Mathematics for voting: rankings and gerrymandering

Mathematics is useful beyond science and technology: we support this claim by showing that the mathematics used in voting goes beyond simply counting ballots.
We first present an activity about *rankings*: we create a setting that is realistic for pupils, namely choosing a movie for a movie night by ranking three movies. The key lesson here is that the rule chosen to determine a winner from the rankings influences the outcome: we have selected numerical values in our example so that each of the three movies can be declared the winner (starting from the same ballots).
The second activity is about *gerrymandering*, which is the practice of grouping voters (creating voting districts) specifically to favor a certain outcome. In short, gerrymandering is like waging a war in which it is possible to select the number of soldiers from each side that participate in various battles (with the constraint that each soldier appears in precisely one battle, and that the total number of soldiers from each side is fixed). Our exploration is algebraic in nature: it starts with a very simple election (nine voters and three districts), and continues with a second part (more challenging but still within reach of the pupils) where one can have any number of districts with the same number of voters. Applying mathematics when choosing voting districts is important, as it can help counter gerrymandering. However, the *impossibility theorem for gerrymandering* states that there is no perfect way to divide voters into districts that is fair under all criteria: conditions such as equal population, compactness of district shape, and respect for communities can conflict with one another.
The two proposed activities are completely independent. They have been chosen for their relevance to real-life situations and because they are very accessible.

Rankings

The students of your school choose a movie for a movie night among three possibilities: Movie A, *All for love* (a romantic comedy), Movie B, *Boulevard police* (action movie), Movie C, *Creepy night* (horror movie). The students vote by ordering the movies in their order of preference (first choice, second choice, third choice). For example, they write $B > C > A$ to mean that B is the first choice, C the second choice, and A the third choice. Suppose that the result of the students' movie ranking is as follows:

$A > B > C$	38%
$A > C > B$	2%
$B > C > A$	18%
$B > A > C$	7%
$C > A > B$	3%
$C > B > A$	32%

In your opinion, which movie should be selected?

- Movie A could be selected because it's the most common first choice (40% of the students have A as first choice, 25% have B as first choice, 35% have C as first choice).
- Movie B could be selected because it wins in head-to-head comparison over Movie A and over Movie C (57% of the students prefer B to A, so B wins over A; 63% of the students prefer B to C, so B wins over C).
- Movie C could be selected, if we discard the students' least favorite first choice and reconsider the ranking (Movie B is the least favorite first option: removing B from the rankings we see that 47% of the students have A as first choice and 53% of the students have C as first choice).

Movie B is the compromise solution because it is the second choice for 70% of the students. Since 95% of the students rank movie B as first choice or second choice, it would be reasonable to suggest to select movie B.

The outcome of an election depends on the voting rule that is applied. Indeed, as we see in the above example, according to the chosen rule, it's either movie A or B or C that gets selected. For this reason, the voting rule must be established before the voting (and before the polls). The voting rule may depend, for example, on whether compromise solutions are well-accepted or not. For example, do we want to: Pick the movie with the most first-place votes? Choose the movie that most people are okay with, even if it's not their favorite? Avoid the movie that many people really don't like? Different rules can lead to different winners!

A different example: Replace B by a movie that is very similar to A (say, B is the movie *Best friends in love*). In this case, the fans of romantic comedies split their votes between A and B. That could make Movie C (the horror movie) win. Suppose that 64 % of students love A or B and dislike horror movie C. If those 64 % divide their votes evenly (32 % for A, 32 % for B), while the remaining 36 % vote for C, then C becomes the most common first choice. This example shows that similar choices can split supporters' votes and allow a different choice to win. It's something to keep in mind when voting or designing a fair election.

Gerrymandering

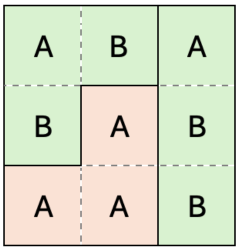
In certain elections, voters are divided into groups; in each group a winner is selected, and to win the election one has to win in most groups. For example, suppose that Alex and Kim compete to become the student representative at your school. Each class could express a preference, and then the candidate who wins in most classes becomes the student representative. *Gerrymandering* is when the groups are drawn to favour one candidate. For example, if polls show Alex does better when students are grouped by school year rather than by class, dividing voters by year instead of by class would give Alex an advantage. If the school president changes the groups this way to favour Alex, that is gerrymandering.

