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# THE GEOMETRY OF HEREDITARY ORDERS AND BEYOND

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# Introduction

Noncommutative algebraic geometry is a potpourri of ideas and techniques from algebra, category theory, geometry and representation theory. The diversity of perspectives means that what one mathematician calls 'noncommutative algebraic geometry' may look substantially different from another's point of view. Therefore, the introduction serves as a prelude, briefly exploring the different leitmotifs from the chapters that follow.

In the context of this thesis the main protagonist is a pair $(X, \mathscr{A})$,

- the first object is usually an integral quasi-projective variety $X$ over a field $\mathbf{k}$,
- the second object $\mathscr{A}$ is a coherent $\mathcal{O}_X$-algebra $\mathscr{A}$.

We call such a 'noncommutative space' $(X, \mathscr{A})$ a *coherent ringed scheme.*

In the first part of the introduction we present three different but overlapping concepts which make a coherent ringed scheme an interesting object for noncommutative algebraic geometry. We then present the main results of the thesis. The last section is devoted to examples and applications of these results.

We should stress at this point that one usually allows $\mathscr{A}$ to be commutative as well. In particular, we will set $\mathscr{A} = \mathcal{O}_X$ from time to time. Thus, the more appropriate expression for 'noncommutative algebraic geometry' in our context would be 'not necessarily commutative algebraic geometry.' The community of noncommutative algebraic geometers has decided to go with the more appealing first term.

## I Links between noncommutative algebra and geometry

Our main focus lies on three concepts. Each of the concepts relates to coherent ringed schemes. We will relate to a coherent ringed scheme $(X, \mathscr{A})$

(I.1) a scheme $\mathsf{BS}(\mathscr{A})$ via a moduli problem,

(I.2) an admissible subcategory of the derived category $\mathsf{D}^{\mathsf{b}}(Y)$ of a scheme $Y$, and

(I.3) a stack $\mathcal{X}$ such that coherent $\mathscr{A}$-modules can be described as coherent sheaves on $\mathcal{X}$.

Coherent ringed schemes $(X, \mathscr{A})$ comprise a vast class of objects. Accordingly, we must restrict this class to make the aforementioned relations precise. We will often start from a noncommutative ringed scheme $(X, \mathscr{A})$, where $\mathscr{A}$ is an $\mathcal{O}_X$*-order*. By an $\mathcal{O}_X$-order we mean a coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ which is

i) torsionfree as an $\mathcal{O}_X$-module, and

ii) an Azumaya algebra on a non-empty open subset $U \subset X$.

## I.1 Brauer–Severi schemes

**Moduli spaces.** We use moduli spaces to classify algebro-geometric objects of a certain type. For example, let $X$ be a scheme and $\mathcal{V}$ a locally free $\mathcal{O}_X$-module of rank $n$. The moduli space of line subbundles $\mathcal{L} \subset \mathcal{V}$ with locally free quotient $\mathcal{V}/\mathcal{L}$ is the projective bundle $\mathbb{P}_X(\mathcal{V}) \to X$ of relative dimension $n-1$.

This example shows a common and desirable phenomenon. Usually, the moduli space is not simply a set, but a scheme. Hence, classification problems which admit a scheme as a moduli space can be studied by algebro-geometric methods.

Conversely, properties of the moduli space are linked to the underlying moduli problem. In the above example, the pushforward of the line bundle $\mathcal{O}_{\mathbb{P}_X(\mathcal{V})}(i)$ along $\pi\colon \mathbb{P}_X(\mathcal{V}) \to X$ is given by

$$\pi_* \mathcal{O}_{\mathbb{P}_X(\mathcal{V})}(i) \cong \mathrm{Sym}^i_X(\mathcal{V}), \tag{1}$$

where $\mathrm{Sym}^i_X(\mathcal{V})$ is the $i$-th symmetric power of $\mathcal{V}$.

**Brauer–Severi schemes.** We can also relate a coherent ringed scheme $(X, \mathscr{A})$ to $\mathbb{P}_X(\mathcal{V})$. The $\mathcal{O}_X$-algebra is given by the Azumaya algebra $\mathscr{A} = \mathcal{E}nd_X(\mathcal{V})$. This time, the projective bundle $\mathbb{P}_X(\mathcal{V})$ is the moduli space parametrizing left ideals of rank $n$ in $\mathscr{A} = \mathcal{E}nd_X(\mathcal{V})$. Therefore, we likewise refer to it as the Brauer–Severi scheme $\mathsf{BS}(\mathcal{E}nd_X(\mathcal{V})) = \mathbb{P}_X(\mathcal{V})$ of $\mathcal{E}nd_X(\mathcal{V})$.

More generally, Van den Bergh [137] showed that the Brauer–Severi scheme $\mathsf{BS}(\mathscr{A})$, parametrizing left ideals $I$ of rank $n$ in $\mathscr{A}$, can be constructed for every $\mathcal{O}_X$-algebra $\mathscr{A}$. Therefore, we have associated with each coherent ringed scheme $(X, \mathscr{A})$ a scheme $\mathsf{BS}(\mathscr{A})$.

Similarly to the isomorphism (1), we can ask whether and how $(X, \mathscr{A})$ determines properties of $\mathsf{BS}(\mathscr{A})$. For example, if $\mathscr{A}$ is an Azumaya algebra on $X$ of rank $n^2$, then $\mathsf{BS}(\mathscr{A}) \to X$ is étale locally a projective bundle of relative dimension $n-1$ [74]. It is a projective bundle if and only if $\mathscr{A} \cong \mathcal{E}nd_X(\mathcal{V})$ as in the example above.

For an $\mathcal{O}_X$-order $\mathscr{A}$, the Brauer–Severi scheme $\mathsf{BS}(\mathscr{A}) \to X$ is only generically étale locally a projective bundle, cf. [5]. Over the ramification locus $\Delta_{\mathscr{A}}$, which is the closed subscheme defined by $p \in \Delta_{\mathscr{A}}$ such that $\mathscr{A}(p)$ is not Azumaya, we obtain singular varieties as the fibers.

The geometry of the ramification locus $\Delta_{\mathscr{A}}$ already contains a lot of information about the Brauer–Severi scheme $\mathsf{BS}(\mathscr{A})$. In Figure 1, we depict the situation for an order $\mathscr{A}$ of rank 4 on the projective plane $\mathbb{P}^2$ ramified along three lines in general position. The Brauer–Severi scheme $\mathsf{BS}(\mathscr{A}) \to \mathbb{P}^2$ is a conic bundle with singular fibers over the ramification locus and non-reduced fibers over the singularities of the ramification locus, [7, 19].

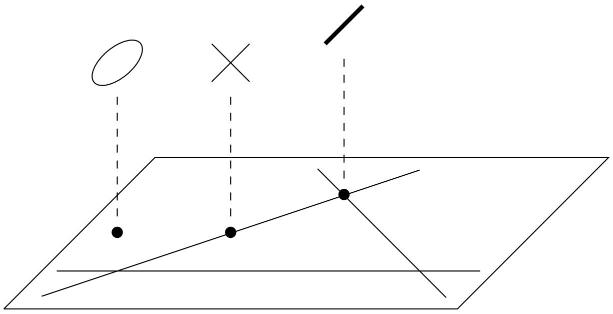


Figure 1: Fibers of $\mathsf{BS}(\mathscr{A})$ of an $\mathscr{O}_{\mathbb{P}^2}$-order $\mathscr{A}$ of rank 4 ramified along $\mathbb{V}(XYZ) \subset \mathbb{P}^2$.

## I.2 Derived categories and admissible components

We now switch perspectives. Instead of starting with a coherent ringed scheme $(X, \mathscr{A})$, we consider a smooth and projective variety $Y$. Gabriel–Rosenberg [68, 124] showed that the abelian category $\mathsf{coh}(Y)$ of coherent sheaves already determines $Y$ up to isomorphism of varieties.

When it comes to calculating invariants of $Y$, most of them like the dimension, K-theory or sums of Hodge numbers, are actually invariants of the bounded derived category $\mathsf{D}^{\mathsf{b}}(Y) := \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(Y))$. Therefore, one may ask whether $\mathsf{D}^{\mathsf{b}}(Y)$ determines the isomorphism class of $Y$.

There is a broad family of varieties [85, 64], including curves and surfaces, which have only finitely many derived equivalent but non-isomorphic varieties. In fact, a huge class of varieties, is determined by $\mathsf{D}^{\mathsf{b}}(Y)$. More precisely, Bondal–Orlov [37] showed that every smooth projective variety with ample (anti)canonical bundle is determined up to isomorphism by its derived category.

The importance of $\mathsf{D}^{\mathsf{b}}(Y)$ for the invariants of $Y$, and its non-trivial behavior with respect to non-isomorphic varieties has lead to a large community of algebraic geometers working on a better understanding of the derived category $\mathsf{D}^{\mathsf{b}}(Y)$ and operations on it.

A common question is how and whether the derived category $\mathsf{D}^{\mathsf{b}}(Y)$ can be ‘decomposed’ into ‘smaller’ pieces, [34, 98, 99]. We are aiming for a so-called semiorthogonal decomposition into components, each being a triangulated category itself. The components are each embedded by a fully faithful functor into $\mathsf{D}^{\mathsf{b}}(Y)$.

**Brauer–Severi schemes.** Let $Y$ be a moduli space. In connection to Section I.1, we ask how the original data of the moduli problem relates to a decomposition of $\mathsf{D}^{\mathsf{b}}(Y)$. We shall make this question more precise in what follows by looking at Brauer–Severi schemes and other well-studied moduli problems.

Denote by $\pi\colon \mathsf{BS}(\mathscr{A}) \to X$ the Brauer–Severi scheme of a coherent ringed scheme $(X, \mathscr{A})$. If $X$ is a smooth projective variety and $\mathscr{A}$ is an Azumaya algebra, a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ was constructed in a series of works [20, 113, 31], each treating the problem in progressively greater generality. The semiorthogonal decomposition takes the form

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{A}_{n-1}, \dots, \mathsf{A}_1, \pi^*\mathsf{D}^{\mathsf{b}}(X)\rangle. \tag{2}$$

The original data of the moduli problem links to the components $\mathsf{A}_i$ in the sense that there is an equivalence $\mathsf{A}_i \simeq \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}^{\otimes i})$ of triangulated categories.

**Admissible subcategories and moduli problems.** The observation that the original data of the moduli problem relates to a semiorthogonal decomposition of the moduli space is a recurring pattern. We illustrate this pattern with three examples.

- Let $C$ be a smooth projective curve of genus $g \geq 2$. Denote by $\mathrm{M}_C(2, \mathcal{L})$ the moduli space of vector bundles on $C$ of rank 2 and determinant $\mathcal{L}$. Then $\mathsf{D}^{\mathsf{b}}(\mathrm{M}_C(2, \mathcal{L}))$ admits a semiorthogonal decomposition with components given by $\mathsf{D}^{\mathsf{b}}(\mathrm{Sym}^i C)$, [65, 25, 133].
- Given a smooth projective surface $S$, denote by $S^{[n]}$ the Hilbert scheme parametrizing collections of $n$ points on $S$. The vanishing of certain Hodge numbers of $S$ is a necessary and sufficient criterion for the embedding of $\mathsf{D}^{\mathsf{b}}(S)$ into $\mathsf{D}^{\mathsf{b}}(S^{[n]})$, [92, 26].
- Similarly for an acyclic finite quiver $Q$, we can look at the moduli space $\mathrm{M}^{\theta}(Q, \mathbf{d})$ parametrizing representations of $Q$ with a fixed dimension vector $\mathbf{d}$. By [24], we often have an embedding of $\mathsf{D}^{\mathsf{b}}(\mathsf{rep}\, Q)$ into $\mathsf{D}^{\mathsf{b}}(\mathrm{M}^{\theta}(Q, \mathbf{d}))$.

In light of these results, we can reconsider the situation of Section I.1. Take a coherent ringed scheme $(X, \mathscr{A})$. Is there a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ which, similarly to (2), has $\mathsf{D}^{\mathsf{b}}(X)$ and $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ as components? We will revisit this question on a motivic level in Chapter 2 for a hereditary order $\mathscr{A}$ over a smooth curve $C$.

A baby case of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ is given by an order of degree 2. Then $\mathsf{BS}(\mathscr{A}) \to C$ defines a conic bundle with singular fibers over the ramified points. It follows from [94, 53] that there is a decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}), \mathsf{D}^{\mathsf{b}}(X)\rangle. \tag{3}$$

We explain in Section III.1 how such a semiorthogonal decomposition relates to the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in the Grothendieck ring of varieties.

## I.3 Orders and stacks

A shared approach in the preceding two sections is to start the analysis of the relation between coherent ringed schemes $(X, \mathscr{A})$ and their geometric counterparts by looking at what happens for Azumaya algebras. In this section, we once again take Azumaya algebras as our starting point.

**A dictionary between orders and stacks.** If $\mathscr{A}$ is an Azumaya algebra, Căldăraru [46] observed that $\mathsf{coh}(X, \mathscr{A})$ is equivalent to the subcategory $\mathsf{coh}^{(1)}(\mathcal{G}_{\mathscr{A}})$ of one-twisted sheaves of a $\mathbb{G}_{\mathrm{m}}$-gerbe $\mathcal{G}_{\mathscr{A}}$.

Working over an algebraically closed field $\mathbf{k}$ of characteristic zero, Chan–Ingalls [51] introduced a dictionary between hereditary orders on a smooth curve $C$, i.e. $\mathcal{O}_C$-orders of global dimension one, and smooth root stacks $\mathcal{C}$. The dictionary takes the following form. Given a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ there exists a unique smooth root stack $\mathcal{C}$ with trivial generic stabilizer and coarse space $C$, such that

$$\mathsf{coh}(C, \mathscr{A}) \simeq \mathsf{coh}(\mathcal{C}) \tag{4}$$

The collection of points of $\mathcal{C}$ with non-trivial stabilizer coincide with the ramification locus $\Delta_{\mathscr{A}}$ of $\mathscr{A}$.

The dictionary was generalized to tame orders in dimension two by [62]. Given a pair $(S, \mathscr{A})$ of a normal quasiprojective surface $S$, and a tame $\mathcal{O}_S$-order $\mathscr{A}$ one does a two-step construction:

i) Replace $S$ by a root stack $\mathcal{S}_{\mathsf{root}}$ with stacky structure along the ramification divisor of $\mathscr{A}$.

ii) Then replace $\mathcal{S}_{\mathsf{root}}$ by the canonical stack $\mathcal{S}_{\mathsf{can}}$, resolving the singularities of $\mathcal{S}_{\mathsf{root}}$.

It culminates in the equivalence

$$\mathsf{coh}(S, \mathscr{A}) \cong \mathsf{coh}(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}) \cong \mathsf{coh}(\mathcal{S}_{\mathsf{can}}, \mathscr{A}_{\mathsf{can}}) \tag{5}$$

of three categories of coherent sheaves, where $\mathscr{A}_{\mathsf{can}}$ is an Azumaya algebra on a stack $\mathcal{S}_{\mathsf{can}}$. We will revisit the dictionaries in detail in Section 4.2.

**Applications of the dictionary.** We can apply the orders-stacks dictionary in several directions, one of which is described in Section III.3. Here we provide a link to Sections I.1 and I.2.

In Section I.2, we asked how $\mathsf{D}^{\mathsf{b}}(Y)$ decomposes for a smooth projective variety $Y$. We can ask the same question replacing $Y$ by a coherent ringed scheme $(X, \mathscr{A})$ (or its stacky equivalent). There are answers [81, 29] to this question for smooth root stacks $\mathcal{X}_{\mathsf{root}}$. They hint towards a decomposition of $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ for their corresponding coherent ringed scheme $(X, \mathscr{A})$, which we will explore in Chapter 3 for hereditary orders on smooth curves over an algebraically closed field $\mathbf{k}$ of characteristic zero.

Let us now turn to the perspective of moduli spaces from Section I.1. The results of [29] show on the stacky side that $\mathsf{D}^{\mathsf{b}}(\mathcal{X}_{\mathsf{root}})$ embeds as an admissible category into $\mathsf{D}^{\mathsf{b}}(Y)$, where $Y$ is a smooth projective scheme. From the moduli problem point of view, it is a natural question whether $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ embeds into $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$. We come back to this question in Chapter 2.

## II Structure and results of the thesis

The four chapters of the thesis can all be read independently and are self-contained. The first chapter can be considered as a background chapter on orders on curves, whereas the other three chapters contain new results.

**Orders on curves.** In Chapter 1, we give a brief introduction to maximal and hereditary orders on curves. The material can be found in standard textbooks such as [119, 144], but is essential for the chapters which follow. We dedicate Section 1.4 to an overview of the non-hereditary orders that will appear in the thesis. The reader familiar with orders may skip the first four sections. Section 1.5 briefly mentions a class of Gorenstein triangular orders which we conjecture to have a stacky counterpart given by a singular root stack.

**The Brauer–Severi scheme of a hereditary order.** In Chapter 2, we study the Brauer–Severi scheme $f\colon \mathsf{BS}(\mathscr{A}) \to C$ of a hereditary order $\mathscr{A}$ on a smooth curve $C$. This links to the discussion in Section I.1. First, we recall Theorem 2.3.15 due to Artin [5] and Frossard [67]. They describe the fiber $\mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec}(\mathbf{k}(p))$ of the moduli space $\mathsf{BS}(\mathscr{A})$ over a ramified point $p \in C$. It turns out that the fiber is a singular reduced variety with several irreducible components.

We extend their work by describing the intersections of the irreducible components in Theorem 2.3.29. This allows us to compute the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in the Grothendieck ring of varieties. As an abelian group, $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ is generated by isomorphism classes of varieties subject to the cut-and-paste relation. This means that for every closed subvariety $Z \subset X$, we have the relation

$$[X] = [Z] + [X \setminus Z] \quad \text{in } \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}). \tag{6}$$

In special cases, we present a formula for $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in terms of the Lefschetz class $\mathbb{L} = [\mathbb{A}^1]$ and $[C] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$.

**Theorem II.1** (Corollary 2.4.7)**.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$, totally ramified over $r$ points and unramified everywhere else. Denote by $f\colon \mathsf{BS}(\mathscr{A}) \to C$ the Brauer–Severi scheme of $\mathscr{A}$. Then*

$$[\mathsf{BS}(\mathscr{A})] = \sum_{i=0}^{n-1} \left( [C] + r \cdot \binom{n}{i} - r \right) \mathbb{L}^i. \tag{7}$$

The theorem allows us to

- write down the Hodge diamond of $\mathsf{BS}(\mathscr{A})$ for a hereditary order $\mathscr{A}$, and
- give heuristics for the decomposition of the derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$.

The reader is invited to Section III.1 for the discussion of Theorem II.1 and its consequences for the Hodge numbers and the derived category of $\mathsf{BS}(\mathscr{A})$ for hereditary orders $\mathscr{A}$ of degree 2.

**Categorical absorption for hereditary orders.** Whereas Chapter 2 is about decomposing $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$, we turn in Chapter 3 to the question how to decompose one of the conjecturally admissible subcategories of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$. More concretely, we present a new approach to the description of the bounded derived category $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ of a hereditary order $\mathscr{A}$ over a smooth curve $C$.

If $\mathscr{A}$ is a hereditary order on a smooth curve $C$, there is an indirect way of decomposing $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$. By appealing to the dictionary between orders and stacks from Section I.3, we have an equivalence of categories $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C})$. The stacky curve $\mathcal{C}$ is a smooth root stack with stacky structure over the ramification locus $\Delta_{\mathscr{A}}$. At this point, we can use the decomposition [81, 29] of $\mathsf{D}^{\mathsf{b}}(\mathcal{C})$.

