behaviour of the Schwarzian derivative - and thus of the differential of the renormalized volume - on long compressible tubes in complex projective surfaces. Furthermore, we establish bounds for its pairing with infinitesimal earthquakes and infinitesimal graftings. We derive how the renormalized volume changes along earthquake and grafting paths.

We define a new version of renormalized volume that adapts to the compressible boundary case, and which, unlike the standard one in this setting, is bounded from below. As a function on Teichmüller space, we show that its differential has infinity norm bounded by a constant depending only on the genus of the surface, as well as the Weil-Petersson norm of its Weil-Petersson gradient. Moreover, we define the adapted renormalized volume for surfaces in the compressible strata of the Weil-Petersson completion of Teichmüller space, and we prove that the adapted renormalized volume extends continuously to these strata, up to a codimension-one subset.

Contents

Introduction

The volume of closed or cusped hyperbolic 3-manifolds has been extensively studied as a topological invariant. However, there exists a wide class of hyperbolic manifolds whose elements have infinite volume, and here is where the notion of renormalized volume comes into play. By the Tameness Theorem, every complete hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact manifold N, possibly with boundary. Whenever N presents a boundary component of genus at least 2, any complete hyperbolic metric on its interior has infinite volume.

The main setting of this thesis is the class of geometrically finite hyperbolic 3-manifold, and, more precisely, the subset of convex co-compact ones. A hyperbolic 3-manifold M is *geometrically finite* if it contains a non-empty convex subset C of finite volume to which M is homotopically equivalent, and it is *convex co-compact* if C is also compact. This is equivalent to requiring that M is geometrically finite without rank one cusps, and that the components of the boundary of its closure $\partial\overline{M}$ are closed surfaces of genus $g \geq 2$. The smallest convex subset $C(M)$ of M whose inclusion is an homotopy equivalence is called the *convex core* of M.

The world of geometrically finite hyperbolic 3-manifold is beautifully interconnected with that of Riemann surfaces and then, by Riemann Uniformization Theorem, with that of hyperbolic surfaces, or even conformal classes of Riemannian metrics. This deep link is due to the fundational work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston, among others. In particular, the deformation space $CC(M)$ of convex co-compact hyperbolic structures on M, considered up to homotopy, is parameterized by a quotient of the space of marked hyperbolic structures, that is, the *Teichmüller space* of $\partial\overline{M}$ [Mar16a, Theorem 5.1.3]. Briefly, the correspondence is realized as follows. The Riemannian metric tensor on a convex co-compact hyperbolic 3-manifold M diverges exponentially on the components of the complement of the convex core, and thus it does not extend to the boundary $\partial\overline{M}$. However, the manifold M can be *conformally compactified*, that is, the hyperbolic metric on M induces a well defined conformal structure on $\partial\overline{M}$, which then defines a point in the Teichmüller space $\mathcal{T}(\partial\overline{M})$. This conformal structure is referred to as the *conformal boundary at infinity* of M, and denoted $\partial_\infty M$.

Within this framework, given a convex co-compact hyperbolic 3-manifold M, the *renormalized volume* is a real-analytic function on $CC(N)$, or, equivalently, on the Teichmüller space of $\partial\overline{M}$. The central idea behind its definition is to renormalize the divergent volumes associated to an exhaustion of M by homotopically equivalent convex compact submanifolds. There are many possible choices for such an exhaustion. Indeed, there is one for each smooth metric in the conformal boundary at infinity [Eps84]. The renormalized volume of M is defined using the exhaustion in convex compacts determined by the unique hyperbolic metric conformal to $\partial_\infty M$. For all other choices of exhaustion, or, equivalently, of conformal metric, the resulting quantity is referred to as a *W-volume.*

The renormalized volume is, in fact, defined in much broader contexts. We provide here a brief overview. It was first introduced in physics by de Haro, Skenderis and Solodukhin as a renormalized gravity action [dHSS01], and further studied in the setting of conformally compact Einstein manifolds by Graham and Witten [GW99]. As the name suggests, such manifolds have a well defined conformal structure on their boundary, which, in certain cases, uniquely determines the Einstein manifold as well [GL91]. Convex co-compact hyperbolic 3-manifolds form a significant subclass of these. The interaction between Einstein manifolds and their conformal boundary are examples of a more general principal in theoretical physics known as *holography*. We mention this here in order to give some motivation and just the flavor of the context in which the renormalized volume originated, although this work does not engage with the topic. According to the holographic principle, a gravitational theory in an $(n+1)$-dimensional manifold should be described in terms of a conformal field theory on its n-dimensional boundary. For Einstein metrics, the gravitational action is proportional to the volume, which diverges. The renormalized volume arises in this setting as a regularized version of this action.

In dimension 3, holography was established for classical Schottky groups by Krasnov in [Kra00]. There, the renormalized volume action (a 3-dimensional data) is shown to be equal to the Liouville action functional introduced by Zograf and Tahktajan in [ZT87], which is intrinsic to the boundary Riemann surface. Here, renormalized volume action refers to the W-volumes associated with a pair consisting of a 3-manifold and a conformal metric on the boundary. This holographic principle was then extended by Tahktajan and Teo in [TT03] to a larger class of Kleinian groups, including finitely generated Schottky groups and Quasi-Fuchsian groups. The *Liouville action* is a functional on the space of smooth conformal metric on a Riemann surface whose critical point is given by the hyperbolic metric. In this way, the action descends on the Teichmüller space. What is particularly interesting about this functional is that it is a Kähler potential for the *Weil-Petersson metric* on the Teichmüller space [TT03], extending McMullen's quasi-Fuchsian reciprocity [McM00]. Very recently, the universal Liouville action was proven to be equal to the renormalized volume associated to quasi-circles on the Riemann sphere [BBPW25].

A more geometric approach to renormalized volume was first treated by Krasnov and Schlenker for *quasi-Fuchsian* manifolds, which are convex co-compact manifolds homeomorphic to $S_g \times (0,1)$, for some closed surface S_g of genus g [KS08]. Many of the results in that paper actually extend to the more general convex co-compact setting. In [GMR15] and [Pal16] the renormalized volume is defined and studied in the setting of geometrically finite hyperbolic 3-manifolds. There are several results highlighting how the renormalized volume is deeply connected to the geometry of the manifold. In [Sch13] Schlenker established an explicit upper bound on the renormalized volume of quasi-Fuchsian manifolds in terms of the *Weil-Petersson distance* between the conformal metrics at infinity. It was observed by Bridgeman, Brock and Bromberg in [BBB23], that the result generalizes to the setting of convex co-compact manifolds with *incompressible boundary*. A non-trivial simple closed curve in the boundary $\partial\overline{M}$ is said to be *compressible* if it bounds a disk in the 3-manifold M. The boundary of M is then called *incompressible* if it does not contain compressible curves. Again in [Sch13] a uniform bound is shown, depending only on the genus of the boundary, on the difference between the renormalized volume and the volume of the convex core. This was generalized by Bridgeman and Canary in [BC17] to convex co-compact manifolds with incompressible boundary, refined in [BBB19], to be then exploited in [KM18, BB16] to derive bounds on the hyperbolic volume of mapping tori.

As a consequence of the holographic principle in dimension 3, fixing a Riemann surface

X, the minimum of the renormalized volumes among all the convex co-compact manifolds M such that $\partial_\infty M = X$ is expected to correspond to the main term in the partition function of the conformal field theory on X [Kra00, SW22]. This leads to the following question, which is attributed to Maldacena.

Question 2.1.1. Given a Riemann surface X, what is the convex co-compact hyperbolic manifold M of smallest renormalized volume, with conformal boundary at infinity X?

This question also appears quite natural from a purely mathematical perspective: in the world of closed orientable hyperbolic 3-manifolds, it is known that the manifold of smallest volume is the Weeks manifold [GMM09]. An analogous result can then be sought in the setting of geometrically finite hyperbolic 3-manifolds, either for the convex core volume, or, perhaps even more interestingly, for the renormalized volume. The main advantage of studying the latter lies in its better analytic properties, a point to which we will return later in this introduction.

In [BBB19], it is shown that the renormalized volume of any convex co-compact structure on a manifold M with incompressible boundary is non-negative. More precisely, for such an M, the infinimum of the renormalized volume over $CC(M)$ coincides with that of the convex core volume. Moreover, the infimum is attained if and only if either M is *acylindrical*, that is, it contains no essential cylinders, or M has the topology of a quasi-Fuchsian manifold. In the first case, the minimum is realized by the convex co-compact structure on M with totally geodesic convex core boundary, that in this case always exists, and it is unique by the Mostow Rigidity Theorem applied to the convex core double. In the quasi-Fuchsian case, the minimum is realized at any *Fuchsian* structure, and it is equal to 0. This result encourages the investigation of the following simplified version of Maldacena's question.

Question 2.1.2. Does every connected Riemann surface X of genus $g \geq 2$ admit a Schottky filling of negative renormalized volume?

A hyperbolic 3-manifold is called a *filling* of X if $\partial_\infty M = X$, and it is said to be *Schottky* if it is homeomorphic to a handlebody. Whenever X satisfies the above question affirmatively, any 3-manifold filling X and minimizing the renormalized volume must have compressible boundary. In particular, given such a Riemann surface X, and letting $\overline{X}$ denote its opposite, the disconnected filling given by two copies of a handlebody with negative renormalized volume would be preferable to the (connected) Fuchsian manifold with conformal boundary $X \cup \overline{X}$.

In the first chapter of this thesis, we positively answer the above question for sub-classes of Riemann surfaces having enough simple closed curves that are short enough, where length is measured with respect to the unique hyperbolic metric in the conformal class. Before stating the results proven in this work, we briefly mention previous partial results in the literature. In [VP25, PMG19], Pallete and Granado gave a positive answer for Riemann surfaces admitting a multicurve of $g-1$ sufficiently short components decomposing the surface into a union of holed tori. In [BP23], Brock and Pallete established a bound for the renormalized volume of classical Schottky groups, which is negative under certain conditions.

Theorem 2.6.1. Let X be a closed Riemann surface of genus $g \geq 2$. Assume that there are k disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ such that $\ell(\gamma_i) \leq 1, 1 \leq i \leq k$, and there are no other geodesic loops of length less or equal than 1 in X. Then there exists a pants decomposition P containing the γ_i's such that, if $M_P(X)$ denotes the Schottky filling X in which all the curves of P are compressible

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}}\sum_{i=1}^{k}\frac{1}{\ell(\gamma_i)}+\left(9+\frac{3}{4}\coth^2\left(\frac{1}{4}\right)\right)k+81\coth^2\left(\frac{1}{4}\right)\pi(3g-3-k)(g-1)^2 \, .$$

The two main steps of the proof of this result are the following. First, in Theorem 2.4.4 and Lemma 2.4.5, we bound from above, by a negative constant, the renormalized volume of Schottky fillings with a 2-dimensional convex core, here referred to as *symmetric*. Then, in Theorem 2.5.4, we analyze how the renormalized volume varies along a path of earthquakes on simple closed geodesics that connects a general Riemann surface with a symmetric one. This particular result, is generalized to the convex co-compact setting in Theorem 2.5.6.

By imposing the right hand side of the estimate in Theorem 2.6.1 to be negative, one obtains the following corollary.

Corollary 2.6.2. For all $g, k, k_1 \in \mathbb{N}$ such that $g \geq 2$, $0 < k \leq 3g-3$ and $0 < k_1 \leq k$, there exists an explicit constant $A = A(g, k_1, k-k_1) > 0$ such that if X is a Riemann surface with k_1 geodesic loops of length less than A and k geodesic loops of length at most 1, then X admits a Schottky filling with negative renormalized volume.

The boundary at infinity $\partial_\infty M$ of any convex co-compact manifold M exhibits an even richer structure. As a consequence of the Riemann sphere $\mathbb{CP}^1$ being the conformal boundary at infinity of the 3-hyperbolic space $\mathbb{H}^3$, and of the fact that the orientation-preserving isometry group of $\mathbb{H}^3$ coincides with $\mathbb{P}SL(2,\mathbb{C})$, the biholomorphisms group of $\mathbb{CP}^1$, the boundary at infinity of a convex co-compact manifold is naturally equipped with a *complex projective structure*. This, in turn, induces the underlying Riemann surface structure. The deformation space of complex projective structures forms a holomorphic vector bundle over the Teichmüller space, whose fibers are parameterized by the *Schwarzian derivative* of the *developing map* f of the complex projective structure [Dum08]. The Schwarzian derivative of a locally injective holomorphic map $f\colon D \to \mathbb{CP}^1$, for an open domain $D \subseteq \mathbb{C}$, is the central object of Chapter 3. It is the *holomorphic quadratic differential* defined as

$$\mathcal{S}(f) = \left(\left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2\right)dz^2 \ ,$$

where z denotes the complex coordinate of $\mathbb{C}$, and $dz^2 = dz \otimes dz$ the associated holomorphic symmetric $(0,2)$-tensor. A projective structure Z on a surface S can be lifted to the universal cover $\widetilde{S} = \mathbb{H}^2 \subseteq \mathbb{C}$, and a *developing map* of Z is an immersion $f\colon \mathbb{H}^2 \to \mathbb{CP}^1$ which restricts to projective charts on small enough opens of $\mathbb{H}^2$. The developing map is unique up to Möbius transformations.

Thanks to the following beautiful result, the Schwarzian derivative plays a key role in the study of the renormalized volume. Before stating it, we need to mention that the cotangent bundle of the Teichmüller space is identified, via the Bers embedding, with the space of holomorphic quadratic differentials [Hub16].

Theorem. ([ZT87], [KS08, Corollary 8.6], [Sch13, Corollary 3.11]) Let M be a convex co-compact hyperbolic 3-manifold, and let $\mathcal{S}(f_M)$ be the holomorphic quadratic differential on $\partial_\infty M$ given by the Schwarzian derivative of the developing map f_M of $\partial_\infty M$. Then, the differential of the renormalized volume at $[\partial_\infty M] \in \mathcal{T}(\partial\overline{M})$ is

$$dV_R = \mathrm{Re}\left(\mathcal{S}(f_M)\right) \ .$$

In the three papers series [BBB19, BBB23, BBVP23], Bridgeman, Brock, Bromberg and Pallete studied the flow of minus the *Weil-Petersson gradient* of the renormalized volume for (relatively) acylindrical manifolds. Thanks to the theorem above, and to the duality between holomorphic quadratic differentials and *harmonic Beltrami differentials*, the Weil-Petersson gradient has a particularly nice and explicit analytic description. They showed that the flow-lines always converge to the unique convex co-compact structure with totally geodesic boundary. To the best of the author's knowledge, ongoing research is progressing toward obtaining similar results in the cylindrical case as well. As the reader may now start to realize, much is known in the incompressible boundary case. This thesis is dedicated instead to the setting in which the boundary of M is compressible.

Depending on whether the image of f is simply connected or not, its Schwarzian derivative $\mathcal{S}(f)$ behaves quite differently. In particular, its supremum norm is bounded by $3/2$ in the first case [Neh49]. In the second case, it is bounded from below by a diverging function of the length of the shortest non-trivial simple closed curve in the image [KM81]. When $f = f_M$ is the developing map of the natural complex projective structure on the conformal boundary $\partial_\infty M$ of a convex co-compact manifold M, these two scenarios correspond, respectively, to the boundary of M being incompressible or not. As a result, the renormalized volume presents a totally different behaviour depending on whether the manifold has incompressible boundary. We have already mentioned that when $\partial\overline{M}$ is incompressible, the renormalized volume is always positive [BBB19]. On the other hand, if $\partial\overline{M}$ is compressible, the works Bridgeman and Canary [BC17], and of Schlenker and Witten [SW22], show that the renormalized volume of a sequence of convex co-compact manifolds associated to the *pinching* of a compressible curve in the boundary diverges to $-\infty$. Pinching a simple closed curve in a Riemann surface means making the length of its geodesic representative go to zero with respect to the hyperbolic metric.

This phenomenon led us to study the behaviour of the Schwarzian derivative on long complex projective tubes, which is the core of Chapter 3. By the Collar Lemma, a short simple curve on a hyperbolic surface indeed presents a long collar [Bus10]. Moreover, the collars of two simple closed curves of length less than $\varepsilon_0 = 2\operatorname{arsinh}(1)$ are disjoint. We say that a complex projective structure Z has a long tube if the image of its developing map contains a simple closed curve of hyperbolic lenght less than ε_0.

Theorem 3.5.4. Let $\mathcal{A}$ be a long tube of a complex projective surface Z, let γ be its core of length $\ell \leq \varepsilon_0$, and let $\tilde{\gamma} \subseteq \widetilde{Z} = \mathbb{H}^2$ be a lift. Let also $f \colon \mathbb{H}^2 \to \mathbb{CP}^1$ be the developing map of Z, and let $\mathcal{S}(f)$ be its Schwarzian derivative. Then, in a neighborhood of $\widetilde{\gamma}$, up to pull-back by a Möbius, the Schwarzian $\mathcal{S}(f)$ behaves as follows

$$\mathcal{S}(f) = \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2 + O\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right) dz^2 \ ,$$

where z is the complex coordinate of $\mathbb{H}^2$, and $O(x)$ stands for a complex valued function such that $\lim_{x\to 0} |O(x)/x|$ is finite.

Combining the above result with the two Stokes-type formulas of Lemmas 2.5.1 and 3.2.1, we obtain the following.

Theorem 3.5.7. Let Z be a complex projective surface, let $\mathcal{S}(f)$ be its Schwarzian, and let $X = \pi(Z)$ its underlying Riemann surface. Let also μ and ν be, respectively, the infinitesimal earthquake and grafting on the simple closed curve $\gamma \subseteq X$ of hyperbolic length $\ell \leq \varepsilon_0$. Then

$$|\mathrm{Re}\,\langle \mathcal{S}(f), \mu\rangle| \leq F_e(\ell)$$

and

$$\left|\operatorname{Re}\langle \mathcal{S}(f), \nu\rangle - \frac{\pi^2}{\ell}\right| \leq \frac{1}{4}\ell + F_{gr}(\ell)$$

with $F_e(\ell)$ and $F_{gr}(\ell)$ two explicit functions such that

$$|F_e(\ell)|\,,\ |F_{gr}(\ell)| \leq C\frac{e^{-\pi^2/\ell}}{\ell}\ ,$$

for some constant $C > 0$.

The main idea to prove the two theorems above is to first establish that the results hold for annuli in $\mathbb{CP}^1$ with concentric round boundaries, that we call *symmetric*. This is done in Lemma 3.3.3 and Proposition 3.3.5. Then, we show that tubes with large *modulus* limit to symmetric ones. This is achieved by exploiting the interpretation of the Schwarzian derviative as the complexification of a certain *Osgood-Stowe tensor*. The Osgood-Stowe tensor was introduced in [OS92], and its deep connection with *Epstein surfaces* and *W-volumes* was extensively investigated in [BB24].

We can then use these results to get information on the renormalized volume behaviour.

Theorem 3.4.1. Let M be a convex co-compact hyperbolic 3-manifold. Let $X_0 \in \mathcal{T}(\partial\overline{M})$, and let $X_t \in \mathcal{T}(\partial\overline{M})$ be the Riemann surface obtained by a parameter $t \in \mathbb{R}$ earthquake on X_0 along a compressible simple closed curve γ in $\partial\overline{M}$, of hyperbolic length $\ell \leq \varepsilon_0$. Then, we have the following estimate for the renormalized volume of the associated convex co-compact manifolds M_0 and M_t:

$$|V_R(M_t) - V_R(M_0)| \leq F(\ell)t\ ,$$

with $F(\ell)$ an explicit function of ℓ such that

$$|F(\ell)| \leq C\frac{e^{-\pi^2/\ell}}{\ell}$$

for some explicit constant $C > 0$.

When ℓ is small enough, Theorem 3.4.1 gives a sharper bound than the one of Theorem 2.5.4 (Theorem 5.4 in [CGS24]), for the change of renormalized volume under earthquake.

As another application of our estimate for the Schwarzian, we obtain the asymptotic behavior of the renormalized volume under *pinching* a compressible curve in the boundary at infinity of a convex co-compact hyperbolic 3-manifold, recovering in particular Theorem A.15 in [SW22]. Pinching γ can be realized by *grafting* along it with a parameter tending to infinity. This gives a bound for the change of renormalized volume under grafting along a short enough curve.

Theorem 3.4.5. Let M_0 be a convex co-compact hyperbolic 3-manifold and let $\gamma \in \partial\overline{M_0}$ be a compressible curve in its boundary of length $\ell_0(\gamma) \leq \varepsilon_0$, with respect to the hyperbolic metric conformal to $\partial_\infty M_0$. The composition of the renormalized volume with the grafting path $(M_s)_{s\in[0,\infty)}$ along γ satisfies

$$V_R(M_s) - V_R(M_0) = -\frac{\pi^3}{\ell_s(\gamma)} + \frac{\pi^3}{\ell_0(\gamma)} + (\ell_s(\gamma) - \ell_0(\gamma))\frac{\pi}{4} + O\left(e^{-\pi s/(2\ell_0(\gamma))}s^3\right)\ .$$

In particular, when $\ell_s(\gamma) \to 0$, the renormalized volume diverges as $-\pi^3/\ell_s(\gamma)$.

In light of the situation that has been described, in Chapter 4, we introduce a new version of renormalized volume *adapted* to the compressible boundary case, preserving some of the properties satisfied by the standard one in the incompressible boundary setting. In particular, we show that the *adapted renormalized volume* is uniformly bounded from below, and that its differential has bounded L^1, L^2 and L^∞ norms. As a corollary, we obtain, in particular, that the Weil-Petersson gradient of the differential of the adapted renormalized volume has bounded Weil-Petersson norm.

Definition 4.4.1. Given M a convex co-compact hyperbolic 3-manifold, we define the *adapted renormalized volume* as the function

$$\widetilde{V_R}\colon \mathcal{T}(\partial\overline{M}) \to \mathbb{R}$$

such that

$$\widetilde{V_R}(X) = V_R(X) + \pi^3 \sum_{\substack{\gamma \text{ compressible} \\ \ell_\gamma(X)<\varepsilon_0}} \frac{1}{\ell_\gamma(X)}$$

where $\varepsilon_0 = 2\mathrm{arsinh}(1)$ is the Margulis constant, γ runs in the set of compressible simple closed curves in $\partial\overline{M}$, and $\ell_\gamma(X)$ denotes its length with respect to the hyperbolic representative in X.

The adapted renormalized volume is smooth outside a codimension-one subset of the Teichmüller space, where the sum in the definition is just lower semi-continuous. In Section 4.5, we discuss how to improve the definition to achieve continuity everywhere, while ensuring that the theorems below still hold.

Theorem 4.4.4. For every convex co-compact hyperbolic 3-manifold M, the adapted renormalized volume $\widetilde{V_R}(\cdot)$ is bounded from below by a constant depending only on the topology of the boundary $\partial\overline{M}$.

Theorem 4.1.1. (Theorems 4.4.5 and 4.4.6) Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the differential of the adapted renormalized volume is bounded in L^1 and L^∞ norms, by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

Corollary 4.4.7. Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the Weil-Petersson gradient of the adapted renormalized volume has Weil-Petersson norm bounded by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

In [GMR15] and [Pal16], the authors proved that the renormalized volume is continuous under a sequence of acylindrical convex co-compact manifolds geometrically converging to a geometrically finite one [Pal16, Theorem 1] or, more generally, under (*admissibly*) pinching an incompressible curve to obtain a rank-1 cusp ([GMR15, Theorem 4]). In [Pal17, Theorem 6.1], additive continuity of the renormalized volume is shown under geometric limits of convex co-compact manifolds with incompressible boundary, and with uniformly bounded convex core volumes. The pinching of a compressible simple closed curve in the boundary limits instead, in the pointed Gromov-Hausdorff topology, to a convex co-compact manifold *marked* at one or two points corresponding to the curve that has been pinched, as described in [SW22, Appendix A.10]. In the Teichmüller space, a sequence of pinching a (multi)curve converges to a point in the stratum corresponding to the same (multi)curve in the *boundary of the Weil-Petersson completion.*

In Chapter 4, we begin by investigating the Epstein surface of a hyperbolic metric with cusp singularities. In particular, we analyze the diverging factors that arise in the associated W-volume (Proposition 4.3.11), in order to define the renormalized volume of convex co-compact manifolds marked at a finite set of points in their boundary (Definition 4.3.12). We then define the adapted version of the renormalized volume of pointed convex co-compact manifolds, as a function on the compressible strata of the Weil-Petersson completion of the Teichmüller space (in Definition 4.4.18). Finally, we prove that the adapted renormalized volume converges, under a sequence of pinching a compressible multicurve, to the sum of the adapted renormalized volumes of the pointed convex co-compact manifolds appearing as Gromov-Hausdorff limits. In this sense, the adapted renormalized volume extends continuously to the strata of the boundary corresponding to compressible multicurves.

Theorem 4.4.20. Let $M_t = (M, g_t)$ be a path of convex co-compact hyperbolic 3-manifolds obtained by pinching a compressible multicurve m in the conformal boundary at infinity. Let $D(m)$ be a union of disks compressing m, and let (M_i, g_i, P_i), for $i = 1, \ldots, k$, be the pointed convex co-compact limits of $(M_t, y_i(t))$ in the Gromov-Hausdorff topology, with $y_i(t)$ in the thick part of the i-th connected component of $C(M_t) \setminus D(m)$. Then, outside a codimension-one set

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = \sum_{i=1}^{k} \widetilde{V_R}(M_i, g_i, P_i) \ .$$

Outline of the thesis

Briefly, the thesis is organized as follows. A more detailed outline can be found at the beginning of each chapter.

In Chapter 1 we furnish all the background and the preliminary knowledge common to the other chapters.

In Chapter 2 we study Maldacena's question (Question 2.1.2), getting an explicit upper bound for the renormalized volume of convex co-compact handlebodies (i.e. a Schottky manifold) in terms of the length of k short simple closed curves, the number k, and the genus (Theorem 2.6.1). Thanks to this, we obtain explicit conditions on the lengths of the curves of the boundary surface X in order to have a Schottky filling of X of negative renormalized volume (Corollary 2.6.2). The content of the chapter can be found in the following joint work with Tommaso Creamschi and Jean-Marc Schlenker.

- Tommaso Cremaschi, Viola Giovannini, and Jean-Marc Schlenker. Filling Riemann surfaces by hyperbolic Schottky manifolds of negative volume, 2024. (Under review) https://arxiv.org/abs/2405.07598

Chapter 3 focuses on the Schwarzian derivative on long complex projective tubes and its application to the renormalized volume. There, we prove Theorem 3.5.4 on the form of the Schwarzian, Theorem 3.5.7 on its pairing with infinitesimal grafting and earthquake; we then we apply these results to Theorems 3.4.1 and 3.4.5, which concern the behaviour of the renormalized volume under complex earthquake. This is the fruit of the following joint work with Tommaso Cremaschi.

- Tommaso Cremaschi and Viola Giovannini. Behaviour of the Schwarzian derivative on long complex projective tubes. 2025. (Submitted) https://arxiv.org/abs/2502.10071

In Chapter 4, we treat the adapted renormalized volume. The first part of the chapter is occupied by the study of the Epstein surface and W-volume associated to a hyperbolic metric with cusp singularities. This leads to the definition of the renormalized volume for pointed convex co-compact manifolds (Definition 4.3.12), and a proof of its finiteness (Theorem 4.3.14). We then proceed to define the adapted renormalized volume, and showing its properties, among which Theorem 4.4.5, Theorem 4.4.6, and Corollary 4.4.7. We conclude proving its continuous extension, in Theorem 4.4.19. As of today, the author believes that the content of this chapter, along with a slightly improved version of the adapted renormalized volume that ensure continuity everywhere (see Section 4.5), will soon be available in the following preprint.

- Viola Giovannini. Adapted renormalized volume for hyperbolic 3-manifolds with compressible boundary. 2025. (Soon on Arxiv)

Chapter 1

Preliminaries and Background

1.1 Basics on Hyperbolic manifolds

We begin with some general basic knowledge of hyperbolic geometry, before turning our attention to the dimensions $n = 2, 3$. For a complete introduction on hyperbolic geometry we refer to [Mar16b, Part 1], [Mar16a, Chapter 1].

The n-dimensional *hyperbolic space* $\mathbb{H}^n$ is the unique, complete, simply connected Riemannian manifold of constant sectional curvature -1. One way to define it is via the so called *Hyperboloid model*:

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n,1} |\ q_{n,1}(x,x) = -1,\ x_{n+1} > 0\}$$

where $q_{n,1}$ denotes the Lorentzian scalar product of the Minkowski space $\mathbb{R}^{n,1}$ defined as

$$q_{n,1}(x,y) = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}$$

which it is easily verified that restricts to a positive definite, non-degenerate scalar product on each tangent space of $\mathbb{H}^n$. It can be shown that this model is isometric to both the *Poincaré disc*

$$\left(\{x \in \mathbb{R}^n |\ \ \|x\| < 1\},\ h(x) = \frac{4}{\left(1 - \|x\|^2\right)^2} g_E(x) \right),$$

and the *Upper half-space model*

$$\left(\{x \in \mathbb{R}^n |\ x_n > 0\},\ h(x) = \frac{1}{x_n^2} g_E(x) \right),$$

where g_E stands for the standard Euclidean metric; in particular the last two models are *conformal* to the Euclidean space. The upper half-space model is the one we will use the most.

A complete Riemannian manifold M^n without boundary is *hyperbolic* if it is locally isometric to the hyperbolic space $\mathbb{H}^n$. Then, when M is connected, it can be expressed as a quotient of $\mathbb{H}^n$ by a subgroup Γ of the orientation-preserving isometry group $\mathrm{Isom}^+(\mathbb{H}^n)$ acting freely and properly discontinuously

$$M = \mathbb{H}^n/\Gamma.$$

In hyperbolic geometry, it is known that a subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ is free if and only if it does not have isometries with a fixed point, and that Γ acts in a proper discontinuous way if and only if it is discrete − by Myers-Steenrod Theorem, isometry groups of Riemannian manifolds are Lie groups.

Every isometry of $\mathbb{H}^n$ extends uniquely to a homeomorphism of the closure $\overline{\mathbb{H}}^n$. An element of $\mathrm{Isom}^+(\mathbb{H}^n)$ is called *hyperbolic*, *parabolic* or *elliptic*, respectively, if it has exactly two fixed points in $\partial\mathbb{H}^n$ and no fixed points in $\mathbb{H}^n$, if it has exactly one fixed point in $\partial\mathbb{H}^n$ and no fixed points in $\mathbb{H}^n$, or if it has at least one fixed point in $\mathbb{H}^n$.

From the two conformal models of $\mathbb{H}^n$ defined above, we notice that the hyperbolic metric tensor h is not well defined on the boundary $\partial\mathbb{H}^n = \mathbb{S}^{n-1}$ of the topological compactification. However, we can renormalize h by a smooth factor $\rho\colon \mathbb{H}^n \to \mathbb{R}$ in such a way that $\rho^2 h$ can be continuously extended to the boundary. The resulting *boundary at infinity* $\partial_\infty\mathbb{H}^n$ is $\partial\mathbb{H}^3$ equipped with the *conformal structure* given by the class $[\rho^2 h]$ of Riemannian metrics conformally equivalent to $\rho^2 h$ (see Section 1.2).

In what follows, we will be interested in the dimensions $n = 2, 3$. In this regard, we point out that

$$\partial_\infty\mathbb{H}^2 = \mathbb{RP}^1 \quad \text{and} \quad \partial_\infty\mathbb{H}^3 = \mathbb{CP}^1 \ .$$

Concerning the isometry groups, using the half-space model, it can be proven that

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathbb{P}\mathrm{SL}(2,\mathbb{R}) \quad \text{and} \quad \mathrm{Isom}^+(\mathbb{H}^3) = \mathbb{P}\mathrm{SL}(2,\mathbb{C}) \ .$$

This phenomenon gives rise to a deep connection between the world of hyperbolic infinite volume *tame* 3-manifolds and that of Riemann surfaces, and also, of complex projective structures. A manifold is called *tame* if it is homeomorphic to the interior of a compact manifold with boundary; by the Tameness Theorem, every complete hyperbolic 3-manifold with finitely generated fundamental group is tame, [CG06, Ago04].

1.2 Hyperbolic and Riemann surfaces

We now focus on surfaces. Any oriented surface of negative Euler characteristic is *hyperbolic*, that is, it can be equipped with a Riemannian metric locally isometric to $\mathbb{H}^2$. In particular, every closed smooth surface S_g of genus $g \geq 2$ can be expressed as a quotient

$$\mathbb{H}^2/\Gamma$$

with $\Gamma < \mathbb{P}SL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ a discrete, torsion-free subgroup of the group of orientation-preserving isometries of $\mathbb{H}^2$.

On the other hand, a *Riemann surface* (of genus $g \geq 2$) is a smooth 2-manifold equipped with an atlas of charts into open subsets (of the upper half-plane $\mathbb{H}^2$) of $\mathbb{C}$, whose transition maps are biholomorphisms. Two complex structures on S are *isomorphic* if there exists a biholomorphic homeomorphism between them, and *marked isomorphic* if the biholomorphism is isotopic to the identity. In the *upper half-plane* model, the two dimensional hyperbolic space is given by the pair

$$\left(\mathbb{H}^2, \rho(z)|dz|^2\right) \ ,$$

where $\mathbb{H}^2 \subseteq \mathbb{C}$ is the set of complex vectors with positive imaginary part, $z = x + iy$ is the complex coordinate of $\mathbb{C}$, $|dz|^2 = dx^2 + dy^2$ is the standard Euclidean metric on the plane and $\rho(z) = 1/y^2$ (see [Mar16b, Chapter 2]).

Notation 1.2.1. In this work, we will use the notation $\mathbb{H}^2$ both for the 2-dimensional hyperbolic space and the subset of complex numbers with strictly positive imaginary part.

There is a strong correspondence between hyperbolic and Riemann surfaces: by Riemann Uniformization Theorem [Hub16, Theorem 1.1.1], the universal cover of any closed Riemann surface of genus $g \geq 2$ is biholomorphic to the open unit disk, which is itself biholomorphic to the upper half-plane. Then, every closed Riemann surface of genus $g \geq 2$ can be realized as a quotient of $\mathbb{H}^2$ by a subgroup of biholomorphisms of $\mathbb{H}^2$ acting freely and properly discontinuously, i.e. by a discrete torsion free subgroup of $\mathbb{P}SL(2, \mathbb{R})$, and it is therefore a hyperbolic surface. Furthermore, two complex structures are isomorphic (resp. marked isomorphic) if and only if their corresponding hyperbolic structures are isometric (resp. isometric via a diffeomorphism isotopic to the identity). A Riemann surface structure on S can also be thought as a *conformal class of Riemannian metrics* $[g]$ on S where

$$g \sim h \ \text{ iff } \ h = e^{2u} g$$

for some smooth function $u \colon S \to \mathbb{R}$, in which there exists a unique hyperbolic representative. In this work, we will use all three viewpoints, and jump from one to another depending on which one is more suited.

Curves

We will often talk about curves and their lengths. In this work, a *closed curve* on a surface S is defined as an immersion of S^1 into S, and it is *simple* if it is an embedding. A fundamental fact in hyperbolic geometry is that in every free homotopy class of a closed curve, there exists a unique closed geodesic. The *length* of a curve is then defined as the length of the unique geodesic representative in the same homotopy class.

A *multicurve* is a finite union of pairwise disjoint simple closed curves. When no confusion arises, we will also refer to its geodesic realization, that is, the union of the corresponding simple closed geodesics, as a multicurve.

1.2.1 Tubes

We wish to dedicate some space here to a somewhat special surface that will play a central role in Chapter 3, and that will also reappear frequently in Chapter 4: the *cylinder*, also referred as the *annulus*, or, more often here, as the *tube*.

Topologically, a tube can be characterized as an orientable surface of Euler characteristic equal to 0 that is not homeomorphic to the torus. Therefore, a tube is homeomorphic to a sphere with two punctures, to a once-punctured sphere with one open disk removed, or to a sphere with two open disks removed. We will consider all these surfaces, and, when we want to emphasize to which one we are referring to, we will use, for the first and the last cases, the terms *infinite tube* and *truncated tube*, respectively.

A peculiar fact about the tube is that it is the unique surface of zero Euler characteristic that admits a complete hyperbolic Riemannian metric, on top of the flat Euclidean one. Moreover, it even admits both a complete hyperbolic metric with two infinite volume *ends*, and a complete hyperbolic metric with just one infinite volume end. Here, an *end* is a connected component of the complement of any compact subset of the tube. To realize these, it is enough to quotient $\mathbb{H}^2$, respectively, by a hyperbolic or a parabolic isometry. The flat Euclidean metric is instead realized by quotienting $\mathbb{R}^2$ by a translation. In this way, every complete constant curvature structure on a tube is realized. The hyperbolic and the flat Euclidean metrics on the truncated tube are incomplete.

A conformal structure X on a tube is determined by its *modulus*.

Definition 1.2.1. Let X be a conformal structure on a surface S, and let γ be a rectifiable curve on S. The *extremal length* of γ with respect to X is

$$ext(\gamma, X) = \inf_{g \text{ admissible}} \text{Area}(S, g) ,$$

where g is *admissible* if and only if it is a Riemannian metric in the conformal class of X and the length $\ell_g(\gamma)$ of γ with respect to g is at least 1, where

$$\ell_g(\gamma) = \inf_{\gamma' \in [\gamma]} \ell_g(\gamma') ,$$

with $[\gamma]$ the homotopy class of γ. The *modulus* of γ with respect to X is defined as

$$m(\gamma, X) = \frac{1}{ext(\gamma, X)} .$$

The definition extends, more generally, to a set of rectifiable curves, but we will not use it. It is straightforward to notice that the modulus is a conformal invariant. In the case of γ being the *core*, i.e. the unique non-trivial simple closed curve of a tube equipped with a conformal structure X, the supremum of Definition 1.2.1 is realized at any Euclidean metric g conformal to X (see [Ahl10, Section 4.2]). We will call the modulus of a conformal tube with respect to the core simply as the modulus of the tube, and we will denote it by m. It is then easy to verify that the modulus of a tube obtained by identifying, via an horizontal translation, the edges of a rectangle in $\mathbb{R}^2$ of base 2π and height h is given by $m = 2\pi/h$. Moreover, by applying the conformal map $e^{-iz}\colon \mathbb{R} \times [0, 2\pi] \to \mathbb{C}^*$, one finds out that the modulus of an annulus $\mathcal{A} = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$, is

$$m = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right) . \tag{1.1}$$

Since any truncated tube is conformal to an annulus $\mathcal{A} = \{z \in \mathbb{C} \mid 1 < |z| < r\}$, for some $r > 1$ (see [Ahl21, Chapter 6, Section 5.2]), by identity (1.1), the modulus determines its conformal structure.

Long tubes

An important role in this work is played by *long tubes*, i.e., conformal tubes with large modulus. We now introduce a particularly important class of long tubes: those furnished by the Collar Lemma, [Bus10, Theorem 4.1.1]. A crucial property of the tubes in this class is that they are pairwise disjoint when their *core* curves are sufficiently *short*, [Bus10, Theorem 4.1.6]. This leads to the *thin-thick decomposition* of hyperbolic surfaces, which in fact is a more general result holding for hyperbolic manifolds (see [Mar16b, Chapter 4]). We first fix the notation for the two dimensional *Margulis constant*

$$\varepsilon_0 = 2\text{arsinh}(1) , \tag{1.2}$$

which will appear frequently in the next chapters.

Definition 1.2.2. A *thin tube* in a hyperbolic surface X, is the set of points $\mathbb{T}(\ell)$ around a simple closed geodesic γ of length $\ell \le \varepsilon_0$ that are at a distance at most

$$L := \text{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right) .$$

We remark that there is no need to assume the length ℓ to be shorter than ε_0 in order for a collar of width L as above to exist. However, the condition is both necessary and sufficient to ensure the collars to be disjoint, [Bus10, Theorem 4.1.6].

1.2.2 The Teichmüller space

Good references for Teichmüller space theory are [FM11, Part 2] and [Hub16, Chapter 6-7]. We recall here what will be mainly needed in this work.

Let S be a closed, connected, orientable surface of genus $g \geq 2$. Following the above discussion on the correspondence between hyperbolic metrics, Riemann surfaces and conformal structures, the *Teichmüller space of* S can be defined in the following three equivalent ways:

$$\mathcal{T}(S) = \{h \mid h \text{ is a hyperbolic metric on } S\}/\mathrm{Diffeo}_0(S) \ ,$$
$$\mathcal{T}(S) = \{c \mid c \text{ is a complex structure on } S\}/\mathrm{Diffeo}_0(S) \ ,$$
$$\mathcal{T}(S) = \{[g] \mid g \text{ is a Riemannian metric on S}\}/\mathrm{Diffeo}_0(S) \ .$$

Here $\mathrm{Diffeo}_0(S)$ is the group of diffeomorphisms of S isotopic to the identity, acting by pull-back. Moreover $g_1 \in [g_2]$ if and only if there exists a smooth function $u\colon S \to \mathbb{R}^+$ such that $g_2 = e^u g_1$, i.e. $[g]$ represent the class of Riemannian metrics conformal to g, in which there exists a unique hyperbolic representative. In what follows, we will use, depending on the set-up, the most suitable.

When S is not connected, the Teichmüller space of S, denoted again by $\mathcal{T}(S)$, is the product of the Teichmüller spaces of each connected component.

Fenchel-Nielsen coordinates

Through the *Fenchel-Nielsen coordinates*, the Teichmüller space of a closed genus g surface S_g is identified with $\mathbb{R}^{6g-6}$. Let us briefly see how (for an exhaustive treatment, see, for example, [Bus10, Sec 6.2], [FM11, Sec 10.6], [Mar16b, Section 7.3]). First, we introduce some terminology. A *pair of pants* is a surface with no genus and three boundary components. A *multicurve* on a surface S is a set of pairwise disjoint homotopically distinct simple closed curves. A *pants decomposition* of S_g is the choice of a maximal multicurve, which then has $3g-3$ components and cuts the surface into $2g-2$ pairs of pants. Any pair of pants Q in a pants decomposition of a hyperbolic surface can be decomposed in right-angled hexagons by the 3 unique length-minimizing *orthogeodesics* connecting every pair of boundary components of Q. Fixed a pants decomposition P of S_g, we can define the set of parameters

$$(\ell_1, \cdots, \ell_{3g-3}, \theta_1, \cdots, \theta_{3g-3}) \in (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} \ ,$$

where, for $1 \leq i \leq 3g-3$, the *length parameter* ℓ_i is the length of the geodesic representative of $\gamma_i \in P$, while the *twist parameter* θ_i measures the twist in the gluing of two pair of pants across the geodesic γ_i. More precisely,

$$\theta_i = \frac{2\pi s_i}{\ell_i} \ ,$$

where s_i is the signed hyperbolic distance in the universal cover $\mathbb{H}^2$ between the two *orthogeodesics* connecting a lift of γ_i to two lifts of other two boundary components of the two pairs of pants containing γ_i (which may coincide). The lifts are determined by the action of the shortest arc joining the two boundaries distinct from γ_i. Since for any triple $a, b, c \in \mathbb{R}_{\geq 0}$ there exists a unique hyperbolic pair of pants with boundaries of length a, b and c, the parameters just defined determine the hyperbolic metric uniquely.

1.2.3 The Weil-Petersson metric

Tangent and cotangent space

The tangent and the cotangent spaces of the Teichmüller space of S at the Riemann surface $X \in \mathcal{T}(S)$, are identified, via the Bers Embedding Theorem (see [Hub16, Sections 6.6 and 6.7], or [Gar00, Chapter 6]), and through a natural pairing, respectively, with the space of *harmonic Beltrami differentials* and *holomorphic quadratic differentials* on X. We denote the spaces of harmonic Beltrami differentials and holomorphic quadratic differentials on X, with $B(X)$ and $Q(X)$, respectively.

Let X be a Riemann surface, and let $z = x + iy$ and $\overline{z} = x - iy$ denote its local holomorphic and anti-holomorphic coordinates. A *Beltrami differential* on X is a $(1, -1)$-tensor μ which can be expressed in local coordinate as

$$\mu(z) = \eta(z)\partial z \otimes d\overline{z} ,$$

where η is a measurable complex-valued function. From now on, we also assume the Beltrami differentials to be bounded in the L^∞ norm:

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in X} |\eta(z)| < \infty ,$$

where the essential supremum means that the set of points in which the norm of η is unbounded has measure zero and is not considered. A *quadratic differential* φ is a symmetric $(0, 2)$-tensor that can be locally written as

$$\varphi(z) = q(z)dz \otimes dz .$$

When $q(z)$ is a holomorphic function, φ is called a *holomorphic quadratic differential.* It is customary to use the notations

$$dz \otimes dz = dz^2 ,$$

and

$$|dz|^2 = -\frac{1}{2i}dz \wedge d\overline{z} = dx^2 + dy^2 .$$

Quadratic differentials and Beltrami differentials on a Riemann structure X on S can be paired by integrating on X their product, which naturally identifies with a 2-form. For any $\mu = \eta\partial_z \otimes d\overline{z}$ and $\varphi = qdz^2$, the natural pairing is then defined as

$$\langle\varphi, \mu\rangle = \int_X (q\eta)dx \wedge dy , \tag{1.3}$$

with $z = x + iy$ the local complex coordinate of X. This establishes a duality between the two spaces. The pairing can be restricted to be between the space of *holomorphic* quadratic differentials $Q(X)$, and the one of *harmonic* Beltrami differentials $B(X)$, i.e. the quotient of the space of Beltrami differentials by the subspace of elements whose pairing with every holomorphic quadratic differential is null. Equivalently, a Beltrami differential μ is harmonic if and only if there exists a holomorphic quadratic differential qdz^2 such that

$$\mu = (\overline{q}/\rho)\partial z \otimes d\overline{z} ,$$

where $\rho|dz|^2$ is the unique hyperbolic metric representative in X. Let $\rho_X = \rho\,|dz|^2$ be the hyperbolic metric conformal to X, with z a local conformal coordinate of X. Then, given $\varphi \in Q(X)$, the quotient

$$\|\varphi(z)\| = |\varphi(z)|\,/\rho_X(z)\ ,$$

is a function that can be used to define L^p norms by integration on X with respect to ρ_X. In particular, the *infinity norm* of a holomorphic quadratic differential $\varphi = qdz^2$ is defined as

$$\|\varphi\|_\infty = \sup_{z\in X} \frac{|\varphi|}{|\rho_X|} = \sup_{z\in X} \frac{|q(z)|}{\rho(z)}\ ,$$

and the L^1-norm as

$$\|\varphi\|_1 = \int_X |q|\,|dz|^2\ .$$

Note that, since we are assuming S to be compact, the space of quadratic differentials with bounded infinity norm coincides with the one with bounded L^1 norm. The natural pairing on $Q(X)$ defined as

$$\langle \varphi_1, \varphi_2 \rangle = \int_X \frac{q_1\overline{q_2}}{\rho}\,|dz|^2$$

induces an isomorphism between $Q(X)$ and the dual of $Q(\overline{X})$, with $\overline{X}$ the conformal surface conjugated to X. The Bers embedding with respect to X induces an isomorphism between the tangent at X of the Teichmüller space $T_X\mathcal{T}(S)$ and $Q(\overline{X})$ (see [Hub16, Theorem 6.6.1]), and therefore an isomorphism

$$T^*_X\mathcal{T}(S) = Q(X)\ .$$

Furthermore, the natural pairing between holomorphic quadratic differentials and harmonic Beltrami differentials (1.3), gives the isomorphism

$$T_X\mathcal{T}(S) = Q^*(X) = B(X)\ .$$

Remark 1.2.3. Given a Riemann surface $X = \mathbb{H}^2/\Gamma$, the spaces $Q(X)$ and $B(X)$ can be identified with the respective spaces of Γ-invariant differentials on the universal cover $\mathbb{H}^2$.

The Teichmüller metric

The L^1 norm on $Q(X)$ induces, as its dual norm, the L^∞ norm on $B(X) = T_X\mathcal{T}(S)$, which is known as the *Teichmüller metric*. This is a Finsler metric, as it is not induced by a scalar product. In turn, up to a factor $1/2$, the Teichmüller metric induces the *Teichmüller distance* d_{Teich} on $\mathcal{T}(S)$ (see [Hub16, Theorem 6.6.5]) defined as the logarithm of the *dilation* of the *quasi-conformal* homeomorphism between the two points considered (see [FM11, Section 11.8]). An important property of the metric space $(\mathcal{T}(S), d_{\text{Teich}})$ is its completeness (see, for example, [FM11, Proposition 11.17]).

The Weil-Petersson metric

The L^2 norm on holomorphic quadratic differentials at X

$$\|\varphi\|_2^2 = \int_X \frac{|\varphi|^2}{\rho_X}\ ,$$

is induced by the inner product

$$\mathrm{Re}\langle\varphi,\psi\rangle = \mathrm{Re}\int_X \frac{\varphi\overline{\psi}}{\rho_X} \ ,$$

which furnishes a Riemannian metric on $T^*\mathcal{T}(S)$, and then, by duality, also on $T\mathcal{T}(S)$, as

$$\mathrm{Re}\langle\mu,\nu\rangle = \int_X \mu\overline{\nu}\rho_X \ .$$

This is called the *Weil-Petersson metric.*

On the other hand, the imaginary part of the *Hermitian inner product* $\langle\cdot,\cdot\rangle$, denoted by ω_{WP} is a closed 2-form [Hub16, Section 7.7], which then endows the Teichmüller space with the structure of a *Kähler* manifold. A beautiful proof of this fact lies in Wolpert's magic formula [Wol82]

$$\omega_{WP} = \frac{1}{2}\sum_{i=1}^{3g-3} d\ell_{\gamma_i}\wedge ds_{\gamma_i} \ ,$$

where the γ_i form a pants decomposition of S, and ℓ and $\theta = 2\pi s/\ell$ are the corresponding Fenchel-Nielsen coordinates.

The boundary of the Weil-Petersson completion

With respect to the Weil-petersson metric, the Teichmüller space is not complete: there exist paths *pinching* a simple closed curve γ - that is, paths along which the hyperbolic length of γ goes to zero - that have finite Weil-Petersson length and exit every compact subset (see, for example, [Wol06, Theorem 3]). This leads to define the *augmented Teichmüller space*, [Ber74], [Abi77], [Wol03, Introduction]. Briefly, as a set, the augmented Teichmüller space of S is the union

$$\mathcal{T}(S)\bigcup_{m\in\mathcal{C}(\mathcal{S})}\mathcal{S}(m)$$

with

$$\mathcal{S}(m) = \{X \mid X \text{ is a marked complete finite area hyperbolic structure on } S\setminus m\} \ ,$$

where $\mathcal{C}(S)$ denotes the set of isotopy classes of multicurves on S (or, equivalently, the set of 0-*skeletons of simplices in the curve complex* of S). The sets $\mathcal{S}(m)$ are called *strata*, and a point in $\mathcal{S}(m)$ can be interpreted as a *nodal surface*, or, equivalently, as a hyperbolic surface where all the curve of m have been *pinched*:

$$\mathcal{S}(m) = \{X \mid \ell_\gamma(X) = 0 \ \text{ iff } \ \gamma\in m\} \ .$$

The topology on the augmented Teichmüller space is furnished by extending the Fenchel-Nielsen coordinates in the following way. Given a multicurve m, complete it to a pants decomposition P. Then, the set $\mathcal{T}(S)\cup\mathcal{S}(m)$ is described by allowing the length parameters of the simple closed curves in $m\subseteq P$ to take values in $\mathbb{R}_{\geq 0}$. The union $\mathcal{T}(S)\cup\mathcal{S}(m)$ is equipped with the coarsest topology which makes the Fenchel-Nielsen coordinates map, forgetting about the twisting parameters of the curves in m, continuous

$$\mathcal{T}(S)\cup\mathcal{S}(m) \longrightarrow (\mathbb{R}_{>0}\times\mathbb{R})^{3g-3-|m|}\times\mathbb{R}_{\geq 0}^{|m|} \ .$$

It is possible to show that this topology does not depend on the choice of the pants decomposition P. Repeating the procedure for each $m \in \mathcal{C}(S)$ defines the topology on the whole space.

In [Ber74] and [Mas76], it is shown that the Weil-Petersson completion of the Teichmüller space coincides with the augmented Teichmüller space. Thus, the boundary of the Weil-Petersson completion of the Teichmüller space $\partial\overline{\mathcal{T}(S)}^{wp}$ is stratified by the space of multicurves $\mathcal{C}(S)$: a point in the boundary is the data of a multicurve m on S and a complete finite area hyperbolic metric on $S \setminus m$. A *stratum* in $\partial\overline{\mathcal{T}(S)}^{wp}$ is then the product of lower dimensional Teichmüller spaces of the connected components of $S \setminus m$. Moreover, the induced metric on $\partial\overline{\mathcal{T}(S)}^{wp}$ coincides with the (product of) Weil-Petersson metrics on the strata.

Definition 1.2.4. We say that a path of Riemann surfaces X_t in the Teichmüller space of S is obtained by *pinching* a multicurve $m \subseteq S$, if the hyperbolic lengths of all the components of m tend to zero, and X_t converges to a point in the Weil-Petersson completion of $\mathcal{T}(S)$.

1.3 Complex projective surfaces

Complex projetive surfaces are the primary subject of Chapter 3. More generally, they play an important role in the study of convex co-compact 3-manifolds, as they naturally arise as structures on their boundary at infinity (see Section 1.4.1).

Let $\mathbb{CP}^1$ denote the Riemann sphere. A *complex projective surface* is a smooth oriented 2-manifold equipped with a maximal atlas consisting of charts into open subsets of $\mathbb{CP}^1$ whose transition maps are restriction of Möbius transformations $\varphi \in \mathbb{P}SL(2,\mathbb{C})$, i.e. restriction of biholomorphisms of $\mathbb{CP}^1$. Analogously to Riemann surfaces, complex projective structures on a surface S are often considered up to *marked isomorphism*:

$$\mathcal{P}(S) = \{Z \mid Z \text{ is a complex projective structure on } S\}/\mathrm{Diffeo}_0(S) \ ,$$

where $\mathrm{Diffeo}_0(S)$ is the group of diffeomorphisms of S isotopic to the identity, which acts by pulling back the atlas of projective charts. Thus, two projective surfaces are the same point in $\mathcal{P}(S)$ if and only if there exists $f \in \mathrm{Diffeo}_0(S)$ pulling back the projective charts of the target surface to those of the first one.

Since Möbius transformations are holomorphic, a complex projective surface Z also induces a Riemann surface $X = \pi(Z)$, where π is the projection

$$\pi \colon \mathcal{P}(S) \to \mathcal{T}(S)$$

defined by forgetting about the projective structure. Vice-versa, the realization of a Riemann surface of genus $g \geq 2$ by a quotient $\mathbb{H}^2/\Gamma$ with $\Gamma < \mathbb{P}SL(2,\mathbb{R}) < \mathbb{P}SL(2,\mathbb{C})$ naturally defines a complex projective structure, which is called *Fuchsian*. The section of π given by the Fuchsian structures, together with the *Schwarzain derivatives of the developing maps* (introduced in the next section) of each complex projective structure make $\mathcal{P}(S)$ a holomorphic vector bundle over $\mathcal{T}(S)$ diffeomorphic to $\mathbb{R}^{12g-12}$ (see [Dum08, Sections 3.2, 3.3]).

Any complex projective structure is uniquely determined by the data of a *developing map* and a *holonomy representation*; equivalently, it is a $(\mathbb{P}SL(2,\mathbb{C}), \mathbb{CP}^1)$-structure. More precisely, a complex projective structure Z on S lifts to a complex projective structure $\widetilde{Z}$ on the universal cover $\widetilde{S}$, and a developing map for Z is an immersion

$$f \colon \widetilde{S} \to \mathbb{CP}^1$$

which restricts to projective charts of $\widetilde{Z}$ on small enough open subsets. Moreover, the developing map is unique up to Möbius transformations. The *holonomy representation* of Z on S is a homomorphism

$$\rho\colon \pi_1(S) \to \mathbb{P}SL(2,\mathbb{C}) \ ,$$

such that, for any $\gamma \in \pi_1(S)$

$$f \circ \gamma = \rho(\gamma) \circ f \ .$$

Again, the holonomy representation is unique up to conjugation by a Möbius transformation (which corresponds to post-composing f with the same Möbius map).

The Thurston metric

An open domain $\Omega \subseteq \mathbb{CP}^1$ is naturally equipped with a complex projective structure induced by the one of $\mathbb{CP}^1$. The domain Ω is assumed to be *hyperbolic*, that is, its universal cover is $\mathbb{H}^2$. The developing map $f\colon \mathbb{H}^2 \to \mathbb{CP}^1$ of Ω equips it with a Riemannian metric called the *Thurston metric*, or also the *projective metric*. This is defined as

$$h_{Th}(z) = \inf_D h_D(z) \ ,$$

where the infimum is taken over the round disks D immersed in Ω, and h_D is the hyperbolic metric on D. The Thurston metric is conformal to the Riemann surface structure naturally induced on Ω.