- *Nine voters choose between candidates A and B. Their votes are as follows:*

A	B	A
B	A	B
A	A	B

Who wins the elections? How can B win the election by gerrymandering, if we group the voters by groups of 3?

Candidate A has 5 votes out of 9 votes, so A wins by direct counting the favorable votes. If we group voters by row (or by column), A still wins the election by winning in 2 out of the 3 groups. Candidate B can gerrymander by choosing the groupings as follows:



In this case, B wins in 2 out of the 3 groups and hence wins the election.

- *Suppose that candidates A and B compete in an election, and there are 81 voters divided into groups of 9. If A has very accurate polls (so that A knows in advance who is going to vote for whom) and A can gerrymander, what is the least number of votes that A needs to have, in order to win the election?*

To win in a group, A needs 5 favorable votes out of the 9 votes. And A needs to win at least 5 groups out of 9 groups. So to win with the least amount of favorable votes, A needs to win precisely 5 groups with precisely 5 favorable votes in each of these groups (and without favorable votes in the remaining groups). This gives a total of 25 favorable voters that are necessary and sufficient for A to win the election with gerrymandering.

- *Suppose that candidates A and B compete in an election, and there are 100 voters divided into groups of 10. If A has very accurate polls (so that A knows in advance who is going to vote for whom) and A can gerrymander, what is the least number of votes that A needs to have, in order to win the election?*

We show that 35 favorable votes are sufficient for A to win the election. Remark that, to win in a group, A needs 6 favorable votes while to have a tie in a group, A needs 5 favorable votes.
Suppose that candidate A wins in w groups and gets a tie in t groups. Then candidate B gets a tie in t groups and wins in $10-w-t$ groups. Candidate A then wins if

$$w>10-w-t$$

which means that the requested condition is $2w+t>10$.

- If t=0, then we must have w≥6 and hence A would need 36 votes (to win in 6 groups). If t is even and strictly positive, then we may decrease t by 1 while keeping the requested inequality, so we can exclude this case.
- If t=1, then we must have w≥5 and hence A would need 35 votes (to win in 5 groups and have a tie in 1 group).
- If t is odd and strictly larger than 1, then we can decrease t by 2 and increase w by 1, leading to sparing 4 votes, so we can exclude this case.

- *In an election there are two competing candidates. Suppose that there are G groups of voters, and that in each group there are V voters (where G≥2 and V≥3). Provided that one candidate has very accurate information from polls and gerrymanders, what is the least number of favorable voters that such candidate needs to win the election? What is roughly the percentage of favorable voters that the candidate needs, if G and V are very large?*

The reasoning is similar to the previous problem, so we only provide the answer: Let g be the smallest integer that is larger than $G/2$ and v be the smallest integer that is larger than $V/2$. The number of needed favorable votes is gv, unless both G and V are even, in which case the number of needed favorable votes is $gv-1$.
The requested percentage is slightly larger than 25% because gv (respectively, $gv-1$ in the special case) is slightly larger than $GV/4$. This shows that gerrymandering is very powerful: one could win an election with only slightly more than 25% of the votes.

- *In fact, if one could gerrymander also by choosing the size of the group of voters, it would be possible to win an election with one billion people having only two favorable votes. How can this be possible?*

Make one group with only the first favorable voter, one group with the second favorable voter, and one group with all of the other voters. Winning two out of three groups means winning the election. This does not look fair at all, right? It's just gerrymandering, taken to the extreme.

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Abstract

In this note, we explore a voting system consisting in the ranking of three candidates: as we will see, it may not be so clear who the winner is. Afterwards, we explore gerrymandering in an algebraic way, concluding with an extreme gerrymandering: it is possible to win an election of one billion voters having only two favorable voters.