Using Kuznetsov–Shinder's deformation absorption [101], we present a direct way of decomposing $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$. Following Le Bruyn's perspective [105] on $(C, \mathscr{A})$, we view the coherent ringed scheme $(C, \mathscr{A})$ as a noncommutative finite covering over $C$ with 'singular fibers' over the ramified points. We depict such a covering in Figure 2.

Let $\Delta_{\mathscr{A}} = \{o\}$ be the ramification locus of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ with ramification index $r$. The first result is based on the study of the fiber $\mathscr{A}(o) = \mathscr{A} \otimes_C \mathbf{k}(o)$.

**Theorem II.2** (Theorem 3.4.7)**.** *The sequence of the simple $\mathscr{A}(o)$-modules $(S_1, \ldots, S_{r-1})$ is semiorthogonal in $\mathsf{D}^{\mathsf{b}}(\mathscr{A}(o))$ and absorbs singularities, i.e. the triangulated subcategory*

$$\mathsf{S} = \langle S_1, \ldots, S_{r-1} \rangle \subset \mathsf{D}^{\mathsf{b}}(\mathscr{A}(o)) \tag{8}$$

*is admissible and both of its complements ${}^{\perp}\mathsf{S}$ and $\mathsf{S}^{\perp}$ are smooth and proper.*

In fact, we will show that the simple $\mathscr{A}(o)$-modules are $\mathbb{P}^{\infty}$-objects. Via a noncommutative version of [101] these objects push forward to exceptional objects in the derived category $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

**Theorem II.3** (Theorem 3.4.11)**.** *Let $\mathfrak{i}_o \colon (\operatorname{Spec} \mathbf{k}(o), \mathscr{A}(o)) \to (C, \mathscr{A})$ be the inclusion of $o \in C$. There is a $C$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathfrak{i}_{o,*}S_1, \ldots, \mathfrak{i}_{o,*}S_{r-1}, \mathsf{D} \rangle, \tag{9}$$

*such that*

*i) the sequence $(\mathfrak{i}_{o,*}S_1, \ldots, \mathfrak{i}_{o,*}S_{r-1})$ is exceptional,*

*ii) the admissible subcategory $\mathsf{D}$ is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$.*

We illustrate the theorems by means of a hereditary $\mathcal{O}_{\mathbb{P}^1}$-order $\mathscr{A}$ of degree 2 in Section III.2.

**Central curves on noncommutative surfaces.** Chapter 4 is joint work [17] with Pieter Belmans and Okke van Garderen. We start from a tame order $\mathscr{A}$ of global dimension two on a quasiprojective normal surface $S$. As explained in Section I.3, there is dictionary between orders and stacks in dimension one and two.

We explain how the two-dimensional dictionary behaves under the restriction along a curve $C \subset S$. By the restriction along $C$ we mean

- on the side of coherent ringed schemes, the coherent ringed scheme $(C, \mathscr{A}|_C)$, where the $\mathcal{O}_C$-algebra $\mathscr{A}|_C$ is the restriction of $\mathscr{A}$ along the closed immersion $C \hookrightarrow S$, and
- on the stacky side, the coherent ringed stack $(\mathcal{C}, \mathscr{A}_{\mathsf{can}}|_{\mathcal{C}})$, where $\mathcal{C} = C \times_S \mathcal{S}$ is the fiber product over the surface $S$ and $\mathscr{A}_{\mathsf{can}}|_{\mathcal{C}}$ is the restriction of $\mathscr{A}_{\mathsf{can}}$ along the closed immersion $\mathcal{C} \hookrightarrow \mathcal{S}$.

The first main result is the following.

**Theorem II.4** (Theorem 4.3.1)**.** *Let $\mathscr{A}$ be a tame order of global dimension 2 on a normal quasiprojective surface $S$, and let $\mathscr{A}_{\mathsf{can}}$ be the Azumaya algebra on the stack $\mathcal{S}$ for which $\mathsf{coh}(S, \mathscr{A}) \simeq \mathsf{coh}(\mathcal{S}, \mathscr{A}_{\mathsf{can}})$. Given a curve $C \subset S$, there exists an equivalence*

$$\mathsf{coh}(C, \mathscr{A}|_C) \simeq \mathsf{coh}(\mathcal{C}, \mathscr{A}_{\mathsf{can}}|_{\mathcal{C}}). \tag{10}$$

The properties of the coherent ringed scheme $(C, \mathscr{A}|_C)$ depend on the intersection of $C$ with the ramification divisor $\Delta_{\mathscr{A}}$. If $C$ does not intersect the singular locus of $\Delta_{\mathscr{A}}$, the stacky curve $\mathcal{C}$ will be a root stack and all properties of the restriction are already controlled by the intermediate root stack $(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}})$. We can therefore link our results to the one-dimensional dictionary between orders and stacks.

**Proposition II.5** (Section 4.3.2)**.** *Let $S, \mathcal{A}, C$ be as in Theorem II.4, and assume moreover that $C$ is integral.*

*i) Then the sheaf of algebras $\mathscr{A}|_C$ is an order if and only if $C$ is not contained in $\Delta$.*

*Assume that $\mathscr{A}|_C$ is an order. Then its ramification locus is $C \cap \Delta$, and in particular:*

*ii) $\mathscr{A}|_C$ is Azumaya if and only if $C \cap \Delta = \emptyset$.*

*iii) If $C$ is smooth and intersects $\Delta$ transversely in its smooth locus, then $\mathscr{A}|_C$ is hereditary.*

A major application of our results is a unified approach to the study of noncommutative plane curves. Such curves are defined by the quotient of a 3-dimensional quadratic Artin–Schelter regular algebra $A$ by a central element $g \in \mathrm{Z}(A)$. We reobtain the classification result [79, 135, 80] of noncommutative plane conics (Theorem 4.5.15) and classify certain skew cubics (Proposition 4.6.1 and Proposition 4.6.2) which were considered in [84].

The perspective of understanding noncommutative plane curves served also as an initial motivation for this article. We explain the motivating example in Section III.3.

# III Examples and applications

Now that we have presented the main results, we dedicate the remainder of the introduction to illustrating them with examples.

On the one hand, the three examples depict how to relate coherent ringed schemes to moduli spaces, decompositions of derived categories, and stacks, as discussed in Section I. We also demonstrate the interplay between the different perspectives on coherent ringed schemes. This is done by looking at similar coherent ringed schemes in every example.

On the other hand, the three examples introduce the reader to the ideas and applications which we have in mind in the later chapters of the thesis, where we present the main results.

## III.1 Conic bundles over curves

Let $C$ be a smooth projective curve over an algebraically closed field $\mathbf{k}$ of characteristic zero. Consider a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ of degree 2. We want to describe the Brauer–Severi scheme $\mathsf{BS}(\mathscr{A})$ of $\mathscr{A}$. Recall from Section I.1 that this is the moduli space parametrizing left ideals of rank 2 in $\mathscr{A}$. By [5, 67], the Brauer–Severi scheme $\mathsf{BS}(\mathscr{A})$ is smooth and projective over $\mathbf{k}$. Below, we describe the fibers of the flat map $\mathsf{BS}(\mathscr{A}) \to C$.

Denote by $\Delta_{\mathscr{A}} \subset C$ the ramification locus, and assume that $r = \#\Delta_{\mathscr{A}}$. Since $\mathscr{A}$ is a hereditary order of degree 2, the fibers $\mathscr{A}(p) = \mathscr{A} \otimes_C \mathbf{k}(p)$ of $\mathscr{A}$ fall into two isomorphism classes

$$\mathscr{A}(p) \cong \begin{cases} \mathrm{Mat}_2(\mathbf{k}(p)) & \text{if } p \notin \Delta_{\mathscr{A}}, \\ \Lambda_2 & \text{if } p \in \Delta_{\mathscr{A}}. \end{cases} \tag{11}$$

We start with the smooth fibers $\mathsf{BS}(\mathscr{A}(p)) \cong \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec} \mathbf{k}(p)$, where $p \notin \Delta_{\mathscr{A}}$. Two-dimensional left ideals in $\mathrm{Mat}_2(\mathbf{k}(p))$ are determined by their first row vector. Therefore, the fiber $\mathsf{BS}(\mathscr{A}(p))$ is isomorphic to a smooth conic or, equivalently, a projective line $\mathbb{P}^1$.

If $p \in \Delta_{\mathscr{A}}$, we first have to describe the algebra $\Lambda_2$. As a $\mathbf{k}$-vector space, it is generated by four elements $e_{11}, e_{12}, e_{21}, e_{22}$. The basis elements satisfy the same multiplication rules as the elementary matrices of $\mathrm{Mat}_2(\mathbf{k}(p))$ with the exception that $e_{12} \cdot e_{21} = e_{21} \cdot e_{12} = 0$. Every two-dimensional left ideal in $\Lambda_2$ must contain $e_{12}$ or $e_{21}$. As a consequence, we obtain two families of left ideals in $\Lambda_2$,

- the first one given by ideals $L = \langle ae_{11} + be_{12}, e_{21} \rangle$ corresponding to a point $[a : b] \in \mathbb{P}^1$,
- the second one given by $L = \langle e_{12}, ce_{21} + de_{22} \rangle$ corresponding to a point $[c : d] \in \mathbb{P}^1$.

It follows that the singular fiber is given by two lines intersecting in one point, described by the left ideal $L = \langle e_{12}, e_{21} \rangle$.

**The motivic description of a conic bundle.** If $\mathscr{A}$ is a hereditary order of degree 2, the discussion above allows us to verify the formula from Theorem II.1 for the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in the Grothendieck ring of varieties. Over each ramified point, the class is $[\mathsf{BS}(\mathscr{A})_p] = 2\mathbb{L} + 1$. Since $[\mathbb{P}^1] = \mathbb{L} + 1$, we have

$$[\mathsf{BS}(\mathscr{A})] = [C \setminus \Delta_{\mathscr{A}}] \cdot (\mathbb{L} + 1) + [\Delta_{\mathscr{A}}] \cdot (2\mathbb{L} + 1) = [C] + ([C] + r)\mathbb{L}. \tag{12}$$

What conclusions can we draw from (12)? Essentially, we want to apply *motivic measures* to obtain invariants of $\mathsf{BS}(\mathscr{A})$. A motivic measure is a ring homomorphism $\mu\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to R$ to a ring $R$.

- **Hodge diamond.** The Hodge polynomial $\mathrm{h}([\mathsf{BS}(\mathscr{A})]) = \sum_{p,q} \mathrm{h}^{p,q}(\mathsf{BS}(\mathscr{A}))x^p y^q \in \mathbb{Z}[x, y]$ is a motivic measure. For a smooth projective curve $C$ of genus $g$, the Hodge polynomial is given by $\mathrm{h}([C]) = 1 + g \cdot x + g \cdot y + xy$. Since $\mathrm{h}(\mathbb{L}) = xy$, it follows that

$$\mathrm{h}([\mathsf{BS}(\mathscr{A})]) = 1 + g \cdot x + g \cdot y + (2 + r) \cdot xy + g \cdot x^2 y + g \cdot xy^2 + x^2 y^2. \tag{13}$$

  The coefficient in front of $x^p y^q$ is the Hodge number $\mathrm{h}^{p,q}(X) = \dim_{\mathbf{k}} \mathrm{H}^q(X, \Omega^p_X)$. For greater clarity, the coefficients can be presented in the Hodge diamond.

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ 0 & & 2+r & & 0 \\ & g & & g & \\ & & 1 & & \end{array}$$

- **The derived category.** Another motivic measure is the map $\mathrm{d}\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}})$ sending $[X]$ to the unique dg enhancement of $[\mathsf{D}^{\mathsf{b}}(X)]$. The Grothendieck ring of dg categories $\mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}})$ consists of equivalence classes of smooth and proper dg categories, where semiorthogonal decompositions serve as the counterparts to the cut-and-paste relation (6). The map d satisfies $\mathrm{d}(\mathbb{L}^i) = 1 = [\mathsf{D}^{\mathsf{b}}(\mathbf{k})]$ for all $i$. Applying d to (12) leads to the following decomposition of the derived category

$$[\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))] = [\mathsf{D}^{\mathsf{b}}(C)] + [\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})]. \tag{14}$$

  The $r$ exceptional objects in $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ can be explained using Theorem II.3. Each ramified point contributes one exceptional object. The motivic decomposition in (14) can be seen as an incarnation of the semiorthogonal decomposition [94]

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}), \mathsf{D}^{\mathsf{b}}(X) \rangle. \tag{15}$$

  for the derived category of a conic bundle $\mathsf{BS}(\mathscr{A}) \to C$.

Our Theorem II.1 provides a general formula for $[\mathsf{BS}(\mathscr{A})]$ and in a similar vein to the discussion for conic bundles, it allows us to

- compute the Hodge numbers of $\mathsf{BS}(\mathscr{A})$,
- propose a conjecture on the decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$.

## III.2 An example of the derived category of a hereditary order

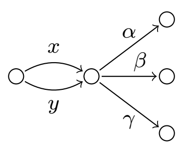
As before, we consider a coherent ringed scheme $(C, \mathscr{A})$, where $C$ is a smooth projective curve over an algebraically closed field $\mathbf{k}$ of characteristic zero, and $\mathscr{A}$ is an $\mathcal{O}_C$-order. In Chapter 3, we investigate the derived category $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

In special cases, the derived category $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ is equivalent to the derived category $\mathsf{D}^{\mathsf{b}}(\Gamma)$ of a finite-dimensional $\mathbf{k}$-algebra $\Gamma$. For the equivalence $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\Gamma)$, it is sufficient that

- the curve $C$ is rational, i.e. $C \cong \mathbb{P}^1$,
- the $\mathcal{O}_C$-order $\mathscr{A}$ is hereditary.

The equivalence $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\Gamma)$ is a classical result by Geigle–Lenzing [69] who studied these spaces under the name of weighted projective lines.

Similarly to Section III.1, let $\mathscr{A}$ be a hereditary $\mathcal{O}_{\mathbb{P}^1}$-order of degree 2 with ramification locus $\Delta_{\mathscr{A}} = \{0, 1, \infty\} \subset \mathbb{P}^1$. Then, by [69, 81, 29], the derived equivalent algebra $\Gamma$ is given by the path algebra of the squid quiver



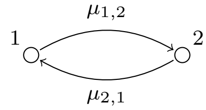

(16)

modulo the admissible ideal $I = (\alpha x, \beta y, \gamma x - \gamma y)$.

**Finite noncommutative coverings.** Obtaining a derived equivalence $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\Gamma)$ does not work if we want to generalize to curves of genus $g \geq 1$, because the non-zero genus prevents $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ from having a so-called tilting object. This is why we take a different approach to the problem. Following Le Bruyn's perspective [105], we consider the coherent ringed scheme $(C, \mathscr{A})$ as a flat family of finite-dimensional $\mathbf{k}$-algebras over $C$.

We describe how this perspective applies to the hereditary $\mathcal{O}_{\mathbb{P}^1}$-order $\mathscr{A}$ with ramification locus $\Delta_{\mathscr{A}} = \{0, 1, \infty\}$. As in (11), the fiber is generically given by $\mathscr{A}(p) = \mathrm{Mat}_2(\mathbf{k}(p))$. Due to the Morita equivalence $\mathsf{mod}(\mathrm{Mat}_2(\mathbf{k}(p))) \simeq \mathsf{mod}(\mathbf{k}(p))$ we say that the generic fiber of the covering is a point.

Over the three ramified points $p = 0, 1, \infty$, the fiber is given by $\mathsf{coh}(\operatorname{Spec} \mathbf{k}(p), \Lambda_2)$. The algebra $\Lambda_2 \cong \mathbf{k}Q/I$ is isomorphic to the path algebra $\mathbf{k}Q$ of the quiver




(17)

modulo the admissible ideal $I = (\mu_{1,2}\mu_{2,1}, \mu_{2,1}\mu_{1,2})$. In the words of [105], the fiber $\mathscr{A}(p)$ consists of two infinitesimally closed points. In contrast to the commutative situation, the points 'communicate' with each other via a non-trivial first extension group. Figure 2 depicts the situation for the chosen hereditary $\mathcal{O}_{\mathbb{P}^1}$-order $\mathscr{A}$ of degree 2 with three ramified points.

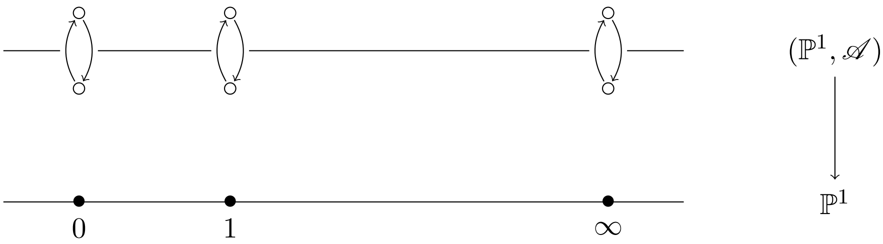



Figure 2: A hereditary $\mathcal{O}_{\mathbb{P}^1}$-order as a finite noncommutative covering.

Let us now apply the results of Chapter 3. Theorem II.2 is applied to the fibers $\mathscr{A}(p)$ for $p \in \Delta_{\mathscr{A}}$. It tells us that the simple $\mathscr{A}(p)$-module $S_1$ corresponding to the first vertex of (17) generates an admissible subcategory $\mathsf{S} \subset \mathsf{D}^{\mathsf{b}}(\mathscr{A}(p))$ such that its complement is smooth and proper. In fact, the complement ${}^{\perp}\mathsf{S}$ is equivalent to the derived category of a point.

Theorem II.3 explains the relation of $S_1$ to the derived category $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A})$. It is an exceptional object in $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A})$. In this way, we obtain three exceptional objects in $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A})$. The complement of the exceptional objects in $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A})$ is $\mathsf{D} \simeq \mathsf{D}^{\mathsf{b}}(\mathbb{P}^1)$.

Studying the derived category $f\colon \mathcal{Y} \to B$ of a flat family of varieties over a base scheme $B$ is an insightful approach [99]. The content of Chapter 3 demonstrates that this is also interesting in the context of coherent ringed schemes $(X, \mathscr{A})$.

## III.3 An example of a noncommutative plane curve

An application of our restriction results from Theorem II.4 and Proposition II.5 is a set of tools for studying of noncommutative plane curves. This point of view was the starting point for our joint work [17] with Pieter Belmans and Okke van Garderen. It was introduced to us by Pieter Belmans, who also wrote a blog post [22] discussing the example we present below.

**Central proj and noncomutative planes.** First, we present another source of coherent ringed schemes $(X, \mathscr{A})$. Let $X$ be a smooth projective variety over $\mathbf{k}$. In light of the Gabriel–Rosenberg theorem [68, 124] we may equivalently study $\mathsf{coh}(X)$ rather than $X$ itself. The variety $X$ comes with a homogeneous coordinate ring $A$. The $\mathbf{k}$-algebra $A$ is a graded connected (commutative) $\mathbf{k}$-algebra generated in degree one. Serre [126] provided a construction for coherent $\mathcal{O}_X$-modules

$$\mathsf{coh}(X) \simeq \mathsf{qgr}(A) := \mathrm{gr}(A)/\mathrm{fd}(A), \tag{18}$$

which does not make use of the topology of $X$. The abelian category $\mathsf{qgr}(A)$ is the Serre quotient of the category $\mathrm{gr}(A)$ of finitely generated graded $A$-modules by $\mathrm{fd}(A)$, the subcategory of $A$-modules of finite length.