1.3.1 The Schwarzian derivative

As already seen in Section 1.2.3, given a Riemann surface X, a holomorphic quadratic differential on X is a holomorphic section of the symmetric square of its holomorphic cotangent bundle, i.e., in local holomorphic coordinate, can be expressed as $\varphi(z)dz \otimes dz = \varphi(z)dz^2$. Let $D \subseteq \mathbb{C}$ be a connected open set, and let $f\colon D \to \mathbb{CP}^1$ be a locally injective holomorphic map. The *Schwarzian derivative* of f is the holomorphic quadratic differential

$$\mathcal{S}(f) = \left(\left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \right) dz^2 \ .$$

The Schwarzian derivative has the following properties:

1. Let f and g be locally injective holomorphic maps such that the composition is well defined, then
$$\mathcal{S}(f \circ g) = g^* \mathcal{S}(f) + \mathcal{S}(g) \ .$$
2. For any holomorphic map $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is an open subset, $\mathcal{S}(f) = 0$ if and only if $f \in \mathbb{P}SL(2,\mathbb{C})$, that is, if and only if f is the restriction to U of a Möbius transformation.

The Schwarzian of a complex projective structure

In this paragraph, we define what is arguably the most important tool of this work.

Given X a connected Riemann surface, as already seen, the Riemann Uniformization Theorem implies that its universal cover $\widetilde{X}$ is conformal to $\mathbb{H}^2$, and that there exists a $\Gamma < \mathbb{P}SL(2,\mathbb{R})$ such that $X = \mathbb{H}^2/\Gamma$. Then, the developing map of each complex projective structure $Z = (f, \rho)$ in the fiber $\pi^{-1}(X)$ has $\mathbb{H}^2$ as domain and, for any $\gamma \in \Gamma$, satisfies the equivariant property

$$f \circ \gamma = \rho(\gamma) \circ f \quad \text{for all } \gamma \in \Gamma \ ,$$

where ρ is the holonomy representation of Z. Thanks to this, since $\mathbb{H}^2 \subseteq \mathbb{C}$, and f is holomorphic and locally injective, the Schwarzain derivative $\mathcal{S}(f)$ of the developing map f of a $Z \in \pi^{-1}(X)$ is a Γ-invariant holomorphic quadratic differential on $\mathbb{H}^2$, and therefore a holomorphic quadratic differential on X. This is called the *Schwarzian of the complex projective structure* Z.

1.4 Convex co-compact 3-manifolds

A hyperbolic 3-manifold M is homeomorphic to $\mathbb{H}^3/\Gamma$ for Γ a discrete, torsion free, subgroup of $\mathbb{P}SL(2, \mathbb{C})$, the isometry group of $\mathbb{H}^3$.

The action of Γ on $\mathbb{H}^3$ can be naturally extended to $\partial_\infty \mathbb{H}^3 = \mathbb{CP}^1$, but it does not remain properly discontinuous: the closure of the orbit of a point $x \in \mathbb{H}^3$ has non-empty set of accumulation points $\Lambda_x(\Gamma)$ in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$. Since Γ acts by isometry on $\mathbb{H}^3$, it is easy to show that actually $\Lambda_x(\Gamma)$ does not depend on x. It is then denoted simply by $\Lambda(\Gamma)$ the *limit set of* Γ

$$\Lambda(\Gamma) = \overline{\Gamma \cdot x} \cap \partial_\infty \mathbb{H}^3 \ ,$$

for any choice of $x \in \mathbb{H}^3$. The complement

$$\Omega(\Gamma) = \partial_\infty \mathbb{H}^3 \setminus \Lambda(\Gamma) = \mathbb{CP}^1 \setminus \Lambda(\Gamma)$$

is called the *domain of discontinuity of* Γ. Note that $\Lambda(\Gamma)$ is closed, and that both $\Lambda(\Gamma)$ and $\Omega(\Gamma)$ are Γ-invariant. Moreover, the action of Γ on $\Omega(\Gamma)$ is properly discontinuous.

The convex core

The *convex core* of $M = \mathbb{H}^3/\Gamma$ is defined as

$$C(M) = \mathrm{Hull}(\Lambda(\Gamma))/\Gamma \ ,$$

where $\mathrm{Hull}(\Lambda(\Gamma))$ is the convex envelope of the points of $\Lambda(\Gamma)$ in $\mathbb{H}^3 \cup \partial \mathbb{H}^3$. The convex core of M is also characterized as the smallest non-empty *strongly geodesically convex* subset of M, that is, the smallest convex subset of M which is also homotopically equivalent to M. Here, strongly indicates that any geodesic segment in M with endpoints in K is entirely contained in K. It is also not difficult to prove that if M has finite volume, then the limit set $\Lambda(\Gamma)$ coincides with $\mathbb{H}^3$, and so $C(M) = M$. In this work, we will be interested in the case of M having infinite volume. The convex core $C(M)$ is generically a 3-dimensional domain, but in some cases, it can be a totally geodesic surface in M, possibly with geodesic boundary.

Definition 1.4.1. A hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ is *convex co-compact* if its convex core $C(M)$ is compact.

1.4.1 The conformal boundary at infinity

Let $M = \mathbb{H}^3/\Gamma$ be a complete hyperbolic 3-manifold with finitely generated fundamental group $\Gamma < \mathbb{P}SL(2,\mathbb{C})$. Thanks to the fact that the action of Γ on the open domain $\Omega(\Gamma)$ is properly discontinuous, the quotient

$$\partial_\infty M = \Omega(\Gamma)/\Gamma$$

is a surface. Moreover, by Ahlfors Finiteness Theorem [Ahl64, Mar06], it is a surface of finite type, that is, compact with a finite number of punctures. Since $\Omega(\Gamma)$ is an open subset of $\mathbb{CP}^1$, and the elements of $\mathbb{P}SL(2,\mathbb{C})$ are biholomorphisms of the Riemann sphere, the surface $\partial_\infty M$ is naturally equipped with a complex projective structure, which in turns induces a Riemann surface structure. For this reason, the surface $\partial_\infty M = \Omega(\Gamma)/\Gamma$ is called *the conformal boundary at infinity of $M = \mathbb{H}^3/\Gamma$.*

When M is convex co-compact, then $\overline{M} = M \cup (\partial_\infty M)$ is its *manifold compactification*, and, recalling that M is tame, the boundary $\partial_\infty M$ is homeomorphic to the closed surface $S = \partial\overline{M}$. Therefore

$$[\partial_\infty M] \in \mathcal{T}(S) \ ,$$

where $\mathcal{T}(S)$ denotes the product of the Teichmüller spaces of the connected components of S.

Definition 1.4.2. Let M be a convex co-compact hyperbolic 3-manifold. We define the *Schwarzian of M*, and we denote it by $\mathcal{S}(f_M)$, as the Schwarzian derivative of the product of the developing maps of the complex projective structures on each connected component of $\partial_\infty M$.

Connected components of $\overline{M \setminus C(M)}$, or, more generally, of the complement of a geodesically convex compact subset of M, are called *ends*. By the Tameness Theorem, an end is homeomorphic to $S^i \times [0, +\infty)$, with S^i a component of $S = \partial_\infty M$. The hyperbolic Riemannian metric on the ends diverges exponentially. In particular, the ends have infinite hyperbolic volume.

1.4.2 A Uniformization Theorem

In this section, we state the beautiful uniformization theorem for the deformation space of convex co-compact hyperbolic manifolds, the fruit of various results in the work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston.

Definition 1.4.3. The *deformation space of convex co-compact hyperbolic structures on M*, considered up to diffeomorphisms homotopic to the identity, is denoted by $CC(M)$.

As the title of this thesis suggests, the following is a crucial definition for this work.

Definition 1.4.4. A non-trivial simple closed curve in the boundary $\partial\overline{M}$ is *compressible* if it bounds a disk in M, i.e. if it is null-homotopic in M. The boundary $\partial\overline{M}$ is said to be *incompressible* if the inclusion

$$\partial\overline{M} \hookrightarrow M$$

is π_1-injective, and *compressible* otherwise.

Analogously, we call a *multicurve* in $\partial\overline{M}$ *compressible* if all of its components are compressible.

Let us denote by $T_0(D)$ the subgroup of the mapping class group of $\partial\overline{M}$ generated by all the *Dehn twists* along compressible simple closed curves in $\partial\overline{M}$. This coincides with the maximal subgroup of non-trivial diffeomorphisms of the boundary surface that extend to a diffeomorphism of M isotopic to the identity. We are now ready to state the Uniformization Theorem (see, for example, [Mar16a, Theorem 5.1.3.] and [MT98, Theorem 5.27]).

Theorem 1.4.5 (Ahlfors, Bers, Kra, Marden, Maskit, Sullivan, Thurston)**.** The deformation space $CC(M)$ is biholomorphically parameterized by the space of conformal structures on the boundary at infinity, as

$$CC(M) = \mathcal{T}(\partial\overline{M})\Big/T_0(D) \ ,$$

where $\mathcal{T}(\partial\overline{M})$ is the product of the Teichmüller spaces of the connected components of $\partial\overline{M}$.

1.5 Epstein surfaces

Epstein surfaces are essential for defining the W-volume of a convex co-compact manifold M with respect to a Riemannian metric g that is conformal to $\partial_\infty M$ (see Section 1.6). They will also be a crucial element of Chapter 4.

Epstein surfaces were introduced for the first time in [Eps84] (the original pre-print dates to 1984, but we reference a LaTex transcription by Bridgeman of 2024). The construction of Epstein produces a surface in $\mathbb{H}^{n+1}$ as an envelope of horospheres, starting from any domain Ω in $\mathbb{S}^n = \partial_\infty\mathbb{H}^{n+1}$ equipped with a conformal metric g:

$$\mathrm{Eps}_{(\Omega,g)} \colon \Omega \to \mathbb{H}^{n+1} \ .$$

Epstein surfaces have been used and studied extensively in recent literature. We present a likely non-exhaustive selection of works. First, in [KS08] the fundamental forms on the Epstein surface induced by $\mathbb{H}^3$were studied, together with their relations to the corresponding forms at infinity (see Section 1.5.3); then, Epstein surfaces were used there to define the W-volumes (see Section 1.6). In [Dum17], Epstein surfaces are used to study limits of holonomy representations of complex projective structures. A comprehensive and elegant treatment of Epstein surfaces, the Osgood-Stowe differential and W-volumes, along with their interrelations, is provided in [BB24]. In [BP23] Epstein surfaces are introduced for domains in $\mathbb{CP}^1$ with round boundaries: a construction we will use in Chapter 4. The work [BBPW25] investigates the Epstein surfaces associated to quasi-circles in $\mathbb{CP}^1$, applying them to the study of renormalized volume and the Liouville action in the universal setting. In [PWW25], is treated the case $n = 1$, exploring the relationship between Epstein circles and the Schwarzian action of diffeomorphisms of the circle.

In this thesis, we are interested in the case $n = 2$, and mainly when the domain in question is a discontinuity domain $\Omega(\Gamma)$ of some convex co-compact manifold $M = \mathbb{H}^3/\Gamma$. In this setting, the resulting Epstein surface is Γ-invariant, so that it descends to a surface in the quotient 3-manifold $\mathbb{H}^3/\Gamma$.

Briefly, the construction is as follows. Let $g = e^{2\varphi}\left|dz\right|^2$ be a smooth metric on a domain $D \subseteq \mathbb{CP}^1 = \partial_\infty\mathbb{H}^3$. For each $z \in D$, consider the *horosphere* $H(z, e^{-\varphi(z)})$ in $\mathbb{H}^3$

pointed at z and of Euclidean radius $e^{-\varphi(z)}$. The *Epstein surface associated to* (Ω, φ) is then obtained by taking the boundary of the convex envelop of the union of this family of horospheres. The metric does not need to be smooth. On the other hand, even when g is smooth, the resulting surface may fail to be immersed. We discuss the regularity in the next sections.

As just presented, the construction might appear somewhat mysterious. In particular, one might naturally question why the radius of the horosphere is chosen to be $e^{-\varphi(z)}$. To provide a better understanding of the situation, and before stating some key properties of the Epstein surfaces, we need to introduce the so called visual metric.

1.5.1 The visual metric

Any point $p \in \mathbb{H}^3$ defines a conformal metric on $\partial_\infty \mathbb{H}^3 = \mathbb{CP}^1$ obtained by pushing forward the induced metric on the unit tangent bundle $T^1_p\mathbb{H}^3$ at p through the exponential map at p, which is a homeomorphism from $T^1_p\mathbb{H}^3$ to $\mathbb{CP}^1$. This metric is called the *visual metric from* p. For example, the visual metric induced from $p = (0,0,1)$ in $\mathbb{H}^3 = \{(z,t) \in \mathbb{C}\times\mathbb{R}^+\}$ to $\partial_\infty \mathbb{H}^3$ is the *spherical metric*

$$g_{\mathbb{S}^2}(z) = \frac{4}{(|z|^2+1)}\,|dz|^2 \ .$$

The visual metric at any other point $p \in \mathbb{H}^3$, is then obtained by pulling back $g_{\mathbb{S}^2}$ via any isometry of $\mathbb{H}^3$ sending p to $(0,0,1)$. More explicitly, given $p = (w,t) \in \mathbb{H}^3$, with $w \in \mathbb{C}$ and $t \in \mathbb{R}^+$, the visual metric from p at $\overline{z}$ is

$$v_p(\overline{z}) = \frac{4t^2}{(|w-\overline{z}|^2+t^2)^2}|dz|^2.$$

Thanks to this, given φ a real valued function

$$H(\overline{z}, e^{-\varphi(\overline{z})}) = \{p \in \mathbb{H}^3 \mid v_p(\overline{z}) = e^{2\varphi(\overline{z})}|dz|^2\} \ ,$$

that is, the set of points in $\mathbb{H}^3$ whose visual metric induced on $\overline{z}$ coincides with the conformal metric $e^{2\varphi(\overline{z})}|dz|^2$ is the horosphere centered at $\overline{z}$ and of Euclidean radius $e^{-\varphi(\overline{z})}$.

Theorem 1.5.1 ([Eps84]). Let (Ω, φ) a domain in $\mathbb{CP}^1$ equipped with a C^k conformal metric $g = e^{2\varphi(z)}\,|dz|^2$. Then, there exists a unique C^{k-1} map

$$\mathrm{Eps}_{(\Omega,\varphi)} \colon \Omega \to \mathbb{H}^3$$

such that

$$v_{\mathrm{Eps}_{(\Omega,\varphi)}(z)}(z) = e^{2\varphi(z)}\,|dz|^2 \ ,$$

and

$$D_z\mathrm{Eps}_{(\Omega,\varphi)} \subseteq T_{\mathrm{Eps}_{(\Omega,\varphi)}(z)}H(z, e^{-\varphi(z)}) \ ,$$

where D_z denotes the image of the tangent map at z.

The image of the map $\mathrm{Eps}_{(\Omega,\varphi)}$ is called *Epstein surface associated to* (Ω, φ). Note that, thanks to the considerations just above on the visual metric, the second condition in the statement implies the first one. Moreover, the Epstein map is natural in the sense that for any Möbius transformation $f \in \mathbb{P}SL(2,\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$

$$\mathrm{Eps}_{(\Omega,\varphi)} = f \circ \mathrm{Eps}_{(f^*\Omega, \varphi\circ f + \log(|df|))} \ .$$

In Section 4.5, are presented the conditions under which the Epstein surface is immersed, as well as those under which it is a convex embedding.

1.5.2 The boundary of the convex core as an Epstein surface

Let us now present another equivalent interpretation of the visual metric. Given a point $z \in \partial \mathbb{H}^3$ the visual metric $v_p(z)$ from p at z coincides with $h_{D(p,z)}(z)$, where $h_{D(p,z)}(z)$ is the hyperbolic metric on the disc $D(p,z)$ detected on $\partial_\infty \mathbb{H}^3 = \mathbb{CP}^1$ by the unique geodesic hyperplane $\mathbb{H}^2 \subseteq \mathbb{H}^3$ containing p and perpendicular to the geodesic ray connecting p to z.

With this viewpoint, when $\Omega = \Omega(\Gamma)$ is a discontinuity domain, it is possible to show [BBB19, Prop 2.13] that the horosphere obtained by imposing the equality between the visual metric and the Thurston metric h_{Th} at $z \in \Omega(\Gamma) \subseteq \partial_\infty \mathbb{H}^3$ is tangent to the lift $\mathrm{Hull}(\Lambda(\Gamma))$ to $\mathbb{H}^3$ of the convex core of $M = \mathbb{H}^3/\Gamma$. In particular, by the previous section

$$h_{Th}(z) = \frac{1}{r_z^2}|dz|^2,$$

where r_z is the radius of the horosphere centered at z and tangent to $\mathrm{Hull}(\Lambda(\Gamma))$.

1.5.3 Fundamental forms at infinity

Let us fix $\Omega \subseteq \mathbb{CP}^1$ a complex projective domain, and let us denote by $\Sigma(g)$ the Epstein surface associated to (Ω, φ), with $g = e^{2\varphi}|dz|^2$. At any immersed point, the *first fundamental form* I of $\Sigma(g)$ is the pull-back via $\mathrm{Eps}_{(\Omega,\varphi)}$ to Ω of the Riemannian metric induced on $\Sigma(g)$ by $\mathbb{H}^3$. The *shape operator* $B\colon T\Sigma(g) \to T\Sigma(g)$ is the bundle morphism defined as

$$B(X) = \nabla_X(N) \ ,$$

where N is the outer unit normal to $\Sigma(g)$, and ∇ is the Levi-Civita connection of $\mathbb{H}^3$. The *second fundamental form* of $\Sigma(g)$ is

$$I\!I(\cdot,\cdot) = I(B(\cdot),\cdot) \ ,$$

and the *third fundamental form* is

$$I\!I\!I(\cdot,\cdot) = I(B(\cdot),B(\cdot)) \ .$$

We use the same notation for their pull-back via $\mathrm{Eps}_{(\Omega,\varphi)}$ to Ω. Similarly, we denote by I_t, B_t, $I\!I_t$ and $I\!I\!I_t$, the pull-back via $\mathrm{Eps}_{(\Omega,\varphi+t)}$ on Ω of the fundamental forms and the shape operator of $\Sigma(e^{2t}g)$.

Definition 1.5.2. ([KS08], [BB24, Section 5]) In the notations above, the *fundamental forms at infinity* of (Ω,φ) are defined as

$$\begin{aligned}
\hat{I} &= I + 2I\!I + I\!I\!I = I((Id+B)(\cdot),(Id+B)(\cdot)) \\
\hat{I\!I} &= I - I\!I\!I = I((Id+B)(\cdot),(Id-B)(\cdot)) \\
\hat{I\!I\!I} &= I - 2I\!I + I\!I\!I = I((Id-B)(\cdot),(Id-B)(\cdot))
\end{aligned}$$

and the *shape operator at infinity* as

$$\hat{B} = (Id+B)^{-1}(Id-B) \ .$$

Proposition 1.5.3. ([Eps84], [KS08, Lemma 5.7, Theorem 5.8]) In the notations above, let I_t denote the pull-back on Ω of the induced metric on $\Sigma(e^{2t}g)$ via $\mathrm{Eps}_{(\Omega,\varphi+t)}$. Then

$$I_t = \frac{1}{4}\left(e^{2t}\hat{I} + 2\hat{I\!I} + e^{-2t}\hat{I\!I\!I}\right) ,$$

in particular

$$\lim_{t\to\infty} 4e^{-2t} I_t = \hat{I} .$$

Moreover

$$\hat{I} = g .$$

The induced fundamental form on a Epstein surfaces can be expressed in terms of the ones at infinity as in Definition 1.5.2 as follows.

Lemma 1.5.4 ([KS08], Lemma 5.6)**.** The fundamental forms $I, I\!I, I\!I\!I$ of an embedded Epstein surface are obtained from $\hat{I}, \hat{I}I, \hat{I\!I\!I}$ as follows

$$\begin{aligned} I &= \frac{1}{4}\left(\hat{I} + 2\hat{I\!I} + \hat{I\!I\!I}\right) = \frac{1}{4}\hat{I}((Id+\hat{B})(\cdot),(Id+\hat{B})(\cdot)) \\ I\!I &= \frac{1}{4}\left(\hat{I} - \hat{I\!I\!I}\right) = \frac{1}{4}\hat{I}((Id+\hat{B})(\cdot),(Id-\hat{B})(\cdot)) \\ I\!I\!I &= \frac{1}{4}\left(\hat{I} - 2\hat{I\!I} + \hat{I\!I\!I}\right) = \frac{1}{4}\hat{I}((Id-\hat{B})(\cdot),(Id-\hat{B})(\cdot)) \end{aligned}$$

and the *shape operator* as

$$B = (Id+\hat{B})^{-1}(Id-\hat{B}) .$$

Remark 1.5.5. Note that any result on Epstein surfaces has an equivariant version. When Ω is the discontinuity domain of $M = \mathbb{H}^3/\Gamma$, then $\Sigma(g)$ is Γ-invariant and descends to a surface in M. We will keep denoting by I, B, $I\!I$ and $I\!I\!I$, respectively, the Riemannian metric, shape operator, second and third fundamental forms induced on $\Sigma(g)/\Gamma$ by M.

Properties of the Epstein surfaces

At points z where $\mathrm{Eps}_{(\Omega,\varphi)}$ is an immersion, the Epstein surface is tangent to the horosphere $H(z, e^{-\varphi(z)})$. One has though to be careful as the map $\mathrm{Eps}_{(\Omega,\varphi)}$ is not always an immersion: for example, the image of $\mathrm{Eps}_{(\mathbb{CP}^1,\varphi)}$, with φ such that $e^{2\varphi}\left|dz\right|^2 = g_{\mathbb{S}^2}$, is the point $p = (0,0,1)$. To avoid the singular cases, the strategy is to consider the Epstein surfaces associated to rescalings of the conformal metric on Ω. More precisely, it is possible to define the map to the unit tangent bundle

$$\widetilde{\mathrm{Eps}}_{(\Omega,\varphi)} \colon \Omega \to T^1\mathbb{H}^3$$

associating to $z \in \Omega$ the unit normal vector at $\mathrm{Eps}_{(\Omega,\varphi)}(z)$ whose geodesic ray ends at z, so that, if $\pi \colon T^1\mathbb{H}^2 \to \mathbb{H}^3$ is the bundle projection, then

$$\mathrm{Eps}_{(\Omega,\varphi)} = \pi \circ \widetilde{\mathrm{Eps}}_{(\Omega,\varphi)} .$$

When the conformal metric at infinity is smooth, the map $\widetilde{\mathrm{Eps}}_{(\Omega,\varphi)}$ is a smooth immersion.

Proposition 1.5.6. ([Eps84], [BB24, Section 5.1, Theorem 5.8], [KS08, Lemma 5.6], or [BB24, Theorem 7.4]) Let $F_t\colon T^1\mathbb{H}^3 \to T^1\mathbb{H}^3$ be the geodesic flow, then

$$F_t \circ \widetilde{\mathrm{Eps}}_{(\Omega,\varphi)} = \widetilde{\mathrm{Eps}}_{(\Omega,\varphi+t/2)} \ .$$

If both the principal curvatures of the image of $\mathrm{Eps}_{(\Omega,\varphi)}$ are different from -1, then it is an immersion. Moreover, in this case, for t big enough it is a locally convex immersion.

Note that the Epstein surface image of $\mathrm{Eps}_{(\Omega,\varphi+t)}$ is the set of t-equidistant points from the image of $\mathrm{Eps}_{(\Omega,\varphi)}$.

Theorem 1.5.7. ([KS08], [BB24, Lemma 8.2]) If the conformal metric on Ω is smooth, and the associated Epstein surface is a locally convex immersion, then it is a convex embedding.

1.6 W-volumes and Renormalized volume

As already seen, convex co-compact hyperbolic 3-manifolds have infinite hyperbolic volume. Then, to meaningfully talk about volumes within this class of hyperbolic structures, some kind of renormalization will be necessary. A possibility is to consider the function

$$V_C\colon CC(M) \longrightarrow \mathbb{R}_{\geq 0} \ ,$$

which assigns to each convex co-compact structure on M the volume of its convex core $\mathrm{Vol}(C(M))$. It is shown in [BC17] that the function V_C is continuous. The renormalized volume is some kind of relative of the function V_C, which presents much better analytic properties.

The main idea is to consider an exhaustion of M by *strongly geodesically convex* compact subsets $\{C_r\}_r$ coming together with an equidistant foliation of the ends, and to renormalize the sequence of associated volumes $\mathrm{Vol}(C_r)$ in order to get a finite number, which does not depend on r.

Before giving the definition of renormalized volume, we need to introduce some preliminary notions.

Definition 1.6.1. Let M be convex co-compact and $C \subseteq M$ be a compact subset with smooth boundary. The *W-Volume of C* is defined as

$$W(C) = \mathrm{Vol}(C) - \frac{1}{2}\int_{\partial C} H dA_{\partial C} \ ,$$

where $\mathrm{Vol}(C)$ is the hyperbolic volume of C with respect to the metric of M, H denotes the mean curvature of ∂C, and $dA_{\partial C}$ is the area form of the induced metric on the boundary ∂C.

The *mean curvature* is half the trace of the shape operator $H = \mathrm{Tr}_I(I\!I)/2$.

In what follows, we assume the compact set C to be strongly geodesically convex, so that it is homotopically equivalent to M and can be used to decompose the manifold M in neighborhoods of its convex co-compact ends, and a compact piece containing the convex core $C(M)$. The additional term involving the mean curvature in the definition just above is the right one to obtain a good renormalization. Indeed, it is shown in [KS08] that if C_r denotes the r-neighborhood of C in M, then, for any $r \geq 0$

$$W(C_r) + r\pi\chi(\partial\overline{M}) = W(C) \ , \tag{1.4}$$

where $\chi(\cdot)$ is the Euler characteristic.

Definition 1.6.2. Let E be an end of a convex co-compact manifold M. An *equidistant foliation* is a foliation $\{S_r\}_{r\geq r_0}$ of a neighborhood of $\partial_\infty M$ in $\overline{M}$ in convex surfaces, such that for any $r' > r > r_0 \geq 0$ the surface $S_{r'}$ lies between S_r and $\partial_\infty M$, and its points stay at constant distance $r' - r$ from S_r.

By definition, the boundaries of the compact sets C_r as above form an equidistant foliation $\{\partial C_r\}_{r\geq 0}$ of the ends in $M \setminus C$.

Given any strongly geodesically convex subset $C \subseteq M$, and any end $E_i = S^i \times [0, +\infty)$ in $M \setminus C$, we can consider the associated equidistant foliation $\{\partial^i C_r\}_r \geq 0$, where ∂^i indicates the connected component of ∂C_r facing the boundary component $S_i \subseteq \partial_\infty M$. Let g_r denote the induced metric on ∂C_r. Then, it is possible to define a metric on the boundary at infinity as

$$g := \lim_{r\to+\infty} 4e^{-2r} g_r \in [\partial_\infty M] \ , \tag{1.5}$$

see [Sch13, Def. 3.2], or [Sch20, Def. 3.2] for a slightly different point of view. A key property of this metric g is that it belongs to the conformal class at infinity of $\partial_\infty M$. Note that this remains true if we change the constant factor 4 in (1.5). Vice-versa, up to scaling by a big enough positive constant, any smooth representative g in $[\partial_\infty M]$ can be realized in this way by constructing the foliation in Epstein surfaces associated to $e^t g$ as in Section 1.5 (see Proposition 1.5.6 and Theorem 1.5.7 in Section 1.5.3). This leads to the following bijective correspondence:

$$\left\{\begin{array}{c}\text{Riemannian metrics g on } S\\ \text{such that g} \in [\partial_\infty M],\\ \text{up to multiplication by } s \in \mathbb{R}^+\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Equidistant convex foliations}\\ \text{of a neighborhood of } \partial_\infty M,\\ \text{up to } \sim_{\mathcal{F}}\end{array}\right\}$$

where two such foliations are $\sim_{\mathcal{F}}$-equivalent if and only if they are equal outside a compact set; rescaling by a positive constant a Riemannian metric in $[\partial_\infty M]$ corresponds to a reindexing of the associated foliation.

Definition 1.6.3. The W-Volume of M with respect to a smooth $g \in [\partial_\infty M]$ is defined as

$$W(M, g) = W(C_r(g)) + \pi r \chi(\partial_\infty M) \ ,$$

where $\{C_r(g)\}_{r\geq r_0}$, for a big enough r_0, is the exhaustion in compact strongly geodesically convex subsets determined by the equidistant foliation associated to g, indexed so that the sequence of induced metrics g_r on $\partial C_r(g)$ satisfies (1.5).

Thanks to equation (1.4) above, the W-volume $W(M, g)$ is well defined. We also point out that choosing the factor 4 in the definition of the metric at infinity (1.5), is the scaling which makes the geodesically convex subset C_0 associated to the hyperbolic metric in the conformal boundary at infinity of a Fuchsian manifold coincide with its 2-dimensional convex core. In this way, if $M = S_g \times \mathbb{R}$ is Fuchsian, then $V_R(M) = 0$ (see the next definition).

We are now ready to define the renormalized volume of M.

Definition 1.6.4. For any convex co-compact hyperbolic 3-manifold M, the *renormalized volume* is defined as

$$V_R(M) = W(M, h) \ ,$$

with $h \in [\partial_\infty M]$ the hyperbolic representative in the conformal boundary at infinity.

Thanks to the parameterization of the deformation space of convex co-compact structures $CC(M)$ provided by Theorem 1.4.5, the renormalized volume can be interpreted as a function on the Teichmüller space of $\partial\overline{M}$

$$V_R \colon \mathcal{T}(\partial\overline{M}) \longrightarrow \mathbb{R} \ .$$

1.6.1 The W-volume of the convex core

Even if it does not have smooth boundary, it is still possible to define the W-volume for the convex core $C(M)$ of M. In this case, the integral of the mean curvature of the boundary is replaced by the length of the *measured bending lamination* (see [Thu80, EM86]):

$$W(C(M)) = \mathrm{Vol}(C(M)) - \frac{1}{4}L(\beta_M) \ .$$

Moreover, the equidistant foliation from $C(M)$ is associated, through the bijective correspondence discussed above, to the Thurston metric $h_{Th} \in [\partial_\infty M]$ (see Section 1.3 and Section 1.5.2), so that

$$W(C(M)) = W(M, h_{Th}) \ .$$

1.6.2 The differential of the renormalized volume

As already anticipated, the renormalized volume is differentiable, and its differential is well-understood and elegantly expressed.

Before stating the result, recall that the space of holomorphic quadratic differentials $Q(X)$ on a Riemann surface $X \in \mathcal{T}(\partial\overline{M})$ is identified with the cotangent bundle $T^*_X\mathcal{T}(\partial\overline{M})$ via the Bers embedding. Moreover, the natural pairing $\langle\cdot,\cdot\rangle$ (1.3) between $Q(X)$ and the space of harmonic Beltrami differentials $B(X)$ on X, identifies the latter with the tangent bundle $T_X\mathcal{T}(\partial\overline{M})$ (see Section 1.2.3). The Riemann surface considered here is the conformal boundary at infinity $\partial_\infty M$, which, we remind, is also naturally equipped with a complex projective structure.

Theorem 1.6.5. ([ZT87], [KS08, Corollary 8.6], [Sch13, Corollary 3.11]) Let M be a convex co-compact hyperbolic 3-manifold, and let $\mathcal{S}(f_M)$ be the holomorphic quadratic differential on $\partial_\infty M$ given by the Schwarzian derivative of the developing map f_M of $\partial_\infty M$. Then, the differential of the renormalized volume at $[\partial_\infty M] \in \mathcal{T}(\partial\overline{M})$ is

$$dV_R(\cdot) = \mathrm{Re}\langle \mathcal{S}(f_M), \cdot\rangle.$$

Theorem 1.6.5 is a consequence of two results. First, in [ZT87], Takhtajan and Zograf proved that the real part of the Schwarzian derivative of a complex projective structure is equal to the traceless part of the second fundamental form at infinity (see also [KS08, Appendix A] for a more geometrical proof); then, Krasnov and Schlenker combined this with a Schläfli type formula for the W-volume [KS08, Corollary 6.2].

Chapter 2

Filling Riemann surfaces by hyperbolic Schottky manifolds of negative volume

In this chapter, we provide conditions under which a Riemann surface X is the asymptotic boundary of a convex co-compact hyperbolic manifold, homeomorphic to a handlebody, of negative renormalized volume. We prove that this is the case when there are on X enough closed curves of short enough hyperbolic length.

2.1 Results and outline of the chapter

The volume of a closed hyperbolic 3-manifold can be considered as a measure of its "complexity", and it is a natural to ask what is the closed, orientable hyperbolic manifold of smallest volume. The answer is the Weeks manifold [GMM09].

Consider now a compact Riemann surface X. We can extend the previous question in the following manner – the case of closed hyperbolic manifolds corresponds to $X = \emptyset$.

Question 2.1.1. Given X, what is the convex co-compact hyperbolic manifold M of smallest volume, with asymptotic boundary X?

As already pointed out, convex co-compact hyperbolic manifolds have infinite volume, but they have a well-defined *renormalized volume* (see Section 1.6).

As discussed in the introduction, beyond the natural mathematical motivation, Question 2.1.1 also occurs naturally from a physical perspective, and specifically from the AdS/CFT correspondence. Very briefly, the AdS/CFT correspondence asserts the equality between the partition function of a conformal field theory (CFT) on a d-dimensional manifold X and a sum, over all $d+1$-dimensional manifolds M_i with boundary X, of a function of the action of a certain (super-)string theory on M_i. In a certain "gravity" limit, where many features disappear, it reduces to a very special and simplified statement: given a Riemann surface X, the partition function of a certain CFT on X should be recovered as a sum of exponential of minus a constant times the renormalized volumes of all convex co-compact hyperbolic manifolds M_i having X as asymptotic boundary:

$$\mathcal{A}(X) = a_0 \sum_{\partial M_i = X} e^{-cV_R(M_i)} \ .$$

where a_0 and c are constants. In this simplified view, the main term on the $d+1$-dimensional "bulk" side corresponds to the convex co-compact manifold M_i with the smallest renormalized volume.

This AdS/CFT correspondence leads to some conjectural statements. For instance, if X is disconnected, the CFT should behave independently on the two connected component, and it might therefore be expected that the convex co-compact manifold of smallest volume "filling" X should also be disconnected (see [SW22] for a more elaborate analysis).

For instance, if $X = X_+ \cup X_-$ is the disjoint union of two connected Riemann surfaces of genus at least 2, with X_- equal to X_+ with opposite orientation, we can compare:

- the Fuchsian manifold M_F with ideal boundary $X_+ \cup X_-$, which has (with the normalization used here) renormalized volume zero,
- any possible filling of $X_+ \cup X_-$ by the disjoint union of two handlebodies M_+ and M_-, with $\partial_\infty M_+ = X_+$ and $\partial_\infty M_- = X_-$.

The heuristics above suggests that one of the disconnected fillings might have negative renormalized volume. This might be a motivation for the following question, attributed to Maldacena (see [VP25]).

Question 2.1.2. Is any connected Riemann surface of genus at least 2 realizable as the asymptotic boundary of a Schottky manifold of negative renormalized volume?

By *Schottky manifold* here we mean a convex co-compact hyperbolic manifold homeomorphic to a handlebody.

2.1.1 Results

In what follows S will always denote a closed orientable surface of genus at least 2.

Existence of fillings of minimal renormalized volume

Before we consider the questions above, it is useful to know that, given a Riemann surface X of finite type, there is at least one convex co-compact filling of X of minimum renormalized volume, and that the set of those minimum volume fillings is finite. Precisely, let $\mathcal{M}(X)$ be the set of convex co-compact (or, more generally, geometrically finite) hyperbolic manifolds with ideal conformal boundary X, then we think the following question to be true.

Question 2.1.3. Let $V := \inf_{M\in\mathcal{M}(X)} V_R(M, X)$, is there a manifold $M_V \in \mathcal{M}(X)$ such that $V_R(M_V) = V$?

This is akin to the hyperbolic manifold case in which the Weeks manifold is the unique smallest volume hyperbolic 3-manifold [GMM09].

An upper bound on the renormalized volume

The main result here is an upper bound on the renormalized volume of a Schottky manifold, when it is obtained from a pants decomposition for which some of the curves are short. We denote by ε_0 the 2-dimensional Margulis constant, equal to $\varepsilon_0 = 2\mathrm{arsinh}(1)$. Given a pants decomposition P of S, we denote by M_P the handlebody with boundary S in which all curves of P are null-homotopic (see Section 2.2), and by $M_P(X)$ the convex co-compact hyperbolic manifold homeomorphic to M_P with complex structure at infinity X. The complex structure has a unique hyperbolic metric in its conformal class and we will take lengths with respect to that. Thus, by $\ell_X(\gamma)$ we mean the length of γ with respect to the hyperbolic structure induced by X. In the case in which there is no ambiguity we will often use $\ell(\gamma)$.

Theorem 2.6.1. Let X be a closed Riemann surface of genus $g \geq 2$. Assume that there are k disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ such that $\ell(\gamma_i) \leq 1, 1 \leq i \leq k$, and there are no other geodesic loops of length less or equal than 1 in X. Then there exists a pants decomposition P containing the γ_i's such that

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \frac{1}{\ell(\gamma_i)} + \left(9 + \frac{3}{4}\coth^2\left(\frac{1}{4}\right)\right)k + 81\coth^2\left(\frac{1}{4}\right)\pi(3g-3-k)(g-1)^2 .$$

By imposing the right hand side of the estimate in Theorem 2.6.1 to be negative we obtain for instance the following corollary.

Corollary 2.6.2. For all $g, k, k_1 \in \mathbb{N}$ such that $g \geq 2$, $0 < k \leq 3g-3$ and $0 < k_1 \leq k$, there exists an explicit constant $A = A(g, k_1, k-k_1) > 0$ such that, if X is a Riemann surface with k_1 geodesic loops of length less than A and k geodesic loops of length at most 1, then X admits a Schottky filling with negative renormalized volume.

Remark 2.1.4. Let us see a couple of examples for Corollary 2.6.2 in the two limit cases.

- **Case $k = k_1 = 1$.** By the inequality of Theorem 2.6.1, we have

$$A(g,1) < \frac{\pi^3}{\sqrt{e}(9 + \frac{3}{4}\coth^2(1/4) + 81\coth^2(1/4)\pi(3g-4)(g-1)^2)} ,$$

  which in the best case of genus $g = 2$ gives a bound of

$$A(2,1) \leq 0.00221.$$

  Note that, for large genus g, we obtain the asymptotic $A(g,1) \sim g^{-3}$.

- **Case $k = k_1 = 3g-3$.** By Theorem 2.6.1, since $k - k_1 = 0$, we are looking for an $A := A(g, 3g-3)$ such that

$$A < \frac{\pi^3}{\sqrt{e}(9 + \frac{3}{4}\coth^2(1/4))} ,$$

  we can then take

$$A(g, 3g-3) = 0.87458 .$$

This statement can be compared to [PMG19, Corollary 5.6], also see [VP25, Theorem 2.1], which states that: if a Riemann surface X of finite type and genus $g \geq 2$ has $g-1$ closed curves $\gamma_1, \cdots, \gamma_{g-1}$ such that the complement of their union is a disjoint union of k-holed tori, and if

$$\frac{1}{\pi - 2}\left(\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)}\right)^2 \leq \pi(g-1) ,$$

then

$$V_R(X,\mathcal{P}) \leq \pi(g-1)\left(3 - \frac{\pi(\pi-2)(g-1)}{\left(\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)}\right)^2}\right)$$

which is negative if

$$\sum_{i=1}^{g-1}\sqrt{\ell_X(\gamma_i)} \leq \left(\frac{\pi(\pi-2)(g-1)}{3}\right)^{\frac{1}{2}} ,$$

and, in the case $g=2$, leads to a better $(2,1) = \frac{\pi(\pi-2)}{3}$.

Convex co-compact fillings

The result in bounding the difference of renormalized volume under earthquake, see Theorem 2.5.4, can also be applied in the more general setting of convex co-compact manifolds. Specifically it makes sense in the setting where $N(X_0) \in CC(M)$ is a convex co-compact hyperbolic 3-manifold, homeomorphic to M, with conformal boundary $X_0 \in \mathcal{T}(\partial M)$. In this more general setting the boundary of M can be disconnected and can be decomposed as $\partial M = F_c \cup F_i$ where F_i does not compress in M and each component of F_c compresses (i.e. it has at least a loop bounding a disk in M).

Let $c_{\mathbf{t}}^{\mathfrak{m}} : [0,1] \to CC(M)$ be an earthquake path (we quake by a parameter t_i, with $\mathbf{t} = (t_1,\dots,t_n)$, along the curve γ_i) along a multicurve $\mathfrak{m} = \{\gamma_i\}_{i=1}^n \subseteq S$ such that with respect to the reference hyperbolic metric X_0 can be subdivided into:

- $\mathfrak{m}_1^c$: the set of compressible geodesic loops γ of $\mathfrak{m}$ with length at most 1;
- $\mathfrak{m}_1$: the set of geodesic loops γ in F_c and not in $\mathfrak{m}_1^c$ such that any compressible geodesic loop α intersecting γ essentially has length at least 1;
- $\mathfrak{m}_\infty$: the set of geodesic loops γ of $\mathfrak{m}$ that are contained in F_i and so incompressible.

Note that not every $\mathfrak{m}$ admits such a decomposition with respect to the given X_0, as there could be a $\gamma_i \in \mathfrak{m}$ in a compressible component, of length more than 1 and intersecting a short compressible loop.

Theorem 2.5.6. Let $X_0 \in \mathcal{T}(\partial M)$ and $\mathfrak{m} = \mathfrak{m}_\infty \cup \mathfrak{m}_1^c \cup \mathfrak{m}_1$ be a multicurve and $c_{\mathbf{t}}^{\mathfrak{m}}$ be an earthquake path terminating at X_1. Then

$$|V_R(X_1) - V_R(X_0)| \leq \sum_{\gamma_i \in \pi_0(\mathfrak{m}_1^c)} (3\ell_i \coth^2(\ell_i/4))t_i + C\sum_{\alpha_j \in \mathfrak{m}_1} t_j\ell_j + 3\sum_{\beta_k \in \mathfrak{m}_\infty} t_k\ell_k ,$$

for $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

2.1.2 Outline of the chapter

The proof of Theorem 2.6.1 follows several steps. First, we introduce in Section 2.3 a notion of *symmetric* Riemann surfaces – those which admit an orientation-reversing involution with quotient a surface with boundary. We prove that given any Riemann surface X of finite-type and any pants decomposition P of X, there is a symmetric surface X_s (for which the involution leaves P invariant component-wise) obtained from X by earthquakes along the curves of P (see Lemma 2.3.7).

Then, in Section 2.4, we estimate the renormalized volume of Schottky fillings of symmetric surfaces with 2-dimensional convex core. In Section 2.5, we provide a formula for the difference of the renormalized volume of a filling under an earthquake path of the boundary surface (see Theorem 2.5.4). The result expresses the estimates in terms of the Schwarzian derivative of the conformal boundary at infinity (see Sections 1.3.1 and 1.4.1) at the core of tubes associated to the pants curves. Finally, Section 2.6 contains the proofs of the main results.

2.2 Preliminaries

In this section we recall the main objects and tools that we will use in this work, which were not already introduced in Chapter 1.

Earthquakes along simple closed geodesics

We recall here some basic facts on earthquakes along closed geodesics, which will be needed. For more background see [FM11, Sec 10.7.3] and [CEM06, Part III].

Given a simple closed geodesic γ on a hyperbolic surface (S, h) a (left) t-earthquake is a map $\varphi_{\gamma,t}$ from S to itself, discontinuous along γ, defined by cutting S along γ, twisting the left-hand side of γ by a fixed length t in the positive direction, and gluing back isometrically the two sides.[1]

Taking the push-forward of the hyperbolic metric by $\varphi_{\gamma,t}$ defines a new hyperbolic metric on S, and in this manner γ and t define a homeomorphism of $\mathcal{T}(S)$, which is also called the right earthquake of length t along γ, and denoted by $E_\gamma(t)$.

By continuously varying the twisting length t, one gets a path of diffeomorphisms of $S \setminus \gamma$, and by pulling back h through such a path, we get a path in $\mathcal{T}(S)$.

Let us now define earthquakes more carefully. Having fixed a simple closed curve γ in S, we consider the unique geodesic on (S, h) in the same isotopy class again by γ. In this way, the operation only depends on the isotopy class of γ. Let ℓ be the length of γ with respect to h, and $N_r \cong S^1 \times [-r, r] \cong \mathbb{R}/\ell\mathbb{Z} \times [-r, r]$ the tubular r-neighborhood of γ parameterized in such a way that $S^1 \times \{0\}$ isometrically identifies with γ and $\{e^{i\theta}\} \times [-r, r] \in S^1 \times [-r, r]$ with a geodesic segment of length $2r$ orthogonal to γ parameterized in unit velocity by the coordinate in $[-r, r]$. We now choose an arbitrary function $f\colon [-r, r] \to \mathbb{R}$ such that f is smooth on $[-r, r] \setminus \{0\}$, increasing on $[-r, 0]$, constantly equal to 0 in a neighborhood of $-r$, constantly equal to 1 in a left neighbourhood of 0, and equal to 0 on $(0, r]$. We then define $\varphi_{\gamma,t}\colon S \to S$, with $t \in [0, \infty)$, as the diffeomorphism of $S \setminus \gamma$ such that $\varphi_{\gamma,t}$ is the identity outside N_r, and

$$\varphi_{\gamma,t}(e^{i\theta}, r) = \left(e^{i(\theta + \frac{2\pi}{\ell} t f(r))}, r\right)$$

for any $(e^{i\theta}, r) \in N_r$. Note that $\varphi_{\gamma,0}$ is the identity, and that $\varphi_{\gamma,\ell}$ extends to a diffeomorphism of S, which is called a *Dehn twist*.

Since $\varphi_{\gamma,t}$ acts by isometry on $S \setminus \gamma$ and it fixes the metric on γ, the push-forward $(\varphi_{\gamma,t})_*(h)$ is a new well defined hyperbolic Riemannian metric on S. We say that $(\varphi_{\gamma,t})_*(h)$ is obtained by a *(left) earthquake of parameter* t along γ.

[1]Note that this definition requires the choice of an orientation of γ, but the result does not depend on which orientation is chosen.

We define $\varphi_\gamma : [0,a] \to \mathcal{T}(S)$, $a > 0$, to be a *earthquake path along* γ by $\varphi_\gamma(t) = (\varphi_{\gamma,t})_*(h)$. The *infinitesimal earthquake along* γ is the derivative of φ_γ in t at $t = 0$, this can also be seen as a vector field v on S by differentiating the path of diffeomorphisms $(\varphi_{\gamma,t})_{t\in[0,\varepsilon]}$ with respect to t and evaluating it at $t = 0$. For more background see [FM11, Sec 10.7.3] and [CEM06, Part III].

Handlebodies.

We will think of an handlebody H_g of genus $g \geq 1$ as the following data. Given a surface $S = S_g$ and a pants decomposition P on S we can form the 3-manifold H_0 by attaching $3g-3$ thickened disk $\mathbb{D}^2 \times I$ to $S \times I$ by gluing each $\partial\mathbb{D}^2 \times I$ to $N_\varepsilon(\gamma) \times \{0\}$ for $\gamma \in P$. The manifold H_0 has then a genus g boundary component and $2g-2$ sphere boundary components. After filling each sphere component with a 3-ball we obtain a handlebody $H_P \cong H_g$, this is unique up to isotopy. We will think of this as the handlebody induced by P.

Definition 2.2.1. Given a conformal structure $X \in \mathcal{T}(S_g)$ and a pants decomposition P on S_g we say that $M_P(X)$ is the *Schottky filling* of X with pants curve P if it is the hyperbolic 3-manifold obtained by uniformising H_P so that its conformal boundary is X. By $CC_P(S)$ we denote the deformation space of a hyperbolic genus g handlebody obtained by gluing disks along P.

Remark 2.2.2. More generally, a handlebody is any irreducible compact 3-manifold M with a unique boundary component such that the map induced by the inclusion $\partial M \hookrightarrow M$ on the fundamental groups is surjective, [Hem76]. Thus, the manifold $M := F \times I$ for F a compact orientable surface with non-empty boundary is also a handlebody with boundary given by the double of F along ∂F. In the case that F is not-orientable then we can consider the twisted I-bundle[2] $N = F \tilde{\times} I$ in which ∂N is given by the orientation double cover of F.

2.3 Earthquakes to symmetric Surfaces

In this section we study conformal structures X on a surface S that admit an orientation-reversing involution $\sigma : X \to X$ such that, if X is equipped with its unique compatible hyperbolic metric, $X_\sigma := X/\sigma$ is a hyperbolic surface with totally geodesic boundary. The main result of this section is Lemma 2.3.7, which states that given $X \in \mathcal{T}(S)$ and $P \subseteq S$ a pants decomposition there exists a symmetric conformal structure $X' \in \mathcal{T}(S)$ and a path in $CC_P(S)$ from $M_P(X)$ to $M_P(X')$ which is obtained by doing earthquakes of bounded length along the curves of P.

Definition 2.3.1. Let $X \in \mathcal{T}(S)$, then X is a *symmetric surface* if S admits an orientation reversing involution $\sigma : S \to S$ that is a local isometry for the hyperbolic metric on X, and such that $X_\sigma := X/\sigma$ is a surface with non-empty boundary. The subset of Teichmüller space of surfaces for which σ is a local isometry will be denoted by $\mathcal{T}_\sigma(S)$ and the subspace of surface admitting an involution σ by $\mathcal{T}_s(S) = \cup_\sigma \mathcal{T}_\sigma(S)$.

Remark 2.3.2. The surface X_σ does not have to be orientable.

[2]Recall that a twisted I-bundle is a non-trivial I-bundle, i.e. $N \ncong F \times I$.

Lemma 2.3.3. Let X be a hyperbolic surface with an orientation reversing involution $\sigma : X \to X$ that is a local isometry. Then, $\partial X_\sigma = \mathrm{Fix}(\sigma)$ is given by a multicurve $\mathfrak{m} \subseteq X$ such that for each $\gamma \in \pi_0(\mathfrak{m})$ we have $\sigma|_\gamma = \mathrm{id}_\gamma$.

Proof. By [Kli95, Theorem 1.10.15] the set of fixed points is a closed totally geodesic submanifold, thus it is the union of a closed multicurve $\mathfrak{m}$ and possibly a finite collection of points. By looking at the action on a small enough ball around an isolated fixed point (so that the centre is the unique fixed point) one can see that, being σ orientation reversing, isolated fixed points are not possible and so the fixed set has to be a geodesic multicurve.

We now want to show that $\mathfrak{m}$ is the boundary of X_σ. Let $B \subseteq X$ be a small enough ball such that $\mathfrak{m} \cap B$ separates B in two balls and $B = \sigma(B)$. Then, B/σ is homeomorphic to a half disk with boundary in $\mathfrak{m}$. By connectedness and continuity this shows that $\mathfrak{m} \subseteq \partial X_\sigma$. The reverse containment follows from the fact that $\sigma : X \setminus \mathfrak{m} \to X \setminus \mathfrak{m}$ is a 2 to 1 cover and so $(X \setminus \mathfrak{m})/\sigma$ is a surface without boundary. ∎

In each pair of pants, a seam is the orthogeodesic connecting two distinct boundary components, so each pair of pants has 3 such arcs, see Figure 2.1. For every pair of pants Q we have on each boundary component γ_i two marked points x_i^1, x_i^2, endpoints of the seams of Q. We define a marked pants decomposition P^m to be P together with a choice of either x_i^1 or x_i^2 for each pair of pants Q and each boundary curve of P.

Let $X \in \mathcal{T}(S)$ be a hyperbolic surface, P be a pants decomposition of X, and $\mathcal{S}$ be the set of the induced seams with marked endpoints, i.e. a marked pants decomposition P^m. Then, we recall that the Fenchel-Nielsen coordinates for X are defined as follows (see Section 1.2.2): $FN(X) = (\ell_i, t_i)_{i=1}^{3g-3}$ where the ℓ_i are the hyperbolic lengths of the pants curve in the hyperbolic structure on X and the t_i are the twist parameters with respect to the two marked points on the curve γ_i.

Thus, if $t_i = 0$, the seams match up and the two marked points are identified. If $t_i = \ell_i/2$, the seams match up but the marked points are opposite to each other.

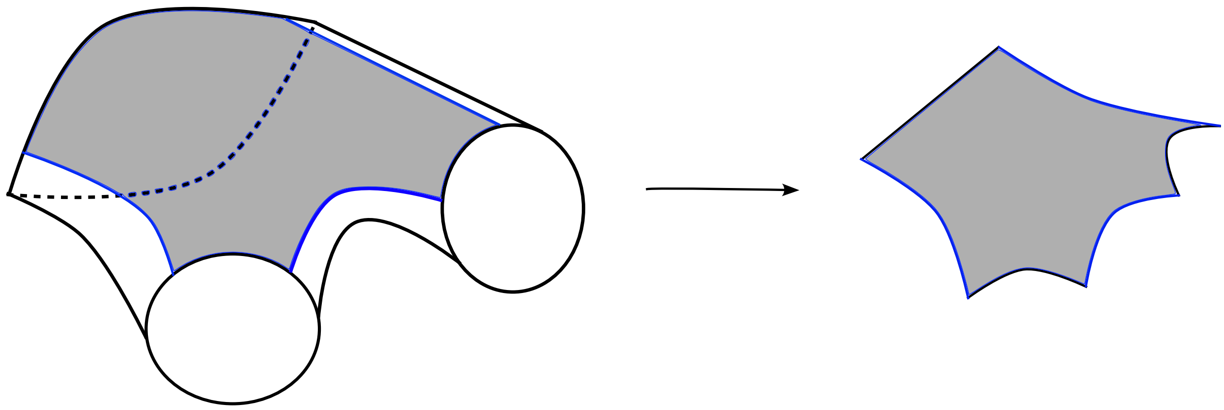

Figure 2.1: The seams (in blue) in a pair of pants with the two hexagons H_1 (shaded), H_2 and the π_Q map. (Image by Tommaso Cremaschi)

Moreover, the seams cut each pair of pants Q into two isometric right-angled hexagons H_1 and H_2. We can then define an orientation-reversing involution $\sigma_Q : Q \to Q$ which maps H_1 to H_2 and H_2 to H_1 isometrically, and is the identity on the seams, see Figure 2.1. The quotient of Q by σ_Q is then a right-angled hexagon E_Q, on which Q projects by a map $\pi_Q : Q \to E_Q$ which is a local isometry outside of the seams.

Remark 2.3.4. The maps $\{\pi_Q\}_{Q\in P}$ glue together to a map $\pi : X \to X$ that is an orientation reversing local isometry (outside of the seams) if all seams match up. Moreover, if that is the case then X_π is a surface, not necessarily orientable, whose boundary is given, by Lemma 2.3.3, by the union of the seams.

Lemma 2.3.5. Let $X \in \mathcal{T}(S)$, and let $P = \{c_1, \cdots, c_{3g-3}\}$ be a marked pants decomposition of X and let (ℓ_i, t_i) be the corresponding Fenchel-Nielsen coordinates. Then, the Riemann surface X_0 with Fenchel-Nielsen coordinates (ℓ_i, t'_i), $t'_i = 0, \ell_i/2$, admits an orientation-reversing isometry which leaves invariant each curve of P.

Proof. We want to show that the surface X_0 defined by (ℓ_i, t'_i) admits an orientation-reversing isometry mapping each geodesic loop in P to itself.

The surface X_0 is obtained by gluing $3g-3$ pairs of pants with boundary lengths given by the ℓ_i's and in the pattern given by P such that if two pairs of pants Q_1 and Q_2 (Q_1 could be equal to Q_2) are glued along a geodesic loop $c_i \in \pi_0(P)$ then the endpoint of the seam $y_1 \in Q_1 \cap c_i$ is glued to $y_2 \in Q_2 \cap c_i$ without any twist. The pairs of pants Q_i, $i = 1, 2$, are obtained by doubling regular hexagons E_i along the seams, and each P_i is equipped with an orientation-reversing isometry map $\pi_i : Q_i \to Q_i$ exchanging the two hexagons. The fixed point set of this map is exactly the seams of Q_i.

Since the seams on $c_i \subseteq Q_1 \cap Q_2$ have endpoints that are $\ell_i/2$ apart and by our glueing condition one of them matches up we know that they both do. Therefore, all the seams with endpoints on c_i match-up and we can glue the maps π_1 and π_2 to obtain an orientation-reversing isometry from $Q_1 \cup Q_2$ to $Q_1 \cup Q_2$. By doing this for all pants we obtain the required statement. ■

Remark 2.3.6. Given X and $\sigma : X \to X$ then, for specific markings in the FN-coordinates the quotient surface is orientable and equal to a thickening of the glueing graph of the pants decomposition in which if a curve c_i has twist parameter equal to $\ell_i/2$ then the quotient edge is glued with an half-twist.

Lemma 2.3.7. Given a pants decomposition P on S and $X \in \mathcal{T}(S)$, there exists $X', X_s \in \mathcal{T}(S)$ such that X_s is symmetric, $M_P(X') \overset{isom}{\cong} M_P(X)$, and X_s is obtained from X', in FN_P coordinates, by twisting at most $\ell(c_i)/4$ (in the positive or negative direction) over each curve in P.

Proof. First note that in $CC_P(S)$ we can do full twists along curves of P and get isometric structures, see [Mar16a, Thm 5.1.3.]. Recall that we denote by $M_P(X) \in CC_P(S)$ the structure corresponding to $X \in \mathcal{T}(S)$ with compressible curves given by P.

We use P to define the Fenchel-Nielsen coordinates by choosing seams $y \in \{x_1^i, x_2^i\} \subseteq c_i$, see Lemma 2.3.5. Also note that a full twist along c_i has length ℓ_i. Let X be the given structure, then $FN_P(X) = (\ell_i(X), t_i(X))_{i=1}^{3g-3}$. By doing full twists along the c_i's we can find a hyperbolic structure X' with the same length parameters, while the twists parameters are between 0 and $\ell_i(X') = \ell_i(X)$, and $M_P(X) \overset{isom}{\cong} M_P(X')$.

By doing twists of length at most $\ell_i(X)/4$ we get a surface X_s with the same length parameters and all seams of pair of pants matching up. Then, the twist parameters are equal to either zero or $\ell_i(X)/2$. ■

2.4 The renormalized volume of symmetric surfaces

In this section we estimate the renormalized volume of a Schottky filling of a surface $X \in \mathcal{T}_s$ corresponding to a "symmetric" pants decomposition. This will be used in the proof of Theorem 2.6.1. In the next two sections, we will bound the variation of the renormalized volume under a variation of the twist parameters in the Fenchel-Nielsen coordinates, and as a consequence we will be able to obtain an upper bound on the renormalized volume of Schottky fillings which are non-symmetric by comparing their renormalized volume to that of a symmetric surface obtained by changing the twist parameters.

In the following lemma we will deal with manifold whose convex core is 2-dimensional. Thus, it will be useful to use a slightly modified definition of convex core boundary which is more compatible with the corresponding conformal boundary. In what follows we denote by $\partial C(M)$ the "boundary" of $C(M)$ for M any convex co-compact hyperbolic manifold and we define:

- $\partial C(M)$ is the boundary of $C(M)$ in the usual sense if $C(M)$ has non-empty interior,
- if $C(M)$ is a totally geodesic orientable surface $\Sigma \subseteq M$, then $\partial C(M)$ is the union of two copies of Σ with opposite orientation, if $\partial\Sigma \neq \varnothing$ then the two copies of Σ are glued along their common totally geodesic boundary.
- if $C(M)$ is a totally geodesic non-orientable surface $\Sigma \subseteq M$, then $\partial C(M)$ is the orientation double-cover of Σ.

In all cases, $\partial C(M)$ is homeomorphic to $\partial_\infty M$. Specifically, the hyperbolic Gauss map, which sends a unit vector normal to a support plane of $C(M)$ to the endpoint at infinity of the geodesic ray it defines, is a homeomorphism from the unit normal bundle of $C(M)$ – which is itself homeomorphic to $\partial C(M)$ – to $\partial_\infty M$.

The "boundary" $\partial C(M)$ is equipped with an induced metric m, which is hyperbolic. However, it is *pleated* along a *measured lamination* β which is geodesic for m, with the transverse measure recording the amount of pleating along the leaves, see [Thu80, EM86]. When $C(M)$ is a totally geodesic surface Σ, the support of β corresponds to the boundary of Σ, with each leave equipped with a weight π.

Let X be the conformal structure at infinity of M. Then X is obtained from m and β by a geometric construction called *grafting*, see e.g. [Dum08]. Given a closed surface S of genus at least 2, this grafting operation defines a map

$$\mathrm{gr} : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S) \ ,$$

where $\mathcal{ML}(S)$ denotes the space of measured laminations on S. The key property that is important to us here is a result of Scannell and Wolf [SW02]: if $\lambda \in \mathcal{ML}(S)$ is fixed, the map $\mathrm{gr}(\cdot, \lambda) : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism.

Now, if M is a convex co-compact hyperbolic manifold with convex core $C(M)$ a totally geodesic surface Σ with boundary, then the lamination β is the fixed-point set of an orientation-reversing involution $\sigma : \partial C(M) \to \partial C(M)$ such that $\Sigma = \partial C(M)/\sigma$, and the induced metric m on $\partial C(M)$ is invariant under σ. Since m and β are both invariant under σ, so is the conformal structure at infinity $X = \mathrm{gr}(m, \beta)$.

Lemma 2.4.1. Let $\sigma : S \to S$ be an orientation-reversing involution with $\mathrm{Fix}(\sigma) \neq \varnothing$ and quotient surface $\Sigma = S/\sigma$. Then, for any invariant conformal structure $X \in \mathcal{T}_\sigma(S)$ there exists a handlebody H with a convex co-compact hyperbolic structure such that the convex core of H is homeomorphic to Σ and the conformal boundary of H is X.

Proof. Let $\beta = \partial\Sigma$. We claim that the restriction map

$$\mathrm{gr}(\cdot, \beta)_{|\mathcal{T}_\sigma(S)} : \mathcal{T}_\sigma(S) \to \mathcal{T}_\sigma(S)$$

is onto. Indeed, let $X \in \mathcal{T}_\sigma(S)$. Since $\mathrm{gr}(\cdot, \beta) : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism, there exists a unique $Y \in \mathcal{T}(S)$ such that $\mathrm{gr}(Y, \beta) = X$. But then

$$\mathrm{gr}(\sigma^* Y, \beta) = \mathrm{gr}(\sigma^* Y, \sigma^* \beta) = \sigma^* \mathrm{gr}(Y, \beta) = \sigma^* X = X = \mathrm{gr}(Y, \beta) \ ,$$

and since Y is unique, $\sigma^* Y = Y$, so that $Y \in \mathcal{T}_\sigma(S)$.

Let $Y_\sigma = Y/\sigma$, homeomorphic to Σ, be the quotient surface of the hyperbolic surface Y by the locally isometric involution σ. Then, Y_σ has a uniformization $\Gamma < \mathbb{P}SL_2(\mathbb{R})$. By considering Γ inside $\mathbb{P}SL_2(\mathbb{C})$, by the natural inclusion, and the corresponding quotient $\mathbb{H}^3/\Gamma$ we obtain a hyperbolic 3-manifolds whose convex core is Y_σ. By the above discussion, we also know that the conformal boundary is X. The fact that $H = \mathbb{H}^3/\Gamma$ is a handlebody follows from the fact that H is homeomorphic to either $Y_\sigma \times I$, if Y_σ has non-empty boundary or is orientable, or to the twisted bundle $Y_\sigma \tilde{\times} I$, if Y_σ is non-orientable with empty boundary. As $\Sigma \cong Y_\sigma$ has boundary by Remark 2.2.2 this yields a handlebody. ∎

Remark 2.4.2. The following remark is not needed in the rest of this paper, however, one should note that Lemma 2.4.1 also works for fixed point free involution if one allows for the topological condition to be that of a twisted I-bundle $K \tilde{\times} I$ over a closed non-orientable surface K.

Remark 2.4.3. In the case we have a pants decomposition P such that, for each $\gamma \in P$, $\sigma(\gamma) = \gamma$, we can also infer from Lemma 2.4.1 and Lemma 2.3.3 that $H \in CC_P(S)$, i.e. P compresses in H and the seams of P form ∂X_σ.

For a convex co-compact hyperbolic 3-manifold M,

$$V_R(M) \leq V_C(M) - \frac{1}{4} L(\beta_M) \ , \tag{2.1}$$

see [Sch13, Lemma 4.1] (and also [BBB19, Theorem 3.7]). In the case considered here, the bending lamination is given by a multicurve with bending measure given by assigning the weight π to each curve, see Lemma 2.4.1. Then, its length is given by:

$$L(\beta_M) = \pi \sum_{\gamma \in \pi_0} \ell_Y(\gamma) \ , \tag{2.2}$$

for Y the hyperbolic structure on the convex-core boundary and π_0 the set of the simple closed curves composing the multicurve. Thus, one has $L(\beta_M) > 0$ and so, by Lemmas 2.3.3 and 2.4.1 we obtain the following statement.

Theorem 2.4.4. Let $X \in \mathcal{T}_s(S)$, and let $\sigma : S \to S$ be such that $X \in \mathcal{T}_\sigma(S)$. Then there exists a handlebody filling H_X such that

$$V_R(H_X) \leq -\frac{\pi}{4} \ell_{X_\sigma}(\partial X_\sigma) < 0 \ .$$

Proof. By equation (2.1) and (2.2), as the convex core volume is zero, we have:

$$V_R(H_X) \leq -\frac{\pi}{4}\ell_{Y_\sigma}(\partial\Sigma)\ ,$$

for H_X the handlebody furnished by Lemma 2.4.1 and Y_σ the hyperbolic structure induced on its convex core, and where $\Sigma = S/\sigma$. Recall that by Lemma 2.3.3 and Remark 2.4.3, as isotopy classes of loops in S, we have that $\partial\Sigma$ and ∂X_σ are the same. By the Schwarz Lemma, the Thurston metric (see Section 1.3) is bigger than the hyperbolic metric at infinity. Moreover, the Thurston metric coincides with the metric obtained by grafting on the lamination of the convex core of H_X, that is, on the boundary of Y_σ, and the length of the bending lamination is preserved through grafting on the same (see Section 1.6.1). Therefore

$$V_R(H_X) \leq -\frac{\pi}{4}\ell_{Y_\sigma}(\partial\Sigma) \leq -\frac{\pi}{4}\ell_{X_\sigma}(\partial X_\sigma).$$

■

If we know some curves are short in X and the pants decomposition is fixed component by component we obtain the following estimate, Lemma 2.4.5. This is the main such estimate we will use in this work. For completeness we also prove the other option in Lemma 2.4.6.

Lemma 2.4.5. There exist universal constants $S, Q \geq 0$ as follows. Let $X \in \mathcal{T}_s(S)$, σ so that $X \in \mathcal{T}_\sigma(S)$, $M = M_P(X)$ be the Schottky manifold corresponding to any pants decomposition for which each curve is invariant under $\sigma : S \to S$, and such that there are $0 \leq k \leq 3g-3$ geodesic loops of P of length $\ell_X(\gamma_i) \leq 1$. Then,

$$V_R(M_P(X)) \leq -\frac{S}{4}\sum_{i=1}^{k}\frac{1}{\ell_X(\gamma_i)} + \frac{Q}{4}k \leq \frac{k}{4}(-S+Q) < 0\ .$$

Specifically, one can take $S = \frac{4\pi^3}{\sqrt{e}} \sim 75.225$ and $Q = 4\pi\log\left(\frac{\pi e^{0.502\pi}}{\mathrm{arsinh}(1)}\right) \sim 35.7901 \leq 36$.

Proof. By Lemma 2.4.1 the convex-core of M is a totally geodesic surface and so $V_C(M) = 0$. However, by Theorem 2' of [BC05], we have

$$\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right) \leq L(\beta_M).$$

Applying it to equation (2.1), one gets:

$$V_R(M_P(X)) \leq -\frac{1}{4}\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right) = -\frac{S}{4}\sum_{i=1}^{k}\frac{1}{\ell_X(\gamma_i)} + \frac{Q}{4}k \leq \frac{k}{4}(-S+Q)\,,$$

concluding the proof. ■

The case in which the pants curves are not fixed component-wise requires introducing some auxiliary functions from [BC03, Corollary 1], these functions will only be needed here. For $m = \cosh^{-1}(e^2)$ we define

$$g(x) = e^{-m}\frac{e^{-\pi^2/2x}}{2}$$

and

$$F(x) = \frac{x}{2} + \sinh^{-1}\left(\frac{\sinh(x/2)}{\sqrt{1-\sinh^2(x/2)}}\right) .$$

Since F is invertible we let $K(x) = \frac{2\pi}{F^{-1}(x)}$ and then define $L(x) = 1 + K(g(x))$.

Lemma 2.4.6. Let $X \in \mathcal{T}_s(S)$, and $M(X)$ be the Schottky manifold with flat convex core and conformal boundary X. Let $\mathfrak{m} = \{\gamma_1, \dots, \gamma_k\}$ be the collection of geodesic loops point-wise invariant by σ and let ρ_X be half of the length of the shortest simple closed compressible geodesic in X. Then,

$$V_R(M(X)) \leq -\frac{\pi}{4L(\rho_X)} \sum_{i=1}^{k} \ell_X(\gamma_i) .$$

Proof. By Lemma 2.4.1 the convex-core of M is a totally geodesic surface and so $V_C(M) = 0$. Moreover, by Lemma 2.3.3 ∂X_σ is given by the multicurve $\mathfrak{m}$ of geodesic loops that are point-wise fixed by σ. Thus, by Corollary 1 of [BC03] we have:

$$\ell_Y(\gamma) \geq \frac{1}{L(\rho_X)} \ell_X(\gamma) .$$

Then, by applying it to equation (2.1) we obtain the required result. ∎

2.5 Variation of the renormalized volume under an earthquake

In this section we compute how the renormalized volume changes under earthquake paths in the deformation space.

2.5.1 First-order variation of the renormalized volume

We start the section with a formula for $\mathrm{d}V_R$ at $M_P(X) = \mathbb{H}^3/\Gamma$. Recall that by $\mathcal{S}(f)$ we are denoting the Schwarzian derivative of the developing map of the domain of discontinuity $\Omega(\Gamma)$ of the Schottky hyperbolic 3-manifold $M_P(X)$, and by S the boundary $\partial\overline{M}$. We will sometimes refer to $\mathcal{S}(f)$ just as the Schwarzian of $M_P(X)$.

Lemma 2.5.1. Let μ be an infinitesimal earthquake (at unit velocity) along a simple closed geodesic on X, parameterized at unit velocity by $\gamma : \mathbb{R}/\ell\mathbb{Z} \to X$. Then, for $q = \mathcal{S}(f)$:

$$\mathrm{d}V_R(\mu) = -\frac{1}{2}\int_{\mathbb{R}/\ell\mathbb{Z}} \mathrm{Re}(q(i\dot\gamma(t), \dot\gamma(t)))dt = \mathrm{Im}\left(\frac{1}{2}\int_{\mathbb{R}/\ell\mathbb{Z}} q(\dot\gamma(t), \dot\gamma(t))\right) .$$

Proof. Let v be a vector field realizing the infinitesimal earthquake along the image of γ. That is, v is the vector field obtained by differentiating at zero, with respect to the time parameter t, the family of diffeomorphisms $\varphi_\gamma(t)$ corresponding to a length t earthquake along γ. We assume that v is discontinuous along $\gamma(\mathbb{R}/\ell\mathbb{Z})$, that is, it has limit zero on the right side and equal to $\dot\gamma(t)$ along $\gamma(\mathbb{R}/\ell\mathbb{Z})$ and is continuous on the left.

The first-order variation of the complex structure associated to v is then determined by the Beltrami differential $\mu = \overline{\partial} v$. Where here by $\overline{\partial} v$ we mean the L^∞ weak* limit of $\overline{\partial} v_n$ for v_n smooth compactly supported vector fields that are C^∞ approximations of v, converging in the uniform topology on compact sets of $S \setminus \gamma$. Specifically, choosing a complex coordinate z, we can write:

$$v = 2\omega \left(\partial_z + \partial_{\overline{z}}\right) \ ,$$

and note that ω vanishes on the right half-neighbourhood of γ.

Consider the area form $dx \wedge dy$ associated to $z = x + iy$, and note that $d\overline{z} \wedge dz = 2i(dx \wedge dy)$. We have

$$\overline{\partial} v = 2(\overline{\partial}\omega)(d\overline{z} \otimes \partial_z + d\overline{z} \otimes \partial_{\overline{z}}) \ ,$$

and so if $q = g(z)dz^2$,

$$\begin{aligned}\langle q, \overline{\partial} v\rangle &= \frac{1}{2i}\int_S 2g(z)(\overline{\partial}\omega(z))(d\overline{z} \wedge (dz^2(\partial_z)) + d\overline{z} \wedge (dz^2(\partial_{\overline{z}}))) \\ &= \int_S 2g(z)(\overline{\partial} w(z))dx \wedge dy\end{aligned}$$

by definition of the duality product commonly used between Beltrami differentials and holomorphic quadratic differentials, see [Hub16].

Consider now the one-form defined by $\alpha = q(v, \cdot) = 2\omega(z)g(z)dz$. Then

$$\begin{aligned}\overline{\partial}\alpha &= \overline{\partial}(2\omega(z)g(z)dz) \\ &= 2(\overline{\partial}\omega(z))g(z)d\overline{z} \wedge dz + 2\omega(z)\overline{\partial} g(z)d\overline{z} \wedge dz \\ &= 4i(\overline{\partial}\omega(z))g(z)dx \wedge dy \ ,\end{aligned}$$

because g is holomorphic, $\overline{\partial} g = 0$, and $d\overline{z} \wedge dz = 2i(dx \wedge dy)$.

The outcome of this discussion is that

$$\int_S \overline{\partial}\alpha = \int_S 4ig(z)(\overline{\partial}\omega(z))dx \wedge dy = 2i\int_S 2g(z)(\overline{\partial}\omega(z))dx \wedge dy = 2i\langle q, \overline{\partial} v\rangle \ .$$

Therefore, we get

$$\langle q, \mu\rangle = -\frac{i}{2}\int_S \overline{\partial}\alpha \ .$$

However, α is a complex 1-form, so that $\partial\alpha = \partial(2gw)dz \wedge dz = 0$, and as a consequence

$$d\alpha = (\partial + \overline{\partial})\alpha = \overline{\partial}\alpha \ .$$

Using Stokes on $S' = S \setminus \gamma(\mathbb{R}/\ell\,\mathbb{Z})$, we obtain that, since α vanishes on one component of $\partial S'$:

$$\langle q, \overline{\partial} v\rangle = -\frac{i}{2}\int_S d\alpha = -\frac{i}{2}\int_{\partial S'} \alpha(\dot\gamma(t))dt = -\frac{i}{2}\int_0^\ell \alpha(\dot\gamma(t))dt \ .$$

However, by definition of α we obtain that

$$\langle q, \overline{\partial} v\rangle = -\frac{i}{2}\int_0^\ell q(v|_{\gamma(t)}, \dot\gamma(t))dt = -\frac{1}{2}\int_0^\ell q(i\dot\gamma(t), \dot\gamma(t))dt \ .$$

The first order variation of the renormalized volume, thanks to Theorem 1.6.5, is equal to:

$$\begin{aligned} \mathrm{d}V_R(\mu) &= \mathrm{Re}\left(\langle q, \overline{\partial} v\rangle\right) = \mathrm{Re}\left(-\frac{1}{2}\int_0^\ell q(i\dot\gamma(t),\dot\gamma(t))dt\right) \\ &= -\frac{1}{2}\int_0^\ell \mathrm{Re}\left(q(i\dot\gamma(t),\dot\gamma(t))\right)dt \ , \end{aligned}$$

completing the proof. ■

Definition 2.5.2. An *earthquake path* $c_{\mathbf{t}} : [0,1] \to CC_P(S)$, with $\mathbf{t} = (t_1, \dots, t_{3g-3})$, is a path which at time $s \in [0,1]$ twists $st_i \in \mathbb{R}$ along each pants curve $\gamma_i \in P$ of $c_{\mathbf{t}}(0)$.

For a compressible loop γ we use $\mathrm{inj}\,|_\gamma$ to denote half of the length of the shortest loop δ such that either $\delta = \gamma$ or δ intersects γ essentially and it bounds a disk in M. Note that if γ is a compressible geodesic loop of length $\leq \varepsilon_0$ in X then $\mathrm{inj}\,|_\gamma = \frac{\ell_X(\gamma)}{2}$.

Lemma 2.5.3. Let $c_t(s)$, for $s \in [0,1]$ and a fixed $t \in \mathbb{R}$, be an earthquake path along a simple geodesic loop γ starting at the Riemann surface X_0. Then, the following bound for $|\mathrm{d}(V_R \circ c_t)|$ holds at any $s \in [0,1]$:

$$|\,\mathrm{d}(V_R \circ c_t)| \leq 3\ell_{X_0}(\gamma)\coth^2\left(\frac{\mathrm{inj}\,|_\gamma}{2}\right) t \ .$$

In particular, if $\mathrm{inj}\,|_\gamma \geq 1/2$ we have

$$|\,\mathrm{d}(V_R \circ c_t)| \leq 3\ell_{X_0}(\gamma)\coth^2\left(\frac{1}{4}\right) t = C\ell_{X_0}(\gamma)t \ , \quad C = 3\coth^2\left(\frac{1}{4}\right) < 50.013 \ .$$

Proof. First, observe that the length of γ remains constantly equal to $\ell_{X_0}(\gamma)$ along the earthquake path $c_t(s)$. Moreover, since earthquaking forms a flow (i.e. $c_t(s_1 + s_2) = c_t(s_1) \circ c_t(s_2)$), the scaling by t of the infinitesimal earthquake μ_s along γ at $X_s = c_t(s)$ coincides with the derivative of $c_t(s)$ at s. Then, at any $s \in [0,1]$, we can use the integration by part of Lemma 2.5.1. Denoting by $S(f_s)(z) = q_s(z)dz^2$ the Schwarzian associated through uniformization to $c_t(s)$, we can estimate $|q_s(z)| \leq 6\coth^2\left(\frac{\mathrm{inj}\,|_\gamma}{2}\right)$ (see [BBB19, Corollary 2.12], and note that the factor 4 comes from the hyperbolic metric), yielding the first bound. The second estimate follows by direct computation. ■

2.5.2 Earthquake paths and V_R estimates

In this section we compute the change of renormalized volume under a path $c_{\mathbf{t}} : [0,1] \to CC_P(S)$ obtained by doing earthquakes along geodesic loops in the pants decomposition P.

Theorem 2.5.4. Let $c_{\mathbf{t}} : [0,1] \to CC_P(S)$ be an earthquake path, and let $\ell_i = \ell_{X_0}(\gamma_i)$. Then

$$|V_R(X_1) - V_R(X_0)| \leq \sum_{i=1}^{k}(3\ell_i\coth^2(\ell_i/4))t_i + C\sum_{i=k+1}^{3g-3} t_i\ell_i \ ,$$

where $\gamma_1, \dots, \gamma_k$ are the geodesic loops of P with $\ell_i < 1$ and for all $j > k$ we have $2\,\mathrm{inj}\,|_{\gamma_j} \geq 1$, and $C = 3\coth^2(1/4)$.

Proof. Pick a 1-thick/thin pants decomposition with k geodesic loops less than 1 and integrate Lemma 2.5.3. ■

Since, by Lemma 2.3.7, to reach a symmetric surface we need to twist at most $\ell_X(\gamma_i)/4$, we can take $t_i \leq \ell_X(\gamma_i)/4$ in the above expression and obtain the following statement.

Corollary 2.5.5. Let $X \in \mathcal{T}(S)$ and $P = \{\gamma_i\}_{i=1}^{3g-3}$ be a pants decomposition in which the first k curves have length less than 1 and the others have injectivity radius at least 1. Then, there exists a symmetric surface X_0 such that

$$|V_R(X) - V_R(X_0)| \leq \frac{3}{4}\sum_{i=1}^{k} \coth^2(\ell_i/4)\,\ell_i^2 + \frac{C}{4}\sum_{i=k+1}^{3g-3} \ell_i^2,$$

with $\ell_i = \ell_{X_0}(\gamma_i)$ and $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

The above estimates also work in the setting of general convex co-compact manifolds. Let $CC(M)$ be the deformation space which is also parameterised by the quotient of $\mathcal{T}(\partial M)$ by Dehn twists along disks. Let $c_{\mathfrak{t}}^{\mathfrak{m}} : [0,1] \to CC(M)$ be an earthquake path along a multicurve $\mathfrak{m} \subseteq S$. Assume that the multicurve $\mathfrak{m}$ can be subdivided, according to the reference metric X_0, in the following way:

- $\mathfrak{m}_1^c$ is the set of geodesic loops γ of $\mathfrak{m}$ that are compressible and have length at most 1;
- $\mathfrak{m}_1$ is the set of geodesic loops γ contained in compressible components of ∂M and not in $\mathfrak{m}_1^c$, and such that any compressible loop intersecting γ essentially has length at least 1;
- $\mathfrak{m}_\infty$ is the set of geodesic loops γ of $\mathfrak{m}$ that are contained in components of ∂M that are incompressible.

Note that not every $\mathfrak{m}$ admits such a decomposition with respect to the given X_0.

Theorem 2.5.6. Let $X_0 \in \mathcal{T}(\partial M)$ and $\mathfrak{m} = \mathfrak{m}_1^c \cup \mathfrak{m}_1 \cup \mathfrak{m}_\infty$ be a multicurve and $c_{\mathfrak{t}}^{\mathfrak{m}}$ be an earthquake path terminating at X_1. Then

$$|V_R(X_1) - V_R(X_0)| \leq \sum_{\gamma_i \in \pi_0(\mathfrak{m}_1^c)} (3\ell_i \coth^2(\ell_i/4))t_i + C \sum_{\alpha_j \in \pi_0(\mathfrak{m}_1)} t_j \ell_j + 3 \sum_{\beta_k \in \pi_0(\mathfrak{m}_\infty)} t_k \ell_k\ ,$$

for $C = 3\coth^2\left(\frac{1}{4}\right) < 50.013$.

Proof. The first two cases follow by the previous computations and integrating Lemma 2.5.3. For the last case we bound the norm of the Schwarzian on the geodesic loops in $\mathfrak{m}_\infty$ by the Kraus-Nehari estimate [Neh49, Kra32] and then integrating gives the result. ■

2.6 Main Results

We now put together the results from the previous sections to prove the main Theorem 2.6.1 and Corollary 2.6.2.

Theorem 2.6.1. Let X be a closed Riemann surface of genus $g \geq 2$. Assume that there are k disjoint simple closed curves $\gamma_1, \cdots, \gamma_k$ such that $\ell(\gamma_i) \leq 1, 1 \leq i \leq k$, and there are no other geodesic loops of length less or equal to 1 in X. Then there exists a pants decomposition P containing the γ_i's such that

$$V_R(M_P(X)) \leq -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \frac{1}{\ell(\gamma_i)} + \left(9 + \frac{3}{4}\coth^2\left(\frac{1}{4}\right)\right) k + 81\coth^2\left(\frac{1}{4}\right) \pi(3g-3-k)(g-1)^2 .$$

Proof. Let P be a pants decomposition containing the k geodesic loops $\gamma_1, \ldots, \gamma_k$ shorter than 1 and the α_i, $i = k+1, \ldots, 3g-3$, being *Bers pants curves* (see [FM11, Theorem 12.8]).

That is, we have:

- $\ell_X(\gamma_i) \leq 1$ for $i \leq k$;
- $1 < \ell_X(\alpha_i) \leq B_g \leq 6\sqrt{3\pi}(g-1)$, see [Bus10, Theorem 5.1.4], and $\operatorname{inj}|_{\alpha_i} \geq 1$ for $k < i \leq 3g-3$;
- P has seams such that in the FN coordinates induced by P, $FN(X)$ has no twists bigger than $\ell_X(\gamma_i)/4$ or $\ell_X(\alpha_i)/4$ (see Lemma 2.3.5 and Lemma 2.3.7).

Let $c_{\mathbf{t}}$ be the path in FN coordinates from X to X_s, the symmetric surface. Then, $c_{\mathbf{t}}$ can be thought of doing $3g-3$ twists along each pants curve, each of length at most $\ell_X(\gamma_i)/4$ or $\ell_X(\alpha_i)/4$, see Lemma 2.3.7. Then, for $C = 3\coth^2\left(\frac{1}{4}\right)$, by Corollary 2.5.5 we get:

$$\begin{aligned}
|V_R(X) - V_R(X_s)| &\leq \frac{3}{4}\sum_{i=1}^{k} \coth^2(\ell_i/4)\,\ell_i^2 + \frac{C}{4}\sum_{i=k+1}^{3g-3} \ell_i^2 \\
&\leq \frac{C}{4}k + \frac{C}{4}\sum_{i=k+1}^{3g-3} B_g^2 \\
&\leq \frac{C}{4}k + \frac{C}{4}(3g-3-k)B_g^2 \\
&\leq \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2,
\end{aligned}$$

where we used the fact that $B_g \leq 6\sqrt{3\pi}(g-1)$ and $\coth^2(x/4)\,x^2$ is an increasing function. Thus, we get that:

$$V_R(X) \leq V_R(X_s) + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2.$$

Since $\ell_i \leq 1$ for $i \leq k$ by using Lemma 2.4.5 to estimate $V_R(X_s)$ we have:

$$V_R(X_s) \leq -\frac{1}{4}\sum_{i=1}^{k}\left(\frac{S}{\ell_X(\gamma_i)} - Q\right),$$

for $S = \frac{4\pi^3}{\sqrt{e}}$ and $Q = 4\pi\log\left(\frac{\pi e^{0.502\pi}}{\operatorname{arsinh}(1)}\right) \sim 35.7901 \leq 36$. Then, we obtain the following bound:

$$\begin{aligned} V_R(X) &\le \sum_{i=1}^{k} \left(-\frac{S}{4\ell_X(\gamma_i)} + \frac{Q}{4} \right) + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2 \\ &\le \sum_{i=1}^{k} \left(-\frac{\pi^3}{\sqrt{e}\ell_X(\gamma_i)} \right) + 9k + \frac{C}{4}k + 27C\pi(3g-3-k)(g-1)^2 \\ &\le -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \left(\frac{1}{\ell_X(\gamma_i)} \right) + \left(9 + \frac{C}{4} \right) k + 27C\pi(3g-3-k)(g-1)^2 \ . \end{aligned}$$

Substituting for $C = 3\coth^2\left(\frac{1}{4}\right)$ concludes the proof. ∎

Corollary 2.6.2. For all $g, k, k_1 \in \mathbb{N}$ such that $g \ge 2$, $0 < k \le 3g-3$ and $0 < k_1 \le k$, there exists an explicit constant $A = A(g, k_1, k-k_1) > 0$ such that, if X is a Riemann surface with k_1 geodesic loops of length less than A and k geodesic loops of length at most 1, then X admits a Schottky filling with negative renormalized volume.

Proof. Let P be a pants decomposition containing the k_1 geodesic loops, $\gamma_1, \dots, \gamma_k$ shorter than A and $k-k_1$ geodesic loops $\gamma_{k_1+1}, \dots, \gamma_k$ of length at most 1 and the α_i, $i = k+1, \dots, 3g-3$ are Bers pants curves.

That is, we have:

- $\ell_X(\gamma_i) < A$ for $1 \le i \le k_1$;
- $\ell_X(\gamma_i) \le 1$ for $k_1 < i \le k$;
- $1 < \ell_X(\alpha_i) \le B_g \le 6\sqrt{3\pi}(g-1)$ and $\mathrm{inj}\,|_{\alpha_i} \ge 1$ for $k < i \le 3g-3$.

Then, by Theorem 2.6.1 we get:

$$V_R(M_P(X)) \le -\frac{\pi^3}{\sqrt{e}} \sum_{i=1}^{k} \frac{1}{\ell(\gamma_i)} + \left(9 + \frac{C}{4} \right) k + 27C\pi(3g-3-k)(g-1)^2 \ ,$$

which can be further decomposed in:

$$V_R(M_P(X)) \le -\frac{\pi^3}{\sqrt{e}} \left(\sum_{i=1}^{k_1} \frac{1}{\ell(\gamma_i)} + \sum_{i=k_1+1}^{k} \frac{1}{\ell(\gamma_i)} \right) + \left(9 + \frac{C}{4} \right) k + 27C\pi(3g-3-k)(g-1)^2 \ .$$

Since for $i \le k_1$ we have that $\frac{1}{\ell_X(\gamma_i)} \ge \frac{1}{A}$ and, similarly, for $k_1 + 1 \le i \le k$ we have that $\frac{1}{\ell_X(\gamma_i)} \ge 1$ we get:

$$V_R(X) \le -\frac{\pi^3}{\sqrt{e}} \left(\frac{k_1}{A} + k - k_1 \right) + \left(9 + \frac{C}{4} \right) k + 27C\pi(3g-3-k)(g-1)^2 \ .$$

We want to find an upper bound on A that makes the above expression negative. Note that

$$B := -\frac{\pi^3}{\sqrt{e}}(k-k_1) + \left(9 + \frac{C}{4} \right) k + 27C\pi(3g-3-k)(g-1)^2 > 2k > 0 \ ,$$

as the smallest case for B is for $k = 3g - 3$ and $k_1 = 0$. Then, to have

$$-\frac{\pi^3}{\sqrt{e}}\frac{k_1}{A} + B < 0 \ ,$$

it suffices to take:

$$A < \frac{\pi^3}{\sqrt{e}}\frac{k_1}{B},$$

concluding the proof. ■

Chapter 3

Behaviour of the Schwarzian derivative on long complex projective tubes

The Schwarzian derivative parametrizes the fibers of the deformation space of complex projective structures on a surface as vector bundle over its Teichmüller space. In this chapter, we study its behaviour on long complex projective tubes, and we obtain estimates for the pairing of its real part with infinitesimal earthquakes and graftings. As the real part of their Schwarzian coincides with the differential of the renormalized volume, we obtain bounds for the variation of renormalized volume under complex earthquake paths, and its asymptotic behaviour under pinching a compressible curve.

3.1 Results and outline of the chapter

We have already talked about how complex projective structures and Riemann surfaces appear in the study of convex co-compact (or, more generally, geometrically finite) hyperbolic 3-manifolds as naturally induced structures on their boundary at infinity (see Section 1.4.1). We also saw how this leads to a parametrization of the deformation space of convex co-compact metrics on a hyperbolic 3-manifold via the conformal boundary at infinity (see Section 1.4.2).

Recall that the deformation space of complex projective structures forms a holomorphic vector bundle over the one of Riemann surfaces (see Section 1.3). Moreover, the fibers are parameterized by the Schwarzian derivative of the developing map f of the complex projective structure (see Section 1.3.1). Depending on the image of f being simply connected or not, its Schwarzian derivative $\mathcal{S}(f)$ behaves quite differently. In this work, we investigate the case in which the image is not simply connected, which is of particular interest for the world of convex co-compact hyperbolic 3-manifolds with compressible boundary, and their renormalized volume.

Let f be the developing map of a domain Ω in $\mathbb{CP}^1$ with universal cover $\mathbb{H}^2$. We denote the upper half-plane model by $(\mathbb{H}^2, \rho\,|dz|^2) \subseteq \mathbb{C}$, with $\rho\,|dz|^2$ the unique complete hyperbolic metric. The *infinity norm* of $\mathcal{S}(f) = q dz^2$ is defined as

$$\|\mathcal{S}(f)\|_\infty = \sup_{z\in\mathbb{H}^2} |q(z)|/\rho(z) \ .$$

If $f\colon \mathbb{H}^2 \to \mathbb{CP}^1$ is univalent, then, by the Kraus-Nehari's bound [Neh49], the infinity norm of $\mathcal{S}(f)$ is bounded:

$$\|\mathcal{S}(f)\|_\infty \leq 3/2 \ .$$

If f is not univalent, the infinity norm is bounded from below by a function of the *injectivity radius* of the image Ω of f, which, in this case, is not simply connected. More precisely, see Kra-Maskit [KM81, Lemma 5.1]:

$$\|\mathcal{S}(f)\|_\infty \geq \frac{1}{2}\coth^2(\delta_\Omega/2) \ ,$$

where the injectivity radius δ_Ω of Ω is defined as the infimum of the injectivity radii $\mathrm{inj}_\Omega(z)$ of its points

$$\delta_\Omega = \inf_{z\in\Omega} \sup\{r \ s.t. \ B_r(z) \subseteq \Omega\} \ ,$$

with $B_r(z)$ the ball centered at z of radius r with respect to the hyperbolic metric pushed forward to Ω by f. In particular, when $\delta_\Omega \sim 0$, then

$$\|\mathcal{S}(f)\|_\infty \gtrsim 2/\delta_\Omega^2 \ .$$

Thus, it is interesting to further investigate the behaviour of the Schwarzian of a projective structure Z on a closed surface S when the image of its developing map Ω is not simply connected, and especially when its injectivity radius δ_Ω is small.

Definition 3.1.1. We say that a complex projective structure Z on a hyperbolic surface has a *long tube* if the image of its developing map contains a simple closed curve of hyperbolic lenght less than ε_0. A long tube equipped with a complex projective structure is called *long complex projective tube.*

On long tubes $\mathcal{A} \subseteq \Omega$, the (restriction of the) injectivity radius coincides with half of the hyperbolic length of the geodesic representative γ of the unique non-trivial simple closed curve in $\mathcal{A}$, which we call the *core* of $\mathcal{A}$. We will say that a simple closed geodesic in Ω is *short* if it has hyperbolic length $\ell \leq \varepsilon_0$. Short simple closed geodesics have disjoint collars of positive length, see Theorem 4.1.6 in [Bus10]. In the following theorem we present the behaviour of the Schwarzian on long tubes in complex projective structures.

Theorem 3.5.4. Let $\mathcal{A}$ be a long tube of a complex projective surface Z, let γ be its core of length $\ell \leq \varepsilon_0$, and let $\tilde{\gamma} \subseteq \widetilde{Z} = \mathbb{H}^2$ be a lift. Let also $f\colon \mathbb{H}^2 \to \mathbb{CP}^1$ be the developing map of Z, and let $\mathcal{S}(f)$ be its Schwarzian derivative. Then, in a neighborhood of $\tilde{\gamma}$, the Schwarzian $\mathcal{S}(f)$, up to pull-back by a Möbius, behaves as follows

$$\mathcal{S}(f) = \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2 + O\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right) dz^2 \ ,$$

where z is the complex coordinate of $\mathbb{H}^2$, and $O(x)$ stands for a complex valued function such that $\lim_{x\to 0} |O(x)/x|$ is finite.

As the space of holomorphic quadratic differentials on a Riemann surface X is identified with the cotangent space $T^*_X\mathcal{T}(S)$ of the Teichmüller space at X, the Schwarzians of the complex projective structures in the fiber of X can be paired with vectors in the tangent space $T_X\mathcal{T}(S)$. Vectors in $T_X\mathcal{T}(S)$ can be expressed as *harmonic Beltrami differentials*, i.e. as objects of the type $(\overline{f}/\rho)\partial z \otimes d\overline{z}$, with f a holomorphic function on X and ρ such that $\rho|dz|^2$ is the unique hyperbolic representative in X. The pairing is then realized by integration on the surface X (see Section 1.2.3).

As already seen, the real part of the Schwarzian derivative, as a holomorphic quadratic differential, is of particular interest as it coincides with the differential of the *renormalized volume* (see Section 1.6.2). In particular, the variation of renormalized volume along paths in the deformation spaces of convex co-compact manifolds with incompressible boundary, i.e. Teichmüller space of their boundary, has been extensively studied in the case of flow-lines of the gradient flow, see ([BBB19], [BBB23], [BBVP23]. For compressible boundary the layout of this problem changes as in general V_R does not converge under gradient flow as its derivative is unbounded, see for example Theorem 3.4.1, and [SW22].

Earthquaking and *grafting* are two natural paths inside the Teichmüller space, and by infinitesimal we mean the tangent vector at time zero of their respective deformation paths (see Section 3.2.2 below). We recall, very briefly, that an earthquake on a simple closed curve $\gamma \subseteq X$ consists in cutting along γ, twisting by a certain parameter $t \in \mathbb{R}$ the left hand side of the surface, and gluing back. Again in short, grafting on γ means to cut over γ and attach an euclidean cylinder of some height $s \in \mathbb{R}^+$, this gives a new well defined conformal structure on X. Infinitesimal earthquake and infinitesimal grafting on the same (multi)curve at X are orthogonal with respect to the complex structure on $\mathcal{T}(S)$ [McM98, Theorem 2.10].

In the following theorem we show that the pairing of the real part of the Schwarzian on a long complex projective tube is almost trivial along *infinitesimal earthquakes*, while, asymptotically, it grows as $\frac{\pi^2}{\ell}$ in the *infinitesimal grafting* direction.

Theorem 3.5.7. Let Z be a complex projective surface, let $\mathcal{S}(f)$ be its Schwarzian, and let $X = \pi(Z)$ its underlying Riemann surface. Let also μ and ν be, respectively, the infinitesimal earthquake and grafting on the simple closed curve $\gamma \subseteq X$ of hyperbolic length $\ell \leq \varepsilon_0$. Then

$$|\mathrm{Re}\,\langle \mathcal{S}(f), \mu\rangle| \leq F_e(\ell)$$

and

$$\left|\mathrm{Re}\,\langle \mathcal{S}(f), \nu\rangle - \frac{\pi^2}{\ell}\right| \leq \frac{1}{4}\ell + F_{gr}(\ell)$$

with $F_e(\ell)$ and $F_{gr}(\ell)$ two explicit functions such that

$$|F_e(\ell)|\,,\ |F_{gr}(\ell)| \leq C\frac{e^{-\pi^2/\ell}}{\ell}\ ,$$

for some constant $C > 0$.

A heuristic of why one should believe Theorem 3.5.7 is presented in Section 3.3, in which we show the result for the *toy model* case of a *symmetric* complex projective tube.

3.1.1 Connection to Renormalized volume

As already pointed out, the boundary at infinity of any convex co-compact hyperbolic 3-manifold M comes naturally equipped with a complex projective structure, and therefore also with its developing map, which we denote with f_M, and its Schwarzian (see Definition 1.4.2). The condition for M of having compressible boundary translates into having a non-univalent f_M. We can then exploit Theorems 3.5.4 and 3.5.7 to get information on the behaviour of the renormalized volume function in the compressible boundary setting.

Thanks to the fact that $dV_R = \mathrm{Re}(\mathcal{S}(f_M))$, Theorem 3.5.4 and, thanks to our key Lemma 3.2.1, also Theorem 3.5.7 can then be rephrased in terms of dV_R.

In [McM98] McMullen introduced the notion of λ-*complex earthquake* with $\lambda = t + is \in \mathbb{H}^2$, which consists in moving inside the Teichmüller space with first a parameter t earthquake, and then a parameter s grafting, on the same multicurve (actually, more generally, on the same measured lamination). With the following two theorems, we bound the change of renormalized volume along a complex earthquake path on a short simple closed geodesic. More specifically, in Theorem 3.4.1 we bound the change of renormalized volume along an earthquake of parameter t, while in Theorem 3.4.5 we calculate the change along a grafting path of parameter s. The next is a Corollary of the first part of Theorem 3.5.7, and the key Lemma 3.2.1.

Theorem 3.4.1. Let M be a convex co-compact hyperbolic 3-manifold. Let $X_0 \in \mathcal{T}(\partial\overline{M})$, and let $X_t \in \mathcal{T}(\partial\overline{M})$ be the Riemann surface obtained by a parameter $t \in \mathbb{R}$ earthquake on X_0 along a compressible simple closed curve γ in $\partial\overline{M}$, of hyperbolic length $\ell \leq \varepsilon_0$. Then, we have the following estimate for the renormalized volume of the associated convex co-compact manifolds M_0 and M_t:

$$|V_R(M_t) - V_R(M_0)| \leq F(\ell)t \ ,$$

with $F(\ell)$ an explicit function of ℓ such that

$$|F(\ell)| \leq C\frac{e^{-\pi^2/\ell}}{\ell}$$

for some explicit constant $C > 0$.

Theorem 3.4.1, as well as Theorem 3.5.7, are stated later with explicit expressions for the functions $F_e(\ell)$, $F_{gr}(\ell)$, $F(\ell)$, and the constants $C > 0$. Deriving this entails some elementary and tedious computations to carry out in Section 3.5.1, that could be made more slender otherwise. On the other hand, this allows for a direct comparison with Theorem 2.5.4 from the previous chapter, other than providing a more precise estimate. It is then possible to check that for a small and explicit ℓ, Theorem 3.4.1 yields a sharper bound, for the change of renormalized volume under earthquake, than the one of Theorem 2.5.4. Such a level of precision was not necessary for the study of Maldacena's question, where the focus was also on curves that are not that short.

If the change in renormalized volume becomes almost trivial when earthquaking along a short curve, it is instead very large when pinching the curve - that is, moving toward the boundary of the Weil-Petersson completion of the Teichmüller space (defined in Section 1.2.3). A way to realize the pinching of a curve is by grafting along it with a parameter tending to infinity (see Section 3.2.2 and Lemma 3.4.2). We can then apply our estimate for the Schwarzian to obtain the asymptotic behavior of the renormalized volume under *pinching* a compressible curve in the boundary at infinity of a convex co-compact hyperbolic 3-manifold, recovering in particular Theorem A.15 in [SW22].

Theorem 3.4.5. Let M_0 be a convex co-compact hyperbolic 3-manifold and let $\gamma \in \partial\overline{M_0}$ be a compressible curve in its boundary of length $\ell_0(\gamma) \leq \varepsilon_0$, with respect to the hyperbolic metric conformal to $\partial_\infty M_0$. The composition of the renormalized volume with the grafting path $(M_s)_{s\in[0,\infty)}$ along γ satisfies

$$V_R(M_s) - V_R(M_0) = -\frac{\pi^3}{\ell_s(\gamma)} + \frac{\pi^3}{\ell_0(\gamma)} + (\ell_s(\gamma) - \ell_0(\gamma))\frac{\pi}{4} + O\left(e^{-\pi s/(2\ell_0(\gamma))}s^3\right) \ .$$

In particular, when $\ell_s(\gamma) \to 0$, the renormalized volume diverges as $-\pi^3/\ell_s(\gamma)$.

3.1.2 Outline of the chapter

Section 3.2 contains the main objects and notions that will be needed and that were not treated in Chapter 1, and the key Lemma 3.2.1. In Section 3.3 we give a motivating example by explicitly calculating the Schwarzian derivative of *symmetric* complex projective tubes. This is done in Lemma 3.3.3 and Proposition 3.3.5, which is the analogue of Theorem 3.5.7.

Section 3.4 contains the applications of the Schwarzian bounds to the renormalized volume for convex co-compact hyperbolic manifolds with compressible boundary.

Section 3.5 is dedicated to the study of the Schwarzian of long tubes in the general case. In Subsection 3.5.1 we prove Theorems 3.5.4 and 3.5.7, assuming the bounds on the correction terms given in Subsections 3.5.2 and 3.5.3. In particular, in Section 3.5.2 we give bounds for the pairing of the real part of the Schwarzian and infinitesimal grafting, while in Section 3.5.3 there are the analogous bounds for the pairing with infinitesimal earthquakes. These are obtained by the tool of the Osgood-Stowe tensor, as explained in Section 3.5.1. There, we recall relations between the Osgood-Stowe tensor [OS92] (and a non-traceless extension) and the Schwarzian derivative, which can also be found in [BB24]. In Section 3.5.4 one can find the Fourier analysis leading to the estimates for the conformal flat factor of long tubes, which describes how far is the tube from being symmetric. This is used to obtain the bounds in Sections 3.5.2 and 3.5.3.

3.2 Preliminaries

We have already introduced tubes and long tubes in the background. However, in this chapter we need some additional specific properties, which are reported in the next section. We then proceed to define complex earthquakes and their infinitesimal counterpart. The section concludes with a Stokes-type lemma, expanding the version for infinitesimal earthquakes of Lemma 2.5.1 in Chapter 2.

3.2.1 Thin tubes

Thin tubes were defined in 1.2.2. Let us see here some more precise information on their geometry.

The hyperbolic metric on the thin tube $\mathbb{T}(\ell)$ of core length ℓ can be written as

$$d\rho^2 + \left(\frac{\ell}{2\pi}\right)^2 \cosh^2(\rho) d\theta^2 \ ,$$

with $\theta \in [0, 2\pi]$ and $\rho \in [-L, L]$, see [Bus10, Collar Lemma: Theorem 4.1.1]. Moreover, in $\mathbb{T}(\ell)$, the *injectivity radius* is bounded as

$$\frac{\ell}{2} \leq \operatorname{inj}(p) = \operatorname{arsinh}\left(\sinh(\ell/2)\cosh(L-d)\right) , \qquad d = d(p, \partial\mathbb{T}(\ell)) \ ,$$

and its maximum is achieved on $\partial\mathbb{T}(\ell)$, see [Bus10, Thm 4.1.6]. In particular we get that each component of the boundary of such a cylinder $\mathbb{T}(\ell)$ has length

$$\ell \cosh\left(\operatorname{arsinh}\left(\frac{1}{\sinh(\ell/2)}\right)\right)$$

and the injectivity radius on points $p \in \partial\mathbb{T}(\ell)$ is given by $\operatorname{arsinh}\left(\sinh(\ell/2)\cosh(L)\right)$, and it is always greater than $\varepsilon/2$ out of the union of the thin tubes.

3.2.2 Complex Earthquakes along simple closed geodesics

We restrict here the treatment to the case of twisting or earthquaking on a simple closed geodesic, but both operations can be extended to the space of measured laminations (see [McM98]).

Let γ be a simple closed geodesic on a Riemann surface $X \in \mathcal{T}(S)$. We fix an identification of the universal cover $\widetilde{X}$ with $\mathbb{H}^2$ such that the geodesic γ lifts to the vertical geodesic $\tilde{\gamma}$ of the upper half-space $\mathbb{H}^2$ through 0 and ∞. Given a hyperbolic surface X and a simple closed geodesic γ the parameter t *earthquake* $\mathrm{eq}_{t,\gamma}(X)$, $t \in \mathbb{R}$, is defined in the lift of a neighborhood of γ as

$$z \longrightarrow \begin{cases} e^t z & \text{if } z \in \mathbb{H}^2_d \\ z & \text{if } z \in \mathbb{H}^2_s \end{cases} ,$$

with $\mathbb{H}^2_d = \{z \in \mathbb{H}^2 \mid \mathrm{Re} z > 0\}$ and $\mathbb{H}^2_s = \{z \in \mathbb{H}^2 \mid -\mathrm{Re} z > 0\}$, while it is the identity outside. This defines a new hyperbolic metric on S, therefore, a new point in $\mathcal{T}(S)$, and also a new Fuchsian projective structure.

Given a hyperbolic surface X and a simple closed geodesic γ a parameter s *grafting* $\mathrm{gr}_{s,\gamma}(X)$, $s \in \mathbb{R}^+$, is obtained by cutting along γ and inserting an Euclidean cylinder of height s and core equal to the hyperbolic length $\ell_\gamma(X)$ of γ. The gluing of the Euclidean and the hyperbolic metric, which remain unchanged in $X \setminus \gamma$, produces a new conformal structure, and therefore a new point in Teichmüller space by uniformizing (and obtaining a new hyperbolic metric). Let now $\widetilde{\mathcal{A}}(s)$ be an s-angular sector in $\mathbb{H}^2$. We equip the inserted Euclidean cylinder with the complex projective structure

$$\widetilde{\mathcal{A}}(s)/\langle z \to e^{\ell_\gamma(X)} z \rangle \ .$$

This, together with the Fuchsian structure on $X \setminus \gamma$, produces a complex projective structure for $\mathrm{gr}_{s\gamma}(X)$. The grafting operation $\mathrm{gr}_{s\gamma}(X)$ lifts to the universal cover by cutting along $\tilde{\gamma}$ with the map

$$z \longrightarrow \begin{cases} e^{is} z & \text{if } z \in \mathbb{H}^2_d \\ z & \text{if } z \in \mathbb{H}^2_s \end{cases} ,$$

and inserting $\widetilde{\mathcal{A}}(s)$. We call, respectively, the *infinitesimal earthquake* and the *infinitesimal grafting* at X along γ, the following two Beltrami differentials $\mu, \nu \in T_X\mathcal{T}(S)$

$$\mu = \partial_t|_{t=0} \left(\mathrm{eq}_{t\gamma}(X)\right) , \qquad \nu = \partial_s|_{s=0} \left(\mathrm{gr}_{s\gamma}(X)\right) \ .$$

Finally, given a parameter $\lambda = t + is \in \mathbb{C}$, $s > 0$, we define the λ-*complex earthquake* on γ in X as the composition of the two operation on the Teichmüller space just defined:

$$\mathrm{Eq}_{\lambda\gamma} = \mathrm{gr}_{s\gamma} \circ \mathrm{eq}_{t\gamma}(X) \ .$$

By McMullen's Theorem 2.10 in [McM98], the complex earthquake map is holomorphic in λ and at any X:

$$\nu = i\mu \ .$$

3.2.3 A Stokes type lemma

In this subsection we state a key lemma for this chapter. This is an application of the Stokes Theorem in the setting of the paring between a holomorphic quadratic differential and the Beltrami differentials μ and ν of, respectively, infinitesimal earthquake and infinitesimal grafting. The first part of the statement is Lemma 2.5.1 (or, Lemma 5.1 in [CGS24]). The proof of the second part follows, by applying the same proof, from the fact that $\nu = i\mu$.

Lemma 3.2.1. Let X be a Riemann surface structure on S, with z its complex coordinate. Let $\gamma\colon \mathbb{R}/\ell\mathbb{Z} \to X$ be a unit length parameterization of a simple closed geodesic with respect to the unique hyperbolic representative in X, of length ℓ. Let also $\mu \in T_X\mathcal{T}(S)$ be the infinitesimal earthquake along γ, $\nu \in T_X\mathcal{T}(S)$ be the infinitesimal grafting along γ, and $q(z)dz^2 \in T^*_X\mathcal{T}(S)$ be a holomorphic quadratic differential. Then the followings hold:

$$\mathrm{Re}\langle q(z)dz^2, \mu\rangle = -\frac{1}{2}\int_{\mathbb{R}/\ell\mathbb{Z}} \mathrm{Re}(q(i\dot\gamma(t), \dot\gamma(t)))dt \ ,$$

and

$$\mathrm{Re}\langle q(z)dz^2, \nu\rangle = -\frac{1}{2}\int_{\mathbb{R}/\ell\mathbb{Z}} \mathrm{Re}(q(\dot\gamma(t), \dot\gamma(t)))dt \ .$$

Proof. We just need to prove the second equality. Let w be the vector field realizing the unit velocity infinitesimal graft ν along the image of γ. It is enough then to notice that, if v is the vector field realizing a unit velocity infinitesimal along the same γ, then, since the complex earthquake map $Eq_{\lambda\gamma}(\cdot)$ is holomorphic (see Theorem 2.10 in [McM98]) and also $gr_{s\gamma}(X) = Eq_{is\gamma}(X)$ and $tw_{t\gamma}(X) = Eq_{t\gamma}(X)$

$$w = iv \ .$$

The proof of the version for infinitesimal earthquakes in [CGS24] then applies straightforward up to substitute v with $w = iv$. ∎

In this chapter, we will apply Lemma 3.2.1 with $q(z)dz^2 = \mathcal{S}(f)$ the Schwarzian of a complex projective surface. This leads to a simpler formula for the variation of the renormalized volume, in the special case of μ equal to an infinitesimal earthquake or an infinitesimal grafting.

3.3 Toy model: the symmetric tube case

In this section we outline the idea behind Theorem 3.5.7. The main observation is that the result is almost trivial in the case of a *symmetric* complex projective tube. Here a complex projective tube is symmetric if its image under its developing map in $\mathbb{CP}^1$ is bounded by two concentric round circles.

Then, the bulk of the remaining work in proving Theorem 3.5.7 is to show that the Schwarzian of a general complex projective long tube behaves like the one in the symmetric case, up to a term that goes to zero in ℓ, with ℓ the hyperbolic length of the core curve.