Following [11], we can also form the category $\mathsf{qgr}(A)$ for a noncommutative graded $\mathbf{k}$-algebra $A$. The central proj construction [104] shows that $\mathsf{qgr}(A)$ is a source of coherent ringed spaces. In other words, we can construct a coherent ringed space $(X, \mathscr{A})$ from $A$ such that $\mathsf{qgr}(A) \simeq \mathsf{coh}(X, \mathscr{A})$.

If $A$ is a 3-dimensional quadratic Artin–Schelter regular algebra, the obtained coherent ringed space is $(\mathbb{P}^2, \mathscr{A})$, where $\mathscr{A}$ is a maximal $\mathcal{O}_{\mathbb{P}^2}$-order. We call such a coherent ringed scheme a *noncommutative plane*.

**Noncommutative plane conics.** An example of a noncommutative plane is given by the graded $\mathbf{k}$-algebra

$$A_q = \frac{\mathbf{k}\langle x, y, z\rangle}{(xy - qyx, xz - qzx, yz - qzy)}, \quad q \in \{\pm 1\}. \tag{19}$$

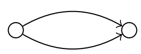
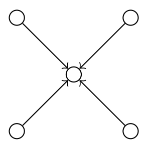
If $q = 1$, the central proj is the projective plane $\mathbb{P}^2 = \operatorname{Proj}(A_1)$. If instead $q = -1$, the central proj of $A_{-1}$ is $(\mathbb{P}^2, \mathscr{A})$, where $\mathscr{A}$ is a maximal $\mathcal{O}_{\mathbb{P}^2}$-order. The ramification locus $\Delta_{\mathscr{A}} = \mathbb{V}(XYZ) \subset \mathbb{P}^2$ is given by three lines in general position.

In both cases, the $\mathbf{k}$-vector space $\langle x^2, y^2, z^2\rangle$ lies in the center $\mathrm{Z}(A_q)$. Hence, we can study the quotient $\mathsf{qgr}(A_q/(f))$ for $f \in \langle x^2, y^2, z^2\rangle$ using quadratic duality. Since $f$ is of degree 2, we say that $\mathsf{qgr}(A_q/(f))$ is a *noncommutative plane conic*.

Ueyama [135, Example 3.25] found for the central element $f = x^2 + y^2 + z^2 \in \mathrm{Z}(A_q)_2$ that

$$\mathsf{D}^{\mathsf{b}}(\mathsf{qgr}(A_q/(f))) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{rep}\, Q_q), \tag{20}$$

where $Q_q$ is a finite acyclic quiver. The two quivers are given by

$Q_1 =$  and $Q_{-1} =$  (21)

the Kronecker quiver, resp. the 4-subspace quiver.

Let us now apply Theorem II.4 and Proposition II.5 to this example. If $q = 1$, the ramification locus is empty and the Azumaya algebra is the structure sheaf $\mathcal{O}_{\mathbb{P}^2}$. Thus, we obtain Beĭlinson's equivalence $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{rep}\, \widetilde{A}_1)$ for projective space [20]. Note that we could have taken any smooth conic curve in $\mathbb{P}^2$ to obtain the derived equivalence with $\mathsf{rep}\, Q_1$.

The case $q = -1$ is more interesting. The central proj construction $\mathsf{coh}(\mathbb{P}^2, \mathscr{A}) \simeq \mathsf{qgr}(A_{-1})$ can be made explicit by using the central elements $X = x^2$, $Y = y^2$ and $Z = z^2$ as coordinates of $\mathbb{P}^2$. For this reason, the central element $f = x^2 + y^2 + z^2 \in A_{-1}$ defines a line in $\mathbb{P}^2$. It intersects the ramification locus $\Delta_{\mathscr{A}} = \mathbb{V}(XYZ)$ transversely in three points.

Therefore, Proposition II.5 tells us that $\mathsf{qgr}(A_{-1}/(f)) \simeq \mathsf{coh}(\mathbb{P}^1, \mathscr{A})$, where $\mathscr{A}$ is a hereditary $\mathcal{O}_{\mathbb{P}^1}$-order of degree 2 with three ramified points. We considered this order before in

Section III.2. It is derived equivalent to the squid algebra (16). The derived equivalence of modules over the squid algebra to $\mathsf{rep}\, Q_{-1}$ is well-known.

Our results show that we could have chosen any line in $\mathbb{P}^2$ intersecting the ramification locus $\Delta_{\mathscr{A}}$ transversely, and we would recover Ueyama's result (20). Moreover, we can ask what happens if the line defined by $f \in \mathrm{Z}(A_q)_2$ intersects the ramification locus $\Delta_{\mathscr{A}}$ in its singular locus. We will see in Section 4.5 that the order is no longer hereditary, and we will describe all isomorphism classes of noncommutative conics in a suitable class of noncommutative planes.

# Chapter 1

# Orders on curves

Let $\mathbf{k}$ be a perfect field. We denote by $X$ a noetherian integral separated normal scheme over $\mathbf{k}$. Following Yekutieli–Zhang [147, Definition 1.1], we call $(X, \mathscr{A})$ a *coherent ringed scheme* if $\mathscr{A}$ is a coherent $\mathcal{O}_X$-algebra.

This chapter is devoted to introducing the reader to a special class of coherent ringed schemes $(X, \mathscr{A})$, which are referred to as *orders*. In Section 1.1 we give the definition of orders, some essential properties, and examples. Afterwards, we specialize to orders over curves, i.e. one-dimensional schemes. Section 1.2 and Section 1.3 are about maximal and hereditary orders. The results are well-known and can mostly be found in standard textbooks such as [119, 144]. We also refer to the unpublished notes [6, 49]. The two sections serve to provide more background to the setting used in our geometric applications.

In Section 1.4, we present some classes of orders which could be referred to as 'singular' coherent ringed schemes. By doing so, we have two goals in mind. The first goal is to provide additional details relevant to the treatment of restrictions to non-hereditary orders in Chapter 4. The second goal is the introduction of non-hereditary orders which we consider as particularly interesting for further study in the realm of noncommutative algebraic geometry. This leads us to Section 1.5, where we present a class of orders suitable for the extension of the dictionary between orders and stacks to singular root stacks.

## 1.1 Definition and properties

Let $X$ be a noetherian integral separated normal scheme over $\mathbf{k}$. Denote by $\mathbf{k}(X)$ the function field of $X$. We start with the definition of an $\mathcal{O}_X$-order.

**Definition 1.1.1.** Let $A$ be a central simple $\mathbf{k}(X)$-algebra. An *$\mathcal{O}_X$-order in $A$* is a coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ which

i) is a torsionfree $\mathcal{O}_X$-submodule of $A$, and

ii) satisfies $\mathscr{A} \otimes_X \mathbf{k}(X) = A$.

The degree of the central simple algebra $A$ is the positive integer $n$ satisfying $n^2 = \dim_{\mathbf{k}(X)} A$. Accordingly, we refer to $n$ as the *degree* of the $\mathcal{O}_X$-order $\mathscr{A}$ in $A$.

**Azumaya algebras.** An important class of examples of $\mathcal{O}_X$-orders is provided by Azumaya algebras. We refer to [110, Chapter IV] for a detailed treatment.

**Definition 1.1.2.** A coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ is an *Azumaya algebra* if there exists an étale covering $f\colon X' \to X$ such that $f^*\mathscr{A} \cong \mathcal{E}nd_{X'}(\mathcal{V}')$ for some locally free $\mathcal{O}_{X'}$-module $\mathcal{V}'$.

We call an Azumaya algebra *split* if we can choose $f = \mathrm{id}_X$.

Since an étale covering is faithfully flat and $\mathcal{V}$ is locally free of finite rank, it follows that an Azumaya algebra $\mathscr{A}$ is a locally free $\mathcal{O}_X$-module. Moreover, the fiber $\mathscr{A}(p) = \mathscr{A} \otimes_X \mathbf{k}(p)$ is a central simple $\mathbf{k}(p)$-algebra for each point $p \in X$. Therefore, we find the following.

**Lemma 1.1.3.** *Let $\mathscr{A}$ be an Azumaya algebra over $X$. Then $\mathscr{A}$ is an $\mathcal{O}_X$-order.*

Azumaya algebras are particularly robust $\mathcal{O}_X$-orders, because they behave well under pullbacks [110, Proposition IV.1.2] and tensor products [110, Corollary IV.1.3].

**Lemma 1.1.4.** *Let $\mathscr{A}, \mathscr{B}$ be Azumaya algebras over $X$.*

*i) For every morphism $f\colon Y \to X$ the pullback $f^*\mathscr{A}$ is an Azumaya algebra over $Y$.*

*ii) The tensor product $\mathscr{A} \otimes_X \mathscr{B}$ is an Azumaya algebra over $X$.*

Lemma 1.1.4 is special to Azumaya algebras. Throughout this thesis, several instances, such as Lemma 3.2.9, Lemma 3.2.5 and Proposition 4.3.5, highlight the subtlety that orders do not necessarily pull back to orders. Example 1.1.21 also illustrates this phenomenon.

**Remark 1.1.5.** Thanks to their robustness, Morita equivalence classes of Azumaya algebras are the elements of the Brauer group $\mathrm{Br}(X)$ of the scheme $X$. The neutral element of $\mathrm{Br}(X)$ is the class of split Azumaya algebras $\mathscr{A} \cong \mathcal{E}nd_X(\mathcal{V})$. Multiplication is given by the tensor product.

**The ramification locus.** An $\mathcal{O}_X$-order is a generalization of an Azumaya algebra, in the sense that there is a maximal non-empty (Zariski-)open subset $U \subset X$ such that the restriction $\mathscr{A}|_U$ along the open immersion $U \hookrightarrow X$ is an Azumaya algebra. This follows from the fact that $X$ is integral and $\mathscr{A} \otimes_X \mathbf{k}(X)$ is a central simple $\mathbf{k}(X)$-algebra. Accordingly, the complement of $U$ becomes an object of interest in the study of the geometry of orders.

**Definition 1.1.6.** Let $\mathscr{A}$ be an $\mathcal{O}_X$-order. We call

$$\Delta_{\mathscr{A}} = \{p \in X \mid \mathscr{A} \otimes_X \mathbf{k}(p) \text{ not central simple}\} \tag{1.1}$$

the *ramification locus of* $\mathscr{A}$.

The ramification locus is a closed subscheme of $\operatorname{codim}_X(\Delta_{\mathscr{A}}) \geq 1$. If $\mathscr{A}$ is Azumaya, the ramification locus is empty. We now consider some examples where the ramification divisor is non-empty.

**Example 1.1.7.** Let $X$ be a noetherian integral separated normal scheme over $\mathbf{k}$, and $D \subset X$ a smooth effective divisor. We fix $m \geq 0$. Then

$$\mathscr{A} = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X \\ \mathcal{O}_X(-mD) & \mathcal{O}_X \end{pmatrix} \tag{1.2}$$

is an $\mathcal{O}_X$-order in the matrix algebra $\operatorname{Mat}_2(\mathbf{k}(X))$ with ramification divisor

$$\Delta_{\mathscr{A}} = \begin{cases} D & \text{if } m > 0, \\ \emptyset & \text{if } m = 0. \end{cases} \tag{1.3}$$

Indeed, the fibers of the order are described by

$$\mathscr{A}(p) \cong \begin{cases} \operatorname{Mat}_2(\mathbf{k}(p)) & \text{if } p \notin D \text{ or } m = 0, \\ \Lambda_2 & \text{if } p \in D \text{ and } m > 0. \end{cases} \tag{1.4}$$

The $\mathbf{k}(p)$-algebra $\Lambda_2$ has a $\mathbf{k}(p)$-basis $e_{11}, e_{12}, e_{21}, e_{22}$ which satisfy the same multiplication rules as the elementary matrices in $\operatorname{Mat}_2(\mathbf{k}(p))$, except that $e_{12} \cdot e_{21} = e_{21} \cdot e_{12} = 0$. Therefore, $\Lambda_2$ is not a simple $\mathbf{k}(p)$-algebra.

**Remark 1.1.8.** If $m \in \{0, 1\}$ and $X$ is a (smooth) curve, then $\mathscr{A}$, as defined in Example 1.1.7, is an example of a hereditary order. We will revisit the fibers $\Lambda_r$ of a hereditary order in a more general context in Section 3.2.2.

We use the next example to introduce new notation. Let $R$ be a (commutative) $\mathbf{k}$-algebra and $a, b \in R$. Assume that the field contains a primitive $n$-th root of unity $\zeta_n \in \mathbf{k}$. We define the $R$-algebra

$$(a, b)_n^R := R\langle i, j\rangle/(i^n - a, j^n - b, ij - \zeta_n ji). \tag{1.5}$$

For example, the $\mathcal{O}_X$-order $\mathscr{A}$ from Example 1.1.7 can be locally written in this way. Assume that 2 is invertible in $\mathcal{O}_X$. Let $U = \operatorname{Spec}(R) \subset X$ be an open neighborhood such that $D$ is cut out by $r \in R$. Then the assignment $i \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $j \mapsto \left(\begin{smallmatrix} 0 & 1 \\ r^m & 0 \end{smallmatrix}\right)$ defines an $R$-algebra isomorphism

$$(1, r^m)_2^R \xrightarrow{\sim} \mathscr{A}|_U. \tag{1.6}$$

We turn now to a two-dimensional affine example.

**Example 1.1.9.** Let $R = \mathbf{k}[x, y]$. Then the $R$-algebra $\Lambda = (x, y)_2^R$ is an $R$-order in the central simple $\mathbf{k}(x, y)$-algebra $(x, y)_2^{\mathbf{k}(x,y)}$ with ramification locus $\Delta_\Lambda = V(xy) \subset \mathbb{A}^2$. Indeed, for a prime ideal $\mathfrak{p} \trianglelefteq R$, we have $\Lambda \otimes_R \mathbf{k}(p) = (x, y)_2^{\mathbf{k}(p)}$. This is a central simple $\mathbf{k}(p)$-algebra if and only if $\mathfrak{p} \notin V(xy)$. We come back to this example in Lemma 1.1.18.

**Maximal orders.** Let $\mathscr{A}$ be an $\mathcal{O}_X$-order in a central simple $\mathbf{k}(X)$-algebra $A$. An $\mathcal{O}_X$-order $\mathscr{B}$ in $A$ is called an *overorder of* $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{B}$. Hence, the set of $\mathcal{O}_X$-orders in $A$ is partially ordered by inclusion.

**Definition 1.1.10.** An $\mathcal{O}_X$-order $\mathscr{A}$ in $A$ is *maximal* if every overorder $\mathscr{B} \supseteq \mathscr{A}$ is equal to $\mathscr{A}$, i.e. $\mathscr{B} = \mathscr{A}$.

It is a non-trivial fact that every $\mathcal{O}_X$-order is contained in a maximal order. A proof of this fact can be found in [119, Corollary 10.4] for example.

**Theorem 1.1.11.** *Let $\mathscr{A}$ be an $\mathcal{O}_X$-order in $A$. Then there is a maximal $\mathcal{O}_X$-order $\mathscr{B}$ containing $\mathscr{A}$ in $A$.*

Now, let us recall some techniques to show maximality of an order, and apply this to the examples we have seen so far. The proposition below is a rephrasing of the affine case presented in [119, Corollary 11.2].

**Proposition 1.1.12.** *Let $\mathscr{A}$ be an $\mathcal{O}_X$-order.*

*i) If $\mathscr{A}$ is maximal, then the restriction $\mathscr{A}|_U := j^*\mathscr{A}$ is maximal for every open immersion $j\colon U \hookrightarrow X$.*

*ii) The $\mathcal{O}_X$-order $\mathscr{A}$ is maximal if and only if there exists an open covering $X = \bigcup_{i\in I} U_i$ such that $\mathscr{A}|_{U_i}$ is maximal $\mathcal{O}_{U_i}$-order for each $i \in I$.*

*iii) It is maximal if and only if the $\mathcal{O}_{X,p}$-order $\mathscr{A}_p$ is maximal at each point $p \in X$.*

Maximality of Azumaya algebras can be verified by starting with maximal orders in $\mathbf{k}(X)$-division algebras. The proof of the next statement can be found in [119, Theorem 8.7].

**Example 1.1.13.** Assume that $\mathscr{A}$ is a maximal $\mathcal{O}_X$-order in $A$. Then $\mathrm{Mat}_n(\mathscr{A})$ is a maximal $\mathcal{O}_X$-order in $\mathrm{Mat}_n(A)$.

Since $X$ is assumed to be normal, it follows that $\mathrm{Mat}_n(\mathcal{O}_X)$ is a maximal order. Using Proposition 1.1.12, we see that every split Azumaya algebra $\mathcal{E}nd_X(\mathcal{V})$, where $\mathcal{V}$ is a locally free sheaf of finite rank, is a maximal $\mathcal{O}_X$-order.

To prove that every Azumaya algebra is maximal, we present the following lemma along with its proof, as it does not appear to be available in the literature.

**Lemma 1.1.14.** *Let $X, Y$ be integral separated normal schemes over $\mathbf{k}$, and $\mathscr{A}$ an $\mathcal{O}_X$-order in a central simple $\mathbf{k}(X)$-algebra $A$. Assume that $f\colon Y \to X$ is a faithfully flat morphism. If $f^*\mathscr{A}$ is maximal over $Y$, then $\mathscr{A}$ is maximal over $X$.*

*Proof.* First, we show that $f^*\mathscr{A}$ is an $\mathcal{O}_Y$-order. The morphism $f$ induces a field extension $\mathbf{k}(Y)/\mathbf{k}(X)$. By [119, Corollary 7.8], the $\mathbf{k}(Y)$-algebra $A_Y := A \otimes_{\mathbf{k}(X)} \mathbf{k}(Y)$ is central simple. The pullback $f^*\mathscr{A}$ is a coherent $\mathcal{O}_Y$-module, [75, Proposition II.5.8]. Since $f$ is flat, $f^*\mathscr{A} \subset A_Y = f^*A$. Therefore, the pullback $f^*\mathscr{A}$ is torsionfree. Moreover,it satisfies $f^*\mathscr{A} \otimes_Y \mathbf{k}(Y) \cong f^*(\mathscr{A} \otimes_X \mathbf{k}(X)) = A_Y$. Hence $f^*\mathscr{A}$ is an $\mathcal{O}_Y$-order in $A_Y$.

Assume that $\mathscr{A} \subseteq \mathscr{B}$ for another $\mathcal{O}_X$-order $\mathscr{B}$ in $A$. Since $f$ is flat, $f^*\mathscr{A} \subseteq f^*\mathscr{B}$. By assumption, $f^*\mathscr{A}$ is maximal over $Y$. Hence, the $\mathcal{O}_Y$-module $f^*(\mathscr{B}/\mathscr{A}) \cong f^*\mathscr{B}/f^*\mathscr{A}$ is zero. The faithful flatness of $f$ implies that $\mathscr{B}/\mathscr{A} = 0$, see [15, Lemma 00HP]. □

**Example 1.1.15.** Let $\mathscr{A}$ be an Azumaya algebra. By Definition 1.1.2 there exists an étale covering $f\colon Y \to X$ such that $f^*\mathscr{A}$ is split. Since an étale covering is faithfully flat, it follows from Example 1.1.13 and Lemma 1.1.14 that an Azumaya algebra $\mathscr{A}$ is a maximal $\mathcal{O}_X$-order.