Consider the half-space model for the 2-hyperbolic space with coordinate $z = \rho e^{i\theta}$, $\rho > 0$, $\theta \in [0, \pi]$, and $f_\ell\colon \mathbb{H}^2 \to \mathbb{C}$ the map uniformizing the infinite cylinder $\mathbb{C}^*$ with a hyperbolic metric of core length ℓ defined as

$$f_\ell(z) = z^{\frac{2\pi i}{\ell}} = e^{-\frac{2\pi\theta}{\ell} + i\frac{2\pi}{\ell}\log(\rho)}.$$

We denote by $\widetilde{\mathcal{A}}(L)$ the L-neighborhood in $\mathbb{H}^2$ of the vertical geodesic between 0 and ∞. Since $\widetilde{\mathcal{A}}(L)$ corresponds to the annular sector $\theta \in [\theta(L), \pi - \theta(L)]$ with (see [Mar16b, Lemma 5.2.7])

$$\frac{1}{\cosh(L)} = \sin(\theta(L)) \ ,$$

the restriction of f_ℓ to $\widetilde{\mathcal{A}}(L)$ is the developing map of the annulus

$$\mathcal{A}_\ell(L) = \left\{ re^{i\theta} \in \mathbb{C} \ \mid \ e^{-\frac{2\pi^2}{\ell} + \frac{2\pi}{\ell}\arcsin\left(\frac{1}{\cosh(L)}\right)} \leq r \leq e^{-\frac{2\pi}{\ell}\arcsin\left(\frac{1}{\cosh(L)}\right)} \right\} \ .$$

Note that this is indeed *symmetric*.

Remark 3.3.1. For any symmetric complex projective tube $\mathcal{A}$, up to composing by a Möbius transformation bringing the center of the concentric round boundaries to the origin, there exist unique $\ell > 0$ and $L > 0$ such that $\mathcal{A}$ is equal to $\mathcal{A}_\ell(L)$. We also recall that composing by Möbius transformations leaves the Schwarzian derivative invariant.

Observe that, if we equip $\mathcal{A}_\ell(L)$ with the restriction of the flat metric $h_0 = \frac{1}{r^2}|dz|^2$ on the infinite cylinder $\mathbb{C}^*$, we get a well truncated euclidean cylinder with core length 2π and of some height m, with m such that $(\mathcal{A}_\ell, h_0)$ is conformal to a truncated hyperbolic cylinder with core of length ℓ and width L.

Remark 3.3.2. The conformal structure of a (truncated) cylinder is uniquely determined by its *modulus*, which, in the case of a symmetric cylinder in $\mathbb{C}$ with boundaries centered at the origin and of radii r_1 and r_2, is equal to $\frac{1}{2\pi}\log(r_2/r_1)$ (see Section 1.2.1). Then, any truncated cylinder equipped with a Riemann surface structure is conformal to a symmetric one of the form $\mathcal{A}_\ell(L)$.

By a direct computation we have:

Lemma 3.3.3. Let $\mathcal{A}$ be a symmetric complex projective tube. Then, there exist unique $\ell > 0$ and $L > 0$ such that the Schwarzian of $\mathcal{A}$ is defined on $\widetilde{A}(L)$ and equal to

$$\mathcal{S}(f_\ell)(z) = \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2 \ .$$

Proof. By remark 3.3.1, we can assume $\mathcal{A}$ to be equal to $\mathcal{A}_\ell(L)$. Then, we just have to compute the Schwarzian derivative of its developing map f_ℓ on $\widetilde{A}(L)$:

$$\begin{aligned} q = \mathcal{S}(f_\ell) &= \left(\left(\frac{(f_\ell)''}{(f_\ell)'}\right)' - \frac{1}{2}\left(\frac{(f_\ell)''}{(f_\ell)'}\right)^2\right) dz^2 \\ &= \left(\left(\left(\frac{2\pi i}{\ell} - 1\right)\frac{1}{z}\right)' - \frac{1}{2}\left(\left(\frac{2\pi i}{\ell} - 1\right)\frac{1}{z}\right)^2\right) dz^2 \\ &= \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2 \ . \end{aligned}$$

Completing the proof. ■

In Section 3.5, the width L will usually be the one given by the Collar Lemma (see Subsection 1.2.1, and Definition 1.2.2): $L = \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right)$. In this case, we denote the symmetric tube simply by $\mathcal{A}_\ell$, and we say that $\mathcal{A}_\ell$ is the *standard symmetric tube* of core length ℓ.

Remark 3.3.4. The expression of $\mathcal{S}(f_\ell)$ in Lemma 3.3.3 is in $\mathbb{H}^2$, however if one wants to use the coordinates of $\mathcal{A}$ instead of seeing it as a neighborhood of the vertical axis in $\mathbb{H}^2$, the formula changes as follows.

We first need to recall that The Schwarzian derivative satisfies the following composition rule:

$$\mathcal{S}(f \circ g) = g^*\mathcal{S}(f) + \mathcal{S}(g) \ , \tag{3.1}$$

whenever f and g are two locally injective holomorphic maps, whose composition $f \circ g$ is well defined. Denoting by w the complex coordinate in $\mathbb{C}$, the local inverse f_ℓ^{-1} of f_ℓ sends w to $z = w^{-\frac{i\ell}{2\pi}}$, and then also $dz(w)^2 = \frac{\ell^2}{4\pi^2} w^{2\left(-\frac{i\ell}{2\pi}-1\right)} dw^2$. Therefore, by applying equation (3.1) with $f = f_\ell$ and $g = f_\ell^{-1}$ we obtain:

$$\mathcal{S}\left(f_\ell^{-1}\right) = -\left(f_\ell^{-1}\right)^* \left(\mathcal{S}(f_\ell)\right)$$

and then

$$\mathcal{S}\left(f_\ell^{-1}\right) = -\frac{1}{2w^2}\left(1 + \frac{\ell^2}{4\pi^2}\right) dw^2 \ .$$

The first part of the statement of the next proposition is the "toy model" version of Theorem 3.5.4, while the second one is the "toy model" version of Theorem 3.5.7. This is because, thanks to the key Lemma 3.2.1, the two integral terms coincide with the pairing of the Schwarzian with, respectively, infinitesimal earthquakes and infinitesimal graftings. Then, in Section 3.5 we show that the result in the general case is equal to the formulas in Proposition 3.3.5 up to an error term decaying exponentially to zero in ℓ.

Proposition 3.3.5. On the core γ of the symmetric complex projective tube $\mathcal{A}_\ell$, the following equalities for its Schwarzian hold:

$$\mathrm{Re}\left(\mathcal{S}\left(f_\ell^{-1}\right)(i\dot\gamma, \dot\gamma)\right) = 0 \ , \qquad \mathrm{Re}\left(\mathcal{S}\left(f_\ell^{-1}\right)(\dot\gamma, \dot\gamma)\right) = \frac{2\pi^2}{\ell^2} + \frac{1}{2} \ .$$

Moreover

$$\frac{1}{2}\int_0^\ell \mathrm{Re}(\mathcal{S}(f_\ell)^{-1}(i\dot\gamma, \dot\gamma))dt = 0 \ , \qquad \frac{1}{2}\int_0^\ell \mathrm{Re}(\mathcal{S}(f_\ell)^{-1}(\dot\gamma, \dot\gamma))dt = \frac{\pi^2}{\ell} + \frac{\ell}{4} \ .$$

Proof. Up to pulling back $\mathcal{A}_\ell$ with f_ℓ, we have $\gamma(t) = ie^t$, $t \in [0, \ell]$, and $\mathcal{S}(f_\ell)$ as in Lemma 3.3.3. Then, $\dot\gamma(t) = e^t \frac{\partial}{\partial y}$, $i\dot\gamma(t) = -e^t \frac{\partial}{\partial x}$, and, since $dz = dx + idy$, on γ, we have

$$\mathrm{Re}\left(\frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2\right)(i\dot\gamma, \dot\gamma) = \mathrm{Re}\left(-\frac{1}{2e^{2t}}\left(1 + \frac{4\pi^2}{\ell^2}\right) ie^{2t}\right) = 0 \ ,$$

and

$$\mathrm{Re}\left(\frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2\right)(\dot\gamma, \dot\gamma) = \mathrm{Re}\left(\frac{1}{2e^{2t}}\left(1 + \frac{4\pi^2}{\ell^2}\right) e^{2t}\right) = \frac{1}{2}\left(1 + \frac{4\pi^2}{\ell^2}\right) \ ,$$

completing the proof of the first part of the statement. The second one follows directly by integration. ∎

3.4 Applications to Renormalized Volume

In what follows, we apply Theorem 3.5.4 and Theorem 3.5.7 to estimate how the renormalized volume changes under complex earthquake (see Section 3.2.2).

3.4.1 Renormalized volume bounds under earthquakes on short compressible curves

In this section, we prove the version of Theorem 3.4.1 with an explicit formula for $F(\ell)$.

Theorem 3.4.1. Let M be a convex co-compact hyperbolic 3-manifold. Let $X_0 \in \mathcal{T}(\partial\overline{M})$, and let $X_t \in \mathcal{T}(\partial\overline{M})$ be the Riemann surface obtained by a parameter $t \in \mathbb{R}$ earthquake on X_0 along a compressible simple closed curve γ in $\partial\overline{M}$ of hyperbolic length $\ell \leq \varepsilon_0$. Then, we have the following estimate for the renormalized volume of the associated convex co-compact manifolds M_0 and M_t:

$$|V_R(M_t) - V_R(M_0)| \leq 113\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell} t + 142\pi^4 \frac{G(\ell)^2}{\ell} e^{-\pi^2/\ell} t \ ,$$

with $G(\ell) = 1 + O\left(e^{-\pi^2/(2\ell)}\right)$ as $\ell \to 0$ and always bounded by $e^{2.8}$.

Proof. We first recall that the differential of the renormalized volume dV_R at a point X_s coincides with $\mathrm{Re}(\mathcal{S}(f_{M_s}))$, with M_s the convex co-compact hyperbolic 3-manifold associated to X_s, and f_{M_s} the developing map of its associated complex projective structure. Let us define the earthquake path $X_s = eq_s(X_0)$, with $s \in [0, t]$, and write the difference of renormalized volumes as

$$V_R(M_t) - V_R(M_0) = \int_0^t (dV_R)_{X_s}(\mu_s) ds \ ,$$

with μ_s the Beltrami obtained by deriving to the earthquaking path at time s, i.e. the infinitesimal earthquake at X_s. Then, the result is a direct corollary of Theorem 3.5.7 in the version of Section 3.5.1 which is on page 77, and the key Lemma 3.2.1. ■

3.4.2 Asymptotic behaviour of V_R along pinching a compressible curve

In this section, we prove Theorem 3.4.5.

Differently from the previous section, in which to prove Theorem 3.4.1 we just integrated the differential given by the first equality in Theorem 3.5.7, we now have to take care of two facts. Firstly, the length of the simple closed geodesic on which we are grafting changes its length along the deformation path. Secondly, while earthquaking on the same geodesic produces a flow, grafting does not.

We will consider the path $X_s = gr_{s\gamma}(X_0)$ obtained by grafting on the short compressible simple closed curve γ in $X_0 = [\partial_\infty M_0]$, and denote by $(M_s)_{s\in[0,\infty)}$ the path of convex co-compact hyperbolic 3-manifolds associated to X_s through the Uniformization Theorem. We will consider here the composition $\ell_\cdot(\gamma)$ of the length function

$$\ell_\gamma \colon \mathcal{T}(\partial\overline{M}) \to \mathbb{R}^+ \ ,$$

with the path $(X_s)_{s\in[0,\infty)}$ which associates to s the length $\ell_s(\gamma)$ of γ with respect to the hyperbolic representative in X_s. For the sake of notation, whenever the dependence on γ is clear, we will just write $\ell_s = \ell_s(\gamma)$. We recall that as s goes to infinity $\ell_s(\gamma)$ tends to 0.

The following Lemma was obtained by Diaz and Kim in [DK12, Proposition 3.4] (or also in [Hen11, Lemma 4.1]). Our contribution is just to explicitly write down the constant appearing in the lower bound.

Lemma 3.4.2. Let $X_0 \in \mathcal{T}(\partial\overline{M})$, and let $X_s \in \mathcal{T}(\partial\overline{M})$, with $s > 0$, be the Riemann surface obtained by a parameter s grafting along a short geodesic γ of length $\ell_0 \leq \varepsilon_0$. Then,

$$\frac{\pi}{\pi+s}\ell_0 \geq \ell_s \geq \frac{\pi}{2(\pi+s)}\ell_0 \ .$$

Proof. The first inequality was proven in [DK12, Proposition 3.4], by a direct computation applying the definition of grafting at the universal cover (see Subsection 3.2.2). In the same statement, we can also find the lower bound

$$\ell_s \geq \frac{2\theta_0}{2\theta_0 + s} \ ,$$

where θ_0 is such that the thin tube around γ (see Definition 1.2.2) in X_0 can be isometrically lifted in $\mathbb{H}^2$ to

$$\{\rho e^{i\theta} \in \mathbb{H}^2 \mid \rho \in [1, e^{\ell}],\ \theta \in [\pi/2 - \theta, \pi/2 + \theta]\} \ .$$

By elementary hyperbolic geometry (see for example [Mar16b, Lemma 5.2.7]), the angle θ_0 satisfies:

$$\theta_0 = \text{artg}(\sinh(L)) = \text{artg}\left(\frac{1}{\sinh(\ell/2)}\right) \ ,$$

were the last equality follows form the fact that the width L of the thin tube is

$$L = \text{arsinh}\left(\frac{1}{\sinh(\ell/2)}\right) \ .$$

Since now $\ell \in [0, \varepsilon_0]$, with $\varepsilon_0 = 2\text{arsinh}(1)$, we can estimate θ_0

$$\pi/4 \leq \theta_0 \leq \pi/2 \ ,$$

from which the right inequality of our statement follows. ∎

Remark 3.4.3. The upper bound in Lemma 3.4.2 is asymptotically sharp (see Theorem 6.6 in [DW08]).

We define:

$$\nu_0^s = \left.\frac{d}{dt}\right|_{t=0} gr_{t\gamma}(X_s) \ ,$$

and

$$\nu_s^0 = \left.\frac{d}{dt}\right|_{t=s} gr_{t\gamma}(X_0) \ .$$

Note that both Beltrami differentials belong to the tangent space $T_{X_s}\mathcal{T}(\partial\overline{M})$, and that the first one corresponds to infinitesimal grafting, while the second one is the derivative of grafting path $gr_{t\gamma}(X_0)$ at $t = s$, and they are not the same.

Lemma 3.4.4. In the notations above:

$$(dV_R)_{X_s}(\nu_s^0) = \left(\frac{\pi}{4} + \frac{\pi^3}{\ell_s^2}\right) d\ell(\nu_s^0) + O\left(\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2}\right) \ ,$$

where $O(x)$ stands for a real function such that $\lim_{x\to 0} |O(x)/x|$ is finite.

Proof. Let us denote by $z = x + iy$ the complex coordinate of the half-space model $\mathbb{H}^2$. Note that the holomorphic quadratic differential dz^2/z^2 is invariant under the action of the hyperbolic isometry $\varphi_{\gamma,s}(z) = e^{\ell_s} z$ relative to the simple closed geodesic γ in X_s. By Gardiner's formula for the differential of the length function of γ (see [Gar75]):

$$d\ell_\gamma(\mu) = \frac{2}{\pi} \mathrm{Re} \left\langle \frac{dz^2}{z^2}, \widetilde{\mu} \right\rangle_{A_{\gamma,s}} \tag{3.2}$$

where μ is a harmonic Beltrami differential in $T_{X_s}\mathcal{T}(\partial \overline{M})$, and $\widetilde{\mu}$ denotes its lift to the annulus $\widetilde{\mathcal{A}_{\gamma,s}} = \mathbb{H}^2/\langle \varphi_{\gamma,s} \rangle$. If we take $\mu = \nu_s^0$, being this equal to zero out of a tubular neighborhood of γ, the pairing is such that

$$\mathrm{Re} \left\langle \frac{dz^2}{z^2}, \widetilde{\nu_s^0} \right\rangle_{\widetilde{\mathcal{A}_{\gamma,s}}} = \mathrm{Re} \left\langle \frac{dz^2}{z^2}, \nu_s^0 \right\rangle \ , \tag{3.3}$$

where the last term stands for the coupling between holomorphic quadratic and Beltrami differentials on the covering of whole surface X_s. Now, recall, see Theorem 1.6.5, that the differential of the renormalized volume function at X_s is the real part of the Schwarzian derivative of the developing map of the boundary at infinity of M_s:

$$(dV_R)_{X_s}(\mu) = \mathrm{Re} \langle \mathcal{S}(f), \mu \rangle \ .$$

Moreover, on a smaller tube (note that we can assume ν_s^0 to be zero outside it), by Theorem 3.5.4 and in the notations of Section 3.3, this can be expressed as a sum of a symmetric part and a correction term:

$$\mathrm{Re} \left(\mathcal{S}(f) \right) = \mathrm{Re} \left(\mathcal{S} \left(f_{\ell_s} \right) + q_{\ell_s}(z) dz^2 \right) \ ,$$

with

$$\mathcal{S} \left(f_{\ell_s} \right) = \frac{1}{2z^2} \left(1 + \frac{4\pi^2}{\ell_s} \right) dz^2 \ , \tag{3.4}$$

and

$$|q_{\ell_s}(z)| = O \left(\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2} \right) \ . \tag{3.5}$$

The identities (3.2), (3.3) and (3.4) give the first term in the right hand side of the equality in the statement:

$$\mathrm{Re} \left\langle \mathcal{S} \left(f_{\ell_s} \right), \nu_s^0 \right\rangle = \mathrm{Re} \left\langle \left(\frac{1}{2} + \frac{2\pi^2}{\ell_s} \right) \frac{dz^2}{z^2}, \nu_s^0 \right\rangle = \left(\frac{\pi}{4} + \frac{\pi^3}{\ell_s^2} \right) d\ell(\nu_s^0) \ .$$

Then, it remains the term

$$\left\langle q_{\ell_s}(z) dz^2, \nu_s^0 \right\rangle \ ,$$

but since, if $\nu_s^0 = \nu \frac{\partial}{\partial z} \otimes d\overline{z}$, with $\nu \in L^\infty(X_s)$, by (3.5) we obtain

$$\left| \mathrm{Re} \left\langle q_{\ell_s}(z) dz^2, \nu_s^0 \right\rangle \right| = \left| \mathrm{Re} \int_{\mathcal{A}_{\gamma,s}} q_{\ell_s} \nu dx dy \right| = O \left(\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2} \right) \ ,$$

which concludes the proof. ∎

We can now prove the main result of this section.

Theorem 3.4.5. Let M_0 be a convex co-compact hyperbolic 3-manifold and let $\gamma \in \partial \overline{M_0}$ be a compressible curve in its boundary of length $\ell_0(\gamma) \leq \varepsilon_0$, with respect to the hyperbolic metric conformal to $\partial_\infty M_0$. The composition of the renormalized volume with the grafting path $(M_s)_{s\in[0,\infty)}$ along γ satisfies

$$V_R(M_s) - V_R(M_0) = -\frac{\pi^3}{\ell_s(\gamma)} + \frac{\pi^3}{\ell_0(\gamma)} + (\ell_s(\gamma) - \ell_0(\gamma))\frac{\pi}{4} + O\left(e^{-\pi s/(2\ell_0(\gamma))}s^3\right) .$$

In particular, when $\ell_s(\gamma) \to 0$, the renormalized volume diverges as $-\pi^3/\ell_s(\gamma)$.

Proof. By integrating Lemma 3.4.4 along the path we have:

$$\begin{aligned} V_R(M_s) &= V_R(M_0) + \int_0^s (dV_R)_{X_s}(\nu_s^0)ds \\ &= V_R(M_0) + \int_0^s \left(\frac{\pi}{4} + \frac{\pi^3}{\ell_s^2}\right)(d\ell)_{X_s}(\nu_s^0)ds + \int_0^s O\left(\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2}\right) ds \\ &= V_R(M_0) + \int_{\ell_0}^{\ell_s} \left(\frac{\pi}{4} + \frac{\pi^3}{u^2}\right) du + \int_0^s O\left(e^{-\pi s/(2\ell_0)}s^2\right) ds \\ &= V_R(M_0) + (\ell_s - \ell_0)\frac{\pi}{4} - \frac{\pi^3}{\ell_s} + \frac{\pi^3}{\ell_0} + O\left(e^{-\pi s/(2\ell_0)}s^3\right) , \end{aligned}$$

where the third equality is obtained by the change of variables $u = \ell(s)$ for the second term, and by using Lemma 3.4.2 in the third term as this implies

$$\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2} \leq \frac{4(\pi+s)^2}{\pi^2\ell_0^2} e^{-\pi^2(\pi+s)/(2\pi\ell_0)} ,$$

and so

$$O\left(\frac{e^{-\pi^2/(2\ell_s)}}{\ell_s^2}\right) = O\left(e^{-\pi s/(2\ell_0)}s^2\right) ,$$

which completes the proof. ∎

3.5 Schwarzian derivative on long tubes

This section is dedicated to the study of the Schwarzian of long complex projective tubes. In the first subsection, we start by introducing the central object used in our analysis: the *Osgood-Stowe tensor of a complex projective structure*. We will denote it by $B(\rho, e^{2u}\rho)$, where ρ stands for the hyperbolic metric of $\mathbb{H}^2$, and $u\colon \mathbb{H}^2 \to \mathbb{R}$ a smooth function which is the *conformal factor* between the two metrics ρ and $e^{2u}\rho$, and depends on the developing map of the projective structure. Right after, we prove Theorem 3.5.4 exploiting the results of Subsection 3.5.4. Still in Subsection 3.5.1, we prove Theorem 3.5.7, first defining two suitable norms for $B(\rho, e^{2u}\rho)$, which coincide, respectively, with the pairing of the real part of the Schwarzian with infinitesimal earthquakes and infinitesimal graftings (see Definitions 3.5.5 and 3.5.6, and Lemma 3.2.1), and using the estimates on these norms of Subsections 3.5.2 and 3.5.3. The main idea behind these estimates, is to express u as a sum of three other conformal factors of different metrics, such that one of them, denoted by u_1, satisfies $\Delta u_1 = 0$. We will then require the Fourier analysis of Subsection 3.5.4, in which we bound the *flat conformal factor* u_1 and its derivatives in terms of the width of the long tube.

3.5.1 The Osgood-Stowe tensor

We present here the main tool of our analysis: the *Osgood-Stowe tensor* presented in [OS92] by the same authors, and whose relation with the 3-dimensional hyperbolic geometry had been largely studied by Bridgeman and Bromberg in [BB24]. The Osgood-Stowe tensor B associates to any pair of conformal metrics g and $g' = e^{2u}g$ on a Riemann surface X, with $u\colon X \to \mathbb{R}$ a smooth function, a real $(0,2)$-tensor, defined through the conformal factor u between the two metrics (see Definition 3.5.1). We will denote it by $B(g, e^{2u}g)$. The key point is then that the real part of the Schwarzian $Re\,(\mathcal{S}(f))$ of a complex projective structure Z (i.e. $f\colon \mathbb{H}^2 \to \mathbb{CP}^1$ is the developing map of Z on S) is equal to $B\,(\rho, e^{2u}\rho)$, with ρ the hyperbolic metric on $\mathbb{H}^2$, and u such that

$$f^*(|dz|^2) = e^{2u}\rho \ ,$$

with z the complex coordinate of $\mathbb{C} \subseteq \mathbb{CP}^1$ (see Theorem 3.5.2). We then say that $B\,(\rho, e^{2u}\rho)$ is the *Osgood-Stowe tensor of the projective structure* Z.

From now on, we will focus on what happens on the restriction of the developing map f of a complex projective structure Z on a closed surface S to a long complex projective tube. More specifically, this means that the image Ω of f is not simply connected and that its shortest geodesic has length less than ε_0. We will keep denoting with f such a restriction. Let γ be the geodesic representative, with respect to the hyperbolic metric h_∞ pulled back on Ω by f, of a non-trivial simple closed curve in Ω of length $\ell \leq \varepsilon_0$. The underlying Riemann structure $X = \pi(Z)$, where, recall, $\pi\colon \mathcal{P}(S) \to \mathcal{T}(S)$ is the vector bundle of projective structures on a closed surface S, defines a natural identification of the universal cover $\widetilde{\Omega}$ with $\mathbb{H}^2$. There are then two discrete subgroups $\Gamma_0 < \mathbb{P}SL(2,\mathbb{R})$ (isomorphic to $\pi_1(S)$) and $\Gamma' < \mathbb{P}SL(2,\mathbb{C})$ (isomorphic to $\pi_1(S)/\pi_1(\Omega)$) such that f is (Γ_0, Γ')-invariant. Up to composing by a Möbius transformation, which does not affect the Schwarzain of f, we can assume $f^{-1}(\gamma)$ to be the geodesic through 0 and ∞ in $\mathbb{H}^2$. Equipping X with its unique hyperbolic metric, we consider the *thin tube* $\mathbb{T}(\ell) \subseteq X$, i.e. the L-neighborhood of $\gamma \subseteq X$, with $L = \operatorname{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right)$ (see Definition 1.2.2). This lifts to the L-neighborhood $\widetilde{\mathcal{A}} \subseteq \mathbb{H}^2$ of $f^{-1}(\gamma)$. Then, its image $\mathcal{A} = f(\widetilde{\mathcal{A}})$ through f is a long complex projective tube in Ω (and then in Z) with core γ. Thus, we have the following diagram:

$$\begin{array}{ccc} \tilde{\mathcal{A}} & \xrightarrow{\subseteq} & \tilde{\Omega} \\ {\scriptstyle /\pi_1(\Omega)}\downarrow & & \downarrow{\scriptstyle /\pi_1(\Omega)} \\ \mathcal{A} & \xrightarrow{\subseteq} & \Omega \\ {\scriptstyle /\Gamma}\downarrow & & \downarrow{\scriptstyle /\Gamma} \\ \mathbb{T} & \xrightarrow{\subseteq} & X \end{array}$$

with Γ isomorphic to $\pi_1(S)/\pi_1(\Omega)$.

We now remind the reader that our main goal is to study $\mathcal{S}(f)$ on $\mathcal{A}$. To this aim, we will widely use the Osgood-Stowe tensor, which we now formally define (see [OS92]).

Definition 3.5.1. Let (X, g) and (X', g') be two Riemannian surfaces, and $f\colon (X,g) \to (X', g')$ be a conformal local diffeomorphism. If $u\colon M \to \mathbb{R}$ is the uniformizing factor such that $f^*g^{'} = e^{2u}g$, then the Osgood-Stowe tensor B of u is defined as

$$B\,(g, f^*g) = B\left(g, e^{2u}g\right) = \mathrm{Hess}(u) - du \otimes du - \frac{1}{2}(\Delta u - ||\nabla u||^2)g$$

where the hessian, the laplacian, and the gradient are with respect to the metric g.

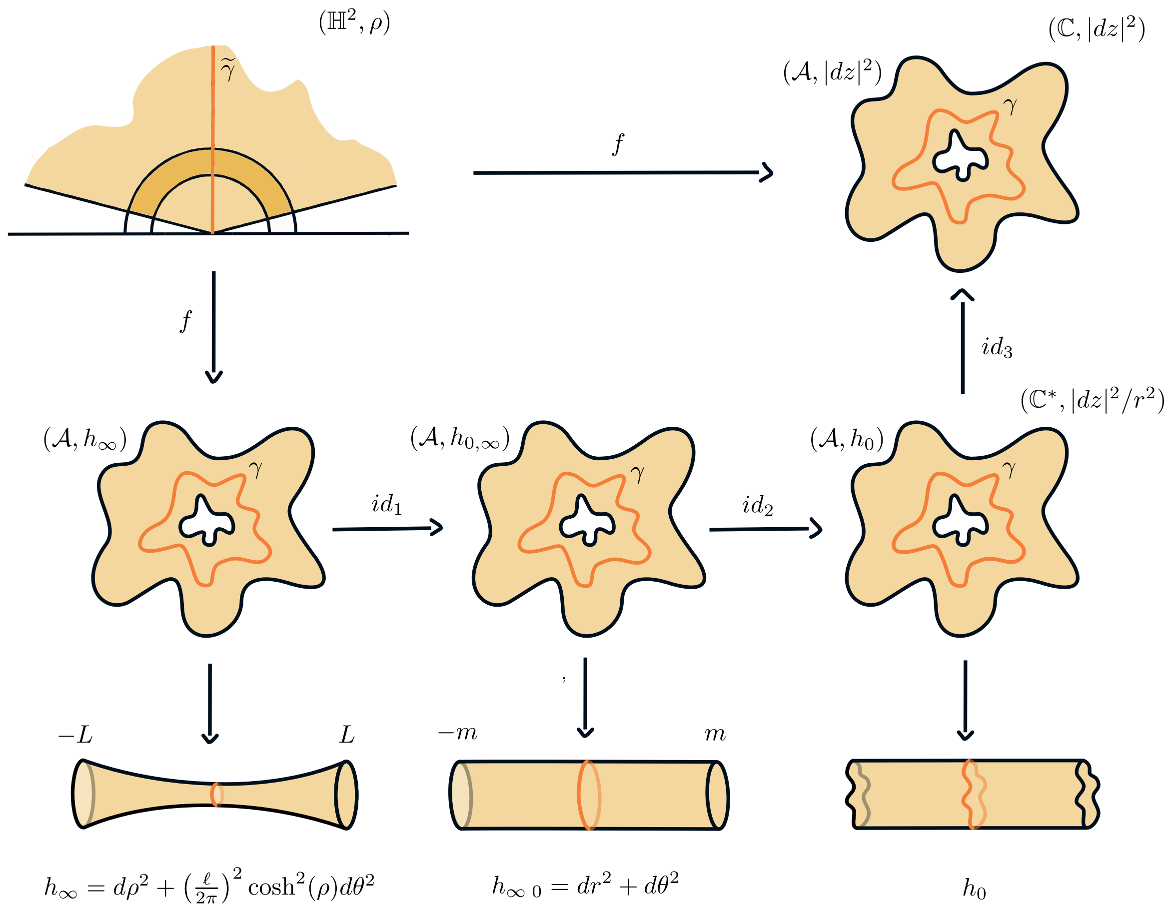

Figure 3.1: The long complex projective tube equipped with different conformal metrics, and its universal cover.

The Osgood-Stowe tensor satisfies analogous properties to those of the Schwarzian derivative, in particular it is additive under composition, and equal to zero when f is a local isometry, see Appendix A in [CG25], [OS92] and [BB24].

We define $h_\infty = f_* \rho$, and note that $f\colon \left(\widetilde{\mathcal{A}}, \rho\right) \to (\mathcal{A}, h_\infty)$ is a local isometry. Let us fix $z = x + iy$ and $|dz|^2 = dx^2 + dy^2$, respectively, the standard complex coordinate and the standard metric on $\mathbb{C}$ contained in $\mathbb{CP}^1$ through stereographic projection.

Theorem 3.5.2. Let $f\colon (\widetilde{\mathcal{A}}, \rho) \to (\mathcal{A}, |dz|^2)$ be the developing map in the notation above, and $u\colon \widetilde{\mathcal{A}} \to \mathbb{R}$ be the conformal factor such that $f^*|dz|^2 = e^{2u}\rho$, then

$$\mathrm{Re}(\mathcal{S}(f)) = B\left(\rho, e^{2u}\rho\right) \ .$$

Proof. This follows from Theorem A.1 in Appendix A in [CG25], or Section 4 of [BB24]. ■

As we are not able to compute $B\left(\rho, e^{2u}\rho\right)$ directly, we split the Osgood-Stowe tensor in a sum of terms, and use the additivity of the Osgood-Stowe tensor to compute $B\left(\rho, e^{2u}\rho\right)$ on $\mathcal{A}$. We will use the following notation for the auxiliary metrics:

- $h_{\infty,0} = e^{2u_0}h_\infty$ where $h_{\infty,0}$ is the conformal flat metric on $\mathcal{A}$ which has the same rotational invariance as h_∞, i.e. such that there exist $(\rho,\theta) \in [-L, L] \times [0, 2\pi]$ and flat coordinates $(r,\theta) \in [-m, m] \times [0, 2\pi]$ on $\mathcal{A}$ such that

$$dr = \frac{2\pi}{\ell \cosh \rho} d\rho \ , \qquad m = \int_0^L \frac{2\pi}{\ell \cosh(\rho)} d\rho \ ,$$

  while h_∞ is written as

$$h_\infty = d\rho^2 + \left(\frac{\ell}{2\pi}\right)^2 \cosh^2(\rho) d\theta^2 \ ,$$

  and

$$h_{\infty,0} = dr^2 + d\theta^2 \ ;$$

  in particular, in this coordinates

$$u_0(\rho,\theta) = \log\left(\frac{2\pi}{\ell\cosh(\rho)}\right) \ .$$

- $h_0 = e^{2u_1}h_{\infty,0}$ is another conformal flat metric on $\mathcal{A}$, with $h_0 = \frac{|dz|^2}{r^2}$ where $r^2 = z\overline{z}$.

Thus, we consider the following locally conformal maps:

$$\begin{array}{ccccc}
\left(\widetilde{\mathcal{A}}, \rho\right) & & \xrightarrow[u]{f|_{\widetilde{\mathcal{A}}}} & & \left(\mathcal{A}, |dz|^2\right) \\
{\scriptstyle f|_{\widetilde{\mathcal{A}}}} \searrow {\scriptstyle v} & & & & \nearrow {\scriptstyle \mathrm{id}_2}\ {\scriptstyle w} \\
 & (\mathcal{A}, h_\infty) \xrightarrow[u_0]{\mathrm{id}_0} & (\mathcal{A}, h_{\infty,0}) & \xrightarrow[u_1]{\mathrm{id}_1} (\mathcal{A}, h_0) &
\end{array}$$

See Figure 3.1 for sketches of the domains.

Clearly, we have that:

$$f|_{\widetilde{\mathcal{A}}} = \mathrm{id}_2 \circ \mathrm{id}_1 \circ \mathrm{id}_0 \circ f|_{\widetilde{\mathcal{A}}} \ .$$

By additivity of the Osgood-Stowe tensor

$$B\left(\rho, e^{2u|_{\widetilde{\mathcal{A}}}}\rho\right) =$$

$$= B\left(\rho, e^{2v|_{\widetilde{\mathcal{A}}}}\rho\right) + f^*B\left(h_\infty, e^{2u_0|_{\mathcal{A}}}h_\infty\right) + f^*B\left(h_{\infty,0}, e^{2u_1|_{\mathcal{A}}}h_{\infty,0}\right) + f^*B\left(h_0, e^{2w|_{\mathcal{A}}}h_0\right) ,$$

where $v = 0$ is the conformal factor of a local isometry, and so its contribution is zero, thus

$$f_*B\left(\rho, e^{2u|_{\widetilde{\mathcal{A}}}}\rho\right) = B\left(h_\infty, e^{2u_0|_{\mathcal{A}}}h_\infty\right) + B\left(h_{\infty,0}, e^{2u_1|_{\mathcal{A}}}h_{\infty,0}\right) + B\left(h_0, e^{2w|_{\mathcal{A}}}h_0\right) \ , \tag{3.6}$$

where the last term can be computed using that $h_0 = \dfrac{|dz|^2}{r^2}$, with r the radial coordinate of $\mathbb{C}^*$. Moreover, thanks to the key Lemma 3.2.1, to prove Theorem 3.5.7, we actually only need these terms on the core γ.

The following remark, together with the proof of Theorem 3.5.4 which follows afterwards, formalizes the heuristic of why the toy model of Section 3.3 is a good model also for general long complex projective tubes.

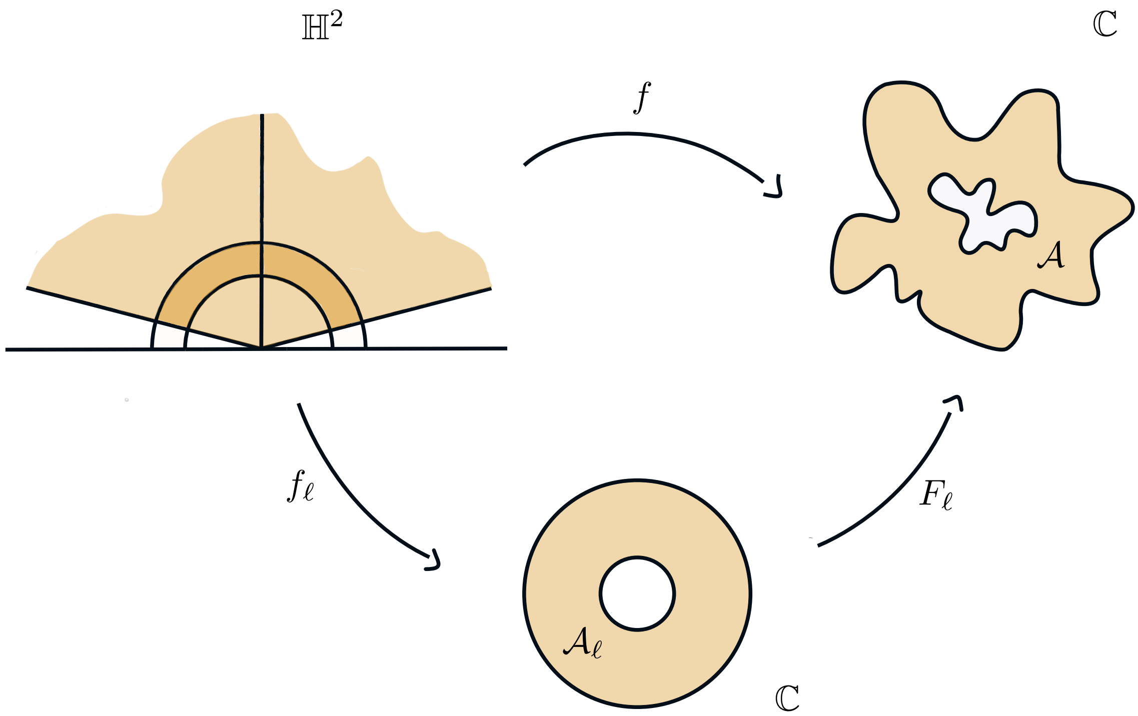

Figure 3.2: Developing a complex projective tube, and its conformal symmetric one.

Remark 3.5.3. If $\mathcal{A}$ is symmetric (as in Section 3.3), the domains in Figure 3.1 all have round boundaries. Moreover $u_1 = 0$, while u_0 and w remain unchanged, as also the respective Osgood-Stowe tensors. For a general long complex projectvie tube $\mathcal{A}$ of core length $\ell \leq \varepsilon_0$ and width $L = \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right)$ (with respect to the underlying hyperbolic structure), in the same notations and definitions of Section 3.3, where we denoted by $\mathcal{A}_\ell$ the standard symmetric tube of core length ℓ, and by f_ℓ its Schwarzian, there exist a conformal map (see Figure 3.2)

$$F_\ell \colon \mathcal{A}_\ell \to \mathcal{A}$$

such that

$$f = F_\ell \circ f_\ell$$

and F_ℓ is a holomorphic isometry between $(\mathcal{A}_\ell, h_0)$ and $(\mathcal{A}, h_{\infty,0})$, i.e.

$$F_\ell^* h_{\infty,0} = h_0 \ .$$

The existence of F_ℓ is guaranteed by the fact that, being the conformal structure of a (truncated) cylinder uniquely determined by its modulus, the underlying Riemann structure of a complex projective tube is conformal to the one of a symmetric tube with same core length and modulus, and, therefore, same width (see Remark 3.3.2).

Theorem 3.5.4. Let $\mathcal{A}$ be a long tube of a complex projective surface Z, let γ be its core of length $\ell \leq \varepsilon_0$, and let $\tilde{\gamma} \subseteq \widetilde{Z} = \mathbb{H}^2$ be a lift. Let also $f \colon \mathbb{H}^2 \to \mathbb{CP}^1$ be the developing map of Z, and let $\mathcal{S}(f)$ be its Schwarzian derivative. Then, in a neighborhood of $\tilde{\gamma}$, the Schwarzian $\mathcal{S}(f)$, up to pull-back by a Möbius, behaves as follows

$$\mathcal{S}(f) = \frac{1}{2z^2}\left(1+\frac{4\pi^2}{\ell^2}\right)dz^2 + O\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right)dz^2 \ ,$$

where z is the complex coordinate of $\mathbb{H}^2$, and $O(x)$ stands for a complex valued function such that $\lim_{x\to 0}|O(x)/x|$ is finite.

Proof. We start by showing how the Osgood-Stowe tensor term $B(h_{\infty,0}, e^{u_1|_{\mathcal{A}}}h_{\infty,0})$ is related to the real part of the Schwarzian derivative of F_ℓ, with F_ℓ as in Remark 3.5.3. We show that they coincide up to pull-back and adding twice the Schwarzian uniformization for the infinite cylinder. By Theorem $A.1$ in the appendix A in [CG25] (or also or Section 4 of [BB24])

$$\mathrm{Re}\left(\mathcal{S}(F_\ell)\right) = B\left(|dz|^2, F_\ell^*|dz|^2\right) \ .$$

By using the properties of the Osgood-Stowe tensor

$$\begin{aligned} B\left(|dz|^2, F_\ell^*|dz|^2\right) &= B\left(|dz|^2, h_0\right) + B\left(h_0, F_\ell^* h_0\right) + B\left(F_\ell^* h_0, F_\ell^*|dz|^2\right) \\ &= B\left(|dz|^2, h_0\right) + F_\ell^* B\left(h_{\infty,0}, h_0\right) - F_\ell^* B\left(|dz|^2, h_0\right) \end{aligned}$$

where for the central term of the second inequality we used that $(F_\ell)^* h_{\infty,0} = h_0$ (as seen in Remark 3.5.3), and we recall that $h_0 = e^{2u_1}h_{\infty,0}$. Then, by the properties of the Schwarzian derivative and the relation just obtained:

$$\begin{aligned} f_*\mathcal{S}(f) &= f_*\mathcal{S}(F_\ell\circ f_\ell) \\ &= f_*\left(f_\ell^*\mathcal{S}(F_\ell) + \mathcal{S}(f_\ell)\right) \\ &= (F_\ell)_*\mathcal{S}(F_\ell) + f_*\mathcal{S}(f_\ell) \end{aligned}$$

with

$$(F_\ell)_*\mathrm{Re}\left(\mathcal{S}(F_\ell)\right) = (F_\ell)_* B\left(|dz|^2, h_0\right) + B\left(h_{\infty,0}, h_0\right) - B\left(|dz|^2, h_0\right) \ . \tag{3.7}$$

We are now going to estimate the terms in the right hand side of equation (3.7). First, by using Definition 3.5.1, or equivalently by Remark 3.5.14, we compute the term $B(|dz|^2, h_0)$ where $h_0 = |dz|^2/r^2$:

$$B(|dz|^2, h_0) = \frac{1}{2r^2}dr^2 - \frac{1}{2}d\theta^2 = \mathrm{Re}\left(\frac{1}{2z^2}dz^2\right) \ ,$$

where $z = re^{i\theta}$ is the complex coordinate of $\mathbb{C}$. Under the isometric change of coordinates $w = \log(z)$ from $(\mathbb{C}^*, h_0)$ to the horizontal infinite strip of height 2π in $(\mathbb{C}, w)$, the Osgood-Stowe tensor $B(|dz|^2, h_0)$ becomes

$$\mathrm{Re}\left(\frac{1}{2}dw^2\right) \ .$$

Secondly, we note that, since F_ℓ is such that $(F_\ell)^* h_{\infty,0} = h_0$, with $h_0 = e^{2u_1}h_{\infty,0}$, the norm of its complex derivative is

$$\|dF_\ell\| = e^{u_1} \ ,$$

where, by Lemma 3.5.24 and Remark 3.5.27, the function e^{u_1} on a neighborhood of the core of $\mathcal{A}$ is such that

$$e^{u_1} = 1 + O\left(e^{-\pi^2/(2\ell)}\right) \ .$$

Then,

$$
\begin{aligned}
(F_\ell)_* B\left(|dz|^2, h_0\right) - B\left(|dz|^2, h_0\right) &= \left(1 + O\left(e^{-\pi^2/(2\ell)}\right)\right)^2 B\left(|dz|^2, h_0\right) - B\left(|dz|^2, h_0\right) \\
&= O\left(e^{-\pi^2/(2\ell)}\right) \mathrm{Re}\left(\frac{1}{2} dw^2\right) \ ,
\end{aligned}
$$

and therefore, denoting by $O_2(x)$, for $x \in \mathbb{R}$, a $(0,2)$-tensor whose coefficients are $O(x)$

$$
(F_\ell)_* B\left(|dz|^2, h_0\right) - B\left(|dz|^2, h_0\right) = O_2\left(e^{-\pi^2/(2\ell)}\right) \ . \tag{3.8}
$$

Concerning the $B\left(h_{\infty,0}, h_0\right)$ term in (3.7), we prove the following claim.

Claim. On smaller tubular neighborhoods of γ contained in $\mathcal{A}$ the Osgood-Stowe tensor $B\left(h_{\infty,0}, h_0\right) \in O_2\left(e^{-\pi^2/(2\ell)}\right)$.

Proof of Claim: Using the isometric change of coordinates (r, θ) in which the cylinder $(\mathcal{A}, h_{\infty,0})$ is expressed as $[-m, m] \times \mathbb{S}^1$, where m is half of the length of $\mathcal{A}$ with respect to $h_{\infty,0}$ (see Figure 3.1), we can calculate the Osgood-Stowe tensor $B\left(h_{\infty,0}, h_0\right) = B\left(h_{\infty,0}, e^{2u_1} h_{\infty,0}\right)$ with Definition 3.5.1 as:

$$
B\left(h_{\infty,0}, e^{2u_1} h_{\infty,0}\right) =
$$

$$
= \begin{bmatrix} \frac{1}{2}(u_1)_{rr} - \frac{1}{2}(u_1)_{\theta\theta} + \frac{1}{2}(u_1)_\theta^2 - \frac{1}{2}(u_1)_r^2 & (u_1)_{\theta r} - (u_1)_\theta (u_1)_r \\ (u_1)_{\theta r} - (u_1)_\theta (u_1)_r & \frac{1}{2}(u_1)_{\theta\theta} - \frac{1}{2}(u_1)_{rr} + \frac{1}{2}(u_1)_r^2 - \frac{1}{2}(u_1)_\theta^2 \end{bmatrix} \ .
$$

The claim now follows by the last block of estimates in Section 3.5.4, which bound the derivatives of u_1 in $\mathcal{A}$, together with the inequality

$$
e^{-m} \leq e^{-\pi^2/(2\ell)} \ ,
$$

as in equation 3.9 of Remark 3.5.12. □

From equation 3.8 and the claim right above

$$
(F_\ell)_* \mathrm{Re}\left(\mathcal{S}(F_\ell)\right) = O_2\left(e^{-\pi^2/(2\ell)}\right) \ .
$$

Therefore, in a neighborhood of the core of $\mathcal{A}$ we have:

$$
f_*\left(\mathrm{Re}(\mathcal{S}(f)\right) = f_*\left(\mathrm{Re}(\mathcal{S}(f_\ell)\right) + O_2\left(e^{-\pi^2/(2\ell)}\right) \ ,
$$

where again the notation $O_2(x)$ stands for a symmetric $(0,2)$-tensor whose coefficients are $O(x)$.

We are interested in the Schwarzian derivative on $\widetilde{\mathcal{A}} \subseteq \mathbb{H}^2$, and we first study its real part

$$
\mathrm{Re}\left(\mathcal{S}(f)\right) = \mathrm{Re}\left(\mathcal{S}(f_\ell)\right) + (f_\ell)^* \mathrm{Re}\left(\mathcal{S}(F_\ell)\right) \ .
$$

Since $df = dF_\ell \circ df_\ell$, and, from Section 3.3, by a direct computation of the derivative of f_ℓ, which is explicit, we can recover that $\|df_\ell\| = O(1/\ell)$ and then also $\|df\| = O(1/\ell)$, we get

$$
(f_\ell)^* \mathrm{Re}\left(\mathcal{S}(F_\ell)\right) = f^*\left((F_\ell)_* \mathrm{Re}\left(\mathcal{S}(F_\ell)\right)\right) \ = O_2\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right),
$$

where we used that $f_\ell^* = f^* \circ (F_\ell)_*$, and therefore

$$\operatorname{Re}\left(\mathcal{S}(f)\right) = \operatorname{Re}\left(\mathcal{S}(f_\ell)\right) + O_2\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right) .$$

To get rid of the real part, given z the coordinate of $\mathbb{H}^2$, we just need to notice that a holomorphic quadratic differential $Q(z) = q(z)dz^2$, where $q(z) = q_0(z) + iq_1(z)$ with q_0 and q_1 real valued functions, and $dz = dx + idy$, satisfies

$$\operatorname{Re}(\mathcal{S}(f)) = q_0(z)(dx^2 - dy^2) - q_1(z)(dx \otimes dy + dy \otimes dx)$$

and

$$\operatorname{Im}(\mathcal{S}(f)) = q_1(z)(dx^2 - dy^2) + q_0(z)(dx \otimes dy + dy \otimes dx) .$$

Then, the coefficients of its real part are $O(x)$ if and only if the ones of its imaginary part are as well. Finally,

$$\mathcal{S}(f) = \operatorname{Re}(\mathcal{S}(f) + i\operatorname{Im}(\mathcal{S}(f)) = \mathcal{S}(f_\ell) + O_2\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right) .$$

Concluding the proof. ■

By using the results of the next two sections, we now prove Theorem 3.5.7. Before, we define the two norms for the Osgood-Stowe tensor $B(g, e^{2u}g)$ that we are going to use.

Definition 3.5.5. Given a smooth loop γ we define:

$$\left\|B\left(g, e^{2u}g\right)\right\|_\gamma := \left|\frac{1}{2}\int_\gamma B\left(g, e^{2u}g\right)(i\dot\gamma, \dot\gamma)\right| .$$

The definition above is so that if μ is the infinitesimal earthquake along a unit length simple closed geodesic γ, we obtain, by Lemma 3.2.1 and Theorem 3.5.2,

$$|\operatorname{Re}\langle\mathcal{S}(f), \mu\rangle| = \left\|B\left(g, e^{2u}g\right)\right\|_\gamma .$$

Analogously,

Definition 3.5.6. Given a smooth loop γ we define:

$$\left\|B\left(g, e^{2u}g\right)\right\|_\gamma^{gr} := \left|\frac{1}{2}\int_\gamma B\left(g, e^{2u}g\right)(\dot\gamma, \dot\gamma)\right| .$$

The definition above is so that if ν is the infinitesimal grafting along a unit length simple closed geodesic γ, we obtain, by Lemma 3.2.1 and Theorem 3.5.2

$$|\operatorname{Re}\langle\mathcal{S}(f), \nu\rangle| = \left\|B\left(g, e^{2u}g\right)\right\|_\gamma^{gr} .$$

We now restate Theorem 3.5.7 in a more expanded form, namely we give explicit expression for the functions $F_e(\ell)$ and $F_{gr}(\ell)$.

Theorem 3.5.7. Let Z be a complex projective surface, let $\mathcal{S}(f)$ be its Schwarzian, and let $X = \pi(Z)$ its underlying Riemann surface. Let also μ and ν be, respectively, the infinitesimal earthquake and grafting on the simple closed curve $\gamma \subseteq X$ of hyperbolic length $\ell \leq \varepsilon_0$. Then

$$|\mathrm{Re}\langle \mathcal{S}(f), \mu\rangle| \leq 142\pi^4 \frac{G^2(\ell)}{\ell} e^{-\pi^2/\ell} + 113\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell} ,$$

and

$$\left|\mathrm{Re}\langle \mathcal{S}(f), \nu\rangle - \frac{\pi^2}{\ell}\right| \leq \frac{1}{4}\ell + 142\pi^4 \frac{G^2(\ell)}{\ell} e^{-\pi^2/\ell} + 18\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell} ,$$

where $G(\ell) = 1 + O\left(e^{-\pi^2/(2\ell)}\right)$ as $\ell \to 0$ and is always bounded by $e^{2.8}$.

Proof. Thanks to Lemma 3.2.1, the norms of the pairings between $\mathcal{S}(f)$ and μ or ν coincide, respectively, with the norms of the Osgood-Stowe differential of Definitions 3.5.5 and 3.5.6. Therefore, the result follows straightforwardly from Theorems 3.5.9 and 3.5.16. ∎

3.5.2 Estimates on the Schwarzian along earthquakes

In this section we are interested in studying $B(g, e^{2u}g)$ on the simple closed geodesic γ.

Remark 3.5.8. For Lemma 3.2.1, in Definitions 3.5.5 and 3.5.6, we need to take γ, to be the core geodesic, which in the metric h_∞ it is parameterised as:

$$\gamma(t) = \left(0, \frac{2\pi}{\ell} t\right), \qquad t \in [0, \ell] ,$$

so that $\|\dot\gamma\|_{h_\infty} = 1$. We want to estimate $\|B(\rho, e^{2u}\rho)\|_\gamma$, with u the conformal factor between h_∞ and $|dz|^2$, through the following equality:

$$f_* B\left(\rho, e^{2u|_{\tilde{\mathcal{A}}}}\rho\right) = B\left(h_\infty, e^{2u_0|_{\mathcal{A}}} h_\infty\right) + B\left(h_{\infty,0}, e^{2u_1|_{\mathcal{A}}} h_{\infty,0}\right) + B\left(h_0, e^{2w|_{\mathcal{A}}} h_0\right) ,$$

since all the maps are the identity the parameterisation of the curve γ never changes. However, the curve is not necessarily geodesic in the other metrics. In particular we have:

$$\|\dot\gamma\|_{h_{\infty,0}} = \frac{2\pi}{\ell},$$

and so even if γ has geodesic image in $h_{\infty,0}$ it is not in unit length parametrisation.

We can finally state our estimate on the norm of $B(\rho, e^{2u}\rho)$. We first define the following auxiliary functions of W and m, with m a constant such that $0 < W \leq 3.7$, which bounds u on the boundary $\partial\mathcal{A}$, and m the width of $(\mathcal{A}, h_{\infty,0})$:

$$G(\ell) := \min\left\{ e^{2.8}, 1 + \sqrt{2} W e^{2.8} \pi \frac{e^m}{(e^m-1)^2} + 2W \frac{e^m}{(e^m-1)^2}\ell + 4\sqrt{2} W e^{2.8} \pi \ell \left(\frac{e^m}{(e^m-1)^2}\right)^2 \right\}$$

and

$$\overline{G}(\ell) = \left(W G(\ell) \left(\frac{16}{15}\right)^2\right)^2 \leq 17.73 G(\ell)^2 ,$$

see Remark 3.5.25.

Theorem 3.5.9. In the notations above, we have the following bound

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma \leq 8\pi^4 \frac{\overline{G}(\ell)}{\ell} e^{-\pi^2/\ell} + 113\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell} ,$$

where ℓ is the length of the core γ with respect to h_∞.

Theorem 3.5.9 will follow from the equality between the Osgood-Stowe differentials

$$f_* B\left(\rho, e^{2u|_{\tilde{\mathcal{A}}}}\rho\right) = B\left(h_\infty, e^{2u_0|_{\mathcal{A}}} h_\infty\right) + B\left(h_{\infty,0}, e^{2u_1|_{\mathcal{A}}} h_{\infty,0}\right) + B\left(h_0, e^{2w|_{\mathcal{A}}} h_0\right) ,$$

bounding, or computing, each term on the right hand-side, in order, in Propositions 3.5.10, 3.5.11, and 3.5.15.

Proposition 3.5.10. For the Osgood-Stowe tensor $B\left(h_\infty, e^{2u_0} h_\infty\right)$: $\left\|B\left(h_\infty, e^{2u_0} h_\infty\right)\right\|_\gamma = 0$.

Proof. We already noted that there exist coordinates (ρ, θ) in which $u_0(\rho, \theta) = \log\left(\frac{2\pi}{\ell \cosh(\rho)}\right)$. Then, we can directly compute the Osgood-Stowe tensor $B\left(h_\infty, e^{2u_0} h_\infty\right)$. The only non trivial Christoffel symbols for h_0 are

$$\Gamma^\theta_{\rho\theta} = \Gamma^\theta_{\theta\rho} = \tanh(\rho) = -\Gamma^\rho_{\theta\theta} ,$$

so

$$\mathrm{Hess}(u_0) = -\frac{1}{\cosh^2(\rho)} d\rho^2 - \tanh^2(\rho) d\theta^2, \qquad du_0 \otimes du_0 = \tanh^2(\rho) d\rho^2 ,$$

and their traces

$$\Delta u_0 = -\frac{1}{\cosh^2(\rho)} - \frac{\tanh^2(\rho)}{\cosh^2(\rho)} \left(\frac{2\pi}{\ell}\right)^2, \qquad \|\nabla u_0\|^2_{h_0} = \tanh^2(\rho) ,$$

therefore, at the core $\rho = 0$

$$B\left(h_\infty, e^{2u_0} h_\infty\right) = -\frac{1}{2} d\rho^2 + \frac{1}{2} \left(\frac{\ell}{2\pi}\right)^2 d\theta^2 .$$

The result now follows from Definition 3.5.5 and the fact that the unit length parametrization with respect to h_∞ of the core γ has derivative $\dot\gamma(t) = \frac{2\pi}{\ell}\frac{\partial}{\partial\theta}$, $t \in [0, \ell]$, and $i\dot\gamma(t) = -\frac{2\pi}{\ell}\frac{\partial}{\partial\rho}$. ■

Proposition 3.5.11. The norm of the Osgood-Stowe tensor $B\left(h_{\infty,0}, e^{2u_1} h_{\infty,0}\right)$ is bounded as:

$$\left\|B\left(h_{\infty,0}, e^{2u_1} h_{\infty,0}\right)\right\|_\gamma \leq 113\pi^2 \frac{e^{-m}}{\ell}, \quad m = \frac{2\pi \arctan(\sinh(L))}{\ell}, \quad L = \operatorname{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right).$$

Proof. We recall that $2m$ is the width of $(\mathcal{A}, h_{\infty,0})$ and that

$$m = \int_0^L \frac{2\pi}{\ell \cosh\rho} d\rho = \frac{2\pi \arctan(\sinh(L))}{\ell} ,$$

with $L = \operatorname{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right)$. Then, e^{-m} is an increasing function of ℓ, and, since $\ell \leq \varepsilon_0 = 2\operatorname{arsinh}(1)$, we can bound e^{-m} by:

$$\begin{aligned} e^{-m} &\leq \exp\left(-\frac{2\pi}{\varepsilon_0}\arctan\left(\sinh\left(\operatorname{arsinh}\left(\frac{1}{\sinh\left(\frac{\varepsilon_0}{2}\right)}\right)\right)\right)\right) \\ &\leq \exp\left(-\frac{2\pi}{\varepsilon_0}\arctan(1)\right) \\ &\leq \frac{1}{16} \ . \end{aligned}$$

The metrics $h_{\infty,0}$ and $h_0 = e^{2u_1}h_{\infty,0}$ are both Euclidean, by the conformal change of curvature:

$$K_0 = e^{-2u_1}(K_{\infty,0} + \Delta u_1),$$

for $K_{\infty,0}$, K_0 the curvatures of $h_{\infty,0}$ and h_0 respectively. Thus, $u_1(r,\theta)$ satisfies

$$\Delta u_1 = 0 \ .$$

Then, we can apply Fourier analysis, see Subsection 3.5.4, on the round cylinder $\mathcal{A} \cong [-m,m] \times \mathbb{S}^1$ to get the following estimates of u_1 on the core $\gamma(\theta) = \left(0, \frac{2\pi}{\ell}\theta\right)$, $\theta \in [0,\ell]$ which has geodesic image in $h_{\infty,0} = dr^2 + d\theta^2$ but is not in unit length parametrisation, see Remark 3.5.8.

Then, by the estimates in Subsection 3.5.4, we have:

- $|(u_1)_r| \leq 4W \sum_{k\in\mathbb{N}} ke^{-km} = 4W\frac{e^m}{(e^m-1)^2}$;
- $|(u_1)_\theta| \leq 2W \sum_{k\in\mathbb{N}} ke^{-km} = 2W\frac{e^m}{(e^m-1)^2}$;
- $|(u_1)_{\theta r}| \leq 4W \sum_{k\in\mathbb{N}} k^2e^{-km} = 4W\frac{e^m(e^m+1)}{(e^m-1)^3}$;

for $0 < W \leq 3.7$ as in Lemma 3.5.22. Since, in the coordinates $(r,\theta) \in [-m,m] \times \mathbb{S}^1$ we can express the Osgood-Stowe tensor as:

$$B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right) =$$

$$= \begin{bmatrix} \frac{1}{2}(u_1)_{rr} - \frac{1}{2}(u_1)_{\theta\theta} + \frac{1}{2}(u_1)_\theta^2 - \frac{1}{2}(u_1)_r^2 & (u_1)_{\theta r} - (u_1)_\theta(u_1)_r \\ (u_1)_{\theta r} - (u_1)_\theta(u_1)_r & \frac{1}{2}(u_1)_{\theta\theta} - \frac{1}{2}(u_1)_{rr} + \frac{1}{2}(u_1)_r^2 - \frac{1}{2}(u_1)_\theta^2 \end{bmatrix},$$

and we need to compute it on $(i\dot{\gamma}, \dot{\gamma})$ with $\gamma(\theta) = \left(0, \frac{2\pi}{\ell}\theta\right)$ we have that:

$$\begin{aligned} B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)(i\dot{\gamma}, \dot{\gamma}) &= B\left(h_{\infty,0}, e^{2u}h_{\infty,0}\right)\left(\left(-\frac{2\pi}{\ell}, 0\right), \left(0, \frac{2\pi}{\ell}\right)\right) \\ &= -\left(\frac{2\pi}{\ell}\right)^2 \left((u_1)_{\theta r} - (u_1)_\theta(u_1)_r\right) \ , \end{aligned}$$

whose norm is bounded by:

$$\begin{aligned}\left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_\gamma &\leq \ell\left(\frac{2\pi}{\ell}\right)^2\left(4W\frac{e^m(e^m+1)}{(e^m-1)^3}+8W^2\left(\frac{e^m}{(e^m-1)^2}\right)^2\right)\\ &= \frac{4\pi^2}{\ell}\left(4We^{-m}\left(\frac{e^{-m}(e^m+1)}{(1-e^{-m})^3}\right)+8W^2e^{-2m}\left(\frac{1}{(1-e^{-m})^2}\right)^2\right)\\ &\leq \frac{16\pi^2}{\ell}\left(1.3We^{-m}+2.6W^2e^{-2m}\right) \qquad e^{-m}\leq 1/16,\\ &\leq \frac{16\pi^2}{\ell}We^{-m}(1.3+2.6We^{-m}) \qquad W\leq 3.7,\\ &\leq 113\pi^2\frac{e^{-m}}{\ell},\end{aligned}$$

which completes the proof. ■

Remark 3.5.12. In Proposition 3.5.11 we get:

$$\left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_\gamma \leq 113\pi^2\frac{e^{-m}}{\ell}\ ,$$

where

$$m=\frac{2\pi\arctan(\sinh(L))}{\ell}\ ,\quad L=\operatorname{arsinh}\left(\frac{1}{\sinh(\ell/2)}\right)\ .$$

Since $\frac{\ell}{2\pi}m$ is a decreasing function in $\ell\in(0,\varepsilon_0]$ its minima is at ε_0 and it is $\frac{\pi}{4}$. Then

$$m\geq\frac{2\pi}{\ell}\frac{\pi}{4}=\frac{\pi^2}{2\ell},$$

and therefore

$$e^{-m}\leq e^{-\pi^2/(2\ell)}\ . \tag{3.9}$$

Thus, we can write:

$$\left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_\gamma \leq 113\pi^2\frac{e^{-\pi^2/(2\ell)}}{\ell}=O\left(e^{-\pi^2/(2\ell)}\ell\right).$$

Moreover, as we have $e^{-m}<\frac{1}{16}$ we also get that:

$$(e^m-1)^{-1}<\frac{16}{15}e^{-m}\ .$$

Before being able to prove the last estimate, we show that γ is almost a round circle in $\mathbb{C}^*$, that is γ is almost geodesic with respect to the h_0 metric. In the next Lemma we have that W is as in Lemma 3.5.22 and D^0 is the Levi-Civita connection with respect to h_0.

Lemma 3.5.13. Given $\gamma:[0,\ell]\to(\mathcal{A},h_{\infty,0})$ the unit length geodesic of the core then:

$$\max_{s\in[0,\ell]}\left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0}\leq 6We^{2.8}\left(\frac{2\pi}{\ell}\right)^2\frac{e^m}{(e^m-1)^2},\quad W\leq 3.7\ .$$

Moreover, by Remark 3.5.12 we have:

$$\max_{s\in[0,\ell]}\left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0}\leq 6We^{2.8}\left(\frac{32\pi}{15}\right)^2\frac{e^{-\pi^2/(2\ell)}}{\ell^2}.$$

Proof. We consider the coordinates (r,θ) such that $\mathcal{A} \cong [-m,m] \times \mathbb{S}^1$, $h_{\infty,0} = dr^2 + d\theta^2$, and $\gamma(s) = \left(0, \frac{2\pi}{\ell} s\right)$ with $\dot\gamma(s) = \left(0, \frac{2\pi}{\ell}\right)$. The metric h_0 is equal to $e^{2u_1} h_{\infty,0}$ and we denote by $D^{\infty,0}$ the Levi-Civita connection for $h_{\infty,0}$.

The change of the Levi-Civita connection under conformal changes of metric, see [Bes87, 1.159 a], is given by the following formula:

$$\begin{aligned} D^0_{\dot\gamma}\dot\gamma(s) &= D^{\infty,0}_{\dot\gamma}\dot\gamma(s) + 2du(\dot\gamma(s))\dot\gamma(s) - \|\dot\gamma(s)\|^2_{h_{\infty,0}} D^{\infty,0}u \\ &= 2\left(\frac{2\pi}{\ell}\right)^2 u_\theta \frac{\partial}{\partial\theta} - \left(\frac{2\pi}{\ell}\right)^2 \left(u_r \frac{\partial}{\partial r} + u_\theta \frac{\partial}{\partial \theta}\right) \\ &= \left(\frac{2\pi}{\ell}\right)^2 \left(u_\theta \frac{\partial}{\partial\theta} - u_r \frac{\partial}{\partial r}\right) . \end{aligned}$$

Taking norms with respect to h_0:

$$\left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0} \le \left(\frac{2\pi}{\ell}\right)^2 \left(|u_\theta| \left\|\frac{\partial}{\partial\theta}\right\|_{h_0} + |u_r| \left\|\frac{\partial}{\partial r}\right\|_{h_0}\right) .$$

Since $\|v\|_{h_0} = e^{u_1} \|v\|_{h_{\infty,0}}$ and $u_1 \le 2.8$ at the core, see Remark 3.5.23, we obtain:

$$\left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0} \le e^{2.8} \left(\frac{2\pi}{\ell}\right)^2 (|u_\theta| + |u_r|) .$$

Finally, by the Fourier analysis bounds of Subsection 3.5.4 we have:

$$|u_r| \le 4W \frac{e^m}{(e^m-1)^2} \qquad |u_\theta| \le 2W \frac{e^m}{(e^m-1)^2} ,$$

where again $0 < W \le 3.7$. This yields:

$$\left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0} \le 6We^{2.8} \left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m-1)^2} .$$

Taking the maximum of the left-hand side over s completes the first part of the statement and the second part follows by the computations in Remark 3.5.12. ■

We now need to compute $\left\|B\left(h_0, e^{2\log(r)} h_0\right)\right\|_\gamma$ which is the last term of our bound for $\left\|B(\rho, |dz|^2)\right\|_\gamma$.

Remark 3.5.14. We remark that by Theorem $A.1$ in [CGS24] (or Section 4 of [BB24]), the Osgood-Stowe tensor $B\left(h_0, e^{2\log(r)}h_0\right)$ is equal to the real part of the Schwarzian derivative of the uniformization map from the infinite flat cylinder, with core 2π, to $\mathbb{C}^*$. This is because $(\mathcal{A}, h_0)$ is locally isometric to the strip equipped with the restriction of the metric $|dz|^2$ in $\mathbb{C}$ (so the relative Osgood-Stowe tensor is zero), and $\exp(z)$ going from the strip to $\mathbb{C}^* \subseteq \mathbb{C}$, has real part of the Schwarzian equal to the Osgood-Stowe tensor with respect to $|dz|^2$ on both sides, which is the standard one for $\mathbb{C}$.

Proposition 3.5.15. We have the following bound:

$$\left\|B\left(h_0, e^{2\log(r)}h_0\right)\right\|_\gamma \le 8\pi^4 W^2 \frac{G(\ell)^2}{\ell} \frac{e^{2m}}{(e^m-1)^4} ,$$

for $G(\ell)$ as in Remark 3.5.25 and $0 < W \leq 3.7$ as in Lemma 3.5.22. Moreover, we have:

$$\left\| B\left(h_0, e^{2\log(r)} h_0\right) \right\|_\gamma \leq 8\pi^4 W^2 \frac{G(\ell)^2}{\ell} \left(\frac{16}{15}\right)^4 e^{-\pi^2/\ell} ,$$

and $G(\ell)$ goes to 1 as ℓ goes to zero.

Proof. Let $\gamma(t) : [0, \ell] \to (\mathcal{A}, h_0)$ be the parameterisation of the loop γ in the flat cylinder $\mathcal{A}$. Note that in this setting $\gamma(t) = (x(t), y(t))$ is not geodesic for h_0. However, by developing to the universal cover $\tilde{\mathcal{A}}$ one sees that $\dot{\gamma}$ has mean $\left(0, \frac{2\pi}{\ell}\right)$ i.e.:

$$\overline{x}' = \frac{1}{\ell} \int_0^\ell \dot{x}(t) dt = 0 \qquad \overline{y}' = \frac{1}{\ell} \int_0^\ell \dot{y}(t) dt = \frac{2\pi}{\ell} .$$

Moreover, by the Mean Value Theorem we have times ξ_1 and ξ_2 such that $\dot{x}(\xi_1) = \overline{x}$ and $\dot{y}(\xi_2) = \overline{y}$. Since, the metric is the flat Euclidean metric the covariant derivative is:

$$D^0_{\dot{\gamma}} \dot{\gamma}(s) = (\ddot{x}(s), \ddot{y}(s)) ,$$

and by Lemma 3.5.13 and Remark 3.5.25:

$$\max_{s \in [0,\ell]} \left\| D^0_{\dot{\gamma}} \dot{\gamma}(s) \right\|_{h_0} \leq 6WG(\ell) \left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m - 1)^2} ,$$

we obtain:

$$\forall s \in [0, \ell] : |\ddot{x}(s)| , |\ddot{y}(s)| \leq 6WG(\ell) \left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m - 1)^2} .$$

Thus,

$$\begin{aligned} |\dot{x}(s) - \overline{x}'| &= |\dot{x}(s) - \dot{x}(\xi_1)| \\ &\leq \ell \, |\ddot{x}(\zeta)| \\ &\leq \ell W G(\ell) \left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m - 1)^2} . \end{aligned}$$

Recall that the cylinder $(\mathcal{A}, h_0)$ is isometric to the vertical euclidean strip, thus the same holds for $|\dot{y}(s) - \overline{y}'|$ so we have:

$$\|\dot{\gamma}(s) - \overline{\gamma}'\|_{h_0} \leq \sqrt{2} \ell W G(\ell) \left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m - 1)^2} = \frac{4\sqrt{2} W G(\ell) \pi^2}{\ell} \frac{e^m}{(e^m - 1)^2} . \tag{3.10}$$

We will now consider the restriction of the uniformization map from the infinite flat cylinder $\mathcal{C}$ of core 2π, which contains isometrically $(\mathcal{A}, h_0)$, to $\mathbb{C}^*$. The map is

$$f(z) = \lambda \exp(z) ,$$

where z is the coordinate on the infinite strip $\mathcal{C} \cong \{z \in \mathbb{C} \,|\, 0 \leq \operatorname{Im} z \leq 2\pi\}$ and λ is a complex number. As $\frac{\ddot{f}}{\dot{f}} = 1$ we have:

$$\mathcal{S}(f)(z) = -\frac{1}{2} dz^2 .$$

Since $B\left(h_0, e^{2\log(r)}h_0\right)$ on the core is equivalent the real part of the Schwarzian right above, see Remark 3.5.14, we need to compute

$$\operatorname{Re}\left(\int_0^\ell \mathcal{S}(f)(i\dot\gamma(s),\dot\gamma(s))ds\right) = \operatorname{Im}\left(\int_0^\ell \mathcal{S}(f)(\dot\gamma(s),\dot\gamma(s))ds\right).$$

Substituting $\mathcal{S}(f)$ and considering $\dot\gamma(s)$ as in $\mathbb{C}$ we have:

$$\begin{aligned} -\operatorname{Im}\left(\int_0^\ell \mathcal{S}(f)(\dot\gamma(s),\dot\gamma(s))ds\right) &= \frac{1}{2}\int_0^\ell \operatorname{Im}(\dot\gamma(s)^2)ds \\ &= \frac{1}{2}\operatorname{Im}\int_0^\ell \dot\gamma(s)^2 ds\ . \end{aligned}$$

Since the mean of $\dot\gamma$ is $\overline{\gamma}' = \frac{2\pi}{\ell}i$ we have:

$$\begin{aligned} \frac{1}{2}\operatorname{Im}\int_0^\ell \dot\gamma(s)^2ds &= \frac{1}{2}\operatorname{Im}\int_0^\ell (\overline{\gamma}')^2 ds + \frac{1}{2}\operatorname{Im}\int_0^\ell (\dot\gamma(s)-\overline{\gamma}')^2ds + \operatorname{Im}\int_0^\ell \overline{\gamma}'(\dot\gamma(s)-\overline{\gamma}')ds \\ &= \frac{1}{2}\operatorname{Im}\int_0^\ell \left(\frac{2\pi}{\ell}i\right)^2 ds + \frac{1}{2}\operatorname{Im}\int_0^\ell (\dot\gamma(s)-\overline{\gamma}')^2 ds \\ &= \frac{1}{2}\operatorname{Im}\int_0^\ell (\dot\gamma(s)-\overline{\gamma}')^2 ds\ . \end{aligned}$$

Then,

$$\begin{aligned} \left\|B\left(h_0, e^{2\log(r)}h_0\right)\right\|_\gamma &= \left|\frac{1}{2}\int_0^\ell B\left(h_0, e^{2\log(r)}h_0\right)(i\dot\gamma,\dot\gamma)ds\right| \\ &= \left|\operatorname{Re}\left(\frac{1}{2}\int_0^\ell \mathcal{S}(f)(i\dot\gamma(s),\dot\gamma(s))ds\right)\right| \\ &\le \frac{1}{4}\int_0^\ell \left\|\dot\gamma(s)-\overline{\gamma}'\right\|_{h_0}^2 ds \\ &\overset{(3.10)}{\le} \frac{\ell}{4}\left(\frac{4\sqrt{2}WG(\ell)\pi^2}{\ell}\frac{e^m}{(e^m-1)^2}\right)^2 \\ &\le 8\pi^4W^2\frac{G(\ell)^2}{\ell}\frac{e^{2m}}{(e^m-1)^4}\ . \end{aligned}$$

The second part follows by the computations in Remark 3.5.12 and 3.5.25. ■

We can now prove the main estimate, and recall that $\overline{G}(\ell) = \left(WG(\ell)\left(\frac{16}{15}\right)^2\right)^2$.

Theorem 3.5.9. We have the following bound

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma \le 8\pi^4\frac{\overline{G}(\ell)}{\ell}e^{-\pi^2/\ell} + 113\pi^2\frac{e^{-\pi^2/(2\ell)}}{\ell}\ ,$$

where ℓ is the length of the core with respect to h_∞.

Proof. We have that:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma = \left|\int_\gamma \left(B\left(h_\infty, e^{2u_0}h_\infty\right) + B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right) + B\left(h_0, e^{2\log(r)}h_0\right)\right)(i\dot\gamma,\dot\gamma)\right|$$

which gives:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_{\gamma} \leq \left\|B\left(h_{\infty}, e^{2u_0}h_{\infty}\right)\right\|_{\gamma} + \left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_{\gamma} + \left\|B\left(h_0, e^{2(-\log(r))}h_0\right)\right\|_{\gamma},$$

which by Proposition 3.5.10, Proposition 3.5.11, Remark 3.5.12, and Proposition 3.5.15 yields:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_{\gamma} \leq 8\pi^4 W^2 \frac{G(\ell)^2}{\ell}\left(\frac{16}{15}\right)^4 e^{-\pi^2/\ell} + 113\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell},$$

f completing the proof. ∎

3.5.3 Estimates on the Schwarzian along grafting

In this section, we repeat the computations done in Section 3.5.2, but this time for the norm of Definition 3.5.6. As before, we are going to estimate the norm of the Osgood-Stowe tensor B at each step. More specifically, we want to prove the following theorem.

Theorem 3.5.16. We have the following bound

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_{\gamma}^{gr} \leq \frac{\pi^2}{\ell} + \frac{1}{4}\ell + 8\pi^4 \frac{\overline{G}(\ell)}{\ell} e^{-\pi^2/\ell} + 18\pi^2 \frac{e^{-\pi^2/(2\ell)}}{\ell},$$

where ℓ is the length of the core γ with respect to h_{∞}.

Theorem 3.5.16 will directly follow, analogously to Theorem 3.5.9, from splitting $\|B\left(\rho, e^{2u}\rho\right)\|_{\gamma}^{gr}$ into the same three terms and bounding each one. This is done in Propositions 3.5.17, 3.5.18, and 3.5.19 respectively.

Proposition 3.5.17. For the Osgood-Stowe tensor $B\left(h_{\infty}, e^{2u_0}h_{\infty}\right)$:

$$\left\|B\left(h_{\infty}, e^{2u_0}h_{\infty}\right)\right\|_{\gamma}^{gr} = \frac{1}{4}\ell,$$

where ℓ is the length of the core γ with respect to h_{∞}.

Proof. We have already seen in the proof of Lemma 3.5.10 that, in the coordinates (ρ, θ) in which $u_0(\rho, \theta) = \log\left(\frac{2\pi}{\ell\cosh(\rho)}\right)$, on the core γ the Osgood-Stowe tensor is given by

$$B\left(h_{\infty}, e^{2u_0}h_{\infty}\right) = -\frac{1}{2}d\rho^2 + \frac{1}{2}\left(\frac{\ell}{2\pi}\right)^2 d\theta^2.$$

As $\dot{\gamma} = \frac{2\pi}{\ell}\frac{\partial}{\partial\theta}$ by a direct computation

$$\left\|B\left(h_{\infty}, e^{2u_0}h_{\infty}\right)\right\|_{\gamma}^{gr} = \left|\frac{1}{2}\int_0^{\ell} B\left(h_{\infty}, e^{2u_0}h_{\infty}\right)(\dot{\gamma}, \dot{\gamma})ds\right| = \left|\int_0^{\ell}\frac{1}{4}ds\right| = \frac{1}{4}\ell.$$

Which concludes the proof. ∎

Proposition 3.5.18. The norm of the Osgood-Stowe tensor $B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)$ is bounded as:

$$\left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_{\gamma}^{gr} \leq 18\pi^2\frac{e^{-m}}{\ell}, \quad m = \frac{2\pi\arctan(\sinh(L))}{\ell}, \quad L = \operatorname{arsinh}\left(\frac{1}{\sinh\left(\frac{\ell}{2}\right)}\right).$$

Proof. As seen in proof of Lemma 3.5.11, from Subsection 3.5.4 we have the following estimates for the derivatives of u_1 at the core:

- $|(u_1)_r| \leq 4W \sum_{k\in\mathbb{N}} ke^{-km} = 4W \frac{e^m}{(e^m-1)^2}$;
- $|(u_1)_\theta| \leq 2W \sum_{k\in\mathbb{N}} ke^{-km} = 2W \frac{e^m}{(e^m-1)^2}$;
- $|(u_1)_{rr}| \leq 2W \sum_{k\in\mathbb{N}} k^2e^{-km} = 2W \frac{e^m(e^m+1)}{(e^m-1)^3}$;
- $|(u_1)_{\theta r}| \leq 4W \sum_{k\in\mathbb{N}} k^2e^{-km} = 4W \frac{e^m(e^m+1)}{(e^m-1)^3}$;
- $|(u_1)_{\theta\theta}| \leq 4W^2 \sum_{k\in\mathbb{N}} k^2e^{-2km} = 4W^2 \frac{e^{2m}(e^{2m}+1)}{(e^{2m}-1)^3}$;

for $0 < W \leq 3.7$ as in Lemma 3.5.22, and the Osgood-Stowe tensor in the coordinates $(r,\theta) \in [-m,m] \times \mathbb{S}^1$ is

$$B\left(h_{\infty,0}, e^{2u}h_{\infty,0}\right) =$$

$$= \begin{bmatrix} \frac{1}{2}(u_1)_{rr} - \frac{1}{2}(u_1)_{\theta\theta} + \frac{1}{2}(u_1)_\theta^2 - \frac{1}{2}(u_1)_r^2 & (u_1)_{\theta r} - (u_1)_\theta (u_1)_r \\ (u_1)_{\theta r} - (u_1)_\theta (u_1)_r & \frac{1}{2}(u_1)_{\theta\theta} - \frac{1}{2}(u_1)_{rr} + \frac{1}{2}(u_1)_r^2 - \frac{1}{2}(u_1)_\theta^2 \end{bmatrix}.$$

We now need to compute $B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)$ on $(\dot\gamma,\dot\gamma)$ with $\gamma(s) = (0, \frac{2\pi}{\ell}s)$:

$$\begin{aligned} B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)(\dot\gamma,\dot\gamma) &= B\left(h_{\infty,0}, e^{2u}h_{\infty,0}\right)\left(\left(0,\frac{2\pi}{\ell}\right),\left(0,\frac{2\pi}{\ell}\right)\right) \\ &= \left(\frac{2\pi}{\ell}\right)^2 \left(\frac{1}{2}(u_1)_{\theta\theta} - \frac{1}{2}(u_1)_{rr} + \frac{1}{2}(u_1)_r^2 - \frac{1}{2}(u_1)_\theta^2\right), \end{aligned}$$

and, remembering that $e^{-m} \leq 1/16$, see Remark 3.5.12, we can bound the norm as follows:

$$\left\|B\left(h_{\infty,0}, e^{2u}h_{\infty,0}\right)\right\|_\gamma^{gr} \leq$$

$$\begin{aligned} &\leq \frac{\ell}{4}\left(\frac{2\pi}{\ell}\right)^2 \left(4W^2\frac{e^{2m}(e^{2m}+1)}{(e^{2m}-1)^3} + 2W\frac{e^m(e^m+1)}{(e^m-1)^3} + 20W^2\left(\frac{e^m}{(e^m-1)^2}\right)^2\right) \\ &= \frac{\pi^2}{\ell}\left(4W^2e^{-2m}\frac{1+e^{-2m}}{(1-e^{-2m})^3} + 2We^{-m}\frac{1+e^{-m}}{(1-e^{-m})^3} + 20W^2e^{-2m}\frac{1}{(1-e^{-2m})^4}\right) \\ &\leq \frac{\pi^2}{\ell}\left(4W^2e^{-2m}\frac{16^4(16^2+1)}{(16^2-1)^3} + 2We^{-m}\frac{17}{16}\left(\frac{16}{15}\right)^3 + 20W^2e^{-2m}\left(\frac{16^2}{16^2-1}\right)^4\right), \\ &\leq \frac{\pi^2}{\ell}We^{-m}\left(\frac{W}{4}1.02 + 2.58 + \frac{5W}{16}\right) \qquad 0 < W \leq 3.7, \\ &\leq 18\pi^2\frac{e^{-m}}{\ell}, \end{aligned}$$

concluding the proof. ■

For $G(\ell)$ as in Remark 3.5.25, $\overline{G}(\ell) = \left(WG(\ell)\left(\frac{16}{15}\right)^2\right)^2$ and $0 < W \leq 3.7$ as in Lemma 3.5.22, we now bound the norm $\|\cdot\|_\gamma^{gr}$ of the Osgood-Stowe tensor $B\left(h_0, e^{2\log(r)}h_0\right)$.

Proposition 3.5.19. The following bound holds:

$$\left\|B\left(h_0, e^{2\log(r)}h_0\right)\right\|_\gamma^{gr} \leq \frac{\pi^2}{\ell} + 8\pi^4\frac{\overline{G}(\ell)}{\ell}e^{-\pi^2/\ell}.$$

Proof. The proof of Lemma 3.5.15 essentially goes through and we just have to use the norm $\|\cdot\|_\gamma^{gr}$. Denoting by $q = -\mathcal{S}(f) = \frac{1}{2}dz^2$ we have:

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}/\ell\mathbb{Z}} \mathrm{Re}(q(\dot\gamma(t),\dot\gamma(t)))dt &= \frac{1}{4}\mathrm{Re}\int_0^\ell \dot\gamma(s)^2 ds\\ &= \frac{1}{4}\mathrm{Re}\int_0^\ell \left(\frac{2\pi}{\ell}i\right)^2 ds + \frac{1}{4}\mathrm{Re}\int_0^\ell (\dot\gamma(s)-\overline{\gamma}')^2 ds\\ &= -\frac{\pi^2}{\ell} + \frac{1}{4}\int_0^\ell (\dot\gamma(s)-\overline{\gamma}')^2 ds\ .\end{aligned}$$

Then,

$$\begin{aligned}\left\|B\left(h_0, e^{2\log(r)}h_0\right)\right\|_\gamma^{gr} &= \left|\frac{1}{2}\int_0^\ell B\left(h_0, e^{2\log(r)}h_0\right)(\dot\gamma,\dot\gamma)ds\right|\\ &= \left|\mathrm{Re}\left(\frac{1}{2}\int_0^\ell \mathcal{S}(f)(\dot\gamma(s),\dot\gamma(s))ds\right)\right|\\ &\le \frac{\pi^2}{\ell} + \frac{1}{4}\int_0^\ell \|\dot\gamma(s)-\overline{\gamma}'\|_{h_0}^2 ds\\ &\overset{(3.10)}{\le} \frac{\pi^2}{\ell} + \frac{\ell}{4}\left(\frac{4\sqrt{2}WG(\ell)\pi^2}{\ell}\frac{e^m}{(e^m-1)^2}\right)^2\\ &= \frac{\pi^2}{\ell} + 8\pi^4 W^2 \frac{G(\ell)^2}{\ell}\frac{e^{2m}}{(e^m-1)^4}\ .\end{aligned}$$

By applying the estimate in Remark 3.5.12

$$e^{-m} \le e^{-\pi^2/(2\ell)}, \qquad (e^m-1)^{-1} < \frac{16}{15}e^{-m}\ ,$$

and looking at the definition of $\overline{G}(\ell)$ right above the statement, we obtain the required result. ■

We can now prove the main estimate of this section.

Theorem 3.5.16. We have the following bound

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma^{gr} \le \frac{\pi^2}{\ell} + \frac{1}{4}\ell + 8\pi^4\frac{\overline{G}(\ell)}{\ell}e^{-\pi^2/\ell} + 18\pi^2\frac{e^{-\pi^2/(2\ell)}}{\ell}\ ,$$

where ℓ is the length of the core γ with respect to h_∞.

Proof. We have that:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma^{gr} = \left|\int_\gamma \left(B\left(h_\infty, e^{2u_0}h_\infty\right) + B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right) + B\left(h_0, e^{2\log(r)}h_0\right)\right)(\dot\gamma,\dot\gamma)\right|$$

which gives:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma^{gr} \le \left\|B\left(h_\infty, e^{2u_0}h_\infty\right)\right\|_\gamma^{gr} + \left\|B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)\right\|_\gamma^{gr} + \left\|B\left(h_0, e^{2\log(r)}h_0\right)\right\|_\gamma^{gr},$$

which by Propositions 3.5.17, 3.5.18, and 3.5.19, and Remark 3.5.12 yields:

$$\left\|B\left(\rho, e^{2u}\rho\right)\right\|_\gamma^{gr} \le \frac{\pi^2}{\ell} + \frac{1}{4}\ell + 8\pi^4\frac{\overline{G}(\ell)}{\ell}e^{-\pi^2/\ell} + 18\pi^2\frac{e^{-\pi^2/(2\ell)}}{\ell},$$

completing the proof. ■

3.5.4 Fourier Analysis

In this section we bound, on a neighborhood of the core γ of the thin tube $\mathbb{T}(\ell)$, u_1 and its derivatives. Recall, that u_1 is the conformal factor between the flat metrics $h_{\infty,0}$ and $h_0 = e^{2u_1}h_{\infty,0}$, and that $\ell \leq \varepsilon_0$ denotes the length of the core γ.

The relationship between $\mathbb{T}(\ell)$ and $\mathcal{A}$ is explained at the beginning of Section 3.5.1, but the relevant fact here is that $\mathbb{T}(\ell)$ and $(\mathcal{A}, h_\infty)$ are isometric. We will use both viewpoints as needed, and specifically, when dealing with hyperbolic geometry, we will use $\mathbb{T}(\ell)$. For our purposes, we need a new metric on $\mathcal{A}$ called the *Thurston metric* (see Section 1.3). Similarly to the definition of the convex core (see Section 1.4), identifying $\mathbb{CP}^1$ with the conformal boundary of $\mathbb{H}^3$, we denote by $CH(\mathbb{CP}^1 \setminus \Omega)$ the smallest closed convex subset of $\mathbb{H}^3$ whose closure in $\mathbb{CP}^1$ is $\mathbb{CP}^1 \setminus \Omega$, and by $\partial CH(\mathbb{CP}^1 \setminus \Omega)$ its boundary. Analogously to the convex core of a hyperbolic 3-manifold, as a pleated surface $\partial CH(\mathbb{CP}^1 \setminus \Omega)$ carries a well defined induced hyperbolic metric (see [BBB19, Section 2.2]).

We start by bounding u_1 on $\partial\mathcal{A}$ in Ω, and, for technical reasons, on a shorter curve of length ε_0 inside the tube. The second estimate is used to have a sharper bound on u_1 at the center of the tube, i.e. on γ. In what follows we will use the Thurston metric h_{Th} and we have:

- $h_{Th} = e^{2u_2}h_0$ on $\mathcal{A}$;
- $h_\infty = e^{2u_3}h_{Th}$ on Ω.

Let α_d be the closed curve in the thin tube $\mathbb{T}(\ell)$ given by one connected component of the set of the points at distance d from $\partial\mathbb{T}(\ell)$, let $\tilde{\alpha}_d \subseteq \partial CH(\mathbb{CP}^1 \setminus \Omega)$ be its pull-back via the normal projection from the boundary $\partial CH(\mathbb{CP}^1 \setminus \Omega)$ to Ω, and $\ell_{\partial CH(\mathbb{CP}^1\setminus\Omega)}(\alpha_d)$ be the length of α_d with respect to the induced hyperbolic metric on $CH(\mathbb{CP}^1 \setminus \Omega)$.

Lemma 3.5.20. For α_d and u_2 as above, the restriction of u_2 to α_d satisfies

$$0 \leq u_2|_{\alpha_d} \leq \frac{\ell_{\partial CH(\Omega)}(\tilde{\alpha}_d)}{2} \leq b(\ell_\infty(\alpha_d)) ,$$

with

$$b(x) = 2\pi e^{0.502\pi} e^{-\frac{\pi^2}{\sqrt{ex}}} \qquad \ell_\infty(\alpha_d) = \ell\cosh(L-d) \qquad L = \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right) .$$

Proof. By definition, the Thurston metric at some point z is given by $\frac{1}{r_{Th}^2}dz^2$ where r_{Th} is the ray of the horosphere in $\mathbb{H}^3$ at z tangent to $\partial CH(\mathbb{CP}^1 \setminus \Omega)$. Up to Möbius transformation, we assume that $\mathbb{CP}^1 \setminus \Omega$ contains 0 and ∞ and that the geodesic $\mathcal{L} \subseteq \mathbb{H}^3$ connecting them is contained in $CH(\mathbb{CP}^1 \setminus \Omega)$ and goes through a compression disk[1] for $\tilde{\alpha}_d$. Then, r_{Th} is less then the radius r_0 of the horosphere at z tangent at $\mathcal{L}$, therefore $u_2 = \log\frac{r_0}{r_{Th}}$ is positive.

We now prove the other inequality. Let us consider the pull-back of α_d through the normal projection $\tilde{\alpha}_d$. We notice that any point $p \in \tilde{\alpha}_d \subseteq \partial CH(\mathbb{CP}^1 \setminus \Omega)$ stays at distance from $\mathcal{L}$ less then $\ell_{\partial CH(\mathbb{CP}^1\setminus\Omega)}(\tilde{\alpha}_d)/2$. For any point $z \in \alpha_d \subseteq \mathbb{C}$ we call $p_z \subseteq \partial CH(\mathbb{CP}^1 \setminus \Omega)$

[1] This is any disk with boundary isotopic to $\tilde{\alpha}_d$.

the point on the horosphere H_z centered at z and tangent to $\partial CH(\mathbb{CP}^1 \setminus \Omega)$. Then we have

$$\ell_{\partial CH(\mathbb{CP}^1\setminus\Omega)}(\tilde{\alpha}_d)/2 > d_{\mathbb{H}^3}(p_z, \mathcal{L}) > d_{\mathbb{H}^3}(H_z, \mathcal{L}) = \log(r_0) - \log(r_{Th}) = u_2 \ .$$

Since L is half the length of the thin tube $\mathbb{T}(\ell)$ and α_d stays at distance $L-d$ from the core of the tube, the length of α_d with respect to h_∞ is

$$\ell_\infty(\alpha_d) = \ell\cosh(L) = \ell\cosh(L-d) \ .$$

The bounds of $\ell_{\partial CH(\mathbb{CP}^1\setminus\Omega)}(\tilde{\alpha}_d)$ by a function of $\ell_\infty(\alpha_d)$ follows from [Can01, Theorem 5.1][2] ■

Remark 3.5.21. In the notation of the previous lemma if α_0 coincide with a component of $\partial\mathbb{T}(\ell)$, its length with respect to h_∞ is

$$\ell_\infty(\alpha_0) = \ell\cosh\left(\operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right)\right) \ .$$

For $\ell \in (0, \varepsilon_0]$ the length $\ell_\infty(\alpha_0)$ is increasing, it is maximized at ε_0 by

$$\varepsilon_0\cosh\left(\operatorname{arcsinh}\left(\frac{1}{\sinh(\varepsilon_0/2)}\right)\right) < 2.5 \ ,$$

and it is greater then its limit at 0, which is 2, therefore

$$2 \leq \ell_\infty(\alpha_0) \leq 2.5 \ .$$

In what follows we let $\alpha = \alpha_{d_{\varepsilon_0}}$ be the simple closed curve on $\mathbb{T}(\ell)$ given by one connected component of the set of points staying at distance d_{ε_0} from $\partial\mathbb{T}(\ell)$, with $d_{\varepsilon_0} \geq 0$ such that $\ell_\infty(\alpha) = \ell\cosh(L - d_{\varepsilon_0}) = \varepsilon_0$.

Lemma 3.5.22. There exists a constant $W = W(\ell) > 0$ such that if $h_0 = e^{2u_1}h_{\infty,0}$, then $\left|u_{1|_{\partial\mathcal{A}}}\right| \leq W \leq 3.7$. Moreover, with the notation above there exists a constant $W_0 = W_0(\ell) > 0$ such that $\left|u_{1|_\alpha}\right| \leq W_0 \leq 2.3$.

Proof. Recall that by α_d we mean the simple closed curve given by one connected component of the set of points at distance d from $\partial\mathbb{T}(\ell)$. Note that, in particular, α_0 is one connected component of $\partial\mathcal{A}$, and $\alpha_{d_{\varepsilon_0}} = \alpha$. We also recall that $L =: \operatorname{arcsinh}(1/\sinh(\ell/2))$ is half of the length of the thin tube $\mathbb{T}(\ell)$. From the proof one can get explicit formulas $W(\ell)$, however for simplicity we just give upper bounds at every step.

We have $u_1 = -(u_2 + u_3 + u_0)$ with $u_{0|_{\alpha_d}} = \log\left(\frac{2\pi}{\ell_\infty(\alpha_d)}\right)$, and $0 \leq u_{2|_{\alpha_d}} \leq b(\ell_\infty(\alpha_d))$, as in Lemma 3.5.20, and

$$\ell_\infty(\alpha_d) = \ell\cosh(L-d) \ .$$

In particular

$$\ell_\infty(\partial\mathcal{A}) = \ell\cosh\left(\operatorname{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right)\right) \ ,$$

and also

$$\ell_\infty(\alpha) = \varepsilon_0 = 2\operatorname{arcsinh}(1) \ .$$

[2] Note that in [Can01, Theorem 5.1] the author requires the length to be less than one, however that is only needed to get a linear bound, see page 11 of [Can01, Theorem 5.1] the second line of equations.

We now consider u_3. By the Schwarz Lemma we have $h_\infty \leq h_{Th}$ and so $u_3 \leq 0$ everywhere. For a point $p \in \alpha_d$, we denote by $\mathrm{inj}(p)$ the injectivity radius at p with respect to h_∞. Then, for any $\varepsilon \leq \mathrm{inj}(p)$, by [BBB19, Theorem 2.8] and [BBB19, Corollary 2.12], we have:

$$h_{Th} \leq h_\infty(1 + 3\coth^2(\varepsilon/2))\coth^2(R_d^\varepsilon/2) \ , \tag{3.11}$$

where R_d^ε is such that the injectivity radius inside a ball of radius R_d^ε centered at any $p \in \alpha_d$ is at least ε. The injectivity radius, see [Bus10, Thm 4.1.6], at a point p at distance d from $\partial\mathbb{T}(\ell)$ satisfies

$$\sinh(\mathrm{inj}(p)) = \sinh(\ell/2)\cosh(L-d) \ ,$$

the condition on R_d^ε, being $\sinh(x)$ increasing for $x > 0$, becomes then

$$\sinh(\varepsilon) \leq \sinh(\ell/2)\cosh(L - d - R_d^\varepsilon) \ . \tag{3.12}$$

Since $\coth^2(x/2)$ is always greater then one, with vertical asymptote at 0 and decreasing for $x > 0$, we need to find suitable ε, d, and R_d^ε, as in equation (3.11) and (3.12), and bound it from below. In order to prove the estimate for $|u_1|$ on $\partial\mathcal{A}$ and α, we are interested in the two cases $d = 0$ and $d = d_{\varepsilon_0}$ respectively.

Case $d = 0$**.** The injectivity radius at any point of the boundary $\partial\mathbb{T}(\ell)$ is at least $\varepsilon_0/2 = \mathrm{arcsinh}(1)$, so we take $\varepsilon = \varepsilon_0/4$ and define our radius to be

$$R_0 = \inf_{\ell\in(0,\varepsilon_0]}\left(\mathrm{arcsinh}\left(\frac{1}{\sinh(\ell/2)}\right) - \mathrm{arcosh}\left(\frac{\sinh(\varepsilon_0/4)}{\sinh(\ell/2)}\right)\right) \ ,$$

as the function we are taking the inf of is decreasing in ℓ the expression is given by its limit in zero which is greater than $\pi/4$. Then, $\coth^2(R_0/2)$ is bounded above by 7.2. Therefore, since

$$\left|u_{3|\partial\mathcal{A}}\right| \leq \frac{1}{2}\log\left(\left(1 + 3\coth^2(\varepsilon_0/8)\right)\coth^2(R_0/2)\right),$$

we get

$$-3.08 \leq u_{3|\partial\mathcal{A}} \leq 0 \ .$$

As $u_1 = -(u_2 + u_3 + u_0)$ by merging the estimates:

$$-b(\ell_\infty(\alpha_0)) - \log\left(\frac{2\pi}{\ell_\infty(\alpha_0)}\right) \leq u_{1|\partial\mathcal{A}} \leq 3.08 - \log\left(\frac{2\pi}{\ell_\infty(\alpha_0)}\right) \ .$$

Recalling the definition of the function $b(x)$ from Lemma 3.5.20, and since $\ell_\infty(\alpha_0) \in [2, 2.5]$, see Remark 3.5.21, the worse bounds for the previous equation are obtained by evaluating the terms at 2.5, hence

$$-3.7 \leq u_{1|\partial\mathcal{A}} \leq 2.16 \ .$$

Case $d = d_\varepsilon$**.** Since $\cosh(L - d_{\varepsilon_0}) = \varepsilon_0/\ell$ the injectivity radius of the points p in α is

$$\mathrm{inj}(p) = \mathrm{arcsinh}(\sinh(\ell/2)\varepsilon_0/\ell) \ ,$$

then, by choosing $\varepsilon = \mathrm{inj}(p)/2$, we take

$$R_{d_{\varepsilon_0}} = \inf_{\ell\in(0,\varepsilon_0]}\left(\mathrm{arcosh}(\varepsilon_0/\ell) - \mathrm{arcosh}\left(\frac{\sinh(\frac{1}{2}\mathrm{arcsinh}(\sinh(\ell/2)\varepsilon_0/\ell))}{\sinh(\ell/2)}\right)\right) \ ,$$

which is greater than 0.77. Then, $\coth^2(R_{d_{\varepsilon_0}}/2)$ is bounded above by 7.5. Thus,

$$\left|u_{3|\alpha}\right| \leq \frac{1}{2}\log\left(\left(1+3\coth^2(\mathrm{inj}(p)/2)\right)\coth^2(R_{d_{\varepsilon_0}}/2)\right) ,$$

and

$$\mathrm{inj}(p) = \operatorname{arcsinh}(\sinh(\ell/2)\varepsilon_0/\ell) \geq \operatorname{arcsinh}(\varepsilon_0/2) ,$$

using these estimate and the fact that $u_3 \leq 0$ we obtain:

$$-2.56 \leq u_{3|_\alpha} \leq 0 .$$

To bound $u_1|_\alpha$ we do as the previous case and we obtain:

$$-b(\varepsilon_0) - \log\left(\frac{2\pi}{\varepsilon_0}\right) \leq u_{1|_\alpha} \leq 2.56 - \log\left(\frac{2\pi}{\varepsilon_0}\right)$$

from which

$$-2.3 \leq u_{1|_\alpha} \leq 1.29 .$$

Defining W and W_0 as the evaluation of the function $-b(x) - \log(2\pi/x)$ at respectively $\ell_\infty(\alpha_0)$ and ε_0 we get the thesis. ■

Now we want to solve the following ODE on $\mathcal{A} \cong [-m, m] \times \mathbb{S}^1$:

$$\Delta u = 0 \qquad |u|_{\partial\mathcal{A}}| \leq W \leq 3.7 ,$$

where m is half of the length of the thin tube as in Proposition 3.5.11 and W is as in Lemma 3.5.22.

Using Fourier analysis, and assuming that the solution is of the form

$$u(r,\theta) = \sum_{k\in\mathbb{N}} u_k(r)e^{-ik\theta} ,$$

and symmetric in r, we get that the coefficient need to satisfy:

$$\ddot{u}_k(r) = k^2 u_k(r), \qquad |u_k(\pm m)| \leq W .$$

Which yields solutions of the form:

$$k = 0 : u_k(r) = C_0 + C_1 r ,$$

and:

$$k \neq 0 : u_k(r) = v_1(k)\cosh(kr) + v_2(k)\sinh(kr) .$$

Then, we get the following linear system of equation:

$$\begin{cases} v_1(k)\cosh(km) + v_2(k)\sinh(km) = u_k(m) \\ v_1(k)\cosh(km) - v_2(k)\sinh(km) = u_k(-m) \end{cases}$$

which can be rewritten as:

$$\begin{cases} v_1(k) = \frac{u_k(m)+u_k(-m)}{2}(\cosh(km))^{-1} \\ v_2(k) = \frac{u_k(m)-u_k(-m)}{2}(\sinh(km))^{-1} \end{cases}$$

which yields for $k \neq 0$, since $e^{-2km} < e^{-m} \leq \frac{1}{16} < \frac{1}{2}$:

$$|v_1(k)| \leq 2We^{-km} \leq 7.4e^{-km} \qquad |v_2(k)| \leq 4We^{-km} \leq 14.8e^{-km},$$

while $|v_1(0)| = |C_0| \leq W_0$. Similarly one can obtain bounds on the n-derivatives of $u(r,\theta)$ of the form $Ck^n e^{-km}$, as:

$$n \text{ even:} \quad \left|\frac{\mathrm{d}^n u_k(r)}{\mathrm{d}r^n}\right| = |v_1(k)k^n \cosh(kr) + v_2(k)k^n \sinh(kr)|\,,$$
$$n \text{ odd:} \quad \left|\frac{\mathrm{d}^n u_k(r)}{\mathrm{d}r^n}\right| = |v_1(k)k^n \sinh(kr) + v_2(k)k^n \cosh(kr)|\,,$$

and the same computation give bounds of the order of $k^n e^{-km}$ which still has exponential decay. Thus, we get that the decay of the coefficients of $u(r,\theta)$ and its n-th derivatives is of the order $k^n e^{-km}$. We have

$$u_k(r) = v_1(k)\cosh(kr) + v_2(k)\sinh(kr)\ ,$$

with

$$|v_1(k)| \leq 2We^{-km} \qquad |v_2(k)| \leq 4We^{-km}\ .$$

Estimates

We can now compute and bound the first and second derivatives of u, which, when $u = u_1$ allow to bound the Osgood-Stowe tensor $B\left(h_{\infty,0}, e^{2u_1}h_{\infty,0}\right)$ for the two flat metrics of the previous sections. As we already seen, it is also necessary to have a good bound on u_1 at the core curve. We will obtain it at the end of this subsection.

Thus, we have:

- $u_r(r,\theta) = \sum_{k\in\mathbb{N}} k\left(v_1(k)\sinh(kr) + v_2(k)\cosh(kr)\right)e^{-ik\theta}$,
- $u_\theta(r,\theta) = \sum_{k\in\mathbb{N}} iku_k(r)e^{-ik\theta}$,
- $u_{\theta r}(r,\theta) = \sum_{k\in\mathbb{N}} ik^2\left(v_1(k)\sinh(kr) + v_2(k)\cosh(kr)\right)e^{-ik\theta}$,
- $u_{rr}(r,\theta) = \sum_{k\in\mathbb{N}} k^2\left(v_1(k)\cosh(kr) + v_2(k)\sinh(kr)\right)e^{-ik\theta}$,
- $u_{\theta\theta}(r,\theta) = \sum_{k\in\mathbb{N}} k^2 u_k^2(r)e^{-ik\theta}$.

Using the bounds on $v_1(k)$ and $v_2(k)$, and since $r \geq 0$, we obtain the following estimates:

- $|u_r(r,\theta)| \leq \sum_{k\in\mathbb{N}} k\left(2We^{-km}\sinh(kr) + 4We^{-km}\cosh(kr)\right)$,
- $|u_\theta(r,\theta)| \leq \sum_{k\in\mathbb{N}} k\left(2We^{-km}\cosh(kr) + 4We^{-km}\sinh(kr)\right)$,
- $|u_{\theta r}(r,\theta)| \leq \sum_{k\in\mathbb{N}} k^2\left(2We^{-km}\sinh(kr) + 4We^{-km}\cosh(kr)\right)$,
- $|u_{tt}(r,\theta)| \leq \sum_{k\in\mathbb{N}} k^2\left(2We^{-km}\cosh(kr) + 4We^{-km}\sinh(kr)\right)$,
- $|u_{\theta\theta}(r,\theta)| \leq \sum_{k\in\mathbb{N}} k^2\left(2We^{-km}\cosh(kr) + 4We^{-km}\sinh(kr)\right)^2$.

If we just care about the geodesic at $r = 0$ (which is the image of γ in the flat coordinates) we have the following estimates,

- $|u_r(0,\theta)| \leq 4W \sum_{k\in\mathbb{N}} ke^{-km} = 4W \frac{e^m}{(e^m-1)^2} \leq 14.8 \frac{e^m}{(e^m-1)^2}$,
- $|u_\theta(0,\theta)| \leq 2W \sum_{k\in\mathbb{N}} ke^{-km} = 2W \frac{e^m}{(e^m-1)^2} \leq 7.4 \frac{e^m}{(e^m-1)^2}$,
- $|u_{\theta r}(0,\theta)| \leq 4W \sum_{k\in\mathbb{N}} k^2e^{-km} = 4W \frac{e^m(e^m+1)}{(e^m-1)^3} \leq 14.8 \frac{e^m(e^m+1)}{(e^m-1)^3}$,
- $|u_{rr}(0,\theta)| \leq 2W \sum_{k\in\mathbb{N}} k^2e^{-km} = 2W \frac{e^m(e^m+1)}{(e^m-1)^3} \leq 7.4 \frac{e^m(e^m+1)}{(e^m-1)^3}$,
- $|u_{\theta\theta}(0,\theta)| \leq 4W^2 \sum_{k\in\mathbb{N}} k^2e^{-2km} = 4W^2 \frac{e^{2m}(e^{2m}+1)}{(e^{2m}-1)^3} \leq (7.4)^2 \frac{e^{2m}(e^{2m}+1)}{(e^{2m}-1)^3}$.

In the more general case:

- $|u_r(r,\theta)| \leq 4W \sum_{k\in\mathbb{N}} ke^{-k(m-r)} = 4W \frac{e^{(m-r)}}{(e^{(m-r)}-1)^2}$,
- $|u_\theta(r,\theta)| \leq 4W \sum_{k\in\mathbb{N}} ke^{-k(m-r)} = 4W \frac{e^{(m-r)}}{(e^{(m-r)}-1)^2}$,
- $|u_{\theta r}(r,\theta)| \leq 4W \sum_{k\in\mathbb{N}} k^2e^{-k(m-r)} = 4W \frac{e^{(m-r)}(e^{(m-r)}+1)}{(e^{(m-r)}-1)^3}$,
- $|u_{rr}(r,\theta)| \leq 4W \sum_{k\in\mathbb{N}} k^2e^{-k(m-r)} = 4W \frac{e^{(m-r)}(e^{(m-r)}+1)}{(e^{(m-r)}-1)^3}$,
- $|u_{\theta\theta}(r,\theta)| \leq 16W^2 \sum_{k\in\mathbb{N}} k^2e^{-2k(m-r)} = 16W^2 \frac{e^{2(m-r)}(e^{2(m-r)}+1)}{(e^{2(m-r)}-1)^3}$.

Note that, in the notation of the previous sections, for any fixed r, when m is the width of the Euclidean cylinder $(\mathcal{A}, h_{\infty,0})$, the last two blocks of estimates are $O(e^{-m}) \leq O(e^{-\pi^2/(2\ell)})$.

Remark 3.5.23. At the core γ:

$$|u(\gamma)| = |u(0,\theta)| = \left|\sum_{k=0}^{\infty} v_1(k)e^{-ik\theta}\right| \leq \sum_{k=0}^{\infty} |v_1(k)| \leq |v_1(0)| + \sum_{k=1}^{\infty} |v_1(k)| \leq W_0 + \sum_{k=1}^{\infty} e^{-km}$$
$$\leq W_0 + \frac{2W}{e^m-1} \leq W_0 + \frac{2W}{\exp\left(\frac{2\pi \arctan(\sinh(L))}{\ell}\right) - 1} \leq 2.3 + 0.5 = 2.8 \ ,$$

where the last inequality is obtained by substituting $L = \operatorname{arsinh}(1/\sinh(\varepsilon_0/2))$, $W \leq 3.7$ and $W_0 \leq 2.3$. Also, we see that for $\ell \to 0$ the bound goes to W_0 as

$$|u(\gamma)| \leq W_0 + \frac{2We^{-\pi^2/(2\ell)}}{1 - e^{-\pi^2/(2\ell)}}$$
$$\leq W_0 + 2.2We^{-\pi^2/(2\ell)} \ .$$

We now want to improve the estimate for u on the core (and on a neighborhood) also by using the bound of Lemma 3.5.13:

$$\max_{s\in\mathbb{S}^1_\ell} \left\|D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0} \leq 6We^{2.8}\left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m-1)^2}, \quad W \leq 3.7 \ ,$$

Lemma 3.5.24. On the core γ the function u satisfies:

$$e^u \leq 1 + \sqrt{2}We^{2.8}\pi\frac{e^m}{(e^m-1)^2} + 2W\frac{e^m}{(e^m-1)^2}\ell + 4\sqrt{2}We^{2.8}\pi\ell\left(\frac{e^m}{(e^m-1)^2}\right)^2 .$$

In particular, $e^u = 1 + O(e^{-\pi^2/(2\ell)})$.