We turn now to Example 1.1.7. The example illustrates that a given order may be contained in several non-isomorphic maximal orders, see also Example 1.2.8.

**Example 1.1.16.** Let $X$ be a noetherian integral separated normal scheme over $\mathbf{k}$ and $D \subset X$ a smooth divisor. Consider the $\mathcal{O}_X$-order

$$\mathscr{A} = \begin{pmatrix} \mathcal{O}_X & \mathcal{O}_X \\ \mathcal{O}_X(-mD) & \mathcal{O}_X \end{pmatrix} \tag{1.7}$$

from Example 1.1.7. If $m > 0$, the order $\mathscr{A}$ is not maximal. Moreover, it is contained in the two maximal $\mathcal{O}_X$-orders

$$\mathscr{B}_1 = \mathcal{E}nd_X(\mathcal{O}_X^{\oplus 2}) \quad \text{and} \quad \mathscr{B}_2 = \mathcal{E}nd_X(\mathcal{O}_X \oplus \mathcal{O}_X(mD)). \tag{1.8}$$

For orders on schemes $X$ of dimension greater than one such as in Example 1.1.9, the Auslander–Goldman [14, Theorem 1.5] criterion simplifies checking maximality. For a prime divisor $D \subset X$, we denote by $\mathcal{O}_{X,D}$ the stalk at the generic point of $D$.

**Theorem 1.1.17** (Auslander–Goldman)**.** *An $\mathcal{O}_X$-order $\mathscr{A}$ is maximal if and only if it is reflexive, and for every prime divisor $D \subset X$, $\mathscr{A}_D = \mathscr{A} \otimes_X \mathcal{O}_{X,D}$ is a maximal $\mathcal{O}_{X,D}$-order.*

The fact that the order from Example 1.1.9 is maximal can be attributed to Chan–Ingalls [52, Proposition 2.8].

**Lemma 1.1.18.** *Let $R = \mathbf{k}[x,y]$. The $R$-order $\Lambda = (x,y)_2^R$ from Example 1.1.9 is a maximal $R$-order.*

To prove the lemma, we use the following criterion [49, Proposition 10.2] for maximality over discrete valuation rings. For this criterion we recall that for a ring $\Gamma$, the *radical* $\operatorname{rad}\Gamma$ is the two-sided ideal

$$\operatorname{rad}\Gamma = \bigcap_{\substack{S \text{ simple} \\ \Gamma\text{-module}}} \operatorname{Ann}(S), \tag{1.9}$$

where $\operatorname{Ann}(S) = \{x \in \Gamma \mid Sx = 0\}$ is the annihilator of $S$ in $\Gamma$. It is the minimal two-sided ideal of $\Gamma$ such that $\Gamma/\operatorname{rad}\Gamma$ is semisimple.

**Theorem 1.1.19.** *Let $R$ be a discrete valuation ring and $\Lambda$ an $R$-order. The following are equivalent:*

*i) The $R$-order $\Lambda$ is maximal.*

*ii) The radical $\operatorname{rad}\Lambda$ is a principal ideal and the quotient $\Lambda/\operatorname{rad}\Lambda$ is a simple Artinian algebra.*

We can now prove Lemma 1.1.18.

*Proof of Lemma 1.1.18.* Theorem 1.1.17 and the calculation of the ramification locus reduce the proof to checking maximality of the $R_{\mathfrak{p}}$-order $\Lambda_{\mathfrak{p}}$ for $\mathfrak{p} \in \{(x), (y)\}$.

Let $\mathfrak{p} = (y)$. We have $\operatorname{rad}\Lambda_{\mathfrak{p}} = j\Lambda_{\mathfrak{p}}$. Indeed, the quotient $\Lambda_{\mathfrak{p}}/j\Lambda_{\mathfrak{p}}$ is a field extension of $\mathbf{k}(x)$ of degree 2. Hence $\operatorname{rad}\Lambda_{\mathfrak{p}} \subseteq j\Lambda_{\mathfrak{p}}$. Vice versa, $(j\Lambda_{\mathfrak{p}})^2 \subseteq (y)\Lambda_{\mathfrak{p}}$. This shows the reverse inclusion. Now the claim follows from Theorem 1.1.19. If $\mathfrak{p} = (x)$ the argument is the same. □

**Remark 1.1.20.** Let $\widehat{R} = \mathbf{k}[\![x, y]\!]$ be the completion of $R = \mathbf{k}[x, y]$ at the maximal ideal $(x, y)$. The $\widehat{R}$-order $\widehat{\Lambda} = \Lambda \otimes_R \widehat{R} = (x, y)_2^{\widehat{R}}$ is maximal. This can be shown using the same strategy as in Lemma 1.1.18. We will return to the $R$-order $\widehat{\Lambda}$ in Section 4.5.

As we will see in Theorem 1.2.1, the maximality of an order on a curve can be verified at the completions of the stalks. However, in higher dimensions this is not true, see [49, Example 4.11].

**Example 1.1.21** (Warning)**.** It is in general not true that maximality is preserved under completion. Let $R = \mathbf{k}[x, y]$ and $\Lambda = (x, y)_2^R$ be the maximal order from Example 1.1.9. Consider a maximal ideal

$$\mathfrak{m}_\lambda \neq (x - \lambda, y) \trianglelefteq R \quad \text{for } \lambda \in \mathbf{k}^*. \tag{1.10}$$

We denote by $S$ the completion of $R$ at $\mathfrak{m}_\lambda$, and by $\Gamma = \Lambda \otimes_R S = (x, y)_2^S$. We can write $S = \mathbf{k}[\![x', y]\!]$ such that $x = x' + \lambda$. Thus, there exists a polynomial $f \in S^*$ such that $x = f^2$. Substituting the generator $i$ by $i/f$, we obtain an embedding

$$\Gamma = (f^2, y)_2^S \cong (1, y)_2^S \hookrightarrow \begin{pmatrix} S & S \\ y & S \end{pmatrix} \tag{1.11}$$

into an order which is strictly contained in the matrix ring $\operatorname{Mat}_2(S)$. The embedding can be defined as in (1.6).

## 1.2 Maximal orders on curves

We specialize now to one-dimensional schemes $C$. More precisely, $C$ denotes a smooth separated curve over $\mathbf{k}$.

We essentially continue the discussion from the previous paragraph, with the aim of classifying maximal orders on $C$. For orders on curves, the following theorem provides a strengthening of Proposition 1.1.12.

**Theorem 1.2.1.** *Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. Then the following are equivalent:*

*i) The $\mathcal{O}_C$-order $\mathscr{A}$ is maximal.*

*ii) The $\mathcal{O}_{C,p}$-order $\mathscr{A}_p$ is maximal at every point $p \in C$.*

*iii) The $\widehat{\mathcal{O}}_{C,p}$-order $\widehat{\mathscr{A}_p}$ is maximal at the completion of every point $p \in C$.*

The proof can be found in [119, Theorem 11.5]. The noteworthy equivalence (ii)⇔(iii) relies on the fact that it suffices to study full lattices over discrete valuation rings. We outline the essential ingredient for this equivalence.

Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m} \trianglelefteq R$ and fraction field $K$. Let $V$ be a finite-dimensional $K$-vector space. Recall that a *full $R$-lattice* in $V$ is a finitely generated torsionfree $R$-submodule $M \subset V$ containing a $K$-basis of $V$.

Denote by $\widehat{R}, \widehat{K}, \widehat{V}$ their completions with respect to the valuation defined by $\mathfrak{m}$. There is an inclusion-preserving one-to-one correspondence

$$\{M \text{ full } R\text{-lattice in } V\} \quad \longleftrightarrow \quad \left\{N \text{ full } \widehat{R}\text{-lattice in } \widehat{V}\right\} \tag{1.12}$$

given by $M \mapsto \widehat{N} := N \otimes_R \widehat{R}$, and $N \mapsto V \cap N$, see [119, Theorem 5.2].

Note that an $R$-order $\Lambda$ is a full lattice in a central simple $K$-algebra. The equivalence (ii)⇔(iii) in Theorem 1.2.1 follows now from (1.12) and the fact that an $R$-order $\Lambda$ gives rise to an $\widehat{R}$-order $\widehat{\Lambda} = \Lambda \otimes_R \widehat{R}$, and vice versa.

**Maximal orders over complete discrete valuation rings.** Theorem 1.2.1 naturally prompts the question of how maximal orders over complete discrete valuation rings can be understood.

Let $R$ be a complete discrete valuation ring with fraction field $K$. An extension of the theory of valuations to $K$-division algebras leads to the construction of a unique maximal $R$-order in a given $K$-division algebra, cf. [119, Theorem 12.8].

**Theorem 1.2.2.** *Let $R$ be a complete discrete valuation ring with fraction field $K = \operatorname{Frac} R$, and $D$ a $K$-division algebra. Then there is a unique maximal $R$-order $\Delta$ in $D$.*

It can be constructed as the integral closure of $R$ in $D$. More concretely,

- let $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ be the valuation associated with $R$,
- and $\det\colon D \to K$ be the reduced norm of $D$.

Then $\Delta$ is given by

$$\Delta = \{a \in D \mid \nu(\det a) \geq 0\}. \tag{1.13}$$

To ensure that $\nu \circ \det \colon D \to \mathbb{Z} \cup \{\infty\}$ is a valuation, we need to assume that $R$ is complete, see [119, Theorem 12.5] and its proof.

The $R$-order $\Delta$ behaves in many ways like a noncommutative version of a discrete valuation ring [119, Theorem 13.2]. The radical of the maximal $R$-order $\Delta$ is a two-sided principal ideal $\mathsf{rad}\,\Delta = \Delta\pi = \pi\Delta$. It is generated by an element $\pi \in \Delta$ such that $\nu(\det(\pi)) > 0$ is minimal. Moreover, $\Delta/\mathsf{rad}\,\Delta$ is a division ring. Every non-zero one-sided ideal in $\Delta$ is a power of $\mathsf{rad}\,\Delta$, and therefore two-sided.

In particular the ideal $\mathfrak{m}\Delta$, where $\mathfrak{m} \trianglelefteq R$ is the maximal ideal of $R$, can be written as a power

$$\mathfrak{m}\Delta = (\mathsf{rad}\,\Delta)^e. \tag{1.14}$$

We call $e$ the *ramification index* of $D$ over $K$, see [119, §13].

**Maximal orders over discrete valuation rings.** Let $R$ be a (not necessarily complete) discrete valuation ring with fraction field $K$.

We are now interested in maximal $R$-orders in an arbitrary central simple $K$-algebra $A$. This links to Theorem 1.2.2 via the Artin–Wedderburn theorem. It says that $A \cong \mathrm{Mat}_n(D)$ for some $n > 0$ and a $K$-division algebra $D$. The statement below summarizes the classification [119, Theorem 18.7] of maximal $R$-orders. We refer to Theorem 17.3 of *op. cit.* for the analogue in the complete local case.

**Theorem 1.2.3.** *Let $R$ be a discrete valuation ring with fraction field $K$, and $A = \mathrm{Mat}_n(D)$ be a central simple algebra. Fix a maximal $R$-order $\Delta$ in $D$.*

*i) Then $\mathrm{Mat}_n(\Delta)$ is a maximal $R$-order in $A$.*

*ii) If $\Lambda$ is a maximal $R$-order in $A$, then there exists an invertible element $u \in A^*$ such that $u\Lambda u^{-1} = \mathrm{Mat}_n(\Delta)$.*

*iii) If $\Lambda$ is a maximal $R$-order in $A$, then there exists a free $\Delta$-module $M$ which is a full $R$-lattice in $D^{\oplus n}$ such that $\Lambda \cong \mathrm{End}_\Delta(M)$. Moreover, $\mathsf{rad}\,\Lambda$ is a principal ideal.*

**Remark 1.2.4.** We briefly explain the distinction to the complete local case.

i) If $R$ is a complete discrete valuation ring, the order $\Delta$ is unique by Theorem 1.2.2.

ii) If $R$ is not complete, then $\Delta$ can be obtained as follows. We pass to the completion $\widehat{R}$ with $\widehat{D} \cong \mathrm{Mat}_m(E)$. By Theorem 1.2.2 there is a unique maximal $\widehat{R}$-order $\Omega \subset E$. Applying Theorem 1.2.3 to $\widehat{R}$, we obtain a maximal $\widehat{R}$-order $\mathrm{Mat}_m(\Omega)$ in $\widehat{D}$. Using the restriction of lattices (1.12), the $R$-order $\Delta = D \cap \mathrm{Mat}_m(\Omega)$ is a maximal $R$-order in $D$.

iii) In general the completion of a $K$-division algebra is not a division algebra. We can use a similar example to the one in Example 1.1.21. Let $K = \mathbf{k}(x, y)$ be the function field of a rational surface. Then $D = (x, y)_2^K$ is a $K$-division algebra. Taking the completion with respect to the valuation of $\mathfrak{p} = (x - \lambda)$ for $\lambda \in \mathbf{k}^*$, we find $f \in K^*$ such that $x = f^2$. Therefore, $\widehat{D} \cong (1, y)_2^K$ is split.

**Maximal orders in the global setting.** In order to pass from the local classification results back to a smooth (projective) curve $C$, we use that we can recover torsionfree coherent $\mathcal{O}_C$-modules from their stalks, as in [44, Lemma 7.2].

**Theorem 1.2.5.** *Let $C$ be a smooth separated curve over $\mathbf{k}$. Denote by $V$ a finite-dimensional $\mathbf{k}(C)$-vector space. Assume that $\mathcal{M}$ is a coherent $\mathcal{O}_C$-submodule of $V$ such that $\mathcal{M} \otimes_C \mathbf{k}(C) = V$. If*

$$\{\mathcal{N}(p) \subset V \mid p \in C \text{ closed}, \mathcal{N}(p) \otimes_{\mathcal{O}_{C,p}} \mathbf{k}(C) = V\} \tag{1.15}$$

*is a collection of finitely generated $\mathcal{O}_{C,p}$-modules in $V$ such that $\mathcal{N}(p) = \mathcal{M}_p$ for almost all $p \in C$, then there exists a unique coherent $\mathcal{O}_C$-submodule $\mathcal{N} \subset V$ such that $\mathcal{N}_p = \mathcal{N}(p)$ for all $p \in C$.*

**Remark 1.2.6.** Due to the correspondence (1.12) between full lattices on discrete valuation rings and their completion, Theorem 1.2.5 also holds if we use a collection $\mathcal{N}(p)$ of finitely generated $\widehat{\mathcal{O}}_{C,p}$-modules in $\widehat{V}$. We refer to [42, Proposition 6.5] for such a version.

**Theorem 1.2.7.** *Let $C$ be a curve with function field $\mathbf{k}(C)$, and $D$ a $\mathbf{k}(C)$-division algebra. Let $\mathscr{D}$ be a maximal $\mathcal{O}_C$-order in $D$, and $V$ an $n$-dimensional right $D$-module. Then every maximal $\mathcal{O}_C$-order in $\mathrm{End}_D(V)$ is of the form*

$$\mathscr{A} = \mathcal{E}nd_{\mathscr{D}}(M), \tag{1.16}$$

*where $M$ is a (right) $\mathscr{D}$-submodule of $V$ which is locally free as an $\mathcal{O}_C$-module.*

The proof of this statement can be done as in the affine case [119, Theorem 21.6] using Theorem 1.2.5 instead of its affine version [119, Theorem 4.22]. We will appeal to this theorem in Section 3.2.2 to construct maximal orders containing hereditary orders.

If $C = \mathrm{Spec}(R)$ is a discrete valuation ring, Theorem 1.2.3 shows that two maximal orders in the same central simple algebra are isomorphic. This does not hold if $C$ is projective.

**Example 1.2.8.** Similar to Example 1.1.16 let $C$ be a smooth projective curve over $\mathbf{k}$ and $p \in C$ a closed point. The two orders

$$\mathscr{B}_1 = \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{O}_C \oplus \mathcal{O}_C) \quad \text{and} \quad \mathscr{B}_2 = \mathcal{E}nd_{\mathcal{O}_C}(\mathcal{O}_C \oplus \mathcal{O}_C(p)). \tag{1.17}$$

are Morita equivalent, because they are both split Azumaya. However, over the affine neighborhood $U_p = C \setminus \{p\}$ both are equal to matrix algebras, whereas for $q \neq p$ and $\mathrm{Spec}(R) = U_q = C \setminus \{q\}$ the are only conjugate to each other. More precisely, let $\mathfrak{m}_p = (r) \trianglelefteq R$ be the maximal ideal corresponding to $p$. Then we have

$$\mathscr{B}_2|_{U_q} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \cdot \mathscr{B}_1|_{U_q} \cdot \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix}, \tag{1.18}$$

and hence the isomorphisms on the two affine patches do not glue.

## 1.3 Hereditary orders

This section is devoted to hereditary orders. They play an important role in all of the subsequent chapters. We start by giving the definition. Let $C$ be a smooth separated curve over $\mathbf{k}$.

**Definition 1.3.1.** An $\mathcal{O}_C$-order $\mathscr{A}$ is *hereditary* if $\operatorname{gldim} \mathscr{A} = 1$.

We follow a similar strategy as in Section 1.2 by first looking at the local situation and then passing to the global case. To do so we start with the analogue of Theorem 1.2.1.

Since $\operatorname{gldim} \mathscr{A} = \sup_{p \in C} \operatorname{gldim} \mathscr{A}_p$ by [119, Theorem 3.29], and the global dimension remains invariant under completion by Theorem 3.30 of *op. cit.*, we can check heredity on (completions of) stalks.

**Theorem 1.3.2.** *Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. The following are equivalent:*

*i) The $\mathcal{O}_C$-order $\mathscr{A}$ is hereditary.*

*ii) The $\mathcal{O}_{C,p}$-order $\mathscr{A}_p$ is hereditary at every point $p \in C$.*

*iii) The $\widehat{\mathcal{O}}_{C,p}$-order $\widehat{\mathscr{A}_p}$ is hereditary at the completion of every point $p \in C$.*

**Example 1.3.3.** As an application of Theorem 1.3.2, a maximal $\mathcal{O}_C$-order $\mathscr{A}$ is hereditary. This is because, by Theorem 1.2.3, we have $\operatorname{gldim} \widehat{\mathscr{A}_p} = \operatorname{gldim}(\operatorname{Mat}_n(\Delta)) = \operatorname{gldim}(\Delta) = 1$.

For an étale map $f \colon C' \to C$, the pullback $f^*\mathscr{A}$ of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ is hereditary by [82, Theorem 4], [67, Proposition 1.7]. This is not true for maximal orders as illustrated in the next example. We use the notation from (1.5).

**Example 1.3.4.** The $\mathbb{R}(\!(t)\!)$-algebra $(-1,t)_2^{\mathbb{R}(\!(t)\!)}$ is a division algebra, cf. [72, Example 1.3.8]. Using Theorem 1.1.19, we find that $\Lambda = (-1,t)_2^{\mathbb{R}[\![t]\!]}$ is a maximal order in $(-1,t)_2^{\mathbb{R}(\!(t)\!)}$. Under the étale map $f \colon \operatorname{Spec}(\mathbb{C}[\![t]\!]) \to \operatorname{Spec}(\mathbb{R}[\![t]\!])$, we obtain the $\mathbb{C}[\![t]\!]$-order

$$\Lambda \otimes_{\mathbb{R}[\![t]\!]} \mathbb{C}[\![t]\!] = (-1,t)_2^{\mathbb{C}[\![t]\!]} \cong \begin{pmatrix} \mathbb{C}[\![t]\!] & \mathbb{C}[\![t]\!] \\ (t) & \mathbb{C}[\![t]\!] \end{pmatrix}. \tag{1.19}$$

This $\mathbb{C}[\![t]\!]$-order is hereditary as can be seen from Theorem 1.3.5 below. It is evidently not maximal because it is strictly contained in $\operatorname{Mat}_2(\mathbb{C}[\![t]\!])$. The reader is referred to [67, Proposition 1.3] for the precise shape of the pullback of a maximal order in a skewfield along an étale map.