Proof. The core is parameterized as $\gamma(s) = \left(0, \frac{2\pi}{\ell}s\right)$ with $s \in [0, \ell]$ with respect to the coordinates (r, θ) of $h_{\infty,0}$ and $h_0 = e^{2u}h_{\infty,0}$. Then, by using that $u(s) = u\left(0, \frac{2\pi}{\ell}s\right)$ we have:

$$\ell_{h_0}(\gamma) = \int_0^\ell \|\dot\gamma\|_0 \, ds = \int_0^\ell e^{u(s)} \|\dot\gamma\|_{h_{\infty,0}} \, ds = \frac{2\pi}{\ell}\int_0^\ell e^{u(s)} ds .$$

We now estimate $\ell_{h_0}(\gamma)$ by using the mean $\overline{\gamma}' = (0, \frac{2\pi}{\ell})$ of $\dot\gamma$ as seen with respect to h_0, see Lemma 3.5.15.

$$\begin{aligned}
\int_0^\ell \|\dot\gamma\|_{h_0} \, ds &\leq \int_0^\ell \|\dot\gamma - \overline{\gamma}'\|_{h_0} \, ds + \int_0^\ell \|\overline{\gamma}'\|_{h_0} \\
&\leq \int_0^\ell \frac{4\sqrt{2}We^{2.8}\pi^2}{\ell}\frac{e^m}{(e^m-1)^2} ds + 2\pi \\
&\leq 2\pi + 4\sqrt{2}We^{2.8}\pi^2\frac{e^m}{(e^m-1)^2} .
\end{aligned}$$

Here the second inequality follows from equation (3.10) in Lemma 3.5.15. Thus,

$$2\pi \leq \frac{2\pi}{\ell}\int_0^\ell e^{u(s)} ds \leq 2\pi + 4\sqrt{2}We^{2.8}\pi^2\frac{e^m}{(e^m-1)^2} ,$$

which gives:

$$\ell \leq \int_0^\ell e^{u(s)} ds \leq \ell + \sqrt{2}We^{2.8}\pi\ell\frac{e^m}{(e^m-1)^2} .$$

Therefore, there must be $\xi \in [0, \ell]$ such that

$$e^{u(\xi)} \leq 1 + \sqrt{2}We^{2.8}\pi\frac{e^m}{(e^m-1)^2} .$$

For all $s \in [0, \ell]$ we have:

$$\begin{aligned}
e^{u(s)} - e^{u(\xi)} &= \int_\xi^s e^{u(s)}\dot u(s) ds \\
&\leq \int_\xi^s e^{u(s)} \max_{s\in[0,\ell]} |\dot u(s)| \, ds \\
&\leq 2W\frac{e^m}{(e^m-1)^2}\int_\xi^s e^{u(s)} ds \\
&\leq 2W\frac{e^m}{(e^m-1)^2}\int_0^\ell e^{u(s)} ds \\
&\leq 2W\frac{e^m}{(e^m-1)^2}\left(\ell + \sqrt{2}We^{2.8}\pi\ell\frac{e^m}{(e^m-1)^2}\right),
\end{aligned}$$

and by using $e^{u(\xi)} \leq 1 + \sqrt{2}We^{2.8}\pi\frac{e^m}{(e^m-1)^2}$ and re-arranging the terms we obtain the required statement. The last equality follows directly by Equation 3.9 in Remark 3.5.12. ∎

Remark 3.5.25. The bound of Lemma 3.5.24 beats the one of Remark 3.5.23 when ℓ is small enough. Then, we can write:

$$\max_{s\in\mathbb{S}^1_\ell} \left\| D^0_{\dot\gamma}\dot\gamma(s)\right\|_{h_0} \leq 6WG(\ell)\left(\frac{2\pi}{\ell}\right)^2 \frac{e^m}{(e^m-1)^2}, \quad W \leq 3.7,$$

with:

$$G(\ell) := \min\left\{e^{2.8}, 1+\sqrt{2}We^{2.8}\pi\frac{e^m}{(e^m-1)^2} + 2W\frac{e^m}{(e^m-1)^2}\ell + 4\sqrt{2}We^{2.8}\pi\ell\left(\frac{e^m}{(e^m-1)^2}\right)^2\right\}.$$

By Equation 3.9 in Remark 3.5.12 $G(\ell)$ is dominated by:

$$\begin{aligned}1+\sqrt{2}We^{2.8}\pi\left(16/15\right)^2 e^{-\pi^2/(2\ell)} + 2W\ell\left(16/15\right)^2 e^{-\pi^2/(2\ell)} + 4\sqrt{2}We^{2.8}\pi\ell\left(16/15\right)^4 e^{-\pi^2/\ell} \\ = 1 + O\left(e^{-\pi^2/(2\ell)}\right) .\end{aligned} \tag{3.13}$$

Remark 3.5.26. The equation (3.13) gives a universal upper bound on the terms in $O(\cdot)$ of Theorem 3.5.4.

Remark 3.5.27. Lemma 3.5.24 together with the last block of estimates for the partial derivatives of u above, imply that the same bound for e^u still holds in a neighborhood of the core γ in $\mathcal{A}$.

Chapter 4

Adapted Renormalized Volume for Hyperbolic 3-Manifolds with Compressible Boundary

As already discussed in the introduction, and evidenced by the previous chapter, depending on whether $\partial\overline{M}$ is incompressible or not, the renormalized volume behaves very differently. When M has incompressible boundary, the Kraus-Nehari bound [Neh49] implies that the *Teichmüller norm* of (minus) the *Weil-Petersson gradient* of the renormalized volume is uniformly bounded (see Section 1.2.3). Since $\mathcal{T}(\partial\overline{M})$ equipped with the Teichmüller norm is complete, the gradient flow exists at any time. In the (relatively) acylindrical case, this has been widely studied in [BBB19], [BBB23] and [BBVP23], where it is proven, in particular, that the renormalized volume is always non-negative, and that the flowlines always converge to the point whose associated convex cocompact structure on M has totally geodesic convex core boundary.

In this chapter, we define a new version of the renormalized volume which adapts to the compressible boundary case, satisfying similar properties to the ones of the classical one in the incompressible setting. In particular, the adapted renormalized volume is bounded from below and its gradient has uniformly bounded Weil-Petersson norm. Moreover, it extends continuously to the compressible strata in the boundary of the Weil-Petersson completion of Teichmüller space. We give a geometric interpretation of the limit quantity by defining a renormalized volume, and its adapted version, for convex co-compact hyperbolic 3-manifolds with a finite set of marked points in the boundary.

4.1 Results and outline of the chapter

Inspired by the asymptotic behaviour of the renormalized volume under the pinching of a compressible curve (see [SW22], or also Theorem 3.4.5), we define the adapted renormalized volume by subtracting to the standard one the divergent terms as follows.

Definition 4.4.1. Given M a convex co-compact hyperbolic 3-manifold, the *adapted renormalized volume* is defined as the function

$$\widetilde{V_R}\colon \mathcal{T}(\partial\overline{M}) \to \mathbb{R}$$

such that

$$\widetilde{V_R}(X) = V_R(X) + \pi^3 \sum_{\substack{\gamma \text{ compressible} \\ \ell_\gamma(X) < \varepsilon_0}} \frac{1}{\ell_\gamma(X)}$$

where $\varepsilon_0 = 2\mathrm{arsinh}(1)$ is the Margulis constant, γ runs in the set of compressible simple closed curves in $\partial\overline{M}$, and $\ell_\gamma(X)$ denotes its length with respect to the hyperbolic representative in X.

Exploiting Theorem 3.4.5 in Chapter 3, we are able to bound the adapted renormalized volume.

Theorem 4.4.4. For every convex co-compact hyperbolic 3-manifold M, the adapted renormalized volume $\widetilde{V_R}(\cdot)$ is bounded from below by a constant depending just on the topology of the boundary $\partial\overline{M}$.

Recall that the differential of the renormalized volume at a point $X \in \mathcal{T}(\partial M)$ coincides with the real part of the Schwarzian derivative of f_M, with f_M the developing map of the natural projective structure on the boundary at infinity $\partial_\infty M$, with $M = M(X)$ the convex co-compact manifold associated to X (see Section 1.4.1, Section 1.3.1 , and Theorem 1.4.5). As already discussed in Chapter 3, the infinity norm of the Schwarzian derivative of a non-univalent map, with image a hyperbolic domain in $\mathbb{CP}^1$, diverges whenever the hyperbolic length of a non-contractible simple closed curve in its image goes to zero, [KM81]. When f is the developing map associated to a convex co-compact manifold M, this corresponds to the situation where the hyperbolic length ℓ of a compressible simple closed curve in $\partial_\infty M$ is going to zero. Thanks to Theorem 3.5.4, a straightforward computation shows that both the L^1-norm and the L^2-norm (i.e., the Teichmüller and Weil-Petersson norm; see Section 1.2.3) of the differential of the renormalized volume diverge as $\ell \to 0$. This divergence does not occur for the differential of the adapted renormalized volume.

Theorem 4.1.1. (Theorems 4.4.5 and 4.4.6) Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the differential of the adapted renormalized volume is bounded in L^1 and L^∞ norm by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

Corollary 4.4.7. Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the Weil-Petersson gradient of the adapted renormalized volume has Weil-Petersson norm bounded by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

As addressed in the introduction, the renormalized volume can actually be defined and studied in the more general setting of *geometrically finite* hyperbolic 3-manifolds, as done in [GMR15], [Pal16], and [Pal17]. These works establish several continuity results for the renormalized volume, among which is the continuity under pinching an incompressible curve to obtain a geometrically finite limit.

The pinching of a compressible simple closed curve in the boundary limits, instead, in the pointed Gromov-Hausdorff topology, to a convex co-compact manifold *marked* at one or two points corresponding to the curve that has been pinched, as described in [SW22, Appendix A.10]. We saw in Section 1.2.3 that a point in the boundary of the Weil-Petersson completion of a closed surface S is the data of a multicurve m and a complete hyperbolic metric on $S \setminus m$. Given M a convex co-compact manifold, we will call *compressible* a stratum in $\overline{\mathcal{T}(\partial\overline{M})}^{wp}$ corresponding to a compressible multicurve.

A natural question to ask is whether the adapted renormalized volume, being bounded, converges, under the pinching of a compressible curve, to the sum of the renormalized volumes of the convex co-compact manifolds arising as the pointed Gromov-Hausdorff limits. The answer is negative. The key reason behind this lies in the fact that the geometric limit of convex cores does not coincide with the convex core of the limit manifold(s), but it is instead the convex hull of the marked point(s) at infinity union the convex core [SW22, Lemma $A.8$]. In fact, the Gromov-Hausdorff limits retains memory of the (multi)curve that has been pinched, encoded through the marked point. Consequently, the limit of the renormalized volume depends on the marked points, other than limit manifold.

In this chapter then, after analyzing the behavior of the Epstein surface in a neighborhood of cusp singularities, we define the *renormalized volume of pointed convex co-compact manifolds* (see Section 4.3.2 and in particular Definition 4.3.12). We then prove that, under a sequence of pinching a compressible (multi)curve, the adapted renormalized volume converges to the sum of the adapted renormalized volumes of the pointed limit convex co-compact manifolds. In this sense, the adapted renormalized volume extends continuously to the strata of the boundary of the Teichmüller space corresponding to compressible multicurves.

Theorem 4.4.20. Let $M_t = (M, g_t)$ be a path of convex co-compact hyperbolic 3-manifolds obtained by pinching an admissible compressible multicurve m in the conformal boundary at infinity. Let $D(m)$ be a union of disks compressing m, and let (M_i, g_i, P_i), for $i = 1, \dots, k$, be the pointed convex co-compact limits of $(M_t, y_i(t))$ in the Gromov-Hausdorff topology, with $y_i(t)$ in the thick part of the i-th connected component of $C(M_t) \setminus D(m)$. Then, outside a codimension-one set

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = \sum_{i=1}^{k} \widetilde{V_R}(M_i, g_i, P_i) \ .$$

The admissibility assumption on the multicurve ensures the Gromov-Hausdorff limits to be convex co-compact (see Theorem 4.4.15, Remark 4.4.16 and Definition 4.4.17).

4.1.1 Outline of the chapter

In Section 4.2, we provide the main objects and knowledge needed in this chapter, that have not been presented in Chapter 1. In Section 4.3, we define the renormalized volume associated to a convex co-compact manifold with marked points at infinity (Definition 4.3.12 and Remark 4.3.16). The main idea is to associate a renormalized volume to the unique hyperbolic representative conformal to the boundary at infinity with a cusp singularity at each marked point. To this aim, we first study the divergence of the W-volume of a truncated hyperbolic cusp (Proposition 4.3.11), by explicitly constructing the associated Epstein surface (in Section 4.3.1).

In Section 4.4, we start by defining the adapted renormalized volume and prove that this is uniformly bounded from below (Theorem 4.4.4), that its differential has bounded infinity norm (Theorem 4.4.6), that its gradient has bounded Weil-Petersson norm (Corollary 4.4.7), and that its variation is arbitrarily small on Bers regions of compressible pants decompositions (Theorem 4.4.9). We then prove its continuity under pinching a compressible multicurve (Theorem 4.4.19). This is done by exploiting Section 4.4.1, in which we study the divergence of the W-volume of long hyperbolic tubes (Theorem 4.4.14).

4.2 Preliminaries

This section provides the background specific to this chapter, that was not covered in Chapter 1.

4.2.1 Conformally equivalent surfaces with marked points

For any pair (X, D) with S a closed Riemann surface and $D \subseteq S$ a finite set of points, by Uniformization Theorem, there exists a unique hyperbolic metric which is conformal to $X \setminus D$ and has cusps at each point in P, [Ahl10, Chapter 10, parabolic case], [Hei62]. Moreover, the behaviour of this conformal metric in a neighborhood of $p \in D$ can be described explicitly, and we are particularly interested in the local expression of the metric tensor. To this end, we report the version from [FSX19], whose proof techniques are well suited to our context.

Theorem 4.2.1 (Theorem 1.2, [FSX19])**.** Let $\mathbb{D}^*$ be the punctured disk in $\mathbb{C}$ of Euclidean radius one centered at the origin, and let $d\rho^2$ be a hyperbolic conformal metric on $\mathbb{D}^*$ with a cusp singularity at 0. Then, there exists a complex coordinate w on $\mathbb{D}(\varepsilon) = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ for some $\varepsilon > 0$ and with $w(0) = 0$, such that

$$d\rho^2 = \frac{1}{|w|^2 \log^2(|w|)} dw^2 \ ,$$

Moreover, the coordinate w is unique up to rotation.

Note that Theorem 4.2.1 does not assume the hyperbolic metric $d\rho^2$ on $\mathbb{D}^*$ to be complete, in which case it would not be necessary to restrict to a smaller disk or to change coordinate (see beginning of Section 4.3.1). We remark that, up to composing with a Möbius transformation, it is always possible to assume the cusp singularity to be at the origin. On the other hand, the change of coordinate from the standard one z to w is conformal but not necessarily projective, that is, it may not be realized by a Möbius transformation. Since we will work with Epstein surfaces, which depend crucially on the complex projective structure (see Section 1.5), it is important to understand how far the aforementioned change of coordinate is from being projective. Examining the proof of Theorem 1.2 in [FSX19], we find that the answer is 'not much', provided we restrict to a sufficiently small neighborhood of the cusp singularity. There is an equivalent analytic definition for a Riemann surface to have a cusp: the Schwarzian of the uniformization map is of the form

$$\frac{1}{2z^2} + \frac{d}{z} + \psi(z) \ ,$$

where d is a number and $\psi(z)$ a holomorphic function, both depending on the choice of coordinate (see Lemma 2.1 in [LLX20]). Thanks to this, as done in [FSX19], by integrating the Schwarzian, up to Möbius transformation, one obtains that the developing map has the form

$$g(z) = \log(z) + \psi_1(z) \ ,$$

with $\psi_1(z)$ a holomorphic function such that $\psi_1(0) = 0$. Then, in the coordinate w such that $\log(w) = \log(z) + \psi_1(z)$, one gets

$$g(w) = \log(w) \ .$$

Note that we can multiply g by $-i$, which, being a Möbius transformation, does not change the Schwarzian derivative (see Section 1.3.1), to get the developing map

$$g_1(w) = -i\log(w) \ ,$$

through which the pull back of the hyperbolic metric of $\mathbb{H}^2$ gives the one in Theorem 4.2.1. The only non projective change of coordinate needed is then

$$w = ze^{\psi_1(z)} \ , \tag{4.1}$$

which, since $\psi_1(z)$ is holomorphic and such that $\psi_1(0) = 0$, behaves at the first order at 0 again like z.

4.2.2 Geometric convergence

There are several equivalent definitions of geometric convergence for *Kleinian groups*, i.e., discrete subgroups of $\mathbb{P}SL(2,\mathbb{C})$, and for their corresponding hyperbolic quotient 3-manifolds (see for example [McM96, Section 2.2], [Mar16b, Chapter 4], [MT98, Chapter 7]). Here, we present the one we will use.

Definition 4.2.2. A sequence of pointed hyperbolic 3-manifold (M_n, x_n) *geometrically converges* to a pointed hyperbolic 3-manifold (M, x) if and only if, for every compact subset $K \subseteq M$ containing x, and for all sufficiently large n, there exists a smooth embedding $f_n\colon K \to M_n$ such that $f_n(x) = x_n$ and f_n converges in the C^∞ topology to an isometric embedding. The induced topology is also called the *pointed Gromov-Hausdorff topology.*

4.2.3 Generalizing Epstein surfaces and W-volumes

Epstein surfaces were already introduced in Section 1.5, however, this chapter requires a more in-depth exploration of the topic.

Let us start by fixing some notations. Let $z = x + iy$ be the complex coordinate of $\mathbb{C}$, then we use the following:

$$\begin{aligned} dz^2 &= dz \otimes dz = dx^2 - dy^2 + i(dx \otimes dy + dy \otimes dx) \\ d\overline{z}^2 &= d\overline{z} \otimes d\overline{z} = dx^2 - dy^2 - i(dx \otimes dy + dy \otimes dx) \\ |dz|^2 &= dx^2 + dy^2 \ . \end{aligned}$$

Given $\Omega \subseteq \mathbb{C}$ an open domain, a *conformal metric* on Ω can be expressed as

$$g = e^{2\varphi}|dz|^2 \ ,$$

with

$$\varphi\colon \Omega \to \mathbb{R}$$

a C^k function, which, if the metric is required to be Riemannian, it must be smooth. In the next chapter, we will sometimes consider conformal factors φ with *cusp singularities* at a finite set of points $P \subseteq \Omega$. In this setting, and in light of Section 4.2.1, it is possible to define cusp singularities for Riemann surfaces as follows.

Definition 4.2.3. Let $g_P = e^{2\varphi}\,|dz|^2$ be a conformal metric on a domain $\Omega \subseteq \mathbb{C}$, where φ is smooth (or just C^k) out of a finite set of points $P \subseteq \Omega$. Then g_P has a *cusp singularity* at $p \in P$ if, for a local coordinate such that $z(p) = 0$, the function $\varphi + \log(|z|\log|z|)$ extends continuously to p.

Epstein surfaces coordiantes

Recall that to any smooth conformal metric $g = e^{2\varphi} |dz|^2$ on a domain $\Omega \subseteq \mathbb{CP}^1 = \partial_\infty \mathbb{H}^3$ it is possible to associate a convex embedded surface $\Sigma(g)$ in $\mathbb{H}^3$ as the image of the Epstein map $\mathrm{Eps}_{\Omega,\varphi} \colon \Omega \to \mathbb{H}^3$ (see Section 1.5). Sometimes, to ensure the regularity of $\Sigma(g)$, it is necessary to rescale g by a large enough constant (see Proposition 1.5.6 and Theorem 1.5.7). The Epstein surface $\Sigma(g)$ is given by the boundary of the convex envelope of the union of horospheres centered at points $z \in \Omega$ and of euclidean radii equal to $e^{-\varphi(z)}$. Thanks to this construction, it is possible to find the explicit coordinates of $\Sigma(g)$ in terms of φ and its first derivatives:

$$\Sigma(g)(z) = \left(z + \frac{2e^{-2\varphi}\nabla\varphi}{1 + e^{-2\varphi} |\nabla\varphi|^2}, \frac{2e^{-\varphi}}{1 + e^{-2\varphi} |\nabla\varphi|^2} \right) , \tag{4.2}$$

where $\nabla(\cdot)$ is the gradient with respect to the Euclidean metric on $\mathbb{C}$. Note that, a rescaling of g by a constant factor e^{2r} produces an Epstein surface that is r-equidistant from the one associated to the original metric.

Fundamental forms

As already seen in Section 1.5.3, there is a beautiful correspondence between the fundamental forms on the Epstein surface induced by $\mathbb{H}^3$ and those of the domain at infinity $\Omega \subseteq \mathbb{CP}^1$ (see Definition 1.5.2). We use here the same notations as in Section 1.5.3. The key properties that will be used in this chapter are summarized below.

Lemma 4.2.4. ([ZT87, KS08]) Let $\hat{I}$ be a Riemannian metric on $\Omega \subseteq \mathbb{CP}^1$, and let $\varphi \colon \Omega \to \mathbb{R}$ be a smooth function such that $\hat{I} = e^{2\varphi} |dz|^2$. Then, the second fundamental form at infinity $\hat{II}$ can be written in terms of φ as

$$\hat{II} = 2q dz^2 + 2\overline{q} d\overline{z}^2 + 4\varphi_{z\overline{z}} |dz|^2$$

with

$$q = \varphi_{zz} - (\varphi_z)^2 .$$

Moreover, the curvature K of $\hat{I}$ satisfies

$$K = -4\varphi_{z\overline{z}} e^{-2\varphi} .$$

Remark 4.2.5. In the coordinates (x, y) such that $z = x + iy$, the equation in Lemma 4.2.4 becomes

$$\hat{II} = (4\varphi_{z\overline{z}} + 4\mathrm{Re}(q))dx^2 + (4\varphi_{z\overline{z}} - 4\mathrm{Re}(q))dy^2 - 4\,\mathrm{Im}(q) dxdy$$

where $dxdy$ denotes twice the symmetric tensor product between dx and dy.

Lemma 4.2.6. ([KS08]) The mean curvature $H = \mathrm{tr}(B)/2$ of an Epstein surface, with B the shape operator, can be expressed in terms of the data at infinity as

$$H = \frac{1 - \det(\hat{B})}{1 + \mathrm{tr}(\hat{B}) + \det(\hat{B})} .$$

Epstein surfaces of domains with boundary

In [BP23], Brock and Pallete extended the definition of Epstein surfaces associated to a k-dimensional open domain in the boundary at infinity $\partial_\infty \mathbb{H}^n$ of $\mathbb{H}^n$ (as introduced in Section 1.5) to domains with boundary. Here, we are interested in the dimensions $k = n = 2$. We report the main facts we will need, restricting to the case of round boundary components, which is also the same setting where it is possible do define an associated W-volume, again introduced in [BP23], and which we discuss in the next subsection.

We now briefly outline the construction in [BP23] (see, in particular, Definition 2.1) and fix some notations. Let $\Omega \subseteq \mathbb{CP}^1$ be a domain with boundary $\partial\Omega$ given by a union of k round circles, and let $e^{2\varphi}|dz|^2$ be a $C^{2,\alpha}$ conformal metric on Ω. It is then possible to associate to the interior of Ω its Epstein surface $\Sigma(\Omega, \varphi)$ in $\mathbb{H}^3$, as in Section 1.5, and whose coordinates are furnished by Equation (4.2). Moreover, it can be defined a surface $\Sigma(\partial\Omega, \varphi)$ parameterized by the normal bundle of $\partial\Omega$ and such that at each point $p \in \partial\Omega$ is assigned the unique horocycle in the horosphere centered at p determined by $\varphi(p)$, which is also tangent to the Epstein surface $\Sigma(\Omega, \varphi)$.

We can then construct a compact region $N(\Omega, \varphi)$ in $\mathbb{H}^3$ as follows: consider the union H of hyperplanes with boundary at infinity in $\partial\Omega$, and connect then $\Sigma(\Omega, \varphi)$ to H via $\Sigma(\partial\Omega, \varphi)$, which intersects $\Sigma(\Omega, \varphi)$ tangentially while H orthogonally. This, by gluing, forms a sphere with piece-wise smooth boundary, which can be filled to obtain the compact region $N(\Omega, \varphi)$. Each of the k connected components of the region connecting the Epstein surface $\Sigma(\Omega, \varphi)$ to the hyperplanes in H is called a *caterpillar region*, and we will denote their union by $C = \cup_{i=1}^k C_i$. Let us remark that C can be parameterized by $\partial\Omega \times I$, for I some compact interval (see [BP23, Section 2.2]):

$$C(s, v) = \tag{4.3}$$

$$\left(\gamma(s) + \gamma'(s)\frac{2\varphi'(s)}{e^{2\varphi(s)} + |\varphi'(s)|^2 + v^2} + i\gamma'(s)\frac{2v}{e^{2\varphi(s)} + |\varphi'(s)|^2 + v^2}, \frac{2e^{\varphi(s)}}{e^{2\varphi(s)} + |\varphi'(s)|^2 + v^2}\right),$$

for $\gamma(s)$ a unit velocity parameterization of $\partial\Omega$ with respect to $|dz|^2$.

Whenever we have a metric $g = e^{2\varphi}|dz|^2$, we will often denote the pair (Ω, φ) as (Ω, g).

W-volume for domains with boundary

During this chapter, we will often talk about W-volumes of compact regions, so we recall the definition here.

Definition 4.2.7. Let M be a convex co-compact manifold, and let $N \subseteq M$ be a compact subset of M with smooth (or piece-wise smooth) boundary. The *W-volume of* N is defined as

$$W(N) = \mathrm{Vol}(N) - \frac{1}{2}\int_{\partial N} H da_{\partial N} ,$$

where $\mathrm{Vol}(N)$ is the hyperbolic volume of N induced by M, H is the mean curvature on the boundary ∂N of N, and $da_{\partial N}$ is the induced area form on ∂N.

Note that we can also take $M = \mathbb{H}^3$ in the definition. When the boundary is piece-wise smooth, the integral of the mean curvature term decomposes into the sum of the integrals over the smooth faces plus the contributions from the bending lines b_j of their intersections. In the case the exterior angles θ_j at each b_j are constant, these terms are given by

$$-\frac{1}{4}\sum_{j=1}^{k}\theta_j \ell(b_j) \ , \tag{4.4}$$

with $\ell(b_j)$ the induced length of b_j.

Definition 4.2.8. In the notations above, the *W-volume associated to a domain with k round boundary components* and equipped with a conformal metric (Ω, g), with $g = e^{2\varphi}|dz|^2$, is defined as

$$W(\Omega, g) = W(N(\Omega,\varphi)) = Vol(N(\Omega,\varphi)) - \frac{1}{2}\int_S H da_S \ ,$$

where $S = \partial N(\Omega,\varphi)$, H is the mean curvature of S, and da_S denotes the area form induced on the boundary of $N(\Omega,\varphi)$ from $\mathbb{H}^3$

Since S is piece-wise smooth with constant exterior angles along its co-dimension one faces, and since $H = 0$ on a hyperplane, the integral of the mean curvature in the definition splits into the sum of the integral over the Epstein surface $\Sigma(\Omega,\varphi)$ of the interior of Ω, the ones over the caterpillar regions $C_j \subseteq \Sigma(\partial\Omega,\varphi)$, and the ones over the faces b_j as in Equation (4.4). We refer to the last two types of contributions as those arising from the j-th boundary component of the domain Ω.

Remark 4.2.9. When $\Omega = \Omega(\Gamma)$ is a discontinuity domain of a convex co-compact manifold $M = \mathbb{H}^3/\Gamma$, the W-volume of the quotient in Definition 4.2.8 coincides with the one in Definition 1.6.1, as the boundary term on the Caterpillar regions $C_j \subseteq \Sigma(\Omega,\varphi)$ cancel out when paired by the action of Γ.

It will be essential for our purposes to understand how the W-volume of a domain with round boundary components changes under a conformal change of the metric. To this aim, we state the following version of the *Polyakov formula* (see [AKR23], [BP23]).

Theorem 4.2.10 ([BP23], Theorem 2.1)**.** Let $g = e^{2\varphi}\left|dz\right|^2$ be a smooth conformal metric on a domain $\Omega \subseteq \mathbb{C}$ with round boundary components, and let $u\colon \Omega \to \mathbb{R}$ be another smooth function, then

$$W(\Omega, e^{2u}g) - W(\Omega, g) = -\frac{1}{4}\int_\Omega \left(\left|\nabla_g u\right|^2 + K(g)u\right) da(g) - \frac{1}{2}\int_{\partial\Omega} k(g)u ds(g) \ ,$$

where $\nabla_g(\cdot)$ is the gradient with respect to g, $K(\cdot)$ and $k(\cdot)$ are, respectively, the scalar curvature on Ω and the induced geodesic curvature of the boundary $\partial\Omega$, and $da(\cdot)$ and $ds(\cdot)$ denote the area and length forms.

Requiring φ and u to be of class $C^{2,\alpha}$ is actually sufficient for the equality in the theorem to hold.

Remark 4.2.11. In the original definition of W-volume for domain with round boundary components (see Definition 2.2 in [BP23]), the following additional term appears:

$$-\frac{3}{2}\int_{\partial\Omega\times[0,1]}(1+H)da_C \ ,$$

whose variation is

$$-\frac{3}{4}\int_{\partial\Omega}\partial_n u ds(g) \ ,$$

where $\partial_n(\cdot)$ denotes the derivative with respect to the outer normal to the boundary $\partial\Omega$. In this formulation, the relative Polyakov formula stated as Theorem 2.1 in [BP23] coincides with the variation of *determinant of the Laplacian* for domain with boundary (see [AKR23]). The three terms in Theorem 4.2.10, instead, arise from an application of the Schläfli formula for the variation of the volume of compact regions with piece-wise smooth boundary (see [Sou04], [RS99], and Section 4 in [KS12] in which the Schläfli formula is expressed in terms of the fundamental forms at infinity). Here, what we do really use of the definition of W-volume for domain with round boundary components, is that the caterpillar regions ensure the intersection between the Epstein surface of the domain and the hyperplanes to always have angle $\pi/2$, so that we do not have to account for its variation.

In the next chapters, it will be crucial to have some form of additivity property for the W-volume.

Lemma 4.2.12. Let (Ω, g) be a domain in $\mathbb{C}$ with round boundary components equipped with a conformal metric g. Suppose $\Omega = \Omega_1 \cup \Omega_2$ to be a decomposition of Ω such that $\Omega_1 \cap \Omega_2$ consists of k round circles γ_i. Then

$$W(\Omega, g) = W(\Omega_1, g) + W(\Omega_2, g) \ .$$

Proof. We follow the notations introduced in this section. The key observation is that the caterpillar regions of $\partial\Omega_1$ and $\partial\Omega_2$ on the common components γ_i coincide, with reversed outer normals, as also the hyperplanes with boundary in $\cup\gamma_i$. Then, the terms of the caterpillar regions associated to γ_i simplify as they are opposite. Moreover, the sum of the hyperbolic volumes of $N(\Omega_1, g)$ and $N(\Omega_2, g)$ is equal to the hyperbolic volume of $N(\Omega, g)$, and the same holds for the integrals of the mean curvature on the Epstein surfaces $\Sigma(\Omega_i, g)$ of the interiors (the exterior angles are considered modulo 2π or π, depending on whether they lie in the interior or on the boundary of $N(\Omega, g)$). ∎

4.3 Renormalized volume for pointed manifolds

In the next section, we will define the adapted renormalized volume. In particular, we will show that this admits limit when approaching points on the boundary of the Weil-Petersson completion of the Teichmüller space (see Section 1.2.3) corresponding to compressible multicurves. Additionally, we aim to provide a geometric interpretation of the limit quantity in such a way that the adapted renormalized volume extends continuously to such boundary points. The limit in the Gromov-Hausdorff convergence (see Section 4.2.2) of a sequence of convex co-compact hyperbolic 3-manifolds given by the pinching of a compressible multicurve is a finite union of new *pointed convex co-compact manifolds*, that is, a disjoint union of pairs (M_i, P_i), where M_i is convex co-compact and $P_i \subseteq \partial_\infty M_i$ is a finite set of points (see [SW22, Appendix A.10]). The goal of this section is then to define the renormalized volume for these objects. The key idea in constructing this extension is to consider the Epstein surface associated to the unique hyperbolic metric conformal to $\partial_\infty M_i$ with a cusp singularity at each $p_i \in P_i$. We begin by analyzing the case where P_i has cardinality one, and then observe that the definition naturally extends to the general case.

4.3.1 Renormalized volume behaviour at cusp singularities

In what follows, we analyze the divergence of the W-volume associated to a domain

$$(\Omega \setminus \{p\}, h_p) \subseteq \mathbb{C} \setminus \{p\} \subseteq \mathbb{CP}^1$$

equipped with a conformal hyperbolic metric with a cusp singularity at p (as in Section 4.2.1). The divergence is caused by the fact that at the cusp singularity the Epstein surface associated to $(\Omega \setminus \{p\}, h_p)$ touches the boundary at infinity $\partial_\infty \mathbb{H}^3$, definitively exiting any r-neighborhood of the geodesic of $\mathbb{H}^3$ through p and ∞ (in the upper half-space model). We start by studying a toy model case: the *hyperbolic cusp on a round punctured disk*. In this case, the metric h_p is explicit, and, up to composing by a Möbius transformation, we can assume $p = 0 \in \mathbb{C}$ and $\Omega = \mathbb{D}^*$, where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ with $\mathbb{D}$ the unit disk centered at the origin. This will allow us to explicitly compute the W-volume of any compact sub-domain of $\mathbb{D}^*$ with two concentric round boundary components (as in Definition 4.2.8), and see how this diverges under shrinking one boundary component to 0. Then, by applying Theorem 4.2.10 together with the argument of Section 4.2.1, we show that the result obtained still holds for a general cusp singularity. This furnishes the right renormalization process needed in order to define the W-volume of a pointed convex co-compact hyperbolic 3-manifold, which will be done in the next section.

Epstein surface of the hyperbolic cusp on a round punctured disk

We fix the notation $\mathbb{D} \subseteq \mathbb{C} \subseteq \mathbb{CP}^1$ for the subset of complex numbers $z \in \mathbb{C}$ of norm $\sqrt{z\overline{z}}$ less than one, and $|dz|^2 = dx^2 + dy^2$, with $z = x + iy$, for the Euclidean flat metric on $\mathbb{C}$. We will use the polar coordinates $(\rho, \theta) \in \mathbb{R}_{\geq 0} \times [0, 2\pi]$ such that $z = \rho e^{i\theta}$. Let us consider the metric tensor

$$\hat{I}_0(z) = \frac{1}{\rho^2 \log^2(\rho)} |dz|^2 \tag{4.5}$$

on the unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\} \subseteq \mathbb{C}$ punctured at the origin, with ρ the norm of $z = \rho e^{i\theta}$. This is a hyperbolic cusp: it can be obtained by pushing the hyperbolic metric on the cusp $\mathbb{H}^2/\langle \psi \rangle$, with ψ the parabolic isometry $\psi(z) = z + 2\pi$, via the diffeomorphism $f \colon \mathbb{H}^2/\langle \psi \rangle \to \mathbb{D}^*$ given by $f(z) = e^{iz}$.

The conformal factor (also called the *Liouville field*) associated to $\hat{I}_0$ is:

$$\varphi_0(z) = -\log(|\rho \log(\rho)|) \tag{4.6}$$

$$= \log(2) - \frac{1}{2}\log(z\overline{z}) - \log(|\log(z\overline{z})|) \tag{4.7}$$

for any $z \in \mathbb{D}^*$.

We consider the upper half-space model $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}^+$ for the 3-hyperbolic space, whose boundary at infinity is conformally identified with $\mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$. A point $p \in \mathbb{H}^3$ then has coordinates $p = (z, t)$, with $z = \rho e^{i\theta} \in \mathbb{C}$ its projection to the boundary at infinity, and $t \in \mathbb{R}^+$ its height.

Lemma 4.3.1. Let $\Sigma(\hat{I}_0) \subseteq \mathbb{H}^3$ be the Epstein surface associated to $(\mathbb{D}^*, \hat{I}_0)$. The point $s = s(z) = (r_0(z), \theta_0(z), t_0(z))$ in $\Sigma(\hat{I}_0)$ whose projection through the geodesic ray based at s and perpendicular to $\Sigma(\hat{I}_0)$ is $z = \rho e^{i\theta} \in \mathbb{D}^*$ has the following coordinates:

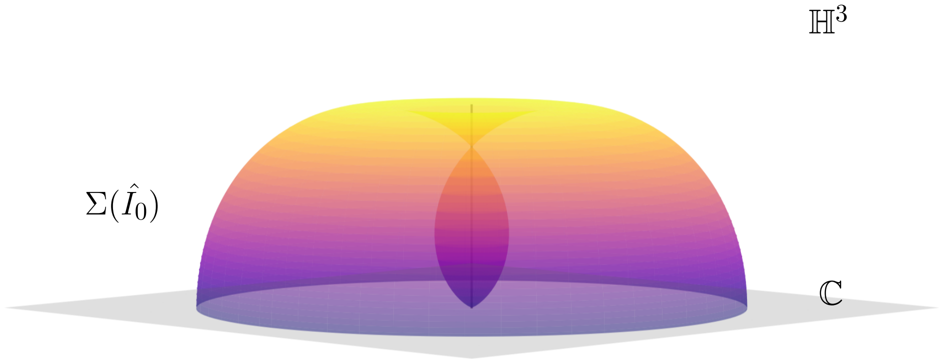

Figure 4.1: Epstein surface of the infinite cusp on the punctured disk with round boundary.

$$\begin{aligned} r_0(z) &= \left| \frac{\rho(\log^2(\rho) - 2)}{A(\rho)} \right| , \\ \theta_0(z) &= \theta + \pi \mathbb{1}_{\rho \leq e^{-\sqrt{2}}} , \\ t_0(z) &= \frac{-2\rho \log(\rho)}{A(\rho)} , \end{aligned}$$

where

$$A(\rho) = \log^2(\rho) + 2\log(\rho) + 2 ,$$

and $\mathbb{1}_{\cdot}$ is the indicator function.

Proof. First, note that since the metric at infinity $\hat{I}_0$ is invariant under rotations at the origin, the surface $\Sigma(\hat{I}_0)$ is invariant under rotations along the vertical geodesic $\{0\} \times \mathbb{R}^+$ in $\mathbb{H}^3$. Therefore $r_0(z) = r_0(\rho)$ and $t_0(z) = t_0(\rho)$, for any $z = \rho e^{i\theta} \in \mathbb{D}^*$. Thanks to Equation (4.2), we can explicitly compute these two coordinates. We use the notation

$$\varphi_{0,\bar{z}} = \frac{\partial \varphi_0}{\partial \bar{z}} ,$$

where we recall that in polar coordiantes

$$\frac{\partial}{\partial \bar{z}} = \frac{e^{i\theta}}{2}\frac{\partial}{\partial \rho} + \frac{ie^{i\theta}}{2\rho}\frac{\partial}{\partial \theta} ,$$

and, since φ_0 depends just on ρ

$$\varphi_{0,\bar{z}} = \frac{e^{i\theta}}{2}\frac{\partial\varphi_0}{\partial\rho} = -\frac{e^{i\theta}}{2}\frac{\log(\rho)+1}{\rho\log(\rho)} = -\frac{e^{i\theta}}{2}(\log(\rho)+1)e^{\varphi_0} \ .$$

Therefore, from Equation (4.2), and using the identity $\nabla\varphi_0 = 2\varphi_{0,\bar{z}}$

$$\begin{aligned} r_0(z) &= \left| z + \frac{4\varphi_{0,\bar{z}}e^{-2\varphi_0}}{1+|2\varphi_{0,\bar{z}}|^2 e^{-2\varphi_0}} \right| = \left| \rho e^{i\theta} - \frac{2e^{i\theta}(\log(\rho)+1)e^{-\varphi_0}}{1+(\log(\rho)+1)^2} \right| \\ &= |e^{i\theta}| \left| \rho - \frac{2(\log(\rho)+1)\rho\log(\rho)}{1+(\log(\rho)+1)^2} \right| = \left| \frac{\rho(2-\log^2(\rho))}{A(\rho)} \right| \ , \end{aligned}$$

and

$$t_0(z) = \frac{2e^{-\varphi_0}}{1+|2\varphi_{0,\bar{z}}|^2 e^{-2\varphi_0}} = \frac{-2\rho\log(\rho)}{1+(\log(\rho)+1)^2} = \frac{-2\rho\log(\rho)}{A(\rho)} \ .$$

Finally, we determine the angular coordinate, again via (4.2), simply by observing that

$$\frac{\rho(2-\log^2(\rho))}{A(\rho)} > 0 \ \text{ iff } \ \rho \geq e^{-\sqrt{2}} \ .$$

■

Remark 4.3.2. Note that in a neighborhood of 0 in $\mathbb{C}$, i.e. for $\rho \sim 0$, the coordinates of the Epstein surface $\Sigma(\hat{I}_0)$ behave like

$$(r_0(z), t_0(z)) \sim \left(\rho, \frac{-2\rho}{\log(\rho)} \right) \ .$$

In particular, the surface $\Sigma(\hat{I}_0)$ exits any r-neighborhood of the vertical geodesic of $\mathbb{H}^3$ based at $0 \in \mathbb{C}$, with tangent going to zero as $-2/\log(\rho)$. Then, it is easy to see that the volume of the region in $\mathbb{H}^3$ between a hyperplane centered at the origin with small radius and $\Sigma(\hat{I}_0)$ is infinite. Moreover, since the angular coordinate satisfies

$$\theta_0(z) = \theta + \pi\mathbb{1}_{\rho \leq e^{-\sqrt{2}}} \ ,$$

for radii $\rho \leq e^{-\sqrt{2}}$, the outer normal is pointing inside the region bounded by the surface, see Figure 4.1.

Definition 4.3.3. Let $\Sigma(\hat{I}_0)$ be the Epstein surface associated to $(\mathbb{D}^*, \hat{I}_0)$ and $0 < \rho_1 < \rho_2 < e^{-\sqrt{2}}$ be two radii. We denote by $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ the subset of $\Sigma(\hat{I}_0)$ obtained by restricting its parameterization to the interior of the annulus

$$\mathbb{D}_{\rho_1}^{\rho_2} = \{ z \in \mathbb{C} \mid \rho_1 \leq |z| \leq \rho_2 \} \ .$$

As a subset of $\mathbb{H}^3$, the surface $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ is obtained by cutting $\Sigma(\hat{I}_0)$ with the two hyperplanes H_i in $\mathbb{H}^3$ with boundary at infinity coinciding with the two circles centered at 0 and radii

$$h_i = \sqrt{r_0(\rho_i)^2 + t_0(\rho_i)^2}$$

respectively for $i = 1, 2$, i.e.

$$\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0) = \Sigma(\hat{I}_0) \cap \{ (z,t) \in \mathbb{H}^3 \mid h_1^2 \leq |z|^2 + t^2 \leq h_2^2 \} \ . \tag{4.8}$$

We will also denote by $N_{h_1}^{h_2}(\hat{I}_0)$ the compact subset of $\mathbb{H}^3$ delimited by $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ and the hyperplanes H_i of radii h_i, for $i = 1, 2$.

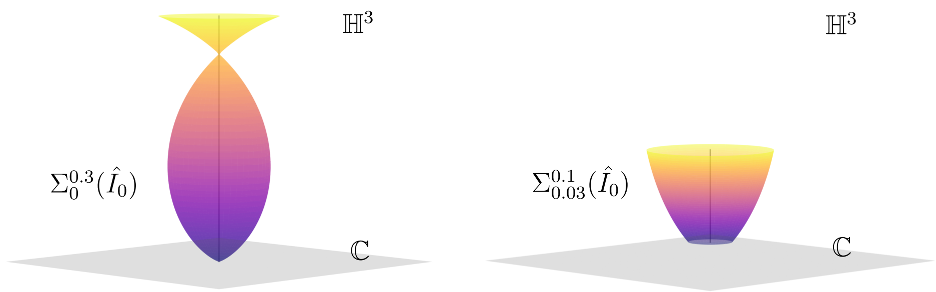

Figure 4.2: The Epstein surfaces associated to the interior of $\mathbb{D}_{\rho_1}^{\rho_2}$ equipped with the hyperbolic metric $\hat{I}_0$, for two different examples of radii ρ_i. Note that $0.3 > e^{-\sqrt{2}} > 0.1$.

In what follows, we first compute the integral of the mean curvature H_0 over the surface $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$, and then the volume of the compact region $N_{h_1}^{h_2}(\hat{I}_0)$. This will be useful in the next section to study the behaviour of the W-volumes associated to the sequence of annuli with round boundary components $(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0)$ as $\rho_1 \to 0$, which approximate a truncation of the infinite cusp $(\mathbb{D}^*, \hat{I}_0)$.

First, we recall how to express the mean curvature $H_0 = \operatorname{tr}(I_0^{-1} I\!I_0)$ in terms of the first and second fundamental forms at infinity $\hat{I}_0$ and $\hat{I\!I}_0$, with I_0 and $I\!I_0$, respectively, the push-forward via the normal projection of the first and second fundamental forms on $\Sigma(\hat{I}_0)$ induced by $\mathbb{H}^3$, (see Section 1.5.3 and Lemma 4.2.6) :

$$H_0 = \frac{1 - \det(\hat{B}_0)}{1 + \operatorname{tr}(\hat{B}_0) + \det(\hat{B}_0)} ,$$

where $\hat{B}_0 = (\hat{I})_0^{-1} \hat{I\!I}_0$.

Lemma 4.3.4. In the notations above, the mean curvature H_0 of the surface $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ satisfies

$$\frac{1}{2} \int_{\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)} H_0 da_{I_0} = \frac{\pi}{12} \left[\log^3(\rho)\right]_{\rho_1}^{\rho_2} := \frac{\pi}{12} \log^3\left(\frac{\rho_2}{\rho_1}\right) ,$$

where da_{I_0} denotes the area form associated to the first fundamental from I_0 induced by $\mathbb{H}^3$ on $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$.

Proof. We start by expressing the area form associated to I_0 in terms of that of $\hat{I}_0$. The pushed-forward through the normal projection of the first fundamental form on the Epstein surface is related to the data at infinity by the identity (see Section 1.5.3)

$$I_0 = \frac{1}{4} \hat{I}_0((\operatorname{id} + \hat{B}_0)(\cdot), (\operatorname{id} + \hat{B}_0)(\cdot)) ,$$

so that

$$\sqrt{\det(I_0)} = \frac{1}{4}\sqrt{\det(\hat{I}_0)}\det(\mathrm{id}+\hat{B}_0) \ ,$$

and therefore

$$da_{I_0} = \frac{1}{4}\det(\mathrm{id}+\hat{B}_0)da_{\hat{I}_0} \ .$$

Since also

$$\det(\mathrm{id}+\hat{B}_0) = 1 + \mathrm{tr}(\hat{B}_0) + \det(\hat{B}_0) \ ,$$

then, from the previous two identities and Lemma 4.2.6

$$\frac{1}{2}\int_{\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)} H_0 da_{I_0} = \frac{1}{8}\int_{\mathbb{D}_{\rho_1}^{\rho_2}} (1-\det(\hat{B}_0))da_{\hat{I}_0} \ . \tag{4.9}$$

It then remains to compute the determinant of the endomorphism $\hat{B}_0$. To this end, we write the second fundamental form at infinity in terms of the derivatives of the Liouville field φ_0 (see Lemma 4.2.4):

$$\hat{I\!I}_0 = 2q_0 dz^2 + 2\overline{q_0}d\overline{z}^2 + 4\varphi_{0,z\overline{z}}|dz|^2$$

with

$$q_0 = \varphi_{0,zz} - (\varphi_{0,z})^2 \ .$$

Since now (4.7)

$$\varphi_0(z) = \log(2) - \frac{1}{2}\log(z\overline{z}) - \log(\log(z\overline{z}))$$

we can compute

$$\begin{aligned}
\varphi_{0,z}(z) &= -\frac{1}{2z}\left(1 + \frac{1}{\log(|z|)}\right) \ , \\
\varphi_{0,z\overline{z}}(z) &= \frac{1}{4|z|^2\log^2(|z|)} \ , \\
\varphi_{0,zz}(z) &= \frac{1}{2z^2}\left(1 + \frac{1}{\log(|z|)} + \frac{1}{2\log^2(|z|)}\right) \ ,
\end{aligned}$$

then also

$$q_0(z) = \frac{1}{4z^2} \ .$$

We observe that

$$\hat{I}_0 = \frac{1}{|z|^2\log^2(|z|)}|dz|^2 = 4\varphi_{0,z\overline{z}}|dz|^2 \ ,$$

moreover, by Remark 4.2.5

$$\hat{I\!I}_0 = (4\varphi_{0,z\overline{z}} + 4\mathrm{Re}(q_0))dx^2 + (4\varphi_{0,z\overline{z}} - 4\mathrm{Re}(q_0))dy^2 - 4\,\mathrm{Im}(q_0)dxdy$$

therefore

$$\hat{B}_0 = (\hat{I}_0)^{-1}\hat{I\!I}_0 = \left(1 + \frac{\mathrm{Re}(q_0)}{\varphi_{0,z\overline{z}}}\right)dx^2 + \left(1 - \frac{\mathrm{Re}(q_0)}{\varphi_{0,z\overline{z}}}\right)dy^2 - \frac{\mathrm{Im}(q_0)}{\varphi_{0,z\overline{z}}}dxdy$$

and so

$$\det(\hat{B}_0) = 1 - \frac{1}{(\varphi_{0,z\overline{z}})^2}\left(\mathrm{Re}(q_0)^2 + \mathrm{Im}(q_0)^2\right) = 1 - \frac{|q_0|^2}{(\varphi_{0,z\overline{z}})^2} = 1 - \log^4(|z|) \ .$$

Finally, thanks to Equation (4.9)

$$\begin{aligned}
\frac{1}{2}\int_{\Sigma^{\rho_2}_{\rho_1}(\hat{I}_0)} H_0 da_{I_0} &= \frac{1}{8}\int_{\mathbb{D}^{\rho_2}_{\rho_1}} (1-\det(\hat{B}_0))da_{\hat{I}_0} \\
&= \frac{\pi}{4}\int_{\rho_1}^{\rho_2} \log^4(\rho)\frac{1}{\rho^2\log^2(\rho)}\rho d\rho \\
&= \frac{\pi}{4}\int_{\rho_1}^{\rho_2} \frac{\log^2(\rho)}{\rho} d\rho \\
&= \frac{\pi}{4}\left[\frac{\log^3(\rho)}{3}\right]_{\rho_1}^{\rho_2} = \frac{\pi}{12}\log^3\left(\frac{\rho_2}{\rho_1}\right) .
\end{aligned}$$

■

In the next lemma we compute the hyperbolic volume of the compact convex domain $N^{h_2}_{h_1}(\hat{I}_0)$ defined in 4.3.3, where we remember that $h_i = \sqrt{r_0(\rho_i)^2 + t_0(\rho_i)^2}$ with $0 < \rho_1 < \rho_2 \leq e^{-\sqrt{2}}$. In particular, we focus on how this diverges for $\rho_1 \to 0$. The proof just consist in an elementary explicit computation and can be skipped.

Lemma 4.3.5. The hyperbolic volume of the compact domain $N^{h_2}_{h_1}(\hat{I}_0)$ is given by

$$\mathrm{Vol}(N^{h_2}_{h_1}(\hat{I}_0)) = \frac{\pi}{12}\log^3\left(\frac{\rho_2}{\rho_1}\right) - \frac{\pi}{2}\log\left(\frac{\rho_2}{\rho_1}\right) + c(\rho_2) - c(\rho_1) .$$

with $c\colon (0,1) \to \mathbb{R}$ a smooth function, bounded on any sub-interval $(0,a)$ with $a<1$, and such that at 0

$$c(\rho) = O\left(\frac{1}{\log(\rho)}\right) ,$$

where $O(x)$ stands for a real function such that the limit $\lim_{x\to 0} O(x)/x$ exists and it is finite.

Proof. First, recall that we use the notation $(r_0(\rho), \theta+\pi, t_0(\rho))$ for the point in $\Sigma(\hat{I}_0)$ associated through the normal projection to the point at infinity $z = \rho e^{i\theta} \in \mathbb{D}^* \subseteq \partial\overline{\mathbb{H}^3}$ (see Section 4.2.3). We compute the volume of $N^{h_2}_{h_1}(\hat{I}_0)$ as the sum of the signed volumes of the following three regions:

$$\begin{aligned}
O(\rho_1,\rho_2) &= \{(z(\rho), t_0(\rho)) \in \mathbb{H}^3 \mid \rho_1 \leq \rho \leq \rho_2,\ 0 \leq |z(\rho)| \leq r_0(\rho)\} \\
S(\rho_i) &= \{(z,t) \in \mathbb{H}^3 \mid 0 \leq |z| + t^2 \leq h_i^2,\ t_0(\rho_i) \leq t \leq h_i\}
\end{aligned}$$

for $i = 1,2$, where

$$h_i = \sqrt{t_0(\rho_i)^2 + r_0(\rho_i)^2}$$

is the euclidean ray of the hyperplane whose intersection with $\Sigma(\hat{I}_0)$ is the circle $\{(z, t_0(\rho_i)) \in \mathbb{H}^3 \mid |z| = r_0(\rho_i)\}$. Then, the hyperbolic volumes satisfy

$$\mathrm{Vol}(N^{h_2}_{h_1}(\hat{I}_0)) = \mathrm{Vol}(O(\rho_1,\rho_2)) - \mathrm{Vol}(S(\rho_1)) + \mathrm{Vol}(S(\rho_2)) .$$

We now explicitly calculate the three terms. The volume form of $\mathbb{H}^3$ in the cylindrical coordinates $(\rho,\theta,t) \in \mathbb{C}\times\mathbb{R}^+$ is

$$d\mathrm{Vol}_{\mathbb{H}^3} = \frac{1}{t^3}\rho d\rho \wedge d\theta \wedge dt ,$$

then, thanks to Lemma 4.3.1:

$$\begin{aligned}
\mathrm{Vol}(O(\rho_1,\rho_2)) &= 2\pi \int_{t_0(\rho_1)}^{t_0(\rho_2)} \int_0^{r_0(\rho)} \frac{1}{t^3}\rho d\rho dt \\
&= 2\pi \int_{\rho_1}^{\rho_2} \frac{r_0(\rho)^2}{2t_0(\rho)^3} t_0^{'}(\rho) d\rho \\
&= \pi \int_{\rho_1}^{\rho_2} \frac{A(\rho)^3}{8\rho^3\log^3(\rho)} \frac{\rho^2(2-\log^2(\rho))^2}{A(\rho)^2} \frac{2(\log(\rho)+1)(\log^2(\rho)+2)}{A(\rho)^2} d\rho \\
&= \frac{\pi}{4} \int_{\rho_1}^{\rho_2} \frac{(2-\log^2(\rho))^2(\log(\rho)+1)(\log^2(\rho)+2)}{\rho\log^3(\rho)(\log^2(\rho)+2\log(\rho)+2)} d\rho \\
&= \frac{\pi}{4}\left[\frac{\log^3(\rho)}{3} - \frac{\log^2(\rho)}{2} - 2\log(\rho) - 4\log\left(\frac{\log(\rho)}{\log^2(\rho)+2\log(\rho)+2}\right) - \frac{2}{\log^2(\rho)}\right]_{\rho_1}^{\rho_2} \\
&= \frac{\pi}{4}\left[\frac{\log^3(\rho)}{3} - \frac{\log^2(\rho)}{2} - 2\log(\rho) + 4\log(\log(\rho))\right]_{\rho_1}^{\rho_2} \\
&+ \frac{\pi}{4}\left[4\log\left(1+\frac{2}{\log(\rho)}+\frac{2}{\log^2(\rho)}\right) - \frac{2}{\log^2(\rho)}\right]_{\rho_1}^{\rho_2} \quad ;
\end{aligned}$$

$$\begin{aligned}
\mathrm{Vol}(S(\rho_i)) &= 2\pi \int_{t_0(\rho_i)}^{h_i} \int_0^{\sqrt{h_i^2-t^2}} \frac{1}{t^3}\rho d\rho dt \\
&= \pi \int_{t_0(\rho_i)}^{h_i} \frac{h_i^2-t^2}{t^3} dt \\
&= \pi \left[-\frac{h_i^2}{2t^2} - \log(t)\right]_{t_0(\rho_i)}^{h_i} \\
&= -\frac{\pi}{2} + \frac{\pi}{2}\left(\frac{h_i}{t_0(\rho_i)}\right)^2 - \pi\log\left(\frac{h_i}{t_0(\rho_i)}\right) \\
&= -\frac{\pi}{2} + \frac{\pi}{2}\left(1+\left(\frac{r_0(\rho_i)}{t_0(\rho_i)}\right)^2\right) - \frac{\pi}{2}\log\left(1+\left(\frac{r_0(\rho_i)}{t_0(\rho_i)}\right)^2\right) \\
&= \frac{\pi(\log^2(\rho_i)-2)^2}{8\log^2(\rho_i)} - \frac{\pi}{2}\log\left(\frac{\log^4(\rho_i)+4}{4\log^2(\rho_i)}\right) \\
&= \frac{\pi}{8}\log^2(\rho_i) - \frac{\pi}{2} + \frac{\pi}{2\log^2(\rho_i)} - \frac{\pi}{2}\log\left(\log^2(\rho_i)\left(1+\frac{4}{\log^4(\rho_i)}\right)\right) + \frac{\pi}{2}\log(4) \\
&= \frac{\pi}{8}\log^2(\rho_i) - \pi\log(\log(\rho_i)) + \frac{\pi}{2}\log(4) - \frac{\pi}{2} + \frac{\pi}{2\log^2(\rho_i)} - \frac{\pi}{2}\log\left(1+\frac{4}{\log^4(\rho_i)}\right) \quad ;
\end{aligned}$$

we then note that

$$\mathrm{Vol}(S_2) - \mathrm{Vol}(S_1) = \left[\frac{\pi}{8}\log^2(\rho) - \pi\log(\log(\rho)) + \frac{\pi}{2\log^2(\rho)} - \frac{\pi}{2}\log\left(1+\frac{4}{\log^4(\rho)}\right)\right]_{\rho_1}^{\rho_2}$$

and therefore

$$\mathrm{Vol}(N_{h_1}^{h_2}(\hat{I}_0)) = \pi\left[\frac{\log^3(\rho)}{12} - \frac{\log(\rho)}{2} + c(\rho)\right]_{\rho_1}^{\rho_2}$$

with

$$c(\rho) = \pi \log\left(1 + \frac{2}{\log(\rho)} + \frac{2}{\log^2(\rho)}\right) - \frac{\pi}{2}\log\left(1 + \frac{4}{\log^4(\rho)}\right) \ , \tag{4.10}$$

which concludes the proof. ∎

W-volume of a truncated cusp on a round annulus

We now consider the truncated cusp of the previous section as a domain with two round boundary components equipped with a hyperbolic metric, and compute its W-volume as in Definition 4.2.8. Let us then consider the annulus

$$\mathbb{D}_{\rho_1}^{\rho_2} = \{z \in \mathbb{C} \mid \rho_1 \leq |z| \leq \rho_2\} \tag{4.11}$$

equipped with the restriction of the conformal metric $\hat{I}_0$, and also the associated Epstein surface $S_{\rho_1}^{\rho_2}(\hat{I}_0)$ for domain with boundary defined as in Section 4.2.3:

$$S_{\rho_1}^{\rho_2}(\hat{I}_0) = \partial N\left(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0\right) \ ,$$

where φ_0 is the Liouville field of $\hat{I}_0$, defined by Equation (4.6). Note that $S_{\rho_1}^{\rho_2}(\hat{I}_0)$ is obtained by connecting $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ (as in Definition 4.3.3) through the caterpillar region to the two hyperplanes whose boundary at infinity coincides with a component of $\partial\mathbb{D}_{\rho_1}^{\rho_2}$, and then capping it with these.