**Hereditary orders over complete discrete valuation rings.** In this paragraph we focus on orders over (complete) discrete valuation rings and classify hereditary orders up to isomorphism in this setting. We will present invariants $(n_1, \dots, n_r)$ and $r$ which are essential to determine their isomorphism (resp. Morita equivalence) class.

Let $R$ be a complete discrete valuation ring with fraction field $K$. The classification of hereditary orders is based on the results for maximal orders.

**Theorem 1.3.5.** *Let $A = \mathrm{Mat}_n(D)$ be a central simple $K$-algebra, where $D$ is a $K$-division algebra. Denote by $\Delta$ the unique maximal $R$-order in $D$. Then every hereditary $R$-order in $A$ is given by*

$$\Lambda \cong \begin{pmatrix} \Delta & \Delta & \dots & \Delta \\ \mathsf{rad}\,\Delta & \Delta & \dots & \Delta \\ \vdots & & \ddots & \vdots \\ \mathsf{rad}\,\Delta & \mathsf{rad}\,\Delta & \dots & \Delta \end{pmatrix}^{(n_1,\dots,n_r)} \subseteq \mathrm{Mat}_n(D), \tag{1.20}$$

*where each entry is an $n_i \times n_j$-matrix with coefficients in $\Delta$, resp.* $\mathsf{rad}\,\Delta$*, and $n_1 + \ldots + n_r = n$.*

The isomorphism (1.20) is obtained by first choosing a maximal overorder $\Gamma \supseteq \Lambda$. Then $\mathsf{rad}\,\Gamma \subseteq \mathsf{rad}\,\Lambda$. By Theorem 1.2.3, there exists a $\Delta$-$\Gamma$-bimodule $M$, which is locally free as left $\Delta$-module such that $\Gamma = \mathrm{End}_\Delta(M) \cong \mathrm{Mat}_n(\Delta)$. The identification of $\Lambda$ inside $\mathrm{Mat}_n(\Delta)$, is obtained by the radical filtration of $M$ with respect to $\mathsf{rad}\,\Lambda$. It leads to the isomorphism (1.20) We refer to [119, Theorem 39.14] for the details.

**Definition 1.3.6.** Let $\Lambda$ be a hereditary $R$-order in $\mathrm{Mat}_n(D)$ isomorphic to an order of the form (1.20) with $(n_1, \dots, n_r) \in \mathbb{N}^r$.

- The integer $r \in \mathbb{N}$ is called the *ramification index of* $\Lambda$.
- The $r$-tuple $(n_1, \dots, n_r) \in \mathbb{N}^r$ is called the *ramification data of* $\Lambda$.

Using the presentation of a hereditary $R$-order $\Lambda$ from Theorem 1.3.5, it is straightforward that the radical of $\Lambda$ is given by

$$\mathsf{rad}\,\Lambda = \begin{pmatrix} \mathsf{rad}\,\Delta & \Delta & \dots & \Delta \\ \mathsf{rad}\,\Delta & \mathsf{rad}\,\Delta & \dots & \Delta \\ \vdots & & \ddots & \vdots \\ \mathsf{rad}\,\Delta & \mathsf{rad}\,\Delta & \dots & \mathsf{rad}\,\Delta \end{pmatrix}^{(n_1,\dots,n_r)} . \tag{1.21}$$

Since $\mathsf{rad}\,\Lambda^r = (\mathsf{rad}\,\Delta) \cdot \Lambda$, we can extend the formula (1.14) to hereditary orders. If $e$ denotes the ramification index of $D$ over $K$, then

$$(\mathsf{rad}\,\Lambda)^{r \cdot e} = \mathfrak{m} \cdot \Lambda, \tag{1.22}$$

where $\mathfrak{m} \trianglelefteq R$ is the maximal ideal of $R$.

Note that the radical $\mathsf{rad}\,\Lambda$ is projective since $\Lambda$ is hereditary. Moreover, it is an invertible $\Lambda$-module. This is a straightforward verification, once we have described the indecomposable projective $\Lambda$-modules. Let $E_{\alpha,\beta}$ denote the elementary matrix of $\mathrm{Mat}_n(\Delta)$ with the $(\alpha, \beta)$-th entry equal to 1, and all other entries equal to 0.

Let $i \in \{1, \ldots, r\}$ and $\alpha_i = n_1 + \ldots + n_i$. Then $P_i = E_{\alpha_i,\alpha_i}\Lambda$ is an indecomposable projective $\Lambda$-module. Moreover, $P_1, \ldots, P_r$ are all indecomposable projective $\Lambda$-modules up to isomorphism, see [119, Theorem 39.23].

Equivalently, we can describe the modules $P_1, \ldots, P_r$ as follows. Let $P_1 = E_{1,1}\Lambda \cong \Delta^{\oplus n}$. Then

$$P_1 = E_{1,1}\Lambda, \quad P_2 \cong P_1 \cdot \operatorname{rad}\Lambda, \quad \ldots, \quad P_r \cong P_1 \cdot (\operatorname{rad}\Lambda)^{r-1}. \tag{1.23}$$

Each of the $\Lambda$-modules $P_i$ is a free $\Delta$-module of rank $n$. It follows from Theorem 1.2.3 that $\Gamma_i = \operatorname{End}_\Delta(P_i)$ is a maximal $R$-order containing $\Lambda$. By [119, Theorem 39.23], we have described all maximal $R$-orders containing $\Lambda$. Moreover, their intersection is

$$\Lambda = \Gamma_1 \cap \ldots \cap \Gamma_r. \tag{1.24}$$

This implies that a hereditary order $\Lambda$ is uniquely determined by the maximal $R$-orders $\Gamma_i$ containing it. Recall that the description (1.20) of the hereditary order depends on the choice of a maximal overorder. A different choice of maximal order permutes the integers $n_1, \ldots, n_r$ cyclically.

We now summarize the consequences for the isomorphism classes (cf. [119, Corollary 39.24]), and Morita equivalence classes (cf. [129, Proposition 2.9]) of hereditary orders.

**Theorem 1.3.7.** *Let $R$ be a complete discrete valuation ring with fraction field $K$ and $D$ be a $K$-division algebra.*

*i) Two hereditary orders in $\operatorname{Mat}_n(D)$ are isomorphic if and only if they have the same ramification index $r$ and the ramification data $(n_1, \ldots, n_r) \in \mathbb{N}^r$ coincide up to a cyclic permutation.*

*ii) Two hereditary $R$-orders $\Lambda_1 \subset \operatorname{Mat}_{n_1}(D)$ and $\Lambda_2 \subset \operatorname{Mat}_{n_2}(D)$ are Morita equivalent if and only if they have the same ramification index $r$.*

**Remark 1.3.8.** Theorem 1.3.5 carries over to not necessarily complete discrete valuation rings if the completion of $\widehat{D}$ of the division algebra $D$ is a division algebra. In Chapters 2 to 4 we will be working over an algebraically closed field $\mathbf{k}$. Therefore, the $K$-division algebra is $D = K$ and its completion remains a field.

Before we move on to the global setting, we record the situation for a discrete valuation ring $R$ over an algebraically closed field $\mathbf{k}$. By Tsen's theorem [72, Theorem 6.2.8], the Brauer group $\operatorname{Br}(\mathbf{k}(C))$ is trivial. This simplifies the discussion.

**Remark 1.3.9.** If $\mathbf{k}$ is algebraically closed, then (1.22) simplifies to $(\operatorname{rad}\Lambda)^r = \mathfrak{m}\Lambda$. Theorem 1.3.7 simplifies to the following statement:

i) Two hereditary $R$-orders are isomorphic if and only if they have the same ramification data $(n_1, \ldots, n_r)$ up to cyclic permutation.

ii) Two hereditary $R$-orders are Morita equivalent if and only if they have the same ramification index $r$.

The isomorphism class will be important for the Brauer–Severi scheme of a hereditary in Chapter 2. In Chapters 3 and 4 we will mostly be interested in the Morita equivalence class of a hereditary order.

**Hereditary orders in the global setting.** As for maximal orders, we appeal to Theorem 1.2.5 for a global characterization. The key ingredient from the (complete) local characterization of hereditary orders is their description (1.24) by maximal orders containing them. The affine version of the next theorem can be found in [119, Theorem 40.8].

**Theorem 1.3.10.** *Let $C$ be a smooth separated curve over $\mathbf{k}$, and $A$ be a central simple $\mathbf{k}(C)$-algebra. Assume that $\mathscr{A}$ is a hereditary $\mathcal{O}_C$-order with ramification locus $\Delta_{\mathscr{A}} = \{p_1, \ldots, p_e\}$ and ramification index $r_j$ at $p_j \in \Delta_{\mathscr{A}}$. Then*

*i) there are precisely $r_1 \cdot \ldots \cdot r_e$ maximal orders containing $\mathscr{A}$,*

*ii) there are precisely $(2^{r_1} - 1) \cdot \ldots \cdot (2^{r_e} - 1)$ hereditary orders containing $\mathscr{A}$.*

The construction of the maximal orders is an application of Theorem 1.2.5. Let $p_j \in \Delta_{\mathscr{A}}$. After passing to the completion at $p_j$, it follows from (1.24) that

$$\widehat{\mathscr{A}_{p_j}} := \mathscr{A}_{p_j} \otimes_{\mathcal{O}_{C,p_j}} \widehat{\mathcal{O}}_{C,p_j} = \widehat{\mathscr{B}}_1(p_j) \cap \ldots \cap \widehat{\mathscr{B}}_{r_{p_j}}(p_j) \tag{1.25}$$

can be uniquely written as the intersection of the maximal orders containing it. Choosing one maximal order $\widehat{B}_{i_j}(p_j)$ for each $p_j \in \Delta_{\mathscr{A}}$, we obtain a unique $\mathcal{O}_C$-order containing $\mathscr{A}$ by Theorem 1.2.5. It is maximal by Theorem 1.2.1.

**Remark 1.3.11.** We will see in Lemma 3.4.15 that for every maximal overorder $\mathscr{B}_{(i_1,\ldots,i_p)} \supseteq \mathscr{A}$, there is a fully faithful functor $\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_{(i_1,\ldots,i_p)}) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

**Corollary 1.3.12.** *Let $C$ be a smooth curve and $A$ a central simple $\mathbf{k}(C)$-algebra. A hereditary $\mathcal{O}_C$-order $\mathscr{A}$ in $A$ is uniquely determined by the set of maximal $\mathcal{O}_C$-orders containing it.*

**Remark 1.3.13.** By Reiten–Van den Bergh [120, Proposition 1.1] there is an analogue of hereditary orders on surfaces in the sense of Corollary 1.3.12. Such orders are called *tame orders* and will be treated in Section 4.2.3.

We have seen in Example 1.2.8 that one can construct many non-isomorphic hereditary $\mathcal{O}_C$-orders which are locally isomorphic. For the question of Morita equivalence, the local data together with the central simple algebra is sufficient, [42, Proposition 7.8]. For hereditary orders we have the following result.

**Theorem 1.3.14.** *Let $C$ be a smooth separated curve over $\mathbf{k}$, and let $\mathscr{A}_i$ be hereditary $\mathcal{O}_{C_i}$-orders in central simple $\mathbf{k}(C)$-algebras $A_i$ for $i = 1, 2$. Then $\mathscr{A}_1$ and $\mathscr{A}_2$ are Morita equivalent if and only if there exists an automorphism $\varphi\colon C \to C$ such that*

*i) the central simple $\mathbf{k}(C)$-algebras $\varphi^* A_2$ and $A_1$ are Morita equivalent,* *ii) we have* $\varphi(\Delta_{\mathscr{A}_1}) = \Delta_{\mathscr{A}_2}$ *for the ramification loci, and*

*iii) for each* $p \in \Delta_{\mathscr{A}_1}$*, the ramification indices satisfy* $r_{\varphi(p)} = r_p$*.*

If $\mathbf{k}$ is algebraically closed, the situation simplifies.

**Remark 1.3.15.** Let $C$ be a smooth separated curve over an algebraically closed field $\mathbf{k}$. Tsen's theorem [72, Theorem 6.2.8] ensures that the Brauer group $\mathrm{Br}(\mathbf{k}(C))$ is trivial and simplifies the situation.

- Using Theorem 1.2.7, every maximal $\mathcal{O}_C$-order is a split Azumaya algebra.
- Using Theorem 1.3.14, two hereditary orders $\mathscr{A}_1, \mathscr{A}_2$ are Morita equivalent if and only if there is an automorphism $\varphi\colon C \to C$ such that
  i) $\varphi(\Delta_{\mathscr{A}_1}) = \Delta_{\mathscr{A}_2}$, and
  ii) for each $p \in \Delta_{\mathscr{A}_1} : r_{\varphi(p)} = r_p$.

Building on Remark 1.3.15, we briefly mention the dictionary between orders and stacks in dimension one. Let $C$ be a curve over an algebraically closed field $\mathbf{k}$ of characteristic zero.

The observation that the Morita equivalence class of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ is determined by a finite number of points $\Delta_{\mathscr{A}} = \{p_1, \dots, p_e\}$, and the ramification indices $(r_1, \dots, r_e)$ over these points, leads to the candidate of its stacky counterpart. By the bottom up construction [71], a smooth separated one-dimensional Deligne–Mumford stack $\mathcal{C}$ with trivial generic stabilizer is uniquely determined (up to isomorphism of stacks) by

- its coarse moduli space $C$,
- the finite set $\Delta_{\mathcal{C}}$ of stacky points,
- the order of the stabilizer group at these points.

The dictionary in dimension one [51, Corollary 7.8] over an algebraically closed field $\mathbf{k}$ of characteristic 0 can then be expressed as follows (cf. Section 4.2.2).

**Theorem 1.3.16** (Chan–Ingalls)**.** *Let* $C$ *be a smooth separated curve over* $\mathbf{k}$*. Moreover, let* $\mathscr{A}$ *be a hereditary* $\mathcal{O}_C$*-order with ramification locus* $\Delta_{\mathscr{A}}$ *and ramification index* $r_p$ *at* $p \in \Delta_{\mathscr{A}}$*. Then there exists a smooth separated Deligne–Mumford stack* $\mathcal{C}$ *with coarse moduli space* $C$ *and trivial generic stabilizer such that*

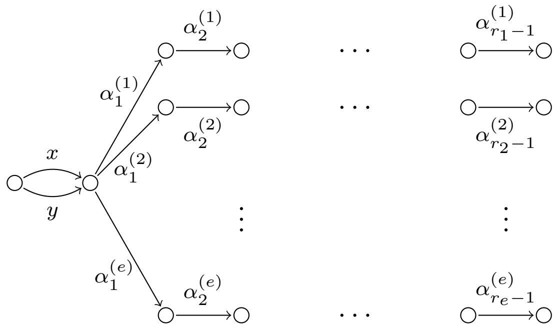
$$\mathsf{coh}(C, \mathscr{A}) \cong \mathsf{coh}(\mathcal{C}). \tag{1.26}$$

*The stack* $\mathcal{C}$ *is a root stack with stacky structure over* $\Delta_{\mathscr{A}}$ *and stabilizer group* $\mu_{r_p}$ *at* $p \in \Delta_{\mathscr{A}}$*. Moreover, it is determined uniquely up to isomorphism by the* $\mathcal{O}_C$*-order* $\mathscr{A}$*.*

*Vice versa, the stacky curve determines a hereditary* $\mathcal{O}_C$*-order* $\mathscr{A}$ *such that* (1.26) *holds. The order* $\mathscr{A}$ *is unique up to Morita equivalence.*

In Chapter 3, we examine the derived category of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$. Under the name of *weighted projective lines*, Geigle–Lenzing [69] described the derived category of a hereditary $\mathcal{O}_{\mathbb{P}^1}$-order. They showed that $\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A}) := \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(\mathbb{P}^1, \mathscr{A}))$ is derived equivalent to the derived category of finitely generated modules over a finite-dimensional $\mathbf{k}$-algebra.

The algebra can be described as $\mathbf{k}Q/I$, where $Q$ is the *squid quiver*




(1.27)

and $I \trianglelefteq \mathbf{k}Q$ is the admissible ideal generated by $\alpha_1^{(j)}(p_j^{(1)}y - p_j^{(2)}x)$, for $p_j = [p_j^{(1)} : p_j^{(2)}] \in \mathbb{P}^1$.

**Theorem 1.3.17.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_{\mathbb{P}^1}$-order ramified at $\Delta_{\mathscr{A}} = \{p_1, \dots, p_e\}$ with ramification index $r_j$ at $p_j \in \Delta_{\mathscr{A}}$. There is an equivalence of derived categories*

$$\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1, \mathscr{A}) \cong \mathsf{D}^{\mathsf{b}}(\mathbf{k}Q/I), \tag{1.28}$$

*where $Q$ is the squid quiver* (1.27) *and $I \trianglelefteq \mathbf{k}Q$ is the admissible ideal as defined above.*

**Serre duality for hereditary orders.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order. It admits a dualizing bimodule $\omega_{\mathscr{A}} = \mathcal{H}om_C(\mathscr{A}, \omega_C)$. We recall from Van den Bergh–Van Geel [136] the Serre duality statement for hereditary orders on the level of the derived category. We start with the description over a complete discrete valuation ring $R$.

**Lemma 1.3.18.** *Let $\Lambda$ be a hereditary $R$-order in $\mathrm{Mat}_n(D)$ as in Theorem 1.3.5 with ramification data $(n_1, \dots, n_r) \in \mathbb{Z}^r_{>0}$. Then there is an isomorphism of $\Lambda$-$\Lambda$-bimodules*

$$\omega_\Lambda \cong \begin{pmatrix} \Delta & (\mathsf{rad}\,\Delta)^{-1} & \dots & (\mathsf{rad}\,\Delta)^{-1} \\ \Delta & \Delta & & (\mathsf{rad}\,\Delta)^{-1} \\ \vdots & & \ddots & \vdots \\ \Delta & \Delta & \dots & \Delta \end{pmatrix}^{(n_1,\dots,n_r)}, \tag{1.29}$$

*where $\Delta$ is the unique maximal $R$-order in $D$.*

The ideal $\mathsf{rad}\,\Delta$ is invertible in $D$ and we denote its inverse by $(\mathsf{rad}\,\Delta)^{-1}$. Using the reduced trace function $\mathrm{trd}\colon \mathrm{Mat}_n(D) \to \mathbf{k}(X)$, see [119, Theorem 10.1 and (9.6a)], one can explicitly calculate $\omega_\Lambda$. By [144, Proposition 15.6.7] there is a $\Lambda$-$\Lambda$-bimodule isomorphism

$$\omega_\Lambda \cong \{x \in \mathrm{Mat}_n(D) \mid \mathrm{trd}(x\Lambda) \subseteq R\}. \tag{1.30}$$

We restrict ourselves to an algebraically closed field $\mathbf{k}$. The following version of Serre duality is essentially due to [136, Theorem 1].