We use the symbol $\sim$ to denote two equivalent functions at 0, i.e.

$$f(x) \sim g(x) \quad \Longleftrightarrow \quad \lim_{x\to 0}\frac{f(x)}{g(x)} = 1 \ .$$

In the following statement, even though it diverges, we separate and highlight the mean curvature terms arising from the boundary of the domain, as this cancels out anytime we sum W-volumes of a decomposition of a domain along round circles, as we will do later.

Proposition 4.3.6. The W-volume associated to $(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0)$ has the following behaviour as $\rho_1 \sim 0$:

$$W(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0) = \frac{\pi}{2}\log(\rho_1) - b(\rho_1) + C'(\rho_2)$$

where $b(\cdot)$ represents the mean curvature boundary term, it is a continuous function, and at $\rho_1 \sim 0$ satisfies

$$b(\rho_1) = -\frac{\pi^2}{8}\log(\rho_1) - \frac{\pi}{2}\log(\rho_1) + O(1/\log\rho_1) \ ,$$

and $C'(\rho_2)$ is a constant depending just on ρ_2. In particular, as $\rho_2 \sim 0$

$$C'(\rho_2) = -\frac{\pi}{2}\log(\rho_2) + b(\rho_2) \ .$$

Proof. In what follows, to streamline the notation, we will sometimes omit the dependence on $\hat{I}_0$, writing, for example, simply $S_{\rho_1}^{\rho_2}$. Also recall that we denote by $\Sigma(\cdot)$ the Epstein surface parameterized by the interior of the domain, while by $S(\cdot)$ the sphere obtained as the union of $\Sigma(\cdot)$, the caterpillar regions, and the hyperplanes whose boundaries match those of the domain. The W-volume associated to a domain with round boundary components is defined as (see Section 4.2.3, Definition 4.2.8):

$$W(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0) = Vol(N\left(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0\right)) - \frac{1}{2}\int_{S_{\rho_1}^{\rho_2}} H da_{S_{\rho_1}^{\rho_2}}$$

where $N(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0)$ is the ball filling $S_{\rho_1}^{\rho_2}$, so that $S_{\rho_1}^{\rho_2} = \partial N(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0)$. First, we note that

$$Vol(N(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0)) = Vol(N_{\rho_1}^{\rho_2}) + Vol(P_2) - Vol(P_1)$$

where $N_{\rho_1}^{\rho_2}$ is the compact delimited by the Epstein surface $\Sigma(\hat{I}_0)$ and the hyperplanes H_i of euclidean radii ρ_i, and P_i, for $i = 1, 2$, is the region of $\mathbb{H}^3$ detected by C_i, the hyperplane H_i, and $\Sigma(\hat{I}_0)$, such that $N(\mathbb{D}_{\rho_1}^{\rho_2}, \varphi_0) = (N_{\rho_1}^{\rho_2} \cup P_2) \setminus P_1$. By Lemma 4.3.5:

$$\mathrm{Vol}(N_{\rho_1}^{\rho_2}) = \frac{\pi}{12}\log^3\left(\frac{x_2}{x_1}\right) - \frac{\pi}{2}\log\left(\frac{x_2}{x_1}\right) + c(x_2) - c(x_1) \ .$$

with $x_i = x_i(\rho_i)$ such that $\rho_i^2 = h^2(x_i) = r_0^2(x_i) + t_0^2(x_i)$, and $c(x)$ a function such that at 0 is $O(1/\log x)$. Since also, by Lemma 4.3.1

$$r_0^2(x) + t_0^2(x) = \frac{x^2(4+\log^4 x)}{(\log^2 x + 2\log x + 2)^2} \ ,$$

then

$$\rho_1 = x_1(1 + O(1/\log x_1))$$

and therefore

$$\mathrm{Vol}(N_{\rho_1}^{\rho_2}) = \frac{\pi}{12}\log^3\left(\frac{x_2}{\rho_1}\right) - \frac{\pi}{2}\log\left(\frac{x_2}{\rho_1}\right) + c(x_2) - O(1/\log\rho_1) \ . \tag{4.12}$$

It is also easy to show (for example, looking at the area of the smallest rectangle containing a vertical section of P_1, seen as a revolution solid) that

$$\lim_{\rho_1\to 0} Vol(P_1) = 0 \ ,$$

while $Vol(P_2)$ is a finite number depending just on ρ_2, which again limits to zero at $\rho_2 \sim 0$. We also notice that the integral of the mean curvature term splits in the following sum:

$$\frac{1}{2}\int_{S_{\rho_1}^{\rho_2}} H da_{S_{\rho_1}^{\rho_2}} = \frac{1}{2}\int_{\Sigma_{\rho_1}^{\rho_2}} H da_{I_0} + \frac{1}{2}\int_{C_1} H da_{C_1} - \frac{1}{2}\int_{C_2} H da_{C_2} + \frac{\pi}{8}\ell(\partial_1 C_1) - \frac{\pi}{8}\ell(\partial_1 C_2) \ ,$$

where $\Sigma_{\rho_1}^{\rho_2}$ denotes the Epstein surface $\Sigma(int(\mathbb{D}_{\rho_1}^{\rho_2}), \hat{I}_0)$, and C_i, for $i = 1, 2$, is the caterpillar region, which meets the hyperplane H_i with an exterior angle of $\pi/2$, and whose intersection with H_i has hyperbolic length $\ell(\partial_1 C_i)$. By Lemma 4.3.4, arguing as for the volume, we already know that

$$\frac{1}{2}\int_{\Sigma_{\rho_1}^{\rho_2}} H da_{I_0} = \frac{\pi}{12}\log^3\left(\frac{x_2}{\rho_1}\right) + O(1/\log\rho_1) \ . \tag{4.13}$$

What remains is to handle the mean curvature terms coming from the boundary $\partial\mathbb{D}_{\rho_1}^{\rho_2}$. We start by studying the integral of the mean curvature on the caterpillar regions. Thanks to the explicit parameterization of the Epstein surface associated to $\partial\mathbb{D}_{\rho_1}^{\rho_2}$ furnished by Equation (4.3), the caterpillar region C_i is parameterized by the subset of the normal bundle of $\partial\mathbb{D}_{\rho_1}^{\rho_2}$ obtained by imposing $s \in [0, 2\pi\rho_i]$ and $v \in \left[\frac{\log(\rho_i)+1}{\rho_i \log(\rho_i)}, \frac{1}{\rho_i}\right]$. The extremes for the parameter v are computed by explicitly solving for the values where the caterpillar region intersects the hyperplane H_i and the Epstein surface $\Sigma(\hat{I}_0)$, respectively. Recalling that $\hat{I}_0 = e^{2\varphi_0}|dz|^2$ and that $k_i = 1/\rho_i$ is the euclidean curvature of the boundary circles of $\partial\mathbb{D}_{\rho_1}^{\rho_2}$, we have [BP23, Section 2.2]:

$$H da_{C_i} = \frac{1}{2} e^{-2\varphi}(2k_i v - v^2) dv ds \ .$$

Therefore,

$$\begin{aligned}
-\frac{1}{2}\int_{C_i} H da_{C_i} =& \int_0^{2\pi\rho_i} \int_{1/\rho_i}^{\frac{log(\rho_i)+1}{\rho_i log(\rho_i)}} \frac{1}{4}\rho_i^2 \log^2(\rho_i)\left(\frac{2v}{\rho_i} - v^2\right) dv ds \\
=& \frac{\pi}{2}\rho_i^3 \log^2(\rho_i)\left[\frac{v^2}{\rho_i} - \frac{v^3}{3}\right]_{1/\rho_i}^{\frac{log(\rho_i)+1}{\rho_i log(\rho_i)}} \\
=& \frac{\pi}{2}\log(\rho_i) - \frac{\pi}{6\log(\rho_i)} \ .
\end{aligned}$$

We remark that, in the W-volume, the term relative to the second boundary component has inverted sign, as the caterpillar region has the opposite induced orientation. It only remains to take care of the mean curvature on the co-dimension one faces of $S_{\rho_1}^{\rho_2}$:

$$\pm\frac{1}{4}\alpha_i \ell(\partial_1 C_i) \ ,$$

where $\alpha_i = \pi/2$ is the exterior angle between the lower boundary $\partial_1 C_i$ of the caterpillar region and the hyperplane H_i, and $\ell(\partial_1 C_i)$ denotes its induced hyperbolic length. Again thanks to Equation (4.3) (see also [BP23, Section 2.2]), and noting that φ_0 is constant on each boundary component of $\partial\mathbb{D}_{\rho_1}^{\rho_2}$, it is possible to find the explicit radial and height coordinates in $\mathbb{H}^3$ of the Epstein surface associated to the boundary:

$$\begin{aligned}
r_b(s,v) &= -\rho_i + \frac{2v}{e^{2\varphi_0} + v^2} \ , \\
t_b(s,v) &= \frac{2e^{\varphi_0}}{e^{2\varphi_0} + v^2} \ ,
\end{aligned}$$

where $e^{2\varphi_0(s)} = 1/(\rho_i^2 \log^2(\rho_i))$. It is now easy to calculate $\ell(\partial_1 C_i)$:

$$\ell(\partial_1 C_i) = \frac{2\pi r_b(v_i)}{t_b(v_i)} = -\pi\log(\rho_i) + O(1/\log(\rho_i)) \ ,$$

where the last inequality holds since $v_i = 1/\rho_i$. The statement now follows by defining $b(\rho_i)$ as

$$b(\rho_i) = -\frac{\pi^2}{8}\log(\rho_i) - \frac{\pi}{2}\log(\rho_i) + O(1/\log(\rho_i)) \ ,$$

and summing up with (4.12) and (4.13). ∎

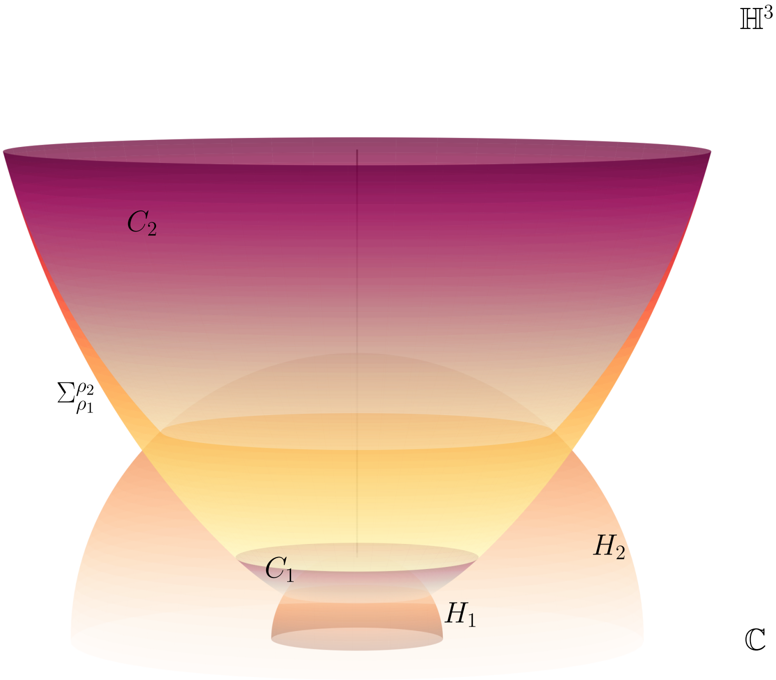

Figure 4.3: The Epstein surface associated to the domain with boundary $(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0)$: in yellow the Epstein surface $\Sigma_{\rho_1}^{\rho_2}(\hat{I}_0)$ of the interior of the annulus, in purple the two caterpillar regions C_i, and in light brown the two hyperplanes H_i of euclidean radii ρ_i.

Remark 4.3.7. The area form of the caterpillar region C_1, and its product with the mean curvature of C_1 diverge with the same rate, as (see [BP23])

$$H da_{C_i} = \frac{1}{2}e^{-2\varphi}(-2k_i v + v^2) dv \wedge ds$$

and

$$da_{C_i} = \left(\frac{1}{2} + \frac{1}{2}e^{-2\varphi}(2k_i v - v^2)\right) dv \wedge ds \ .$$

Thanks to this, the additional term appearing in Definition 2.2 in [BP23] of W-volume coming from the innermost boundary of $(\mathbb{D}_{\rho_1}^{\rho_2}, \hat{I}_0)$ vanishes in the limit $\rho_1 \to 0$ as:

$$\frac{3}{2}\int_{C_1}(1+H)da_{C_1} = 2\pi\rho_1\frac{3}{4}\int_{1/\rho_1}^{\frac{\log(\rho_1)+1}{\log(\rho_1)}} dv = \frac{3\pi}{2\rho_1\log(\rho_1)} \to 0 \ .$$

Let us fix some notation. The hyperbolic cusp $[0, 2\pi] \times \mathbb{R}^+ \subseteq \mathbb{H}^2$ is foliated by the curves given by the intersection between the horocycles at height $t \in \mathbb{R}^+$ centered at infinity and the cusp itself. This corresponds to foliate $(\mathbb{D}^*, \hat{I}_0)$ by the circumferences centered at the origin of $\mathbb{C}$ and Euclidean radii $\rho = e^{-t}$. Equivalently, we are foliating a cusp in simple closed curves of hyperbolic length $\varepsilon = 2\pi/|\log(\rho)| = 2\pi/t$. Then, the domain $\mathbb{D}(\varepsilon) \subseteq \mathbb{D}^*$ defined as

$$\mathbb{D}(\varepsilon) = \{\rho e^{i\theta} \in \mathbb{C} | 0 < \rho \leq e^{-2\pi/\varepsilon}\} \tag{4.14}$$

equipped with the restriction of the metric tensor $\hat{I}_0$ is a *truncated cusp* with boundary of length ε. We also define the function

$$\rho(\varepsilon) = e^{-2\pi/\varepsilon} \ . \tag{4.15}$$

Definition 4.3.8. The *renormalized volume of a truncated cusp on a punctured disk with round boundary* of radius $\overline{\rho}$ is defined as

$$V_R(\mathbb{D}(\overline{\varepsilon})) = \lim_{\rho \to 0} \left(W(\mathbb{D}^{\overline{\rho}}_{\rho}, \hat{I}_0) - \frac{\pi}{2} \log(\rho) + b(\rho) \right) \ ,$$

with $b(\rho) = -\pi^2 \log(\rho)/8 - \pi \log(\rho)/2 + O(1/\log \rho)$ the boundary term as in Proposition 4.3.6.

Proposition 4.3.9. For any $\overline{\varepsilon} > 0$, the renormalized volume of the truncated cusp with round boundary $V_R(\mathbb{D}(\overline{\varepsilon}))$ exists and it is finite.

Proof. This is a direct corollary of Proposition 4.3.6. ∎

Remark 4.3.10. Using the identity of Equation (4.15), the renormalized volume of Definition 4.3.8 can be expressed as a limit in the hyperbolic length ε of the horocycles as

$$V_R(\mathbb{D}(\overline{\varepsilon})) = \lim_{\varepsilon \to 0} \left(W\left(\mathbb{D}^{\rho(\overline{\varepsilon})}_{\rho(\varepsilon)}, \hat{I}_0\right) + \frac{\pi^2}{\varepsilon} + b(\rho(\varepsilon)) \right) \ ,$$

where

$$b(\rho(\varepsilon)) = \frac{\pi^3}{4\varepsilon} + \frac{\pi^2}{\varepsilon} + O(\varepsilon) \ .$$

General case

In the previous section, we analyzed the divergence of the W-volume of a cusp on a punctured disk with round boundary, assuming the conformal metric to be equal to $\hat{I}_0$. We turn here to the general case of a cusp singularity at infinity, proving that, if we restrict to a small enough neighborhood, the same behaviour holds. Up to Möbius transformation, we can assume the cusp singularity being at $p = 0$.

Proposition 4.3.11. Let $\Omega \subseteq \mathbb{C}$ be a hyperbolic complex projective domain with $0 \in \Omega$, and let h_0 be the unique conformal hyperbolic metric with a cusp singularity at 0. Then, for $\overline{\rho} > 0$ small enough, the limit

$$\lim_{\rho \to 0} \left(W(\mathbb{D}^{\overline{\rho}}_{\rho}, h_0) - \frac{\pi}{2} \log(\rho) + b(\rho) \right) \ ,$$

with $b(\rho)$ the boundary term as in Proposition 4.3.6, exists and it is finite.

Proof. As seen in Section 4.2.1, in a small enough neighborhood $U \subseteq \Omega$ of 0, there exists a (non projective) change of coordinate $w = ze^{\psi(z)}$, with ψ a holomorphic function such that $\psi(0) = 0$, in which the metric h_0 coincides with $\hat{I}_0$ (see Theorem 4.2.1, the discussion below, and Equation (4.1)). Then, in the projective coordinate z of Ω, in U, we can write h_0 as

$$\begin{aligned} h_0(z) &= \frac{|e^{\psi(z)} + ze^{\psi(z)}\psi'(z)|^2}{|z|^2|e^{2\psi(z)}|\log^2(|ze^{\psi(z)}|)}|dz|^2 \\ &= \frac{|1 + z\psi'(z)|^2}{(1 + Re(\psi(z))/(\log(|z|))^2}\hat{I}_0(z) \ . \end{aligned}$$

Let us define $\nu\colon\ : \Omega \to \mathbb{R}$ as

$$\nu(z) = \log\left(\frac{|1 + z\psi'(z)|}{1 + Re(\psi(z))/(\log(|z|)}\right) \ , \tag{4.16}$$

so that

$$h_0 = e^{2\nu}\hat{I}_0 \ .$$

By applying Polyakov formula to estimate the difference of the W-volumes $W(\mathbb{D}^{\overline{\rho}}_{\rho}, e^{2\nu}\hat{I}_0)$ and $W(\mathbb{D}^{\overline{\rho}}_{\rho}, \hat{I}_0)$ (see Theorem 4.2.10), since $\nu(z) = O(|z|)$ and $\nu'(z) = \varphi'(0) + O(|z|)$, the statement follows from Proposition 4.3.9. ∎

4.3.2 Renormalized volume of pointed hyperbolic 3-manifolds

Building on the previous section, we are going to define the renormalized volume of the triple (M, g, p), where $(M, g) = \mathbb{H}^3/\Gamma$ is a convex co-compact hyperbolic 3-manifold, and $p \in \Omega(\Gamma)/\Gamma = \partial_\infty M$ a marked point at infinity. We will call the triple (M, g, p) a *pointed convex co-compact manifold*. Let h be the hyperbolic metric in the conformal class of the boundary at infinity $[\partial_\infty M]$, and let h_p be the unique conformal hyperbolic metric on $\partial\overline{M} \setminus \{p\}$ with a cusp singularity at p (see Section 4.2.1). With some abuse of notation, we denote again by p, h and h_p their Γ-invariant lifts respectively to $\Omega(\Gamma)$ and $\Omega(\Gamma) \setminus \Gamma \cdot p$. Up to Mobius transformation, we can assume that $p = 0 \in \mathbb{C} \subseteq \mathbb{CP}^1 = \partial_\infty\mathbb{H}^3$, so that the restriction of h_0 to a small enough neighborhood of 0 coincides with $\hat{I}_0$ (see (4.5)), up to lower order terms described by Equation (4.16). Given a sufficiently small $\varepsilon > 0$, we can cut M with the hyperplane $H(\varepsilon)$ centered at 0 and such that the intersection between its boundary at infinity and the domain of discontinuity $\Omega(\Gamma)$ has length ε with respect to h_0. We sometimes denote by $D(\varepsilon)$, with an abuse of notation, both the ball with boundary $\partial H(\varepsilon)$ containing 0 in $\Omega(\Gamma)$ and in $\partial_\infty M$. This disk coincides with $\mathbb{D}(\varepsilon')$ (as in (4.14)) where

$$\varepsilon = \int_{\partial\mathbb{D}(\varepsilon')} e^{\nu} ds(\hat{I}_0) \ ,$$

with ν defined by Equation (4.16). We fix the following notations:

$$\Omega(\Gamma, \varepsilon) := \Omega(\Gamma) \setminus (\Gamma \cdot D(\varepsilon)) \ ,$$

and analogously

$$\partial_\infty M(\varepsilon) := \Omega(\Gamma, \varepsilon)/\Gamma \ .$$

We then consider the Γ-invariant Epstein surface $\widetilde{\Sigma}(h_0)$ in $\mathbb{H}^3$ associated to $(\Omega(\Gamma), h_0)$, and the Γ-invariant Epstein surface $\widetilde{\Sigma}(h_0, \varepsilon)$ of $(\Omega(\Gamma, \varepsilon), h_0)$ obtained by restricting the parameterization of $\widetilde{\Sigma}(h_0)$ to the open domain $\Omega(\Gamma, \varepsilon)$, and connecting it to $H(\varepsilon)$ through the caterpillar region (as in 4.2.3). We also denote by $\Sigma(h_0)$ and $\Sigma(h_0, \varepsilon)$ the corresponding quotient Epstein surfaces in M. Finally, we define $W(\partial_\infty M(\varepsilon), h_0)$ to be the W-volume of the compact region in M bounded by $\Sigma(h_0, \varepsilon)$ and $H(\varepsilon)$.

Definition 4.3.12. Let (M, g, p) be a convex co-compact hyperbolic 3-manifold pointed at $p \in \partial\overline{M}$. Let also $\Gamma < PSL(2, \mathbb{C})$ be a representation of $\pi_1(M)$ such that M is isometric to $\mathbb{H}^3/\Gamma$ and $p = 0 \in \Omega(\Gamma)$, and let h_0 be the unique conformal hyperbolic metric on $\partial_\infty M$ with a cusp singularity at 0. In the notations above, we define the *renormalized volume of* (M, g, p) as

$$V_R(M, g, p) = \lim_{\varepsilon \to 0} \left(W(\partial_\infty M(\varepsilon), h_0) + \frac{\pi^2}{\varepsilon} + b(\varepsilon) \right) ,$$

with $b(\varepsilon) = \pi^3/(4\varepsilon) + \pi^2/\varepsilon + O(\varepsilon)$.

Remark 4.3.13. Even if it is always possible to assume the marking point to be 0, given two different marking points p_1 and p_2 in $\Omega(\Gamma)$, except in very special cases, there is no Möbius transformation that restricts to a homeomorphism of $\Omega(\Gamma)$ and sends p_1 to p_2. Even if two pointed convex co-compact manifold (M, g, p_1) and (M, g, p_2) are isometric, their renormalized volume strictly depends on the choice of the marking point.

Theorem 4.3.14. The renormalized volume of a convex co-compact manifold (M, g) pointed at $p \in \partial\overline{M}$ exists and it is finite.

Proof. Let us fix a small enough $\overline{\varepsilon} > 0$, so that, thanks to the additive property of the W-volume of Lemma 4.2.12, for any $0 < \varepsilon < \overline{\varepsilon}$, we can split $W(\partial_\infty M(\varepsilon), h_0)$ as

$$W(\partial_\infty M(\varepsilon), h_0) = W\left(D(\varepsilon, \overline{\varepsilon}), h_0\right) + W(\partial_\infty M(\overline{\varepsilon}), h_0) ,$$

with $D(\varepsilon, \overline{\varepsilon}) = D(\overline{\varepsilon}) \setminus \mathrm{int}(D(\varepsilon))$, and where, we recall, the boundary of $D(x)$ has length x with respect to h_0. Then, using the notations introduced in (4.11) and (4.14), there exist two radii $0 < \rho < \overline{\rho}$ depending respectively on ε and $\overline{\varepsilon}$, such that

$$D(\varepsilon, \overline{\varepsilon}) = \mathbb{D}_\rho^{\overline{\rho}} ,$$

and

$$D(\varepsilon) = \mathbb{D}(-2\pi/\log(\rho)) .$$

Moreover, since $h_0 = e^{2\nu}\hat{I}_0$, with $\left|\nu(\rho e^{i\theta})\right| = O(|\rho|)$ at $\rho \sim 0$, then, for small radii

$$\varepsilon = \int_{\partial\mathbb{D}(-2\pi/\log(\rho))} e^\nu ds(\hat{I}_0) \sim -\frac{2\pi}{\log(\rho)} .$$

Therefore, since also the boundary term $b(\cdot)$ defined in Proposition 4.3.6 is continuous

$$\lim_{\varepsilon \to 0} \left(W(D(\varepsilon, \overline{\varepsilon}), h_0) + \frac{\pi^2}{\varepsilon} + b(\varepsilon) \right) = \lim_{\rho \to 0} \left(W(\mathbb{D}_\rho^{\overline{\rho}}, h_0) - \frac{\pi}{2}\log(\rho) + b(\rho) \right) ,$$

and the right hand side exists and is finite by Proposition 4.3.11. The renormalized volume of (M, g, p) can then be expressed as

$$V_R(M, g, p) = W(\partial_\infty M(\overline{\varepsilon}), h_0) + \lim_{\varepsilon \to 0} \left(W\left(D(\varepsilon, \overline{\varepsilon}), h_0\right) + \frac{\pi^2}{\varepsilon} + b(\varepsilon) \right) .$$

It then remains to show that the first term in the equality is finite. First, note that $\overline{\varepsilon}$ can also be chosen such that h_0 and h stay at uniformly bounded small distance on $\partial_\infty M(\overline{\varepsilon})$, where h is the unique hyperbolic representative in $[\partial_\infty M]$ (see Section 4.2.1). Therefore, the difference between $W(\partial_\infty M(\overline{\varepsilon}), h_0)$ and $W(\partial_\infty M(\overline{\varepsilon}), h)$ is bounded. Moreover, the last one is less than the renormalized volume of M, plus the boundary term $b(\overline{\varepsilon})$ coming from $\partial D(\overline{\varepsilon})$, which, for a fixed $\overline{\varepsilon}$, is also finite. ■

Remark 4.3.15. We point out that, in the classical definition of the W-volume, it is often necessary to rescale the metric at infinity by a constant factor to ensure the associated Epstein surface is embedded (see Section 1.6), and the well-definedness is guaranteed by the identity $W(M, g) = W(M, e^{2r}g) + \pi r\chi(\partial_\infty M)$. This continues to work in the setting of W-volumes for domain with round boundary components Ω, since, thanks to the Polyakov formula (Theorem 4.2.10), for any $r > 0$

$$W(\Omega, e^{2r}g) - W(\Omega, g) = -\frac{1}{4}\int_\Omega K(g)tda_\Omega - \frac{1}{2}\int_{\partial\Omega} k(g)tds_{\partial\Omega} = -r\pi\chi(\Omega) ,$$

where the last equality follows by the Gauss-Bonnet Theorem and the fact that the scalar curvature $K(g)$ is twice the Gaussian curvature.

Remark 4.3.16. We can extend Definition 4.3.12 to a finite set of marked points $P \subseteq \partial_\infty M$ of cardinality $|P|$ as

$$V_R(M, g, P) = \lim_{\varepsilon \to 0}\left(W(\partial_\infty M(\varepsilon), h_P) + |P|\left(\frac{\pi^2}{\varepsilon} + b(\varepsilon)\right)\right) ,$$

where h_P is the unique complete hyperbolic metric conformal to $\partial_\infty M$ with cusps at each point in P, and

$$\partial_\infty M(\varepsilon) = \Omega(\Gamma, \varepsilon)/\Gamma \quad \text{with} \quad \Omega(\Gamma, \varepsilon) = \Omega(\Gamma) - \bigcup_{i=1}^{|P|}\Gamma \cdot D_i(\varepsilon) ,$$

where $D_i(\varepsilon)$ denotes the euclidean ball of boundary coinciding with the one of the hyperplane $H(\varepsilon)$ used to cut the i-th puncture. The proof of Theorem 4.3.14 for the finiteness of $V_R(M, g, P)$ applies straightforwardly.

4.4 Adapted renormalized volume

In this section, we finally define the adapted renormalized volume for hyperbolic manifold with compressible boundary. In Theorem 4.4.4, we show that, unlike the classical version, it is bounded from below. Moreover, we prove that its differential has uniformly bounded norm (Theorems 4.4.5 and 4.4.7). We then proceed to prove that the adapted renormalized volume extends by continuity to points of the boundary of the Teichmüller space representing the pinching of a compressible multicurve − whose renormalized volume was defined in the previous section (see Definition 4.3.12), and for which we define the adapted version later in this section.

We recall that the deformation space of the convex co-compact structures on a tame manifold $\overline{M}$ is identified with the quotient of the Teichmüller space of the boundary $\partial\overline{M}$ by the subgroup generated by all Dehn twists along compressible simple closed curves (see

Theorem 1.4.5). Accordingly, we will use the notation $V_R(X)$ for $V_R(M(X))$, where $M(X)$ is the convex co-compact hyperbolic 3-manifold associated to the Riemann surface X via uniformization theorem.

Definition 4.4.1. Given M a convex co-compact hyperbolic 3-manifold, we define the *adapted renormalized volume* as the function

$$\widetilde{V_R}\colon \mathcal{T}(\partial \overline{M}) \to \mathbb{R}$$

such that

$$\widetilde{V_R}(X) = V_R(X) + \pi^3 \sum_{\substack{\gamma \text{ compressible} \\ \ell_\gamma(X) < \varepsilon_0}} \frac{1}{\ell_\gamma(X)}$$

where $\varepsilon_0 = 2\text{arsinh}(1)$ is the Margulis constant, γ runs in the set of compressible simple closed curves in $\partial \overline{M}$, and $\ell_\gamma(X)$ denotes its length with respect to the hyperbolic representative in X.

Remark 4.4.2. The simple length spectrum is invariant under the action of the mapping calss group, and Dehn twists along compressible simple closed curves preserve the property of a curve being compressible or not. Therefore, the adapted renormalized volume of a convex co-compact manifold is well defined, through uniformization, as

$$\widetilde{V_R}(M(X)) = \widetilde{V_R}(X) \ .$$

Remark 4.4.3. Outside of the union of codimension-one manifolds in $\mathcal{T}(\partial \overline{M})$, each corresponding to the set of surfaces with a compressible simple closed geodesic of length equal to ε_0, the adapted renormalized volume is real analytic.

Theorem 4.4.4. For every convex co-compact hyperbolic 3-manifold M, the adapted renormalized volume $\widetilde{V_R}(\cdot)$ is bounded from below by a constant depending just on the topology of the boundary $\partial \overline{M}$.

Proof. We consider the *δ-compressible thick part* of the Teichmüller space $\mathcal{T}^c_\delta(\partial \overline{M})$, i.e. the subset of marked Riemann surfaces having length of the shortest (with respect to the hyperbolic metric) compressible curve bounded below by δ. Observe that if $\delta \geq \varepsilon_0$, then $\widetilde{V_R}(X) = V_R(X)$ for any $X \in \mathcal{T}^c_\delta(\partial \overline{M})$. Then, merging Theorem 2.16 in [BBB19], which furnishes an upper bound on the length of the bending lamination, and Theorem A.6 in [SW22], which states that the W-volume of the convex core (see Section 1.6.1) and the renormalized volume have distance bounded by a constant depending just on the genus of the boundary, for any $X \in \mathcal{T}^c_\delta(\partial \overline{M})$

$$\widetilde{V_R}(X) = V_R(X) \geq V_C(X) - \frac{3}{2}\pi |\chi(\partial \overline{M})| \coth^2(\delta/4) - C \ ,$$

where $V_C(\cdot)$ is the function associating to the convex co-compact manifold $M(\cdot)$ the volume of its convex core, and C is a constant depending just on the Euler characteristic of the boundary $\chi(\partial \overline{M})$. Let us now fix $\delta = \varepsilon_0 = 2\text{arsinh}(1)$, let X be a point in the complement of $\mathcal{T}^c_{\varepsilon_0}(\partial \overline{M})$, and let m be the set of compressible curves in X of length less than δ. The set m has cardinality at most $3g-3$, with g the genus of the surface (see Lemma 4.1 [Bus10]). We order the k components γ_i of m, and we consider the path in $\mathcal{T}(\partial \overline{M})$ given by the concatenation of k (negative) grafting paths, each of whom on γ_j and terminating in a surface X_j in which the geodesic representatives of the first j curves in m have length ε_0, and $X_0 = X$. Then, the ending point X_k belongs to $\mathcal{T}^c_{\varepsilon_0}(\partial\overline{M})$. By Theorem 3.4.5 (Theorem 1.4 in [CG25]), we can estimate the variation of the renormalized volume under the just mentioned grafting path:

$$\begin{aligned}\left|\widetilde{V_R}(X_k) - \widetilde{V_R}(X)\right| &\leq \sum_{j=1}^{k-1}\left|\widetilde{V_R}(X_j) - \widetilde{V_R}(X_{j-1})\right| = \sum_{j=1}^{k-1}\left|V_R(X_j) - V_R(X_{j-1}) - \frac{\pi^3}{\ell_j(X_{j-1})}\right| \\ &\leq \sum_{j=1}^{k-1}\left|\frac{\pi^3}{\ell_j(X_j)} - \frac{\pi}{4}\left(\ell_j(X_j) - \ell_j(X_{j-1})\right) + O\left(e^{-\pi s_j/(2\ell_j(X_j))}s_j^3\right)\right| \\ &= \sum_{j=1}^{k-1}\left|\frac{\pi^3}{\varepsilon_0} - \frac{\pi}{4}\left(\varepsilon_0 - \ell_j(X_{j-1})\right) + O\left(e^{-\pi s_j/(2\varepsilon_0)}s_j^3\right)\right|\end{aligned}$$

where $\ell_j(X_i)$ denotes the length of the curve γ_j with respect to the hyperbolic metric in X_i, and s_j is the parameter of the j-th grafting path. It is easy to see that the last expression is uniformly bounded, in particular it is always smaller than

$$(3g-3)\left(\frac{\pi^3}{\varepsilon_0} + \frac{\pi\varepsilon_0}{4} + A\left(\frac{6\varepsilon_0}{e\pi}\right)^3\right) ,$$

for some universal constant A (see Remark 3.5.26). Since for X_k holds the first estimate, the statement follows. ■

In the next theorem, we prove that the L^1-norm of the differential of the adapted renormalized volume is bounded. We recall that the L^1-norm of a covector of the Teichmüller space at $X = \mathbb{H}^2/\Gamma$, i.e. of a Γ-invariant holomorphic quadratic differential $q(z)dz^2$ on $\mathbb{H}^2$, is defined as

$$\left\|q(z)dz^2\right\|_1 = \int_X |q| \ . \tag{4.17}$$

The dual of this norm is the *Teichmüller norm* on the space of harmonic Beltrami differential (see Section 1.2.3).

Theorem 4.4.5. Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the differential of the adapted renormalized volume is bounded in the L^1 norm by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

Proof. Similarly to the proof of Theorem 4.4.4, first, we observe that at any point in the δ-compressible thick part $X \in \mathcal{T}^c_\delta(\partial\overline{M})$

$$(d\widetilde{V_R})_X = (dV_R)_X = \mathrm{Re}(\mathcal{S}(f_X)) ,$$

where f_X is the developing map of the boundary at infinity of $M(X)$. Then, by Lemma 5.1 in [KM81] (see also Corollary 2.12 in [BBB19]):

$$\left\|(d\widetilde{V_R})_X\right\|_1 \leq 2\pi\,|\chi(X)|\left\|(d\widetilde{V_R})_X\right\|_\infty = 2\pi\,|\chi(X)|\,\|\mathrm{Re}(\mathcal{S}(f_X))\|_\infty \leq 3\pi\,|\chi(X)|\coth^2(\delta/4) \ .$$

Let us now fix $\delta = \varepsilon_0$, and study the differential of the adapted renormalized volume at a point X in the complementary of $\mathcal{T}^c_\delta(\partial\overline{M})$, and of the codimension-one subset where

$\widetilde{V_R}$ is not continuous (see Remark 4.4.3). Then, there exists a non-empty compressible multicurve with $k \leq 3g - g$ components γ_j of length strictly less than ε_0, with g the genus of X, such that in a neighborhood of X

$$\widetilde{V_R}(\cdot) = V_R(\cdot) + \sum_{j=1}^{k} \frac{\pi^3}{\ell_j(\cdot)} \ ,$$

so that

$$(d\widetilde{V_R})_X = \mathrm{Re}(\mathcal{S}(f_X)) - \sum_{j=1}^{k} \frac{\pi^3}{\ell_j^2(X)}(d\ell_j)_X \ , \tag{4.18}$$

where $\ell_j(\cdot)$ is the hyperbolic length function of γ_j. Let us fix some notation. We denote by $X_{thick} \subseteq X$ the thick part of X, that is, the maximal subsurface in X that has injectivity radius bigger than $\varepsilon_0/2$ (this splitting is always possible, see for example Chapter 4 in [Mar16b]). The subsurface X_{thick} contains the complement of the union of the thin tubes around any simple closed curve of lenght $< \varepsilon_0$ (see Definition 1.2.2, and [Bus10, Theorem 4.1.6]). We now define X_{ε_0} as the subsurface of X given by the union of the thin tubes $\mathcal{A}_j$ around γ_j, for $1 \leq j \leq k$. Let also D be a fundamental domain for Γ such that $X = \mathbb{H}^2/\Gamma$. We denote by φ_j the hyperbolic isometry corresponding to γ_j, and by $X_j = \mathbb{H}^2/\langle\varphi_j\rangle$ the respective quotient annulus, which is a covering of X. Anytime j is fixed and we focus on a tubular neighborhood $\mathcal{A}_j$ of γ_j, up to conjugation, we can choose the lift $\widetilde{\gamma_j}$ of γ_j given by the geodesic of $\mathbb{H}^2$ thorough $\{0, \infty\}$. Moreover, we can identify X_j with the annulus

$$\{z \in \mathbb{H}^2 \mid 1 \leq |z| \leq e^{\ell_j(X)}\} \ ,$$

and assume D to contain $\widetilde{\mathcal{A}_j}$, with $\widetilde{\mathcal{A}_j}$ the lift of $\mathcal{A}_j$ through $\pi_j \colon X_j \to X$ intersecting $\widetilde{\gamma_j}$.

The L^1-norm of $(d\widetilde{V_R})_X$ can be split into the sum of the L^1-norms of its restriction to X_{ε_0} and its complement $X_{\varepsilon_0}^c$:

$$\left\|(d\widetilde{V_R})_X\right\|_1 = \int_X \left|(d\widetilde{V_R})_X\right| = \int_{X_{\varepsilon_0}} \left|(d\widetilde{V_R})_X\right| + \int_{X_{\varepsilon_0}^c} \left|(d\widetilde{V_R})_X\right| \ . \tag{4.19}$$

We start by bounding the L^1-norm in $X_{\varepsilon_0}^c$, which is contained in the union of the thick part X_{thick} and the incompressible thin tubes. The norm of the Schwarzian in this region is bounded as before by $3\pi\coth^2(\varepsilon_0/4)$. We need to bound the L^1-norm of $(d\ell_j)_X/\ell_j^2(X)$. To this end, let us recall the Gardiner's formula for the differential of the length function of γ_j (see [Gar75]):

$$d\ell_j(\mu) = \frac{2}{\pi}\mathrm{Re}\left\langle \frac{dz^2}{z^2}, \widetilde{\mu} \right\rangle_{X_j} \tag{4.20}$$

where μ is a harmonic Beltrami differential in $T_X\mathcal{T}(\partial\overline{M})$, and $\widetilde{\mu}$ denotes its lift to the annulus X_j. Note that, in the setting fixed above, the differential $2dz^2/(\pi z^2)$ is φ_j-invariant. It might be tempting to assume $(d\ell_j)_X$ to be equal to $2dz^2/(\pi z^2)$ by the non-degeneracy of the Weil-Petersson pairing, but, since the pairing in (4.20) is on the whole annulus X_j, this is not true. In fact, the differential dz^2/z^2 is not invariant under Γ, as instead (the lift of) $(d\ell_j)_X$ is. Gardiner's formula can though be rephrased as a pairing on the fundamental domain D by taking the holomorphic quadratic differential giving by the Θ-series of $2dz^2/(\pi z^2)$ with respect to the projection $\pi_j \colon X_j \to X$ (see [Hub16, Chapter 5], [Gar75, Theroem 3]), so that:

$$(d\ell_j)_X = (\pi_j)_*\left(\frac{2dz^2}{\pi z^2}\right) = \frac{2}{\pi}\sum_{\beta\in\Gamma/\langle\varphi_j\rangle} \beta^*\left(\frac{dz^2}{z^2}\right) \ . \tag{4.21}$$

It is now easy to see that the L^1-norm of $(d\ell_j)_X$ on D, and therefore on X, is bounded by the L^1-norm of $2dz^2/(\pi z^2)$ on X_j:

$$\left\|(\pi_j)_*\left(\frac{dz^2}{z^2}\right)\right\|_1 = \int_D \left|\sum_{\beta\in\Gamma/\langle\varphi_j\rangle} \beta^*\left(\frac{dz^2}{z^2}\right)\right| \leq \sum_{\beta\in\Gamma/\langle\varphi_j\rangle} \int_D \left|\beta^*\left(\frac{dz^2}{z^2}\right)\right| =$$
$$= \sum_{\beta\in\Gamma/\langle\varphi_j\rangle} \int_{\beta(D)} \left|\frac{dz^2}{z^2}\right| = \int_{X_j} \left|\frac{dz^2}{z^2}\right| = \left\|\left(\frac{dz^2}{z^2}\right)\right\|_1 .$$

It is then enough to bound the norm of $dz^2/(\ell_j^2(X)z^2)$ on the complement of $\widetilde{\mathcal{A}_j}$ in X_j. There exists a lift of $X^c_{\varepsilon_0}$ to X_j that is contained in the subset (see [Mar16b, Lemma 5.2.7])

$$D\setminus\widetilde{\mathcal{A}_j} = \{z=\rho e^{i\theta}\in\mathbb{H}^2 \mid 1\leq\rho\leq e^{\ell(\gamma_j)},\ 0<\sin(\theta_j)<1/\cosh(L_j)\} ,$$

where

$$L_j = \operatorname{arsinh}\left(1/\sinh(\ell_j(X)/2)\right) ,$$

is the width of the thin tube $\mathcal{A}_j$. Then, omitting the dependence by X of the lengths

$$\int_{X^c_{\varepsilon_0}} \frac{|(d\ell_j)_X|}{\ell_j^2} \leq \frac{1}{\ell_j^2}\int_{D\setminus\widetilde{\mathcal{A}_j}} |(d\ell_j)_X| \leq \frac{2}{\pi\ell_j^2}\int_{X_j\setminus\widetilde{\mathcal{A}_j}} \left|\frac{dz^2}{z^2}\right| = \frac{4\theta_j}{\pi\ell_j} \leq \frac{2}{\pi} \tag{4.22}$$

where the last inequality follows from

$$\theta = \arcsin(1/\cosh(L_j)) \leq \ell_j/2 .$$

Therefore, from Equation (4.18)

$$\int_{X^c_{\varepsilon_0}} \left|(d\widetilde{V_R})_X\right| \leq 3\pi\coth^2(\varepsilon_0/4) + 2\pi^2(3g-3) . \tag{4.23}$$

We now proceed to estimate the L^1-norm of $(d\widetilde{V_R})_X$ on the union X_{ε_0} of the compressible thin tubes. Let us fix a j and use the same notations as above. Thanks to Theorem 3.5.4 (Theorem 1.1 in [CG25]), again denoting $\ell_{\gamma_j}(X)$ simply by ℓ_j, the Schwarzain derivative on $\widetilde{\mathcal{A}_j}$ satisfies

$$\mathcal{S}(f) = \frac{1}{2z^2}\left(1+\frac{4\pi^2}{\ell_j^2}\right)dz^2 + q_{\ell_j}(z)dz^2 ,$$

with

$$\left|q_{\ell_j}\right| = O\left(\frac{e^{-\pi^2/(2\ell_j)}}{\ell_j^2}\right) \leq A ,$$

where A is a universal constant (see Remark 3.5.26). Therefore, by splitting $(d\ell_j)_X$ as

$$(d\ell_j)_X = \frac{2}{\pi}\sum_{\substack{\beta\in\Gamma/\langle\varphi_j\rangle\\ \beta\neq[\varphi_j]}} \beta^*\left(\frac{dz^2}{z^2}\right) + \frac{2}{\pi}\frac{dz^2}{z^2} ,$$

on $\widetilde{\mathcal{A}_j}$

$$(d\widetilde{V_R})_X = \frac{1}{2}\frac{dz^2}{z^2} + q_{\ell_j}(z)dz^2 - \frac{2\pi^2}{\ell_j^2}\sum_{\substack{\beta\in\Gamma/\langle\varphi_j\rangle\\ \beta\neq[\varphi_j]}} \beta^*\left(\frac{dz^2}{z^2}\right) - \sum_{\substack{i=1\\ i\neq j}}^{k} \frac{\pi^3}{\ell_i^2}(d\ell_i)_X . \tag{4.24}$$

It is easy to see that the L^1-norm of the first two terms is bounded on $\widetilde{\mathcal{A}_j}$ by a universal constant B. Moreover, the L^1-norm on $\widetilde{\mathcal{A}_j}$ of the first sum, that we denote now by $\Theta_0(dz^2/z^2)$, satisfies

$$\frac{2\pi^2}{\ell_j^2}\left\|\Theta_0\left(\frac{dz^2}{z^2}\right)\right\|_1 \leq \frac{2\pi^2}{\ell_j^2}\int_{X_j\setminus D}\left|\frac{dz^2}{z^2}\right| \leq \frac{2\pi^2}{\ell_j^2}\int_{X_j\setminus\widetilde{\mathcal{A}_j}}\left|\frac{dz^2}{z^2}\right| \leq 2\pi^2 \ ,$$

where the last inequality follows from the proof of the other case. Lastly, observing that, for any $i \neq j$, $\pi_i^{-1}(\mathcal{A}_j) \subseteq X_i \setminus \widetilde{\mathcal{A}_i}$, the L^1-norm on $\widetilde{\mathcal{A}_j}$ of each term $(d\ell_i)_X/\ell_i^2(X)$ in the last sum is bounded again as in (4.22). We finally obtain the estimate

$$\int_{X_{\varepsilon_0}}\left|(d\widetilde{V_R})_X\right| = \sum_{j=1}^{k}\int_{\mathcal{A}_j}\left|(d\widetilde{V_R})_X\right| \leq (B + 2\pi^2 + 2\pi^2(k-1))k. \tag{4.25}$$

Since $k \leq 3g-3$, the bounds in (4.23) and (4.25) depend only on the genus g of X, and the statement of the theorem follows from (4.19). ■

In the next theorem, we prove that the differential of the adapted renormalized volume is also bounded with respect to the infinity norm. Before stating it, we recall the definition

$$\|q\|_\infty = \sup_{z\in X}\|q\| = \sup_{z\in X}\left|\frac{q}{\rho_X}\right| \ ,$$

for $q \in Q(X)$ and where ρ_X stands for the hyperbolic metric conformal to X. This result is of particular interest, as it implies that minus the Weil-Petersson gradient of the adapted renormalized volume

$$(\nabla\widetilde{V_R})_X = -\frac{\overline{dV_R}}{\rho_X}$$

has bounded L^∞ norm, i.e., bounded Teichmüller norm, with respect to which Teichmüller space is complete (see Section 1.2.3), opening the possibility of studying its flow lines in the regular regions.

Theorem 4.4.6. Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the differential of the adapted renormalized volume is bounded in the L^∞ norm by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

Proof. As in the previous proof, if X belongs to the ε_0-compressible thick part $\mathcal{T}^c_{\varepsilon_0}(\partial\overline{M})$, then, since $(d\widetilde{V_R})_X = (dV_R)_X$, the statement follows directly from the already used bound on the infinity norm of the Schwarzian derivative given in Lemma 5.1 in [KM81]:

$$\left\|(d\widetilde{V_R})_X\right\|_\infty \leq \frac{3}{2}\coth^2(\varepsilon_0/4) \ .$$

Otherwise, following the same strategy and notations as before, we start by bounding the L^∞ norm on the complement $X^c_{\varepsilon_0}$ of the union of the compressible thin tubes $\mathcal{A}_j$, for the indexes $j = 1, \dots, k \leq 3g-3$. Here, the norm of the Schwarzian is again bounded by $3\coth^2(\varepsilon_0/4)/2$. Next, we have to bound the L^∞ norm of $\pi^3 d\ell_j/\ell_j^2$, on the lift $\widetilde{X^c_{\varepsilon_0}}$ of $X^c_{\varepsilon_0}$ to a fundamental domain $D \subseteq X_j$, with $X_j = \mathbb{H}^2/\langle z \to e^{\ell_j}z\rangle$ the covering associated to the core of the thin tube $\mathcal{A}_j$. In this setting (see Equation 4.21)

$$\pi^3\frac{d\ell_j}{\ell_j^2} = 2\pi^2\sum_{\beta\in\Gamma/\langle\varphi_j\rangle}\beta^*\left(\frac{dz^2}{z^2}\right) \ ,$$

with ℓ_j the hyperbolic lengths of the simple closed curves γ_j in X of length less than ε_0, and φ_j the corresponding hyperbolic isometries. Before going on with the proof, we remark that the bound on the infinity norm on the thick part of X follows directly from Theorem 4.4.5, as in general, if $q \in Q(X)$, then on X_{thick}

$$\sup_{z \in X_{thick}} \|q\| \leq C \|q\|_1 \ ,$$

for some constant C depending just on the topology of the surface X. Nevertheless, we provide a more explicit argument, as it will be needed for the second part of the proof, where we handle the estimate on the thin tubes.

For any fixed $j \leq k$, consider the lift $\widetilde{\gamma_j} = \{0, \infty\}$ of the core of $\mathcal{A}_j$, and choose D to be the fundamental domain in X_j intersecting $\widetilde{\gamma_j}$. Denoting by $\widetilde{\mathcal{A}_j}$ the lift of $\mathcal{A}_j$ contained in D

$$\sup_{z \in D \setminus \widetilde{X}_\varepsilon^c} \frac{\pi^3}{\ell_j^2} \|d\ell_j\| \leq \frac{2\pi^2}{\ell_j^2} \sum_{\beta \in \Gamma/\langle\varphi_j\rangle} \sup_{z \in \beta(D \setminus \widetilde{A}_j)} \left\| \frac{dz^2}{z^2} \right\| = \frac{2\pi^2}{\ell_j^2} \sum_{\beta \in \Gamma/\langle\varphi_j\rangle} \sup_{z \in \beta(D \setminus \widetilde{A}_j)} \left| \frac{y^2}{z^2} \right| \ ,$$

where $z = x + iy = \rho e^{i\theta}$, so that y is the imaginary part of z and θ is its angular coordinate, and then

$$\sup_{z \in \beta(D \setminus \widetilde{A}_j)} \left| \frac{y^2}{z^2} \right| = \sup_{z \in \beta(D \setminus \widetilde{A}_j)} \left| \sin^2(\theta) \right| \ .$$

In the case $\beta = [\varphi_j] \in \Gamma/\langle\varphi_j\rangle$, we compute

$$\sup_{z \in D \setminus \widetilde{A_j}} \frac{2\pi^2}{\ell_j^2} \left| \frac{y^2}{z^2} \right| = \sup_{z \in D \setminus \widetilde{A_j}} \frac{2\pi^2}{\ell_j^2} \left| \frac{\rho^2 \sin^2(\theta)}{\rho^2} \right| \leq \frac{2\pi^2}{\ell_j^2} \left(\frac{\sinh(\ell_j/2)}{\cosh(\ell_j/2)} \right)^2 \leq \pi^2 \ ,$$

where the first inequality follows from

$$\sin(\theta) \leq \frac{1}{\cosh(L_j)} = \left(\cosh\left(\operatorname{arsinh}\left(1/\sinh(\ell_j/2)\right)\right)\right)^{-1} = \left(\left(\frac{1}{\sinh(\ell_j/2)} \right)^2 + 1 \right)^{-1} \ ,$$

with L_j the width of the thin tube $\mathcal{A}_j$ (see Definition 1.2.2, and [Mar16b, Lemma 5.2.7]). For all the other elements $\beta \in \Gamma/\langle\varphi_j\rangle$, the Prime Geodesic Theorem, [Hub59], implies that, asymptotically, for a given natural number N, there are approximately e^N/N translates of the fundamental domain D at hyperbolic distance $\asymp L_j + N$ from $\widetilde{\gamma_j}$. Therefore

$$\frac{2\pi^2}{\ell_j^2} \sum_{\beta \in \Gamma/\langle\varphi_j\rangle} \sup_{z \in \beta(D \setminus \widetilde{A}_j)} \left| \sin^2(\theta) \right| \lesssim \frac{2\pi^2}{\ell_j^2} \sum_N \left| \frac{e^N}{N \cosh^2(L_j + N)} \right| \lesssim \frac{2\pi^2 e^{-2L_j}}{\ell_j^2} \sum_N \left| \frac{e^{-N}}{N} \right| \ ,$$

and, since the series converges and e^{-2L_j}/ℓ_j^2 is bounded as before, we get

$$\sup_{z \in D \setminus \widetilde{X}_\varepsilon^c} \frac{\pi^3}{\ell_j^2} \|d\ell_j\| \leq A \ ,$$

for some constant $A > 0$. Therefore, we proved

$$\sup_{z \in X_{\varepsilon_0}^c} \left\| (d\widetilde{V_R})_X \right\| \leq 3\coth^2(\varepsilon_0)/2 + (3g-3)A \ .$$

Let us now consider the union X_{ε_0} of the thin compressible tubes. As already shown in the proof of Theorem 4.4.5, on any thin tube $\mathcal{A}_j \subseteq X_{\varepsilon_0}$, the diverging term of the Schwarzian derivative simplifies with the one of the holomorphic quadratic differential $\pi^3 d\ell_j/\ell_j^2$, so that (see Equation (4.24)), on $\widetilde{\mathcal{A}_j}$:

$$(d\widetilde{V_R})_X = \frac{1}{2}\frac{dz^2}{z^2} + q_{\ell_j}(z)dz^2 - \frac{2\pi^2}{\ell_j^2} \sum_{\substack{\beta\in\Gamma/\langle\varphi_j\rangle \\ \beta\neq[\varphi_j]}} \beta^*\left(\frac{dz^2}{z^2}\right) - \sum_{\substack{i=1 \\ i\neq j}}^{k} \frac{\pi^3}{\ell_i^2}(d\ell_i)_X \ .$$

A direct and easy computation shows that the infinity norm of the first two terms is uniformly bounded. Furthermore, each differential appearing in the last sum can be estimated as in the previous case, thanks to the containment $\widetilde{\mathcal{A}_j} \subseteq X_i \setminus \widetilde{\mathcal{A}_i}$. The same holds for the remaining terms in the Θ-series of $d\ell_j$, since for any $\beta \neq [\varphi_j]$, the translates satisfy $\beta(\widetilde{\mathcal{A}_j}) \subseteq X_j \setminus \widetilde{\mathcal{A}_j}$.

■

As a corollary of the previous two theorems, we obtain also the bound on the L^2 norm, that is, on the Weil-Petersson norm (see Section 1.2.3).

Corollary 4.4.7. Let M be a convex co-compact hyperbolic manifold. At the points where it exists, the Weil-Petersson gradient of the adapted renormalized volume is bounded in the Weil-Petersson norm by a constant that depends only on the topology of the boundary $\partial\overline{M}$.

Proof. The Weil-Petersson norm on harmonic Beltrami differentials coincides with the L^2-norm on holomoprhic quadratic differentials. That is, for any $q \in Q(X)$

$$\|q\|_2^2 = \int_X \frac{q\overline{q}}{\rho_X} = \left\langle q, \frac{\overline{q}}{\rho_X}\right\rangle = \left\|\frac{\overline{q}}{\rho_X}\right\|_2^2 \ ,$$

where ρ_X denotes the conformal hyperbolic metric on X, and $\langle\cdot,\cdot\rangle$ the Weil-Petersson pairing (see Section 1.2.3). Moreover, the L^2-norm is bounded by the L^∞ norm and the L^1-norm as

$$\|q\|_2^2 \leq \|q\|_\infty \|q\|_1 \ . \tag{4.26}$$

Therefore, the statement follows directly from Theorems 4.4.5 and 4.4.6.

■

Another corollary of Theorem 4.4.6 is the following.

Corollary 4.4.8. Let M be a convex co-compact hyperbolic manifold, and let μ denote the infinitesimal earthquake or the infinitesimal grafting associated to a simple closed curve $\gamma \subseteq X$, with $X \in \mathcal{T}(\partial\overline{M})$. Then

$$\left|(d\widetilde{V_R})_X(\mu)\right| \leq C(\partial M)\ell \ ,$$

where $C(\partial M)$ is a constant depending only on the topology of ∂M, and ℓ is the length of γ with respect to the hyperbolic metric in the conformal class of X.

Proof. By Theorem 4.4.5, there exists $C_0(\partial M)$ such that, at any X

$$\left\|d\widetilde{V_R}\right\|_\infty \leq C(\partial M) \ .$$

The statement now follows by applying Lemma 3.2.1. ■

Fixed a pants decomposition P of $\partial\overline{M}$ and a small $\delta > 0$, a δ-*Bers region* is the subset $\mathcal{B}_\delta$ of $\mathcal{T}(\partial\overline{M})$ such that

$$\mathcal{B}_\delta = \{X \in \mathcal{T}(\partial\overline{M}) \mid \ell_\gamma(X) \leq \delta \ \ \forall \gamma \in P\} \ .$$

Theorem 4.4.9. Let P be a compressible pants decomposition of the boundary $\partial\overline{M}$ of a convex co-compact hyperbolic 3-manifold. For any $\varepsilon \leq \varepsilon_0$ there exists a $\delta > 0$ such that for any X_0 and X_1 in the δ-Bers region of P

$$\left|\widetilde{V_R}(X_1) - \widetilde{V_R}(X_0)\right| \leq \varepsilon \ .$$

Proof. Let X_0 and X_1 be two points of $\mathcal{B}_\delta$. We can modify the Fenchel-Nielsen coordinates with respect to P (see Section 1.2.2) to move from X_0 to X_1 with a complex earthquake path on P, i.e. first we modify the twist parameter by earth-quaking on each component of P, and then the lengths by grafting (see [McM98]). Let us consider $\gamma \in P$, we start by changing the twist parameter by a t-earthquake, reaching the surface X_t whose Fenchel-Nielsen coordiantes differ from the one of X_0 just by the twist of γ, which coincides with the one of X_1. In this way the standard renormalized volume changes, by Theorem 3.4.1, as

$$|V_R(X_t) - V_R(X_0)| \leq F(\delta)t \ ,$$

with $F(\delta)$ such that

$$|F(\delta)| \leq C\frac{e^{-\pi^2/\delta}}{\delta} \leq C \ ,$$

for some explicit constant $C > 0$. Since, by uniformization theorem of the deformation space of convex co-compact structures, the renormalized volume descends to a function on the quotient of $\mathcal{T}(\partial\overline{M})$ by the group generated by Dehn twists on compressible simple closed geodesic, we can assume $t \leq \delta$ (see Remark 4.4.2). Moreover, remaining the length of any short compressible simple closed curve unchanged

$$\left|\widetilde{V_R}(X_t) - \widetilde{V_R}(X_0)\right| = |V_R(X_t) - V_R(X_0)| \ .$$

We repeat this on any $\gamma \in P$, obtaining the surface $X_{1/2}$. Finally, when changing the length of any $\gamma_i \in P$, for $i = 1, \ldots, k$, by a parameter s_i grafting on it, By Theorem 3.4.5, the difference of the adapted renormalized volumes is bounded as

$$\left|\widetilde{V_R}(X_1) - \widetilde{V_R}(X_{1/2})\right| = \frac{\pi}{4}\sum_{i=1}^{k}\left[\left|(\ell_{\gamma_i}(X_1) - \ell_{\gamma_i}(X_{1/2}))\right| + O\left(e^{-\pi s_i/(2\delta)}s_i^3\right)\right] \leq \frac{k\pi\delta}{4} + \left(\frac{6\delta}{e\pi}\right)^3 \ .$$

We proved that the total variation of the adapted renormalized volume is bounded by a linear function of δ, therefore the statement follows. ■

Note that the hypothesis of the theorem above imply that M is an handlebody.

4.4.1 Renormalized volume of a long tube

In this section, we are going to study how the renormalized volume diverges on long tubes. This will be useful in the next section, where we study the sequence of adapted renormalized volumes under the pinching of a compressible (multi)curve, in order to prove its continuous extension to the boundary of the Teichmüller space. We point out that Theorem 3.4.5 already shows the asymptotic behaviour of the renormalized volume under the pinching of a compressible curve. Moreover, by looking at the proof, we see that the divergent contribution comes from the long tubes, as it is precisely there that the differential of the renormalized volume cannot be uniformly bounded. However, here we will need a more explicit control on the divergence, which also takes into account *where* the tube has been truncated. More precisely, we aim to express its W-volume, as a domain with boundary equipped with a hyperbolic metric, in terms of the length of the boundary curves, other than the length of its core curve.

Similarly to what we have done in Section 3.3, we start by studying a class of examples in which the developing map of the tube and the metric are explicit. In the subsection that follows the next, we will generalize the result, analyzing the lower order terms that arise.

Toy model

Let us consider the map $f_\ell \colon \mathbb{H}^2 \to \mathbb{C}$ from the half-space model of the 2-hyperbolic space, with coordinate $z = \rho e^{i\theta}$, where $\rho > 0$ and $\theta \in [0, \pi]$, defined by

$$f_\ell(z) = z^{\frac{2\pi i}{\ell}} = e^{-\frac{2\pi\theta}{\ell} + i\frac{2\pi}{\ell}\log(\rho)} \ .$$

The restriction $f_{\ell,\varepsilon}$ of f_ℓ to the neighborhood of the vertical axis given by the cone of angle $\pi - 2\arcsin(\ell/\varepsilon)$ is the developing map of the symmetric complex projective annulus

$$\mathcal{A}_\ell(\varepsilon) = \left\{ \rho e^{i\theta} \in \mathbb{C} \ \middle|\ e^{-\frac{2\pi^2}{\ell} + \frac{2\pi}{\ell}\arcsin\left(\frac{\ell}{\varepsilon}\right)} \leq \rho \leq e^{-\frac{2\pi}{\ell}\arcsin\left(\frac{\ell}{\varepsilon}\right)} \right\} \ . \tag{4.27}$$

By *symmetric* here we mean that $\mathcal{A}_\ell(\varepsilon)$ has concentric round boundary components. Note that, equipping $\mathcal{A}_\ell(\varepsilon)$ with the restriction of the flat metric $h_{Eu} = \frac{1}{r^2}|dz|^2$ on the infinite tube $\mathbb{C} \setminus \{0\}$, one obtains a truncated euclidean tube with core length 2π and modulus such that the pair $(\mathcal{A}_\ell(\varepsilon), h_{Eu})$ is conformally equivalent to a truncated hyperbolic tube with core length ℓ and boundaries of length ε. Moreover, up to Möbius transformations, any symmetric complex projective tube is realized as some $\mathcal{A}_\ell(\varepsilon)$ for suitable $\ell > 0$ and $\varepsilon > 0$, and any tube is conformal to a symmetric one (as any modulus is achievable, and this is a complete conformal invariant, see Section 1.2.1).

The hyperbolic metric $\dfrac{1}{t^2}(dx^2 + dt^2)$ on $\mathbb{H}^2 = \mathbb{R} \times \mathbb{R}^+$, with coordinates (x, t), can be pushed forward via f_ℓ, obtaining the following metric tensor on $\mathcal{A}_\ell(\varepsilon)$:

$$\hat{I}_\ell(z) = \frac{\ell^2}{4\pi^2\rho^2\sin^2\left(\frac{\ell}{2\pi}\log(\rho)\right)}\,|dz|^2 \tag{4.28}$$

at $z = \rho e^{i\theta}$.

Remark 4.4.10. For any fixed $0 < \rho_1 < \rho_2 < 1$, the metric $\hat{I}_\ell$ converges uniformly to $\hat{I}_0$ (as in (4.5)) on the compact $\mathbb{D}_{\rho_1}^{\rho_2}$ (as in (4.11)) as $\ell \to 0$.

Remark 4.4.11. The Schwarzian derivative of f_ℓ is:

$$\begin{aligned} q = \mathcal{S}(f_\ell) &= \left(\left(\frac{(f_\ell)''}{(f_\ell)'}\right)' - \frac{1}{2}\left(\frac{(f_\ell)''}{(f_\ell)'}\right)^2\right)dz^2 \\ &= \left(\left(\left(\frac{2\pi i}{\ell} - 1\right)\frac{1}{z}\right)' - \frac{1}{2}\left(\left(\frac{2\pi i}{\ell} - 1\right)\frac{1}{z}\right)^2\right)dz^2 \\ &= \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right)dz^2 \ . \end{aligned}$$

Lemma 4.4.12. Let $(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell)$ be half of the symmetric tube $(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell)$, with boundaries given by the core γ of $\mathcal{A}_\ell(\varepsilon)$ and one of its boundary components. Then

$$W(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell) = 2W(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell) + 2b(\gamma, \hat{I}_\ell) \ ,$$

where $b(\gamma, \cdot)$ denotes the integral of the mean curvature terms arising from the boundary component γ:

$$b(\gamma, \hat{I}_\ell) = \frac{1}{2}\int_{C(\gamma)} H da_{C(\gamma)} + \frac{\pi}{8}\ell(\partial_1 C(\gamma)) \ ,$$

where $C(\gamma)$ is the associated caterpillar region, and $\ell(\partial_1 C(\gamma))$ is the hyperbolic length of the lower boundary component of $C(\gamma)$.

Proof. First, note that the core of $\mathcal{A}_\ell(\varepsilon)$, being the image under f_ℓ of the vertical geodesic between 0 and ∞ of $\mathbb{H}^2$, corresponds to the circle of radius $e^{-\pi^2/\ell}$ in $\mathbb{C}$. We consider the hyperplane of boundary at infinity coinciding with the core, and the inversion $r(z) = (\overline{z})^{-1}e^{-2\pi^2/\ell}$ along it, which is a Möbius anti-transformation and thus an orientation reversing isometry of $\mathbb{H}^3$. In particular, r reverses the induced orientation on γ, so that

$$W(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell) = W\left(r(\mathcal{C}_\ell(\varepsilon)), r^*\left(\hat{I}_\ell\right)\right) - 2b(\gamma, \hat{I}_\ell) \ .$$

The thesis now follows, by additivity of the W-volume (see Lemma 4.2.12), just by observing that

$$(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell) = (\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell) \cup (r((\mathcal{C}_\ell(\varepsilon)), r^*(\hat{I}_\ell)) \ .$$

■

Let us denote by $N_\ell(\varepsilon)$ the compact subset of $\mathbb{H}^3$ bounded by the Epstein surface associated to the domain with boundary $(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell)$ capped with the hyperplanes whose boundary union coincides with $\partial\mathcal{A}_\ell(\varepsilon)$ (as in Section 4.2.3).

Proposition 4.4.13. The W-volume of $N_\ell(\varepsilon)$, for any fixed $\varepsilon > \ell$ small enough, has the following behavior at $\ell \sim 0$:

$$W(N_\ell(\varepsilon)) = W(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell) = -\frac{\pi^3}{\ell} + \frac{2\pi^2}{\varepsilon} + 2b(\varepsilon) + O(\ell, \varepsilon) \ ,$$

with

$$b(\varepsilon) = \frac{\pi^3}{4\varepsilon} + \frac{\pi^2}{\varepsilon} + O(\varepsilon) \ ,$$

and where $O(\ell, \varepsilon)$ is a function of ℓ and ε such that the limit $\lim_{\ell\to 0} \frac{O(\ell,\varepsilon)}{\ell}$ is finite.

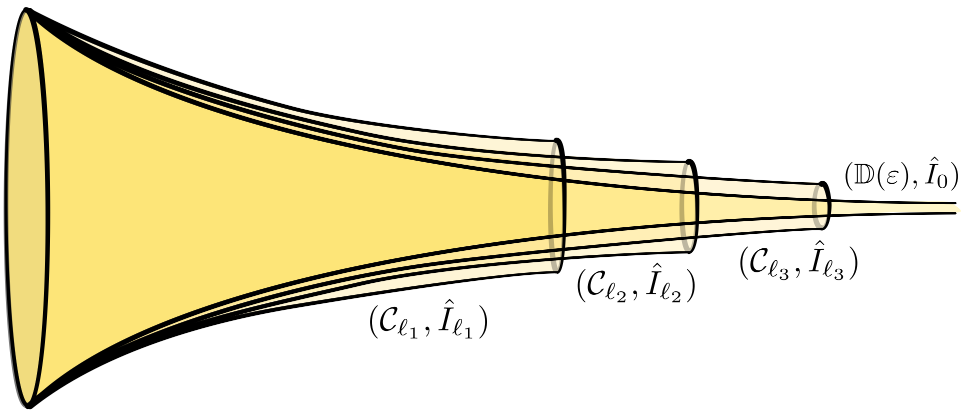

Figure 4.4: Approximating a cusp with long truncated half-tubes of core lengths $\ell_1 > \ell_2 > \ell_3$. (Image by David Fisac)

Proof. We are going to study the W-volume of half of the tube $(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell)$, and then use Lemma 4.4.12. First observe that, without loss of generality,

$$\mathcal{C}_\ell(\varepsilon) = \mathbb{D}_{\rho_1(\ell,\varepsilon)}^{\rho_2(\ell,\varepsilon)} , \quad \rho_1(\ell,\varepsilon) = e^{-\frac{\pi^2}{\ell}} \quad \text{and} \quad \rho_2(\ell,\varepsilon) = e^{-\frac{2\pi}{\ell}\arcsin\left(\frac{\ell}{\varepsilon}\right)} .$$

Then, thanks to Proposition 4.3.6:

$$W(\mathcal{C}_\ell(\varepsilon), \hat{I}_0) = -\frac{\pi}{2}\log\left(\frac{\rho_2}{\rho_1}\right) - b(\rho_1) + b(\rho_2) = -\frac{\pi^3}{2\ell} + \frac{\pi^2}{\varepsilon} - b(\rho_1) + b(\rho_2) ,$$

where $b(\rho_i)$ is the boundary term given by the integral of the mean curvature for the caterpillar region, and

$$b(\rho_i) = -\frac{\pi^2}{8}\log(\rho_i) - \frac{\pi}{2}\log(\rho_i) + O(1/\log(\rho_i)) .$$

In particular

$$b(\rho_2) = b(\varepsilon) = \frac{\pi^3}{4\varepsilon} + \frac{\pi^2}{\varepsilon} + O(\varepsilon) . \tag{4.29}$$

We now apply the Polyakov formula for the W-volume associated to domain with round boundaries (see Theorem 4.2.10), in order to estimate the difference between $W(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell)$ and $W(\mathcal{C}_\ell(\varepsilon), \hat{I}_0)$ (see Figure 4.4). To this end, let us denote by w_ℓ the smooth function on $\mathcal{C}_\ell(\varepsilon)$ such that

$$\hat{I}_\ell = e^{2w_\ell}\hat{I}_0 .$$

We need to bound w_ℓ and its derivatives: denoting $\mathcal{C}_\ell(\varepsilon)$ simply by $\mathcal{C}$, Theorem 4.2.10 states:

$$W(\mathcal{C}_\ell(\varepsilon), e^{w_\ell}\hat{I}_0) - W(\mathcal{C}_\ell(\varepsilon), \hat{I}_0) =$$
$$= -\frac{1}{4}\int_{\mathcal{C}}\left(\left|\nabla_{\hat{I}_0} w_\ell\right|^2 + K(\hat{I}_0)w_\ell\right) da(\hat{I}_0) - \frac{1}{2}\int_{\partial\mathcal{C}} k(\hat{I}_0)w_\ell ds(\hat{I}_0)$$

where $\nabla_{\hat{I}_0}$, $K(\hat{I}_0)$, $k(\hat{I}_0)$, $da(\hat{I}_0)$ and $ds(\hat{I}_0)$ denote, respectively, the gradient, scalar curvature, geodesic curvature of the boundary, area form, and length form, all with respect to the metric tensor $\hat{I}_0$. In particular, being $\hat{I}_0$ a hyperbolic cusp, it has scalar curvature $K(\hat{I}_0) = -1$. Moreover, the round circles of the boundary of $\mathcal{C}$ have $\hat{I}_0$-geodesic curvature $k(\hat{I}_0) = 1$, as they are horocycles. To lighten up the notation, we define the quantity

$$a(\ell) = \frac{\ell}{2\pi} \ ,$$

and we observe that w_ℓ can be made explicit, at $z = \rho e^{i\theta}$, as

$$w_\ell(z) = w_\ell(\rho) = \log\left(\frac{a(\ell)\log(\rho)}{\sin\left(a(\ell)\log(\rho)\right)}\right) \ .$$

Note that $w_\ell(\rho_2) = O(\ell^2)$, and $w_\ell(\rho_1) = \log(\pi/2)$ is bounded for any ℓ. Fixed $\varepsilon > 0$ small enough, we can expand $w_\ell(\rho)$ and its derivative $(w_\ell(\rho))'$ near $\ell \sim 0$, getting

$$\begin{aligned} w_\ell(\rho) &\sim a^2(\ell)\log^2(\rho)/6 \\ (w_\ell(\rho))' &\sim \frac{a^2(\ell)\log(\rho)}{3\rho} \end{aligned}$$

First, we study the boundary term. We parameterize the two connected components of $\mathcal{C}$ as $\alpha_i(s) = \rho_i e^{si/\rho_i}$, with $s \in [0, 2\pi\rho_i]$, for $i = 1, 2$. Then $||\dot{\alpha}_i(s)||_{\hat{I}_0} = 1/(\rho_i \log(\rho_i))$ and

$$-\frac{1}{2}\int_{\partial\mathcal{C}} k(\hat{I}_0) w_\ell ds(\hat{I}_0) = \frac{1}{2}\int_0^{2\pi\rho_1} \frac{w_\ell(\rho_1)}{\rho_1\log(\rho_1)} ds - \frac{1}{2}\int_0^{2\pi\rho_2} \frac{w_\ell(\rho_2)}{\rho_2\log(\rho_2)} ds = \pi\left(\frac{w_\ell(\rho_1)}{\log(\rho_1)} - \frac{w_\ell(\rho_2)}{\log(\rho_2)}\right)$$

but now, since $w(\rho_2) = O(\ell^2)$ and $w_\ell(\rho_1)/\log(\rho_1) = \ell\log(\pi/2)/\pi^2 = O(\ell)$, this is $O(\ell)$.

Let us move on to analyzing the integrals over $\mathcal{C}$. We have

$$\frac{1}{4}\int_{\mathcal{A}} K(\hat{I}_0) w_\ell da(\hat{I}_0) = -\frac{\pi}{2}\int_{\rho_1}^{\rho_2} \frac{w_\ell(\rho)}{\rho\log^2(\rho)} d\rho \sim -\frac{\pi a^2(\ell)}{12}\int_{\rho_1}^{\rho_2} \frac{1}{\rho} d\rho \ ,$$

and

$$-\frac{\pi a^2(\ell)}{12}\int_{\rho_1}^{\rho_2} \frac{1}{\rho} d\rho = -\frac{\pi a^2(\ell)}{12}\log\left(\frac{\rho_2}{\rho_1}\right) \sim \frac{\pi}{12}\left(\frac{\ell}{2\pi}\right)^2\left(-\frac{2\pi}{\varepsilon} + \frac{\pi^2}{\ell}\right)$$

which is of order $O(\ell)$ for any fixed $\varepsilon > \ell$. Concerning the term in the gradient, since w_ℓ depends just on the radial coordinate

$$\left|\nabla_{\hat{I}_0} w_\ell\right|^2 da(\hat{I}_0) = (\rho\log(\rho))^2\left(\frac{\partial w_\ell}{\partial\rho}\right)^2 \frac{d\rho d\theta}{\rho\log^2(\rho)} \sim \left(\rho\log(\rho)\frac{a^2(\ell)\log(\rho)}{3\rho}\right)^2 \frac{d\rho d\theta}{\rho\log^2(\rho)} \ ,$$

so that

$$\int_{\mathcal{C}} \left|\nabla_{\hat{I}_0} w_\ell\right|^2 da(\hat{I}_0) \sim \frac{2\pi a^4(\ell)}{9}\int_{\rho_1}^{\rho_2} \frac{\log^2(\rho)}{\rho} d\rho = \frac{2\pi a^4(\ell)}{27}\left(-\frac{2\pi}{\varepsilon} + \frac{\pi^2}{\ell}\right)^3 \ ,$$

which again is of order $O(\ell)$ for any fixed $\varepsilon > \ell$. Putting everything together, we conclude:

$$W(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell) = -\frac{\pi^3}{2\ell} + \frac{\pi^2}{\varepsilon} - b(\rho_1) + b(\rho_2) + O(\varepsilon, \ell) \ .$$

Now, by Lemma 4.4.12

$$W(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell) = 2W(\mathcal{C}_\ell(\varepsilon), \hat{I}_\ell) + 2b(\alpha_1, \hat{I}_\ell) \ ,$$

where

$$b(\alpha_1, \hat{I}_\ell) = b(\rho_1)$$

as

$$\alpha_1(s) = \rho_1 e^{si/\rho_1}$$

for $s \in [0, 2\pi\rho_1]$ and $\rho_1 = e^{-\pi^2/\ell}$. Therefore, thanks also to Equation (4.29), the statement follows

$$W(\mathcal{A}_\ell(\varepsilon), \hat{I}_\ell) = -\frac{\pi^3}{\ell} + \frac{2\pi^2}{\varepsilon} + 2b(\varepsilon) + O(\varepsilon, \ell) \ .$$

■

General case

Let us now consider an hyperbolic domain $\Omega \subseteq \mathbb{C}$ containing a short geodesic γ of length $\ell \leq \varepsilon_0$ with respect to the hyperbolic conformal metric h on Ω. For any $\varepsilon > 0$ small enough, let then $\mathcal{A}_h(\varepsilon)$ be a tubular neighborhood of γ obtained by cutting Ω with two hyperplanes whose boundaries at infinity have length ε with respect to h.

Theorem 4.4.14. The W-volume associated to $(\mathcal{A}_h(\varepsilon), h)$, with $\mathcal{A}_h(\varepsilon)$ the tubular neighborhood of a short simple closed curve in a domain $\Omega \subseteq \mathbb{C}$ of length ℓ with respect to the conformal hyperbolic metric h of Ω, with round boundary components of length a small enough ε, has the following behaviour:

$$W(\mathcal{A}_h(\varepsilon), h) = -\frac{\pi^3}{\ell} + \frac{2\pi^2}{\varepsilon} + 2b(\varepsilon) + O(\ell, \varepsilon) \quad ,$$

where $O(\ell, \varepsilon)$ is a function of ℓ and ε such that the limit $\lim_{\ell\to 0} \frac{O(\ell,\varepsilon)}{\ell}$ si finite.

Proof. Thanks to Theorem 3.5.4 (that is, Theorem 1.1 in [CG25]), for any $\varepsilon > 0$ small enough, we can assume $\mathcal{A}_h(\varepsilon)$ to be contained in a neighborhood of γ whose developing map f is defined on some neighborhood of the geodesic through 0 and ∞ in $\mathbb{H}^2$, and such that

$$\mathcal{S}(f) = \frac{1}{2z^2}\left(1 + \frac{4\pi^2}{\ell^2}\right) dz^2 + O\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right) dz^2 \ .$$

Note that, by Remark 4.4.11, the first term in $\mathcal{S}(f)$ coincides with the Schwarzian derivative of the developing map of the symmetric tube $\mathcal{A}_\ell(\varepsilon)$. We can then integrate the Schwarzians $\mathcal{S}(f_\ell)$ and $\mathcal{S}(f)$ (see [Dum08, Section 3.2]) to obtain, up to Möbius transformation, the developing maps of $\mathcal{A}_h(\varepsilon)$ and $\mathcal{A}_\ell(\varepsilon)$, respectively. In general, every holomorphic quadratic differential $\psi(z)dz^2$ can be realized as the Schwarzian derivative of a locally injective holomorphic function equal, up to Möbius maps, to the quotient of a base of the two-dimensional space of solutions of the following ODE (see [Hub16, Section 6.3])

$$f(z)'' + \frac{1}{2}\psi(z)f(z) = 0 \ .$$

Since the solutions of the just above ODE depend continuously on $\psi(z)$, the supremum norm of the difference of the developing maps of $\mathcal{A}_h(\varepsilon)$ and $\mathcal{A}_\ell(\varepsilon)$ is bounded by a factor of order $O\left(\frac{e^{-\pi^2/(2\ell)}}{\ell^2}\right)$. By continuity of the W-volume, the statement follows now from Proposition 4.4.13. ■

4.4.2 Convergence of the adapted renormalized volume

We consider $M = \mathbb{H}^3/\Gamma$, a convex co-compact hyperbolic 3-manifold with compressible boundary, and we denote by $h \in [\partial_\infty M]$ the unique hyperbolic representative in its conformal boundary at infinity. Let also $\gamma \subseteq \partial_\infty M = \Omega(\Gamma)/\Gamma$ be a compressible simple closed curve of length ℓ with respect to h, and consider its lift to $\Omega(\Gamma)$, which we still denote by γ. For any small enough $\varepsilon > 0$, we furthermore consider the topological tube $\mathcal{A}_h(\varepsilon) \subseteq \Omega(\Gamma)$ containing γ obtained by cutting the boundary at infinity of M with the boundaries of the two hyperplanes $H_i(\varepsilon)$ of length ε with respect to h. Note that, if h coincides with $\hat{I}_\ell$, then also $\mathcal{A}_h(\varepsilon) = \mathcal{A}_\ell(\varepsilon)$, with $\hat{I}_\ell$ and $\mathcal{A}_\ell(\varepsilon)$ as in Section 4.4.1, defined, respectively, by Equations (4.28) and (4.27). As a direct consequence of Theorem 3.5.4 (Theorem 1.1 in [CG25]), when ℓ is sufficiently small, the two tubes are almost projectively equivalent, as seen in the proof of Theorem 4.4.14. We denote by $N_h(\varepsilon)$ the compact region in M bounded by the Epstein surface associated to the domain with boundary $(\mathcal{A}_h(\varepsilon), h)$ and the two hyperplanes $H_i(\varepsilon)$, as described in Section 4.2.3.

The goal of this section is to show that the function $\widetilde{V_R}$ extends continuously to the strata in the boundary of the Weil-Petersson completion $\overline{\mathcal{T}(\partial\overline{M})}^{wp}$ corresponding to compressible multicurves (see Section 1.2.3), and that the limit is equal to the sum of the adapted renormalized volumes of convex co-compact manifolds pointed at the pinched components of the multicurve of the corresponding stratum, defined in 4.4.18. Recall that a pointed convex co-compact manifold is a pair (M, P), where $P \subseteq \partial\overline{M}$ is a finite set of points, and that its renormalized volume was defined in 4.3.12 (see also Remark 4.3.16).

To this end, we first need to recall the following result on the geometric convergence of convex co-compact hyperbolic manifolds under the pinching of compressible (multi)curves (see Section 4.2.2).

Theorem 4.4.15. ([SW22, Section $A.10$]) Let $M_n = (M, g_n)$ be a sequence of convex co-compact hyperbolic 3-manifolds, and let m be a compressible multicurve in $\partial\overline{M}$ such that the conformal boundaries $\partial_\infty M_n$ converge to a conformal structure X_m in the stratum of $\overline{\mathcal{T}(\partial\overline{M})}^{wp}$ corresponding to m. Let also $D(m)$ be a union of disks compressing m. For each n, fix a point $y_i(n)$ in the thick part of every of the i connected components of $C(M_n) \setminus D(m)$. Then $(M_n, y_i(n))$ converges in the Gromov-Hausdorff topology to a pointed complete hyperbolic manifold (M_i, P_i), whose boundary at infinity is the union of the connected components of X_m facing the one of $y_i(n)$ in $C(M_n) \setminus D(m)$. Moreover, each limit M_i is either convex co-compact, a solid torus, or a 3-ball.

In the notation of the theorem above, we denote by h_{P_i} the complete hyperbolic metric in the conformal class of the connected component(s) of X_m corresponding to (M_i, P_i) that has a cusp at each point $p \in P_i$. We also remark that the manifold M_i is diffeomorphic to the i-th connected component of $M \setminus D(m)$.

Remark 4.4.16. From now on, we will always assume that all pointed Gromov-Hausdorff limits are convex co-compact. However, we remark that even in the cases where M_i is diffeomorphic to a solid torus or a 3-ball, the Euler characteristic of $\partial M_i \setminus P_i$ is negative, and therefore the hyperbolic metric h_{P_i} still exists. The definition of renormalized volume of a pointed convex co-compact manifold seems to be extendable to these cases, as well as the result that follows.

Definition 4.4.17. Let $m \subseteq S$ be a multicurve on a closed surface. We say that m is *admissible* if every component of its complement $S \setminus m$ is a surface of genus $g \geq 2$.

We now define the adapted version for the renormalized volume of pointed convex co-compact manifolds (M,P) (as in Definition 4.3.12), as a function on the strata of the boundary of the Teichmüller space. We first fix some notation. Given M a convex cocompact manifold diffeomorphic to the interior of N a tame manifold, and $m \subseteq \partial N$ a compressible multicurve, we denote by N_i, with $i = i, \cdots, k$, the connected components of $N \setminus D(m)$. Moreover, recall that a point in the stratum $\mathcal{T}(\partial\overline{M} \setminus m)$ can be seen as the union (X,P) of pointed Riemann surfaces (X_i, P_i), and each of them is homeomorphic to ∂N_i. We also denote by $M_i(X_i)$ the convex cocompact manifold diffeomorphic to the interior of N_i and with conformal boundary X_i.

Definition 4.4.18. Let M be a convex co-compact manifold, let $m \subseteq \partial\overline{M}$ be an admissible compressible multicurve, and let $\mathcal{T}(\partial\overline{M} \setminus m)$ be the corresponding stratum in $\overline{\mathcal{T}(\partial\overline{M})}^{wp}$. In the notation above, we define the *adapted renormalized volume on the stratum of* m as the function

$$\widetilde{V_R} \colon \mathcal{T}(\partial\overline{M} \setminus m) \to \mathbb{R}$$

such that

$$\widetilde{V_R}(X,P) \coloneqq \sum_{i=1}^{k} \widetilde{V_R}(M_i(X_i), P_i) \coloneqq \sum_{i=1}^{k} \left[V_R(M_i(X_i), P_i) + \pi^3 \sum_{\substack{\gamma \text{ compressible} \\ \ell_\gamma(X_i) < \varepsilon_0}} \frac{1}{\ell_\gamma(X_i)} \right]$$

where $\varepsilon_0 = 2\mathrm{arsinh}(1)$ is the Margulis constant, γ runs in the set of compressible simple closed curves in $\partial\overline{M_i}$, and $\ell_\gamma(X_i)$ denotes its length with respect to the hyperbolic representative in X_i.

A simple closed curve on a surface is called *separating* if its complement is disconnected. The two cusps that arise when a curve is pinched, depending on whether this is separating or not, lie in either two different or the same connected component of the limit Riemann surface X_m. In the case of a multicurve one has to be more careful, as a union of non-separating curve can still have a disconnected complement. In any case P_i corresponds to the subset of the $2|m|$ cusp singularities in the components of X_m in the boundary of M_i. To avoid the technicalities in the notation due to this phenomenon, we first state and prove the theorem for the pinching of a single separating compressible curve, whose compressing disk also disconnect the 3-manifold.

Theorem 4.4.19. Let $M_t = (M, g_t)$ be a path of convex co-compact hyperbolic 3-manifolds with connected boundary obtained, via uniformization theorem, by the pinching of an admissible separating compressible simple closed curve γ in the conformal boundary at infinity, with separating compressing disk $D(\gamma)$. Let also (M_1, g_1, p_1) and (M_2, g_2, p_2) be the two pointed convex cocompact limits of $(M_t, y_{i,t})$ in the Gromov-Hausdorff topology, with $y_i(t)$, for $i = 1, 2$, lying in the thick part of the two different connected components of $C(M_t) \setminus D(\gamma)$. Then, outside a codimension-one set

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = \widetilde{V_R}(M_1, g_1, p_1) + \widetilde{V_R}(M_2, g_2, p_2)\ .$$

Before proceeding with the proof, we remark that the continuous extension fails precisely at the intersection between the closure of the codimension-one locus in which $\widetilde{V_R}$ is just lower semi-continuous (see Remark 4.4.3) and the boundary of the Teichmüller space.

Proof. Since the convex core of any $M_t = \mathbb{H}^3/\Gamma_t$ contains a hyperbolic geodesic $\mathcal{L}_t$ going through the the (unique up to isotopy) disk $D(\gamma)$ that compresses γ, up to composition with a path of Möbius transformations φ_t, denoting by Ω_t the domain of discontinuity of M_t, we can assume $\Omega_t \subseteq \mathbb{CP}^1 - \{0, \infty\}$ and $\mathcal{L}_t$ to be the geodesic through 0 and ∞. Let us denote by $\ell(t)$ the length of γ with respect to the hyperbolic representative $h_t \in [\partial_\infty M_t]$ in the conformal class of the boundary at infinity of M_t. We have then that $\ell(t) \to 0$.

For any $\varepsilon > 0$ small enough, we also denote by $\mathcal{A}_{h_t}(\varepsilon)$ the tube containing γ obtained by cutting the boundary at infinity with two hyperplanes with boundary in (Ω_t, h_t) of length ε with respect to h_t. Again up to a path of Mobius transformations, we can assume that $\mathcal{A}_{h_t}(\varepsilon)$ is contained in $\mathbb{D}^*$ for any t big enough. With an abuse of notation, we denote by $\mathcal{A}_{h_t}(\varepsilon)$ also its image via the projection to $\partial_\infty M_t = \Omega_t/\Gamma_t$. Since by hypothesis $\partial \overline{M}_t$ is connected and γ is separating, $\partial_\infty M_t - \mathcal{A}_{h_t}(\varepsilon)$ has two connected components which we denote by $(\partial_\infty M_t)_i(\varepsilon)$, for $i = 1, 2$. Let also $\Gamma_{t,i} < \mathbb{P}SL(2, \mathbb{C})$ denote the fundamental groups of the two connected manifolds obtained by cutting along the compressible disk $D(\gamma)$, whose free product gives Γ_t.

We recall that the conformal boundary at infinity $\partial_\infty M_t$ is naturally equipped with a complex projective structure, and so also $(\partial_\infty M_t)_i(\varepsilon)$ is equipped with the complex projective structure induced by the restriction. Analogously, we also denote by $\Omega_{i,t}(\varepsilon)$ the components of $\Omega_t - \mathcal{A}_{h_t}(\varepsilon)$, which cover $(\partial_\infty M_t)_i(\varepsilon)$. In this way, the Epstein surface associated to $((\partial_\infty M_t)_i(\varepsilon), h_t)$, is the quotient of the one associated to $(\Omega_{i,t}(\varepsilon), h_t)$ by the action of the group of Möbius transformation $\Gamma_{t,i}$. Let us fix now a small $\varepsilon > 0$. By additivity of the W-volume (see Lemma 4.2.12), we can split the renormalized volume of M_t as

$$V_R(M_t) = W((\partial_\infty M_t)_1(\varepsilon), h_t) + W((\partial_\infty M_t)_2(\varepsilon), h_t) + W(\mathcal{A}_{h_t}(\varepsilon), h_t) \ .$$

Then, the adapted renormalized volume of M_t splits as

$$\widetilde{V_R}(M_t) = W((\partial_\infty M_t)_1(\varepsilon), h_t) + W((\partial_\infty M_t)_2(\varepsilon), h_t) + W(\mathcal{A}_{h_t}(\varepsilon), h_t) + \frac{\pi^3}{\ell(t)} + L(\partial_\infty M_t, \gamma) \ ,$$

where every time we consider h_t restricted to the domain we are referring to, and with

$$L(X, \gamma) = \pi^3 \sum_{\substack{\alpha \neq \gamma \text{ compressible} \\ \ell_\alpha(X) < \varepsilon_0}} \frac{1}{\ell_\alpha(X)} \ .$$

We denote respectively by h_1 and h_2 the hyperbolic representatives in the conformal boundary at infinity of the convex co-compact manifolds (M_1, g_1) and (M_2, g_2), and by h_{p_1} and h_{p_2} the corresponding unique conformal hyperbolic metrics with a cusp, respectively, at p_1 and p_2. We also denote by $\Omega_i(\varepsilon)$, for $i = 1, 2$, the complement in the discontinuity domain of M_i of a ball at p_i with boundary of length ε with respect to h_{p_i}. Now, observe that the Gromov-Hausdorff convergence implies the one of the domains $\Omega_{i,t}(\varepsilon) \to \Omega_i(\varepsilon)$. Therefore, by continuity of the W-volume

$$\lim_{t\to\infty} W((\partial_\infty M_t)_1(\varepsilon), h_t) = W(\partial_\infty M_1(\varepsilon), h_{p_1})$$

and equivalently

$$\lim_{t\to\infty} W((\partial_\infty M_t)_2(\varepsilon), h_t) = W(\partial_\infty M_2(\varepsilon), h_{p_2})$$

where again all the metrics are considered restricted to the domains indicated. Moreover, since $\ell(t) \to 0$, by Theorem 4.4.14

$$\lim_{t\to\infty} W(\mathcal{A}_{h_t}(\varepsilon), h_t) + \frac{\pi^3}{\ell(t)} = \frac{2\pi^2}{\varepsilon} + 2b(\varepsilon) \ .$$

Therefore, for any $\varepsilon > 0$

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = W(\partial_\infty M_1(\varepsilon), h_{p_1}) + \frac{\pi^2}{\varepsilon} + b(\varepsilon) + W(\partial_\infty M_2(\varepsilon), h_{p_2}) + \frac{\pi^2}{\varepsilon} + b(\varepsilon) + \lim_{t\to\infty} \text{Ł}(\partial_\infty M_t, \gamma) \ .$$

Since $\partial_\infty M_t$ converges to the union of $(\partial_\infty M_1, p_1)$ and $(\partial_\infty M_2, p_2)$, taking the limit for $\varepsilon \to 0$ of the above equality, by Definition 4.3.12 and Definition 4.4.18, up to the codimension-one set where the function $L(X, \gamma)$ is only lower semi continuous

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = \widetilde{V_R}(M_1, g_1, p_1) + \widetilde{V_R}(M_2, g_2, p_2) \ .$$

■

The proof of Theorem 4.4.19 applies by iteration to the case of pinching a compressible multicurve, using the definition of renormalized volume for pointed convex co-compact manifolds of remark 4.3.16.

Theorem 4.4.20. Let $M_t = (M, g_t)$ be a path of convex co-compact hyperbolic 3-manifolds obtained by pinching an admissible compressible multicurve m in the conformal boundary at infinity. Let $D(m)$ be a union of disks compressing m, and let (M_i, g_i, P_i), for $i = 1, \dots, k$, be the pointed convex co-compact limits of $(M_t, y_i(t))$ in the Gromov-Hausdorff topology, with $y_i(t)$ in the thick part of the i-th connected component of $C(M_t) \setminus D(m)$. Then, outside a codimension-one set

$$\lim_{t\to\infty} \widetilde{V_R}(M_t) = \sum_{i=1}^{k} \widetilde{V_R}(M_i, g_i, P_i) \ .$$

4.5 On the definition of adapted renormalized volume

We dedicate this section to presenting two alternative possibilities for the definition of adapted renormalized volume: one that fails, and another that appears promising and for which seems that the proofs of the main theorems of this chapter can be easily adjusted. The main aim would be to avoid the discontinuities of Definition 4.4.1, see Remark 4.4.3.

Infinite sum

The discontinuities in definition 4.4.1 arise from the use of an upper bound - specifically ε_0 - on the lengths of curves included in the sum. This restriction is necessary due the well known fact that the sum of the inverses of the lengths of all the simple closed curves on a hyperbolic surface diverges. One might hope that restricting the sum to only compressible curves would ensure convergence, but this is not the case, as the following example illustrates.

Let M be a convex co-compact handlebody of genus 2. Consider simple closed curves $a, a_1, b, b_1 \in \partial_\infty M$ representing generators of the first homology group of $\partial\overline{M}$. Suppose that a and b are compressible, while a_1 and b_1 are incompressible, and that their geometric intersection numbers satisfy $i(a, a_1) = 1 = i(b, b_1)$, and vanish otherwise. Then, it is easy to verify that the commutator $[a, b_1] = ab_1a^{-1}b_1^{-1}$ is simple. Moreover, the image of $[a, b_1]$ in $\pi_1(M)$ is trivial, and therefore $[a, b_1]$ is compressible. Now, one can perform Dehn twists on b to get infinitely many simple closed compressible curves, whose inverse length sum diverges.

Since the space of convex co-compact hyperbolic 3-manifolds is parameterized by the Teichmüller space modulo the subgroup generated by Dehn twists along compressible simple closed curves, one might consider modifying the sum in Definition 4.4.1 by summing over all compressible simple closed curves up to this equivalence. Unfortunately, even after this identification, the sum still diverges. Let us show an example. Consider again the curves a, a_1, b, b_1 as before, and the family of simple closed (incompressible) curves $b_1^n b$, for $n \in \mathbb{N}$. Now, for any n, the commutators $[a, b_1^n b]$ are all simple closed compressible curves, that cannot be obtained by a Dehn twsits on a compressible curve from one another. Moreover, the sum over the inverses of their lengths diverges again.

Maximum on multicurves

Another way to avoid the discontinuities is to modifying our definition of adapted renormalized volume by taking the maximum over the compressible multicurves of the same sum:

$$\widetilde{V_R}(X) = V_R(X) + \max_{m \text{ compressible}} \sum_{\gamma \in m} \frac{\pi^3}{\ell_\gamma(X)} .$$

A first remark is that the maximum is the same as the one over the *maximal* compressible multicurves. Once proven the following lemmas, the proofs of Theorems 4.4.4, 4.4.5, 4.4.7, 4.4.6 and 4.4.20 should work similarly.

1. The maximum exists
2. There exists an $\varepsilon > 0$ such that the multicurves realizing the maximum contain all the simple compressible closed curves of length less than $\varepsilon > 0$
3. At any point X in the Teichmüller space there are a finite number of multicurves realizing the maximum.

We also remark that with this definition $\widetilde{V_R}$ is continuous and Lipschitz, as the standard renormalized volume is , and the additional term is now maximum of continuous functions. Moreover, the points in which it is not smooth correspond to the ones in which the multicurves realizing the maximum is not unique.

Bibliography

[Abi77] William Abikoff. Degenerating families of Riemann surfaces. *Ann. of Math. (2)*, 105(1):29–44, 1977.

[Ago04] Ian Agol. Tameness of hyperbolic 3-manifolds. http://arxiv.org/abs/math/0405568, 2004.

[Ahl64] Lars V. Ahlfors. Finitely generated kleinian groups. *American Journal of Mathematics*, 86(2):413–429, 1964.

[Ahl10] Lars V. Ahlfors. *Conformal invariants*. AMS Chelsea Publishing, Providence, RI, 2010. Topics in geometric function theory, Reprint of the 1973 original, With a foreword by Peter Duren, F. W. Gehring and Brad Osgood.

[Ahl21] Lars Ahlfors. *Complex analysis—an introduction to the theory of analytic functions of one complex variable*. AMS Chelsea Publishing, Providence, RI, third edition, [2021] ©2021. Reprint of the 1978 original [0510197].

[AKR23] Clara L. Aldana, Klaus Kirsten, and Julie Rowlett. Polyakov formulas for conical singularities in two dimensions. https://arxiv.org/abs/2010.02776, 2023.

[BB16] Jeffrey F. Brock and Kenneth W. Bromberg. Inflexibility, Weil-Peterson distance, and volumes of fibered 3-manifolds. *Math. Res. Lett.*, 23(3):649–674, 2016.

[BB24] Martin Bridgeman and Kenneth Bromberg. Epstein surfaces, w-volume, and the osgood-stowe differential. https://arxiv.org/abs/2402.15065, 2024.

[BBB19] Martin Bridgeman, Jeffrey Brock, and Kenneth Bromberg. Schwarzian derivatives, projective structures, and the weil–petersson gradient flow for renormalized volume. *Duke Mathematical Journal*, 168(5):867 – 896, 2019.

[BBB23] Martin Bridgeman, Jeffrey Brock, and Kenneth Bromberg. The Weil-Petersson gradient flow of renormalized volume and 3-dimensional convex cores. *Geom. Topol.*, 27(8):3183–3228, 2023.

[BBPW25] Martin Bridgeman, Kenneth Bromberg, Franco Vargas Pallete, and Yilin Wang. Universal liouville action as a renormalized volume and its gradient flow. https://arxiv.org/abs/2311.18767, 2025.

[BBVP23] Martin Bridgeman, Kenneth Bromberg, and Franco Vargas Pallete. Convergence of the gradient flow of renormalized volume to convex cores with totally geodesic boundary. *Compos. Math.*, 159(4):830–859, 2023.

[BC03] Martin J. Bridgeman and Richard Canary. From the boundary of the convex core to the conformal boundary. *Geometriae Dedicata*, 96:211–240, 2003.

[BC05] Martin Bridgeman and Richard D. Canary. Bounding the bending of a hyperbolic 3-manifold. *Pacific Journal of Mathematics*, 218(2):299–314, 2005.

[BC17] Martin Bridgeman and Richard D. Canary. Renormalized volume and the volume of the convex core. *Annales de l'Institut Fourier*, 67(5):2083–2098, 2017.

[Ber74] Lipman Bers. *Spaces of degenerating Riemann surfaces*, pages 43–56. Princeton University Press, Princeton, 1974.

[Bes87] Arthur Besse. *Einstein Manifolds*. Springer, 1987.

[BP23] Jeffrey Brock and Franco Vargas Pallete. W-volume for planar domains with circular boundary. https://arxiv.org/abs/2312.00230, 2023.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces*. Modern Birkhäuser Classics. Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1992 edition.

[Can01] R. D. Canary. The conformal boundary and the boundary of the convex core. *Duke Mathematical Journal*, 106(1):193 – 207, 2001.

[CEM06] Richard D. Canary, David Epstein, and Albert Marden. *Fundamental of Hyperbolic Manifolds*. Cambridge University Press, 2006.

[CG06] Danny Calegary and David Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *Journal of American Mathematical Society*, 19(2):385–446, 2006.

[CG25] Tommaso Cremaschi and Viola Giovannini. Behaviour of the schwarzian derivative on long complex projective tubes. https://arxiv.org/abs/2502.10071, 2025.

[CGS24] Tommaso Cremaschi, Viola Giovannini, and Jean-Marc Schlenker. Filling riemann surfaces by hyperbolic schottky manifolds of negative volume. https://arxiv.org/abs/2405.07598, 2024.

[dHSS01] Sebastian de Haro, Kostas Skenderis, and Sergey N. Solodukhin. Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence. *Comm. Math. Phys.*, 217(3):595–622, 2001.

[DK12] Raquel Díaz and Inkang Kim. Asymptotic behavior of grafting rays. *Geom. Dedicata*, 158:267–281, 2012.

[Dum08] David Dumas. Complex projective structures. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 455–508. Eur. Math. Soc., Zürich, 2008.

[Dum17] David Dumas. Holonomy limits of complex projective structures. *Adv. Math.*, 315:427–473, 2017.

[DW08] David Dumas and Michael Wolf. Projective structures, grafting and measured laminations. *Geometry & Topology*, 12(1):351 – 386, 2008.

[EM86] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces. In D. B. A. Epstein, editor, *Analytical and geometric aspects of hyperbolic space*, volume 111 of *L.M.S. Lecture Note Series*. Cambridge University Press, 1986.

[Eps84] Charles L. Epstein. Envelopes of horospheres and weingarten surfaces in hyperbolic 3-space. https://arxiv.org/abs/2401.12115, 1984.

[FM11] Benson Farb and Dan Margalit. *Primer on Mapping Class Groups*. Princeton Mathematical Series, 2011.

[FSX19] Yu Feng, YiQian Shi, and Bin Xu. Isolated singularities of conformal hyperbolic metrics. *Chinese Ann. Math. Ser. A*, 40(1):15–26, 2019.

[Gar75] Frederick P. Gardiner. Schiffer's interior variation and quasiconformal mapping. *Duke Math. J.*, 42(1):371–380, 1975.

[Gar00] Nikola Gardiner, Frederick P.; Lakic. *Quasiconformal Teichmüller Theory*, volume 76 of *Mathematical Surveys and Monographs*. American Mathematical Society, first edition, 2000.

[GL91] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.

[GMM09] David Gabai, Robert Meyerhoff, and Peter Milley. Minimum volume cusped hyperbolic three-manifolds. *Journal of the American Mathematical Society*, 22(4):1157–1215, October 2009.

[GMR15] Colin Guillarmou, Sergiu Moroianu, and Frédéric Rochon. Renormalized volume on the teichmüller space of punctured surfaces. *Annali scuola normale superiore - classe di scienze*, 2015.

[GW99] C. Robin Graham and Edward Witten. Conformal anomaly of submanifold observables in AdS/CFT correspondence. *Nuclear Phys. B*, 546(1-2):52–64, 1999.

[Hei62] Maurice Heins. On a class of conformal metrics. *Nagoya Math. J.*, 21:1–60, 1962.

[Hem76] John Hempel. *3-Manifolds*. Princeton University Press, 1976.

[Hen11] Sebastian W. Hensel. Iterated grafting and holonomy lifts of teichmüller space. *Geometriae Dedicata*, 155:31–67, 2011.

[Hub59] Heinz Huber. Zur analytischen theorie hyperbolischer raumformen und bewegungsgruppen. *Mathematische Annalen*, 138(1):1–26, 1959.

[Hub16] John H Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics*, volume 1-2. Matrix Editions, 2016.

[Kli95] Wilhelm P. A. Klingenberg. *Riemannian geometry*, volume 1 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 1995.

[KM81] Irwin Kra and Bernard Maskit. *Remarks on Projective Structures*, pages 343–360. Princeton University Press, Princeton, 1981.

[KM18] Sadayoshi Kojima and Greg McShane. Normalized entropy versus volume for pseudo-Anosovs. *Geom. Topol.*, 22(4):2403–2426, 2018.

[Kra32] W. Kraus. Über den zusammenhang einiger charakteristiken eines einfach zusammenhängenden bereiches mit der kreisabbildung. *Mitteilungen des Mathematischen Seminars Giessen*, 21:1–28, 1932.

[Kra00] Kirill Krasnov. Holography and Riemann surfaces. *Adv. Theor. Math. Phys.*, 4(4):929–979, 2000.

[KS08] Kirill Krasnov and Jean-Marc Schlenker. On the renormalized volume of hyperbolic 3-manifolds. *Communications in Mathematical Physics*, (279):637–668, 2008.

[KS12] Kirill Krasnov and Jean-Marc Schlenker. The Weil-Petersson metric and the renormalized volume of hyperbolic 3-manifolds. In *Handbook of Teichmüller theory. Volume III*, volume 17 of *IRMA Lect. Math. Theor. Phys.*, pages 779–819. Eur. Math. Soc., Zürich, 2012.

[LLX20] Bo Li, Long Li, and Bin Xu. Bounded projective functions and hyperbolic metrics with isolated singularities. *Ann. Acad. Sci. Fenn. Math.*, 45(2):687–698, 2020.

[Mar06] Albert Marden. A proof of the Ahlfors finiteness theorem. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 247–257. Cambridge Univ. Press, Cambridge, 2006.

[Mar16a] Albert Marden. *Hyperbolic Manifolds: An introduction in 2 and 3 dimensions.* Cambridge University Press, 2016.

[Mar16b] Bruno Martelli. An introduction to geometric topology. arXiv:1610.02592v1, 2016.

[Mas76] Howard Masur. Extension of the Weil-Petersson metric to the boundary of Teichmuller space. *Duke Math. J.*, 43(3):623–635, 1976.

[McM96] Curtis McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle.* Princeton University Press, 1996.

[McM98] Curt McMullen. Complex earthquakes and teichmüller theory. *Journal of the American Mathematical Society*, 11:283–320, 1998.

[McM00] Curtis T. McMullen. The moduli space of Riemann surfaces is Kähler hyperbolic. *Ann. of Math. (2)*, 151(1):327–357, 2000.

[MT98] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic Menifolds and Kleinian Groups.* Oxford University Press, 1998.

[Neh49] Zeev Nehari. The schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.*, 55:545–551, 1949.

[OS92] Brad Osgood and Dennis Stowe. The schwarzian derivative and conformal mapping of riemannian manifolds. *Duke Mathematical Journal*, 67(1):57 – 99, 1992.

[Pal16] Franco Vargas Pallete. Continuity of the renormalized volume under geometric limits. https://arxiv.org/abs/1605.07986, 2016.

[Pal17] Franco Vargas Pallete. Additive continuity of the renormalized volume under geometric limits. arXiv:1708.04009, 2017.

[PMG19] Franco Vargas Pallete and Dídac Martínez-Granado. Comparing hyperbolic and extremal lengths for shortest curves. https://arxiv.org/abs/1911.09078, 2019.

[PWW25] Franco Vargas Pallete, Yilin Wang, and Catherine Wolfram. Epstein curves and holography of the schwarzian action. https://arxiv.org/abs/2503.05690, 2025.

[RS99] Igor Rivin and Jean-Marc Schlenker. The Schläfli formula in Einstein manifolds with boundary. *Electron. Res. Announc. Amer. Math. Soc.*, 5:18–23, 1999.

[Sch13] Jean-Marc Schlenker. The renormalized volume and the volume of the convex core of quasifuchsian manifolds. *Math. Res. Lett.*, 20(4):773–786, 2013. Corrected version available as arXiv:1109.6663v4.

[Sch20] Jean-Marc Schlenker. Volumes of quasifuchsian manifolds. *Surveys in Differential Geometry*, 25(1):319–353, 2020. arXiv:1903.09849.

[Sou04] Rabah Souam. The Schläfli formula for polyhedra and piecewise smooth hypersurfaces. *Differential Geom. Appl.*, 20(1):31–45, 2004.

[SW02] Kevin P. Scannell and Michael Wolf. The grafting map of Teichmüller space. *J. Amer. Math. Soc.*, 15(4):893–927 (electronic), 2002.

[SW22] Jean-Marc Schlenker and Edward Witten. No ensemble averaging below the black hole threshold. *J. High Energy Phys.*, (7):Paper No. 143, 50, 2022.

[Thu80] William P. Thurston. Three-dimensional geometry and topology. Originally notes of lectures at Princeton University, 1979. Recent version available on http://www.msri.org/publications/books/gt3m/, 1980.

[TT03] Leon A. Takh and Lee-Peng Teo. Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography. *Comm. Math. Phys.*, 239(1-2):183–240, 2003.

[VP25] Franco Vargas Pallete. Upper bounds on renormalized volume for Schottky groups. *Math. Ann.*, 392(1):733–750, 2025.

[Wol82] Scott Wolpert. The Fenchel-Nielsen deformation. *Ann. of Math. (2)*, 115(3):501–528, 1982.

[Wol03] Scott A. Wolpert. Geometry of the Weil-Petersson completion of Teichmüller space. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, volume 8 of *Surv. Differ. Geom.*, pages 357–393. Int. Press, Somerville, MA, 2003.

[Wol06] Scott A. Wolpert. Convexity of geodesic-length functions: a reprise. In *Spaces of Kleinian groups*, volume 329 of *London Math. Soc. Lecture Note Ser.*, pages 233–245. Cambridge Univ. Press, Cambridge, 2006.

[ZT87] P. G. Zograf and L. A. Takhtadzhyan. On the uniformization of Riemann surfaces and on the Weil-Petersson metric on the Teichmüller and Schottky spaces. *Mat. Sb. (N.S.)*, 132(174)(3):304–321, 444, 1987.