**Theorem 1.3.19.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order. Then*

$$\mathbb{S}_{\mathscr{A}} = - \otimes_{\mathscr{A}} \omega_{\mathscr{A}}[1] \colon \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \tag{1.31}$$

*is a Serre functor, i.e. there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(M, N) \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(N, \mathbb{S}_{\mathscr{A}}(M))^* \tag{1.32}$$

*for all $M, N \in \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.*

Here $(-)^* = \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k})$ stands for the $\mathbf{k}$-dual.

*Proof.* Let $M, N \in \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$, and denote by $M^\vee = \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathscr{A}) \in \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(M, N) &\cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C)}(\mathcal{O}_C, M^\vee \otimes^{\mathbf{L}}_{\mathscr{A}} N) \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C)}(M^\vee \otimes^{\mathbf{L}}_{\mathscr{A}} N, \omega_C[1])^* \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(N, \mathbf{R}\mathcal{H}om_C(M^\vee, \omega_C[1]))^* \\ &\cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(N, \mathbb{S}_{\mathscr{A}}(M))^*. \end{aligned} \tag{1.33}$$

The first isomorphism follows from the isomorphism $\mathcal{H}om_{\mathscr{A}}(M, N) \cong M^\vee \otimes_{\mathscr{A}} N$ of $\mathcal{O}_C$-modules. We use that $M \in \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ can be represented by a bounded cochain complex of locally free $\mathscr{A}$-modules since $\mathscr{A}$ is flat and of finite global dimension (cf. Lemma 3.3.2). The second is Serre duality for $\mathsf{D}^{\mathsf{b}}(C)$, the third is the tensor-hom adjunction, and the last follows from the following chain of identifications

$$\mathbf{R}\mathcal{H}om_C(M^\vee, \omega_C[1]) \cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M^\vee, \mathcal{H}om_C(\mathscr{A}, \omega_C[1])) \cong M \otimes_{\mathscr{A}} \omega_{\mathscr{A}}[1]. \tag{1.34}$$

using again the tensor-hom adjunction and the isomorphism $M^{\vee\vee} \cong M$. □

## 1.4 Beyond hereditary orders

This chapter serves to give a brief overview of the non-hereditary orders which appear in the thesis. Mostly, we will encounter Bass, Gorenstein and nodal orders in the context of noncommutative plane curves in Chapter 4. Throughout this section we will assume that $\mathbf{k}$ is algebraically closed. By $C$ we mean a smooth separated curve over $\mathbf{k}$.

**Bass and Gorenstein orders.** Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. For hereditary orders, we have already used the dualizing bimodule $\omega_{\mathscr{A}} = \mathcal{H}om_C(\mathscr{A}, \omega_C)$ to formulate Serre duality in Theorem 1.3.19. Similarly to the situation of schemes, we define Gorenstein orders to be orders with invertible dualizing bimodule.

**Definition 1.4.1.** An $\mathcal{O}_C$-order $\mathscr{A}$ is called *Gorenstein* if $\omega_{\mathscr{A}}$ is invertible, i.e. $\omega_{\mathscr{A}} \cong \mathscr{A}$ as a left and as a right $\mathscr{A}_p$-module.

Note that this condition is strictly weaker than $\omega_{\mathscr{A}} \cong \mathscr{A}$ as $\mathscr{A}$-$\mathscr{A}$-bimodules. We present an example from [112, Example 4], where the dualizing bimodule is not isomorphic to the order as a bimodule.

**Example 1.4.2.** Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$. We will compute the dualizing bimodule $\omega_\Lambda$ of the Gorenstein $R$-order

$$\Lambda = \begin{pmatrix} R & R & R \\ \mathfrak{m}^2 & R & R \\ \mathfrak{m}^2 & \mathfrak{m}^2 & R \end{pmatrix} \tag{1.35}$$

in Proposition 1.5.1. It is not isomorphic to $\Lambda$ as a $\Lambda$-$\Lambda$-bimodule.

**Definition 1.4.3.** An $\mathcal{O}_C$-order $\mathscr{A}$ in a central simple algebra $A$ is *Bass* if every $\mathcal{O}_C$-order $\mathscr{B} \supseteq \mathscr{A}$ in $A$ containing $\mathscr{A}$ is Gorenstein.

We have seen that maximality and heredity are (complete) local properties for orders on curves (cf. Theorems 1.2.1 and 1.3.2). The same is true for Bass and Gorenstein orders.

**Lemma 1.4.4.** *Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. Then the following are equivalent:*

*i) The $\mathcal{O}_C$-order $\mathscr{A}$ is Bass, resp. Gorenstein.*

*ii) The $\mathcal{O}_{C,p}$-order $\mathscr{A}_p$ is Bass, resp. Gorenstein, at every point $p \in C$.*

*iii) The $\widehat{\mathcal{O}}_{C,p}$-order $\widehat{\mathscr{A}_p}$ is Bass, resp. Gorenstein, at the completion of every point $p \in C$.*

The proof of this lemma will be given in Lemma 4.5.11. It relies on the correspondences given in Theorem 1.2.5 and (1.12).

We recall the local classification of Bass orders from [59, 76] over an algebraically closed field $\mathbf{k}$. To do so we start with a class of examples. Denote by $R$ a discrete valuation ring with maximal ideal $\mathfrak{m}$ and let $K$ be its fraction field.

**Example 1.4.5.** Denote by $\mathfrak{m} = (t)$ the maximal ideal of $R$. Recall the notation from (1.5). Then for each $e \in \mathbb{Z}_{\geq 0}$, the $R$-order

$$\Gamma_e = (t, t^e)_2^R \tag{1.36}$$

is Bass. Since hereditary orders are Bass, the statement is clear for $e = 0$. If $e > 0$, then $\mathsf{rad}(\Gamma_e) = (i, j)$. By [55, Corollary 1.3] an $R$-order of degree 2 is Bass if and only if its radical can be generated by two elements.

Now we can come to the classification result from [59, Theorem 11.3] modified for $\mathbf{k}$ algebraically closed.

**Theorem 1.4.6.** *Let $\Lambda$ be a Bass $R$-order in* $\mathrm{Mat}_n(K)$*. Then $\Lambda$ is isomorphic to one of the following three types:*

*i) a hereditary order of the form* (1.20)*,*

*ii) a non-hereditary order of the form*

$$\begin{pmatrix} R & R \\ \mathfrak{m}^d & R \end{pmatrix}^{(m,n-m)} \subset \operatorname{Mat}_n(R) \quad \textit{for } m, d \in \mathbb{N}, m \leq n, d \geq 2, \tag{1.37}$$

*iii) an order of the form* $\operatorname{Mat}_{n/2}(\Gamma_e)$ *such that* $e > 0$ *is a positive integer and* $n$ *is even.*

By [59, Theorem 11.3], the first two classes (i) and (ii) are Bass. Note that the order in (1.37) is denoted by $B_n(m, d)$ in *op. cit.*

The remaining isomorphism class consists of matrix rings in so-called primary Bass orders. By definition, a Bass $R$-order $\Lambda$ is *primary* if it is indecomposable as a $\Lambda$-module. Since $\mathbf{k}$ is algebraically closed, it follows that every primary Bass $R$-order $\Gamma$ lies in $\operatorname{Mat}_2(K)$, see [59, §6]. Moreover, such an order is primary if and only if $\Gamma/\operatorname{rad}\Gamma \cong \mathbf{k}$.

Each of the $R$-orders from Example 1.4.5 is primary for $e > 0$, and by [144, §24.5.12] these cover all possible primary orders.

**Example 1.4.7.** The order $\Lambda$ from Example 1.4.2 is a Gorenstein order which is not Bass. This can be seen from the fact that it is not hereditary, nor of type (iii) for degree reasons. On the other hand it has three indecomposable projective non-isomorphic $\Lambda$-modules. This is impossible for orders of type (ii).

Regarding Gorenstein orders, we do not know of a general classification result. For tiled orders over a discrete valuation ring, there is a combinatorial criterion by Kirichenko [86, Lemma 3.2] for when a tiled order is Gorenstein. We come back to the criterion after the general discussion on tiled orders in Theorem 1.4.11.

**Tiled orders.** Tiled orders form a large family of orders with varying behavior for the global dimension, [131, Section 3]. They were first defined in [131] and studied over discrete valuation rings in a series of papers [131, 86, 89, 122, 88, 87]. They also appear under the name *graduated order* in [117, 118, 66]. That being said, we do not know of attempts of treating tiled orders globally on a curve. In Section 1.5 we present evidence supporting the interest in such a global theory from the viewpoint of the dictionary between orders and stacks.

For our purposes, we keep the assumption that $\mathbf{k}$ is algebraically closed. Moreover, $C$ is a smooth separated curve over $\mathbf{k}$.

The understanding of hereditary orders in Theorem 1.3.5, relies on the fact that they can locally be concretely realized as a subalgebra of a matrix algebra (with entries in a maximal order of a division algebra). Moreover, hereditary orders contain at each point the full set of idempotents $E_{ii} \in \operatorname{Mat}_n(\mathcal{O}_{C,p})$ for $i = 1, \ldots, n$. Therefore, we propose the following definition of a tiled order.

**Definition 1.4.8.** An $\mathscr{O}_C$-order $\mathscr{A}$ in $\mathrm{Mat}_n(\mathbf{k}(C))$ is *tiled* if at every point $p \in C$, the stalk $\mathscr{A}_p$ contains the maximal number of $n$ orthogonal (nonzero) idempotents.

The next lemma relates this notion to the definition of tiled orders in [131, 87].

**Lemma 1.4.9.** *Let $R$ be a discrete valuation ring with fraction field $K$, and $\Lambda$ an $R$-order in the central simple $K$-algebra $A = \mathrm{Mat}_n(K)$ of $\deg \Lambda = n$. The following are equivalent:*

*i) The order $\Lambda$ is tiled.*

*ii) There exist $u \in A^*$ and ideals $0 \neq \mathfrak{a}_{ij} \trianglelefteq R$ satisfying*

*a) for all $i = 1, \ldots, n : \mathfrak{a}_{ii} = R$,*

*b) and $\mathfrak{a}_{ij} \cdot \mathfrak{a}_{jk} \subset \mathfrak{a}_{ik}$ for all $1 \leq i, j, k \leq n$*

*such that*

$$u\Lambda u^{-1} = \begin{pmatrix} \mathfrak{a}_{11} & \mathfrak{a}_{12} & \ldots & \mathfrak{a}_{1n} \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & \ldots & \mathfrak{a}_{2n} \\ \vdots & & \ddots & \vdots \\ \mathfrak{a}_{n1} & \mathfrak{a}_{n2} & \ldots & \mathfrak{a}_{nn} \end{pmatrix} \subset \mathrm{Mat}_n(K). \tag{1.38}$$

*Proof.* By Theorem 1.2.3, there exists $u \in A^*$ such that the tiled order $u\Lambda u^{-1}$ lies in the maximal order $\mathrm{Mat}_n(R)$. with the standard idempotents $E_{ii} = ue_iu^{-1} \in \mathrm{Mat}_n(R)$ expressed by the elementary matrices.

Assume that $\Lambda$ is tiled. Then $\mathfrak{a}_{ij} \subset E_{ii}\Lambda E_{jj} \subset \Lambda$ is a non-zero ideal in $R$. The properties (a) and (b) follow from the fact that $u\Lambda u^{-1}$ is an algebra. This shows (i)⇒(ii). The other direction is straightforward. □

Over a discrete valuation ring $R$ every ideal is a power of the maximal ideal $\mathfrak{m}$. This can be used to characterize tiled orders by an integer matrix, [87, Definition 1.2].

**Definition 1.4.10.** Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = (t)$ and $\Lambda = (\mathfrak{a}_{ij})_{i,j} \subset \mathrm{Mat}_n(K)$ be a tiled order defined by ideals $\mathfrak{a}_{ij} \subset R$. The *exponent matrix* $\mathcal{E}(\Lambda) = (\alpha_{ij})_{i,j} \in \mathrm{Mat}_n(\mathbb{Z})$ consists of integers $e_{ij}$ such that $\mathfrak{a}_{ij} = (t^{\alpha_{ij}})$.

The exponent matrix provides a combinatorial criterion for determining whether a tiled order is Gorenstein, see [86, Lemma 3.2] (or [87, Theorem 3] for an English version).

**Theorem 1.4.11.** *Let $R$ be a discrete valuation ring and $\Lambda$ a tiled $R$-order with exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})_{i,j} \in \mathrm{Mat}_n(\mathbb{Z}_{\geq 0})$ such that $\alpha_{ij} + \alpha_{ji} > 0$ for all $1 \leq i \neq j \leq n$. Then $\Lambda$ is Gorenstein if and only if there exists a permutation $\sigma \in \mathrm{S}_n$ such that*

$$\alpha_{ij} + \alpha_{j\sigma(i)} = \alpha_{i\sigma(i)} \quad \textit{for all } i, j = 1, \ldots, n. \tag{1.39}$$

**Example 1.4.12.** The primary $R$-orders $(t, t^e)_2^R$ are Bass orders which are not tiled. They appear in Theorem 4.5.15 for the construction of noncommutative plane conics.

It is a straightforward application of Theorem 1.4.11 that the following tiled order is not Gorenstein.

**Example 1.4.13.** Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$. The $R$-order

$$\Lambda = \begin{pmatrix} R & R & R \\ \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m} & R \end{pmatrix} \tag{1.40}$$

is a tiled order. It is not Gorenstein, because there is no permutation $\sigma \in \mathrm{S}_3$ such that $\alpha_{3,1} + \alpha_{1,\sigma(3)} = \alpha_{3,\sigma(3)}$ for the exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})_{i,j}$. This order was also presented (up to conjugation) as an example of a non-Gorenstein order in [112, Example 5].

**Triangular orders.** Triangular orders will appear in the context of Theorem 4.5.15 and Proposition 4.6.2. We will revisit a special class of these orders in Section 1.5.

**Definition 1.4.14.** A tiled $\mathcal{O}_C$-order $\mathscr{A}$ in $A = \mathrm{Mat}_n(K)$ is *triangular* if for each $p \in C$ there exists $u \in A^*$ such that

$$u\mathscr{A}_p u^{-1} = \begin{pmatrix} \mathcal{O}_{C,p} & \mathcal{O}_{C,p} & \cdots & \mathcal{O}_{C,p} \\ \mathfrak{a}_{21} & \mathcal{O}_{C,p} & \cdots & \mathcal{O}_{C,p} \\ \vdots & & \ddots & \vdots \\ \mathfrak{a}_{n1} & \mathfrak{a}_{n2} & \cdots & \mathcal{O}_{C,p} \end{pmatrix} \subset \mathrm{Mat}_n(\mathcal{O}_{C,p}). \tag{1.41}$$

**Example 1.4.15.** Let $C$ be a smooth separated curve over $\mathbf{k}$, and $p \in C$ be a closed point. The $\mathcal{O}_C$-order

$$\mathscr{A} = \begin{pmatrix} \mathcal{O}_C & \mathcal{O}_C \\ \mathcal{O}_C(-mx) & \mathcal{O}_C \end{pmatrix} \subset \mathrm{Mat}_2(\mathcal{O}_C) \tag{1.42}$$

is a triangular order of degree 2.

In fact every tiled order of degree 2 is triangular, as we will see now.

**Example 1.4.16.** Every tiled order of degree two is Gorenstein and triangular, because locally a tiled $\mathcal{O}_C$-order $\mathscr{A}$ is of the form

$$\mathscr{A} \otimes_C \mathcal{O}_{C,p} = \begin{pmatrix} \mathcal{O}_{C,p} & (a) \\ (b) & \mathcal{O}_{C,p} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{O}_{C,p} & \mathcal{O}_{C,p} \\ (ab) & \mathcal{O}_{C,p} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \tag{1.43}$$

where $a, b \in \mathcal{O}_{C,p}$ are non-zero elements. We can apply Kirichenko's criterion from Theorem 1.4.11 to see that the order is Gorenstein. Moreover, it is not hard to see that the global dimension of the order $\mathscr{A}$ is either 1 or $\infty$. We will give a proof of this in Proposition 1.5.1.

**Remark 1.4.17.** In general, the dichotomy $\operatorname{gldim} \mathscr{A} \in \{1, \infty\}$ fails for triangular orders of degree greater than 2. In fact, if $\mathscr{A}$ is a triangular order of finite global dimension, Tarsy

[132, Theorem 1] shows that the maximal global dimension of $\mathscr{A}$ grows linearly with its degree. We have listed the maximal global dimension in Table 1.1 for orders of low degree. More precisely, we have $\operatorname{gldim} \mathscr{A} \leq 2 \cdot (\deg \mathscr{A} - 2)$ for $\deg \mathscr{A} \geq 3$.

| $\deg \mathscr{A}$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\max \operatorname{gldim} \mathscr{A}$ | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

Table 1.1: Maximal global dimension of triangular orders of finite global dimension.

**Example 1.4.18.** Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$. The $R$-order

$$\Lambda = \begin{pmatrix} R & R & R \\ \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m} & R \end{pmatrix} \tag{1.44}$$

from Example 1.4.13 is an example of a triangular order of $\operatorname{gldim} \Lambda = 2$, [131, Theorem 10]. Indeed, one finds that $\operatorname{projdim}(E_{22}\Lambda / \mathsf{rad}\, E_{22}\Lambda) = 2$, whereas the other simple modules have projective dimension one.

**Nodal orders.** We keep the assumption that $\mathbf{k}$ is algebraically closed. Nodal orders appear in the description of noncommutative plane conics in Theorem 4.5.15. They were introduced in [60, 40]. We recall their definition and classification from [41].

**Definition 1.4.19.** Let $R$ be a discrete valuation ring. An $R$-order $\Lambda$ is *nodal* if there exists a hereditary overorder $\Gamma \supseteq \Lambda$ such that

i) the Jacobson radicals coincide, i.e. $\mathsf{rad}\, \Gamma = \mathsf{rad}\, \Lambda$, and

ii) for every finitely generated simple $\Lambda$-module $S$ the length of $S \otimes_\Lambda \Gamma$ as a $\Lambda$-module is at most 2.

Following [41, Definition 4.1], we say that an $\mathcal{O}_C$-order $\mathscr{A}$ on a smooth separated curve $C$ over $\mathbf{k}$ is *nodal* if for every point $p \in C$ the stalk $\mathscr{A}_p$ is a nodal $\mathcal{O}_{C,p}$-order.

We present now the complete local classification over smooth curves from [41, §3.2]. Let $R = \mathbf{k}[\![t]\!]$ with maximal ideal $\mathfrak{m} = (t)$ and fraction field field $K = \mathbf{k}(\!(t)\!)$. In contrast to *op. cit.*, we are only interested in nodal orders in central simple algebras.

A nodal order will be described by the following data:

i) a finite subset $\Omega = \{1, \ldots, r\} \subset \mathbb{N}$ such that $r \leq n$,

ii) a symmetric but not necessarily reflexive binary relation $\approx$ on $\Omega$ such that every element is related to at most one element in $\Omega$, and

iii) a weight function $\mathsf{wt}\colon \Omega \to \mathbb{Z}_{>0}$, the definition of which we explain in the following.

First note that $\Omega$ decomposes into a disjoint union of

- simple elements, i.e. elements which are not related to any other element,
- reflexive elements, i.e. $\omega \in \Omega$ such that $\omega \approx \omega$,
- tied elements, i.e. $\omega \in \Omega$ such that $\omega \approx \omega'$ and $\omega \neq \omega'$.

Then a weight function on $\Omega$ is a map

$$\mathsf{wt}\colon \Omega \to \mathbb{Z}_{>0}, \tag{1.45}$$

such that

i) for every reflexive element $\mathsf{wt}(\omega) > 1$,

ii) for every pair of tied elements $\omega \approx \omega'$, one has $\mathsf{wt}(\omega) = \mathsf{wt}(\omega')$,

iii) and $\sum_{\omega\in\Omega} \mathsf{wt}(\omega) = n$.

Moreover, if $\omega \in \Omega$ is a reflexive element, we replace it by two unrelated elements $\omega_+, \omega_-$, and we require that the weight function extends such that $\mathsf{wt}(\omega) = \mathsf{wt}(\omega_+) + \mathsf{wt}(\omega_-)$.

From the data, one can fix the hereditary overorder $\Gamma$ to be of the form (1.20) with ramification data $(\mathsf{wt}(\omega_1), \dots, \mathsf{wt}(\omega_r)) \in \mathbb{N}^r$. Thus, the weight function determines the ramification data of the hereditary overorder with the same radical as the nodal order attached to the data $(\Omega, \approx, \mathsf{wt})$.

**Example 1.4.20.** It is clear from the definition that hereditary orders are nodal. We describe the data $(\Omega, \approx, \mathsf{wt})$ attached with a hereditary order. Let $\Lambda$ be a hereditary order in $\mathrm{Mat}_n(R)$ with ramification data $(n_1, \dots, n_r) \in \mathbb{N}^r$. Then $\Omega = \{\omega_1, \dots, \omega_r\}$. Each of the elements $\omega_i \in \Omega$ is simple. The weight is given by $\mathsf{wt}(\omega_i) = n_i$.

Let $R = \mathbf{k}[\![t]\!]$. In the following denote the $\mathsf{wt}(\omega_i) \times \mathsf{wt}(\omega_j)$-block of a matrix $M \in \mathrm{Mat}_n(R)$ by $M^{(\omega_i,\omega_j)}$.

**Definition 1.4.21.** Fix $(\Omega = \{\omega_1, \dots, \omega_r\}, \approx, \mathsf{wt})$ as above, and let $\Gamma$ be the hereditary order with ramification data $(\mathsf{wt}(\omega_1), \dots, \mathsf{wt}(\omega_r))$. The *nodal order* $\Lambda := \Lambda(\Omega, \approx, \mathsf{wt})$ *associated with the data* $(\Omega, \approx, \mathsf{wt})$ is the $R$-subalgebra $\Lambda \subset \Gamma$ such that for an element $M \in \Gamma$ we have $M \in \Lambda$ if and only if the following hold:

i) For every tied pair $\omega \approx \omega' \in \Omega$: $M^{(\omega,\omega)} \equiv M^{(\omega',\omega')} \mod t$.

ii) For every reflexive $\omega \in \Omega$: $M^{(\omega_+,\omega_-)} \equiv 0 \mod t$ and $M^{(\omega_-,\omega_+)} \equiv 0 \mod t$.

**Example 1.4.22.** Let $\Omega = \{\omega_1, \omega_2\}$ with $\omega_1 \approx \omega_2$ and $\mathsf{wt}(\omega_1) = \mathsf{wt}(\omega_2) = 1$. Then

$$\Lambda(\Omega, \approx, \mathsf{wt}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \left(\begin{smallmatrix} R & R \\ \mathfrak{m} & R \end{smallmatrix}\right) \mid a \equiv d \mod t \right\} \cong (t,t)_2^R. \tag{1.46}$$

This is the primary Bass $R$-order $\Gamma_1$ from Example 1.4.5.

**Example 1.4.23.** Let $\Omega = \{\omega_1, \omega_2\}$ such that $\omega_1$ is reflexive and $\omega_2$ is simple. Let $\mathsf{wt}(\omega_1) = 2$, and $\mathsf{wt}(\omega_2) = 1$. Then

$$\Lambda(\Omega, \approx, \mathsf{wt}) = \begin{pmatrix} R & \mathfrak{m} & R \\ \mathfrak{m} & R & R \\ \mathfrak{m} & \mathfrak{m} & R \end{pmatrix} = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 1 \\ 0 & t & 0 \end{pmatrix} \cdot \begin{pmatrix} R & R & R \\ \mathfrak{m} & R & R \\ \mathfrak{m}^2 & \mathfrak{m} & R \end{pmatrix} \cdot \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 0 & t^{-1} \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.47}$$

The nodal order $\Lambda(\Omega, \approx, \mathsf{wt})$ is therefore isomorphic to the tiled order from Example 1.4.13.

Burban–Drozd [41, Theorem 3.12 and Theorem 3.19] show that $\Lambda(\Omega, \approx, \mathsf{wt})$ is always a nodal order, and present the characterization of all nodal $R$-orders in $\mathrm{Mat}_n(K)$ as follows.

**Theorem 1.4.24.** *Let $\Lambda$ be a nodal $R$-order. Then there exist $(\Omega, \approx, \mathsf{wt})$ as above such that $\Lambda \cong \Lambda(\Omega, \approx, \mathsf{wt})$. Moreover,*

*i) for two triples $(\Omega_1, \approx_1, \mathsf{wt}_1)$ and $(\Omega_2, \approx_2, \mathsf{wt}_2)$ the following are equivalent:*

  *a) The nodal orders $\Lambda(\Omega_1, \approx_1, \mathsf{wt}_1)$ and $\Lambda(\Omega_2, \approx_2, \mathsf{wt}_2)$ are isomorphic.*

  *b) One has $\Omega_1 = \Omega_2$ and there exists a cyclic permutation $\varphi \in \langle (1\ 2\ \ldots\ r) \rangle \subset \mathrm{S}_r$ such that*

  - *for all $\omega, \omega' \in \Omega_1 \colon \varphi(\omega) \approx_2 \varphi(\omega') \;\Leftrightarrow \omega \approx_1 \omega'$,*
  - *for all $\omega \in \Omega_1 \colon \mathsf{wt}_2(\varphi(\omega)) = \mathsf{wt}_1(\omega)$.*

*ii) Every nodal order is Morita equivalent to a nodal order $\Lambda(\Omega, \approx, \mathsf{wt}_0)$ with*

$$\mathsf{wt}_0(\omega) = \begin{cases} 2 & \text{if } \omega \text{ is reflexive,} \\ 1 & \text{otherwise.} \end{cases} \tag{1.48}$$

The results presented in this theorem follow from [41, Chapter 3], which is done in the utmost generality for orders in semisimple algebras. We give a quick explanation how the statements simplify in our setting.

*Proof.* By [41, Theorem 3.19] every nodal order in $\mathrm{Mat}_n(K)$ is isomorphic to an order associated with the data $(\Omega', \tau, \approx', \mathsf{wt}')$. The isomorphism is given by conjugation with an element in $\mathsf{GL}_n(R)$. The element $\sigma \in \mathrm{S}_r$ is a cyclic permutation of length $r = \#\Omega$ acting on $\Omega$.

We explain how to obtain an isomorphism to some $\Lambda(\Omega, \sigma, \approx, \mathsf{wt})$ with $\Omega = \{\omega_1, \ldots, \omega_r\}$ and $\sigma = (1\ 2\ \ldots\ r)$. Fix $\omega_1' \in \Omega'$ and a bijection $\varphi \colon \Omega \to \Omega'$ by $\varphi(\tau^\ell(\omega_1')) = \omega_\ell$. Use the induced binary relation for $\approx$ from $\approx'$ via $\varphi$, and set $\mathsf{wt}(\omega_\ell) := \mathsf{wt}'(\tau^\ell(\omega_1'))$.

By [41, Theorem 3.12] the orders $\Lambda(\Omega, \sigma, \approx, \mathsf{wt})$ and $\Lambda(\Omega', \tau, \approx', \mathsf{wt}')$ are isomorphic. Since we can always use the same permutation $\sigma$, we have decided to not mention it in Definition 1.4.21.

Statement (i) follows now from the fact that any bijection $\varphi\colon \Omega \to \Omega$ such that $(1\ 2\ \dots\ r) \circ \varphi = \varphi \circ (1\ 2\ \dots\ r)$ must lie in $\langle(1\ 2\ \dots\ r)\rangle$. Hence (i) is a consequence of the last point, and (ii) is a consequence of the fourth point of Theorem 3.12 in *op. cit.* □

## 1.5 Gorenstein triangular orders and singular stacks

We have seen in Theorem 1.3.16 that there is a dictionary between hereditary orders and smooth root stacks. Our joint work in Chapter 4 shows that the dictionary can be extended to singular root stacks. More precisely, Theorem 4.5.15 provides evidence that root stacks in a non-reduced divisor correspond to a class of Gorenstein triangular orders, which we present in this section. Due to its connection to Sections 4.4 and 4.6 we invite the reader to come back to this section when reading Chapter 4.

Recall that the definition of a Gorenstein order, Lemma 1.4.4, and the definition of a triangular order, Definition 1.4.14, are both of local nature. Therefore, we may study Gorenstein triangular orders locally.

Let $\mathbf{k}$ be an algebraically closed field of characteristic 0. Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = (t)$ such that $R/\mathfrak{m} = \mathbf{k}$. Denote by $K$ the fraction field of $R$.

We focus on a class which satisfies $\operatorname{gldim} \Lambda \in \{1, \infty\}$. As for hereditary orders in Theorem 1.3.5, let $r > 0$ be a positive integer, $(n_1, \dots, n_r) \in \mathbb{Z}^r_{>0}$. Given a non-negative integer $a \in \mathbb{Z}_{\geq 0}$, we define

$$\Lambda = \begin{pmatrix} R & R & R & \cdots & R \\ \mathfrak{m}^a & R & R & \cdots & R \\ \mathfrak{m}^a & \mathfrak{m}^a & R & \cdots & R \\ \vdots & & & \ddots & \vdots \\ \mathfrak{m}^a & \mathfrak{m}^a & \mathfrak{m}^a & \cdots & R \end{pmatrix}^{(n_1,\dots,n_r)} \subset \operatorname{Mat}_n(R). \tag{1.49}$$

Unless $a = 0$, where $\operatorname{rad} \Lambda = \mathfrak{m}\Lambda$, the radical is given by

$$\operatorname{rad} \Lambda = \begin{pmatrix} \mathfrak{m} & R & R & \cdots & R \\ \mathfrak{m}^a & \mathfrak{m} & R & \cdots & R \\ \mathfrak{m}^a & \mathfrak{m}^a & \mathfrak{m} & \cdots & R \\ \vdots & & & \ddots & \vdots \\ \mathfrak{m}^a & \mathfrak{m}^a & \mathfrak{m}^a & \cdots & \mathfrak{m} \end{pmatrix}^{(n_1,\dots,n_r)}. \tag{1.50}$$

Assume that $a \geq 1$, and denote by $E_{i,j}$ the elementary matrices of $\operatorname{Mat}_n(R)$. The isomorphism classes of indecomposable projective $\Lambda$-modules are given by the row vectors

$$P_k = E_{\alpha_k,\alpha_k}\Lambda, \tag{1.51}$$

where $\alpha_k = n_1 + \ldots + n_k$ for $k = 1, \dots, r$. In contrast to (1.23) for a hereditary order, they cannot always be represented by repeated multiplication of the first row with the radical. It follows that there are $r$ isomorphism classes of simple $\Lambda$-modules, represented by $S_k = P_k/\operatorname{rad} P_k$. We denote by $\pi_k\colon P_k \to S_k$ the natural projection.

A free generating set of $\Lambda$ as an $R$-module is given by

$$e_{i,j} = \begin{cases} E_{i,j} & \text{if } i \le j, \\ t^a E_{i,j} & \text{if } i > j. \end{cases} \tag{1.52}$$

Depending on $a$ there are two different situations.

- If $a = 1$, each simple admits a finite projective resolution

$$0 \to P_{k+1} \xrightarrow{e_{k,k+1}} P_k \xrightarrow{\pi_k} S_k \to 0, \tag{1.53}$$

  where $k = 1, \ldots, r$ (counted modulo $r$).

- If $a > 1$, each simple admits an infinite projective resolution

$$\ldots \to P_k \oplus P_{k+1} \xrightarrow{d^{-3}} P_k \oplus P_{k+1} \xrightarrow{d^{-2}} P_k \oplus P_{k+1} \xrightarrow{d^{-1}} P_k \xrightarrow{\pi_k} S_k \to 0, \tag{1.54}$$

  where $k = 1, \ldots, r$ (counted modulo $r$) and the differential is given by

$$d^{-i} = \begin{cases} \begin{pmatrix} t\cdot\mathrm{id}_{P_k} & e_{k,k+1} \end{pmatrix} & \text{if } i = 1, \\ \begin{pmatrix} t^{a-1}\cdot\mathrm{id}_{P_k} & e_{k,k+1} \\ -e_{k+1,k} & -t\cdot\mathrm{id}_{P_{k+1}} \end{pmatrix} & \text{if } i > 0 \text{ even}, \\ \begin{pmatrix} t\cdot\mathrm{id}_{P_k} & e_{k,k+1} \\ -e_{k+1,k} & -t^{a-1}\cdot\mathrm{id}_{P_{k+1}} \end{pmatrix} & \text{if } i > 1 \text{ odd}, \\ 0 & \text{else}. \end{cases} \tag{1.55}$$

**Proposition 1.5.1.** *Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m} = (t)$ and fraction field $K$. Let $\Lambda \subset \mathrm{Mat}_n(K)$ be a triangular order as in (1.49) with $(n_1, \ldots, n_r) \in \mathbb{N}^r$.*

*i) It is maximal if and only if $a = 0$.*

*ii) It is hereditary if and only if $a \in \{0, 1\}$.*

*iii) It is Bass if and only if $a \in \{0, 1\}$ or $r \le 2$.*

*iv) It is a one-dimensional Gorenstein ring with dualizing bimodule*

$$\omega_\Lambda = \begin{pmatrix} R & \mathfrak{m}^{-a} & \mathfrak{m}^{-a} & \cdot & \mathfrak{m}^{-a} \\ R & R & \mathfrak{m}^{-a} & \cdot & \mathfrak{m}^{-a} \\ R & R & R & \cdot & \mathfrak{m}^{-a} \\ \vdots & & & \ddots & \vdots \\ R & R & R & \cdot & R \end{pmatrix}^{(n_1,\ldots,n_r)} . \tag{1.56}$$

*v) The global dimension is*

$$\operatorname{gldim} \Lambda = \begin{cases} 1 & \text{if } a \in \{0, 1\}, \\ \infty & \text{otherwise}. \end{cases} \tag{1.57}$$

*Proof.* The first three statements follow from the local structure of maximal (Theorem 1.2.3), resp. hereditary (Theorem 1.3.5), resp. Bass orders (Theorem 1.4.6).

The order being Gorenstein can be attributed to [112]. Let $S_k$ be a simple $\Lambda$-module. By applying $\mathrm{Hom}_\Lambda(-,\Lambda)$ to the projective resolution (1.53) or (1.54) respectively, one calculates that the extension groups are

$$\mathrm{Ext}^i_\Lambda(S_k,\Lambda) \cong \begin{cases} \mathbf{k} & \text{if } i = 1, \\ 0 & \text{else.} \end{cases} \tag{1.58}$$

Therefore, $\Lambda$ is a one-dimensional Gorenstein order by [112, Theorem 1.1].

For the explicit presentation of the dualizing bimodule we use the isomorphism $\omega_\Lambda \cong \{x \in \mathrm{Mat}_n(K) \mid \mathrm{trd}(x\Lambda) \subseteq R\}$ from (1.30). Since we consider an order in $\mathrm{Mat}_n(K)$, the reduced trace $\mathrm{trd}(x) = \mathrm{tr}(x) \in K$ is the trace of the matrix $x \in \mathrm{Mat}_n(K)$. Using the free generating set (1.52) the presentation (1.56) follows.

The global dimension was calculated in [132, Theorem 1]. The theorem implies that the global dimension of a triangular order is infinite as soon as the entries in the first subdiagonal are of the form $\mathfrak{m}^a$ for $a > 1$. We can also calculate the global dimension directly from the projective resolutions (1.53) and (1.54). □

The dichotomy of the global dimension for Gorenstein triangular orders is analogous to what happens for root stacks. Moreover, coherent sheaves on singular root stacks in Theorem 4.5.21 can be described by modules over a Gorenstein triangular $\mathcal{O}_C$-order. Therefore, we conjecture that there is the following extension of Theorem 1.3.16.

**Conjecture 1.5.2.** *Let $C$ be a smooth separated curve over $\mathbf{k}$. Let $\mathcal{C}$ be a root stack in the divisor $\Delta = a \cdot [p]$ for $a \geq 1$, where $p \in C$ is a closed point. Denote by $\mathfrak{m}_p \trianglelefteq \mathcal{O}_{C,p}$ the maximal ideal at $p \in C$. The Gorenstein triangular $\mathcal{O}_C$-order $\mathscr{A}$ with ramification divisor $\Delta_{\mathscr{A}} = \{p\}$ given by*

$$\mathscr{A}_p \cong \begin{pmatrix} \mathcal{O}_{C,p} & \mathcal{O}_{C,p} & \cdots & \mathcal{O}_{C,p} \\ \mathfrak{m}_p^a & \mathcal{O}_{C,p} & \cdots & \mathcal{O}_{C,p} \\ \mathfrak{m}_p^a & \mathfrak{m}_p^a & \cdots & \mathcal{O}_{C,p} \\ \vdots & & & \ddots \\ \mathfrak{m}_p^a & \mathfrak{m}_p^a & \cdots & \mathcal{O}_{C,p} \end{pmatrix} \tag{1.59}$$

*at $p$, relates to $\mathcal{C}$ via the equivalence*

$$\mathsf{coh}(\mathcal{C}) \simeq \mathsf{coh}(C, \mathscr{A}) \tag{1.60}$$

*of abelian categories.*

# Chapter 2

# The Brauer–Severi scheme of a hereditary order

## 2.1 Introduction

Given an Azumaya algebra $\mathscr{A}$ of degree $n$ on an integral variety $X$, one can associate a smooth $X$-scheme $\mathsf{BS}(\mathscr{A})$ with it, called the *Brauer–Severi scheme* of $\mathscr{A}$, [74, 72]. This construction generalizes the notion of a $\mathbb{P}^{n-1}$-bundle in the sense that $\mathsf{BS}(\mathscr{A})$ becomes a projective bundle in the étale topology of $X$.

One way to construct the Brauer–Severi scheme is as the fine moduli space representing the functor $\mathcal{BS}(\mathscr{A})\colon \mathbf{Sch}_X^{\mathrm{op}} \to \mathbf{Sets}$ which assigns to an $X$-scheme $h\colon Y \to X$ the set of left ideals $L \subset h^*\mathscr{A}$ of rank $n$ such that the quotient $h^*\mathscr{A}/L$ is locally free. Continuing from this point of view, [5, 137] construct a fine moduli space $\underline{\mathsf{BS}}(\mathscr{A})$ representing $\mathcal{BS}(\mathscr{A})$ in the more general setting, where $\mathscr{A}$ is an arbitrary coherent $\mathcal{O}_X$-algebra.

For non-Azumaya algebras $\mathscr{A}$, the geometry of $\underline{\mathsf{BS}}(\mathscr{A})$ is more involved. As already observed by Artin [5, Example 3.4], the moduli space $\underline{\mathsf{BS}}(\mathscr{A})$ is usually disconnected (cf. Example 2.3.35) and singular. To make the situation more tractable we follow Artin's approach, and restrict ourselves to

i) torsionfree coherent $\mathcal{O}_X$-algebras $\mathscr{A}$ which are generically Azumaya, and

ii) the connected component $\mathsf{BS}(\mathscr{A}) \subset \underline{\mathsf{BS}}(\mathscr{A})$, which contains the fiber over the generic point $\underline{\mathsf{BS}}(\mathscr{A})_\eta := \underline{\mathsf{BS}}(\mathscr{A}) \times_X \operatorname{Spec} \mathbf{k}(X)$.

A coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ satisfying (i) goes under the name $\mathcal{O}_X$-*order*. It comes with a ramification locus $\Delta_{\mathscr{A}}$, the complement of the maximal (non-empty) open subset $U$, such that the restriction of $\mathscr{A}$ along the inclusion $U \subset X$ is Azumaya. The fibers of the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$ (and its connected component $\mathsf{BS}(\mathscr{A})$) are singular over the ramification locus $\Delta_{\mathscr{A}}$.

By specializing to a smooth curve $C$, it is possible to describe the fibers of $\mathsf{BS}(\mathscr{A})$ for a special class of $\mathcal{O}_C$-orders $\mathscr{A}$. This allows us to explicitly compute motivic invariants of the component $\mathsf{BS}(\mathscr{A})$ of the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$, which we will outline in the next two paragraphs.

**The Brauer–Severi scheme for orders.** Focusing on the setting of an order $\mathscr{A}$ has proven insightful for understanding the geometry $\mathsf{BS}(\mathscr{A})$, especially in degrees 2 and 3. For example, it has played an important role in rationality questions of $\mathsf{BS}(\mathscr{A})$, [7, 109, 91], and the study of the derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$, [30, 31].

In general, an order $\mathscr{A}$ of degree $n \geq 3$ gives rise to a scheme $\mathsf{BS}(\mathscr{A})$ with a highly intricate geometry. By restricting to orders on a smooth curve $C$, the situation becomes tractable. For a maximal $\mathcal{O}_C$-order $\mathscr{A}$, Artin [5] provided a description of the fiber $\mathsf{BS}(\mathscr{A})_p := \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec} \overline{\mathbf{k}(p)}$ if $\mathbf{k}(p)$ is perfect. He also showed that $\mathsf{BS}(\mathscr{A}) \subsetneq \underline{\mathsf{BS}}(\mathscr{A})$.

Subsequently, Frossard [67] generalized Artin's arguments to describe the singular fibers of $\mathsf{BS}(\mathscr{A})$, where $\mathscr{A}$ is a hereditary $\mathcal{O}_C$-order $\mathscr{A}$, i.e. an $\mathcal{O}_C$-algebra of global dimension one. To distinguish between the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$, and its connected component $\mathsf{BS}(\mathscr{A})$ containing the fiber over the generic point of $C$, we follow [67] in referring to $\mathsf{BS}(\mathscr{A})$ as the *Artin model* of the $\mathcal{O}_C$-order $\mathscr{A}$.

For our purposes, we restrict ourselves to curves over an algebraically closed field $\mathbf{k}$. Then, the isomorphism class of a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ at a point $p \in C$ is determined by

- the *ramification index* $e \in \mathbb{Z}_{>0}$ of $\mathscr{A}$ at $p$, and
- the *ramification data* $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{Z}^e_{>0}$ of $\mathscr{A}$ at $p$.

The observation that the local isomorphism class of $\mathscr{A}$ is determined by these numerical data is a key ingredient to the description of the fiber $\mathsf{BS}(\mathscr{A})_p$.

In Section 2.3.3, we will recall this description, which is due to Artin [5, Theorem 1.4] and Frossard [67, Proposition 2.3]. Given $\mathscr{A}$ a hereditary $\mathcal{O}_C$-order of degree $n$, they showed that the morphism $f\colon \mathsf{BS}(\mathscr{A}) \to C$ is flat and projective. Moreover, if $p \in C$ is a point such that $\mathscr{A}$ has ramification index $e$ at $p$,

i) the fiber $\mathsf{BS}(\mathscr{A})_p$ is a reduced projective variety of pure dimension $n-1$, and

ii) it consists of $e$ irreducible components $V_{1,\mathbf{n},e}, \ldots V_{e,\mathbf{n},e}$ which intersect transversely.

We refer to Theorem 2.3.15 for the details of the statement. In Section 2.3.2, we present the construction of the irreducible components $V_{i,\mathbf{n},e}$. They are smooth rational varieties. As these varieties were introduced by [5] to describe fibers of the Artin model, we call them *Artin auxiliary varieties*. For a point $p \in C$ with ramification index $e$, there are $e^2$ such varieties $V_{i,\mathbf{n},j}$, where $i, j = 1, \ldots, e$. We recall from [5, 67] how $V_{1,\mathbf{n},e}, \ldots, V_{e,\mathbf{n},e}$ form the fiber $\mathsf{BS}(\mathscr{A})_p$. Our presentation of the proof differs from [5, 67] in the sense that we will always keep track of how $V_{i,\mathbf{n},e}$ is embedded into the fiber, cf. Remark 2.3.13.

In Section 2.3.5, we make use of our presentation to describe the intersection of an arbitrary number of irreducible components $V_{i,\mathbf{n},e}$ inside $\mathsf{BS}(\mathscr{A})_p$.

**Theorem 2.A** (Theorem 2.3.29)**.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order and $p \in \Delta_{\mathscr{A}}$ such that $\mathscr{A}$ has ramification index $e$ at $p$. Then*

$$\bigcap_{j=1}^{k} V_{i_j,\mathbf{n},e} = V_{i_2,\mathbf{n},i_2-i_1} \times V_{i_3,\mathbf{n},i_3-i_2} \times \ldots \times V_{i_1,\mathbf{n},e-i_k+i_1} \tag{2.1}$$

*for $1 \leq i_1 < \ldots < i_k \leq e$.*

This generalizes the results of [5, Proposition 3.10], who described the non-zero coordinates in the intersection, and Frossard's result [67, Proposition 2.4], who showed that the intersection of *all* $e$ irreducible components is a product of projective spaces. To facilitate the computation of intersections for Artin auxiliary varieties we present indicator matrices in Section 2.3.4.

**A motivic description of the Artin model.** The results of Section 2.3 are intended to provide a presentation of the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in the Grothendieck ring of varieties. The Grothendieck ring of varieties $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ is generated by isomorphism classes $[X]$ of varieties over $\mathbf{k}$, subject to the cut-and-paste relation. This means that for every closed subvariety $Z \subseteq X$, we have $[X] = [Z] + [U]$, where $[U]$ is the complement of $Z$ in $X$.

For a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ we provide in Proposition 2.4.2 a recursive method to compute the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in terms of the Lefschetz class $\mathbb{L} = [\mathbb{A}^1]$ and the class $[C]$ of the underlying curve. In a specific case, namely when the ramification index $e$ equals the degree $n = \deg \mathscr{A}$, we provide a closed formula.

**Theorem 2.B** (Corollary 2.4.7)**.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ such that for each $p \in \Delta_{\mathscr{A}}$ the ramification index of $\mathscr{A}$ at $p$ is $n$. If $r = \#\Delta_{\mathscr{A}}$, then*

$$[\mathsf{BS}(\mathscr{A})] = \sum_{i=0}^{n-1} \left( [C] + r \cdot \binom{n}{i} - r \right) \mathbb{L}^i = [C] \cdot [\mathbb{P}^{n-1}] + \sum_{i=1}^{n-1} r \cdot \left( \binom{n}{i} - 1 \right) \mathbb{L}^i. \tag{2.2}$$

**Applications of the motivic description.** It is a recurring motif in the study of moduli spaces to compute their class in $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ in order to extract geometric information. For example, this approach has been taken for the moduli space of vector bundles on a curve [25] or for the Hilbert scheme of points [73]. If $C$ is a smooth projective curve, the motivic description of $\mathsf{BS}(\mathscr{A})$ turns out to be powerful tool, as it is flexible enough to handle the singular fibers of $\mathsf{BS}(\mathscr{A})$, and also provides important motivic invariants of the smooth total space $\mathsf{BS}(\mathscr{A})$.

Working over an algebraically closed field $\mathbf{k}$ with characteristic zero, we will particularly be interested in two invariants, captured by *motivic measures*, i.e. a ring homomorphism with $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ as a domain, see [108].

- The first motivic measure is the Hodge polynomial $\mathrm{h}\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to \mathbb{Z}[u,v]$, which has the Hodge numbers $\mathrm{h}^{p,q}(X) := \dim \mathrm{H}^q(X, \Omega_X^p)$ of $X$ as coefficients. A corollary of Theorem 2.B is the computation of the Hodge numbers of $\mathsf{BS}(\mathscr{A})$, see Corollary 2.4.8.

- The second motivic measure, due to [38], assigns to the class $[X]$ of a smooth projective variety $X$, the class $[\mathsf{D}^{\mathsf{b}}(X)] \in \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}})$ of (the unique dg enhancement of) the derived category $\mathsf{D}^{\mathsf{b}}(X)$. The motivic decomposition allows us to give heuristics for a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(X)$, see Section 2.4.4.

**Notation.** We denote by $\mathbf{k}$ an algebraically closed field. A curve $C$ is a one-dimensional smooth quasiprojective scheme over $\mathbf{k}$ with function field $\mathbf{k}(C)$.

## 2.2 Preliminaries

Let $C$ be a smooth quasiprojective curve over an algebraically closed field $\mathbf{k}$. A coherent $\mathcal{O}_C$-algebra $\mathscr{A}$ is an *$\mathcal{O}_C$-order* if

- it is torsionfree as an $\mathcal{O}_C$-module, and
- at the generic point $\eta \in C$, the $\mathbf{k}(C)$-algebra $\mathscr{A}_\eta := \mathscr{A} \otimes_C \mathbf{k}(C)$ is central simple.

We will restrict ourselves to the situation of hereditary orders. The necessary background on hereditary orders is provided in Section 2.2.1. We refer to Chapter 1 for a more extensive treatment of hereditary orders. In Section 2.2.2, we recall the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$ and the Artin model $\mathsf{BS}(\mathscr{A})$ of a hereditary order $\mathscr{A}$, which is connected component of $\underline{\mathsf{BS}}(\mathscr{A})$.

### 2.2.1 Hereditary orders

Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order. Since $\mathbf{k}$ is assumed to be algebraically closed, it follows from Tsen's theorem [72, Theorem 6.2.8] that $\mathscr{A} \otimes_C \mathbf{k}(C) \cong \mathrm{Mat}_n(\mathbf{k}(C))$. We call $n$ the *degree* of $\mathscr{A}$. As $\mathscr{A}$ is torsionfree over a curve, it is a locally free $\mathcal{O}_C$-module of rank $n^2$.

Central simplicity of $\mathscr{A} \otimes_C \mathbf{k}(C)$ implies that there is a maximal non-empty open subset $U \subset C$ such that the restriction $\mathscr{A}|_U$ is an Azumaya algebra over $U$. We call the complement $\Delta_{\mathscr{A}} = C \setminus U$ the *ramification locus* of $\mathscr{A}$. It is a set of finitely many closed points of $C$.

Our focus lies on *hereditary orders*, i.e. $\mathcal{O}_C$-orders $\mathscr{A}$ of global dimension one. It is a standard result that heredity is a local property, see [119, Corollary 3.24] or Theorem 1.3.2.

**Lemma 2.2.1.** *Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. Then $\mathscr{A}$ is a hereditary $\mathcal{O}_C$-order if and only if for every $p \in C$ the stalk $\mathscr{A}_p$ is a hereditary $\mathcal{O}_{C,p}$-order.*

We recall the classification of hereditary orders over a discrete valuation ring from [119, Theorem 39.14]. Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$. For each point $p \in C$ there exist $e \in \mathbb{Z}_{>0}$ and an $e$-tuple $\mathbf{n} \in \mathbb{Z}^e_{>0}$ of positive integers with $n = |\mathbf{n}| := \sum_{i=1}^e n_i$ such that

$$\mathscr{A}_p \cong \mathscr{A}_{(\mathbf{n},e)}(\mathcal{O}_{C,p}) := \begin{pmatrix} \mathcal{O}_{C,p} & \mathcal{O}_{C,p} & \dots & \mathcal{O}_{C,p} \\ \mathfrak{m}_p & \mathcal{O}_{C,p} & & \mathcal{O}_{C,p} \\ \vdots & \ddots & \ddots & \vdots \\ \mathfrak{m}_p & \mathfrak{m}_p & & \mathcal{O}_{C,p} \end{pmatrix}^{\mathbf{n}} \subseteq \mathrm{Mat}_n(\mathbf{k}(C)), \tag{2.3}$$

where $\mathfrak{m}_p \subset \mathcal{O}_{C,p}$ is the maximal ideal. The superscript $\mathbf{n}$ indicates that the $(i,j)$-th coordinate of the matrix in (2.3) stands for an $n_i \times n_j$-matrix with entries in $\mathfrak{m}_p$, resp. $\mathcal{O}_{C,p}$.

**Remark 2.2.2.** The result [119, Theorem 39.14] is for orders over a complete discrete valuation ring. But since $\mathcal{O}_{C,p}$ is the unique $\mathcal{O}_{C,p}$-order in $\mathbf{k}(C)$ by normality of $C$, it already holds over the discrete valuation ring $\mathcal{O}_{C,p}$.

It follows from (2.3) that the ramification locus of $\mathscr{A}$ is given by

$$\Delta_{\mathscr{A}} = \{p \in C \mid \mathscr{A} \otimes_C \mathcal{O}_{C,p} \cong \mathscr{A}_{(\mathbf{n},e)}(\mathcal{O}_{C,p}) \text{ with } e > 1\} \subseteq C. \tag{2.4}$$

For $p \in \Delta_{\mathscr{A}}$, we say that $\mathscr{A}$ is *ramified at* $p$ with *ramification index* $e$ and we call $\mathbf{n} \in \mathbb{N}^e$ (resp. the tuple $(\mathbf{n}, e)$) the *ramification data* of $\mathscr{A}$ at $p$. Unless $\Delta_{\mathscr{A}}$ is empty, it is a reduced divisor. If $p \notin \Delta_{\mathscr{A}}$, the associated ramification data is $n = \deg \mathscr{A}$. In this case, $\mathscr{A}_p$ is isomorphic to $n \times n$-matrices, denoted by $\mathscr{A}_{(n,1)}(\mathcal{O}_{C,p})$.

**The stalk and the fiber of a hereditary order.** Let $p \in C$ be a point. We recall the free generating set of $\mathscr{A}_p$ as an $\mathcal{O}_{C,p}$-module from [5, §2]. It encodes the algebra structure in a useful way for the description of the Artin model $\mathsf{BS}(\mathscr{A})$.

For this we fix the ramification index $e > 0$ and the ramification data $\mathbf{n} \in \mathbb{Z}^e_{>0}$ associated with $\mathscr{A}_p$. In addition we need to introduce

- the $(e+1)-$*tuple* $\mathbf{r} := (r_0, \dots, r_e) \in \mathbb{Z}^{e+1}_{\geq 0}$ *of partial sums of* $\mathbf{n}$ with $i$-th entry given by $r_i := \sum_{j=1}^i n_j$, and
- the *index function* $\mathrm{ind}\colon \{1, \dots, n\} \to \{1, \dots, e\}$, which assigns to $\alpha$ the unique integer $\mathrm{ind}(\alpha)$ such that $r_{\mathrm{ind}(\alpha)-1} + 1 \leq \alpha \leq r_{\mathrm{ind}(\alpha)}$.
- Moreover, we fix the following subset

$$F_3 := \left\{(i,j,k) \in \{1, \dots, e\}^{\times 3} \mid i \leq j \leq k \text{ or } j \leq k < i \text{ or } k < i \leq j\right\}. \tag{2.5}$$

Frossard [67, Proposition 2.3] observed that the above data is convenient to describe the algebra structure of $\mathscr{A}_p$ and $\mathscr{A}(p) := \mathscr{A}_p \otimes_{\mathcal{O}_{C,p}} \mathbf{k}(p)$ with ramification data $(\mathbf{n}, e)$.

To make this precise, we denote by $E_{\alpha,\beta}$ the elementary matrix of $\mathrm{Mat}_n(\mathcal{O}_{C,p})$ which is zero in every coordinate, except 1 at the position $(\alpha, \beta)$. Let $\mathfrak{m}_p = (t) \subset \mathcal{O}_{C,p}$ be the maximal ideal with uniformizer $t$. Using that $\mathscr{A}_{(\mathbf{n},e)}(\mathcal{O}_{C,p}) \subset \mathrm{Mat}_n(\mathcal{O}_{C,p})$, we define

$$E^{(p)}_{\alpha\beta} = \begin{cases} E_{\alpha\beta} & \text{if } \mathrm{ind}(\alpha) \leq \mathrm{ind}(\beta), \\ tE_{\alpha\beta} & \text{otherwise.} \end{cases} \tag{2.6}$$

From the presentation (2.3) it follows immediately that $\{E^{(p)}_{\alpha\beta} \mid 1 \leq \alpha, \beta \leq n\}$ defines a free generating set of $\mathscr{A}_{(\mathbf{n},e)}(\mathcal{O}_{C,p})$ as an $\mathcal{O}_{C,p}$-module.

The multiplication of the elements $E^{(p)}_{\alpha\beta}$ is given by $E^{(p)}_{\alpha\beta} \cdot E^{(p)}_{\beta\gamma} = T_{\alpha\beta\gamma} E^{(p)}_{\alpha\gamma}$, where the element $T_{\alpha\beta\gamma} \in \mathcal{O}_{C,p}$ satisfies

$$T_{\alpha\beta\gamma} = \begin{cases} 1 & \text{if } (\mathrm{ind}(\alpha), \mathrm{ind}(\beta), \mathrm{ind}(\gamma)) \in F_3, \\ t & \text{otherwise.} \end{cases} \tag{2.7}$$

Moreover, we have $E^{(p)}_{\alpha\beta} E^{(p)}_{\beta'\gamma} = 0$ if $\beta \neq \beta'$.

Restricting along $\mathrm{Spec}(\mathbf{k}(p)) \to \mathrm{Spec}(\mathcal{O}_{C,p})$ to the closed point, we obtain a $\mathbf{k}(p)$-basis $e_{\alpha\beta}$ of $\mathscr{A}(p) = \mathscr{A}_p \otimes_{\mathcal{O}_{C,p}} \mathbf{k}(p)$. The algebra structure is given by $e_{\alpha\beta} \cdot e_{\beta\gamma} = \tau_{\alpha\beta\gamma} e_{\alpha\gamma}$. The element $\tau_{\alpha\beta\gamma} \in \mathbf{k}(p)$ is the projection of $T_{\alpha\beta\gamma} \in \mathcal{O}_{C,p}$ to $\mathbf{k}(p)$, i.e.

$$\tau_{\alpha\beta\gamma} = \begin{cases} 1 & \text{if } (\mathrm{ind}(\alpha), \mathrm{ind}(\beta), \mathrm{ind}(\gamma)) \in F_3, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

**Remark 2.2.3.** In Lemma 3.2.9, it is shown that $\mathscr{A}(p)$ is Morita equivalent to $\mathbf{k}Q/I$, where $Q$ is the cyclic quiver with $e$ vertices and $I \trianglelefteq \mathbf{k}Q$ is the admissible ideal generated by all paths of length $e$. The elements $e_{\alpha,\beta}$ with $\mathrm{ind}(\alpha) = i$ and $\mathrm{ind}(\beta) = i + 1$ correspond to the (unique) arrow $i \to i + 1$ in $Q$ (where $i$ is counted modulo $e$).

### 2.2.2 The Brauer–Severi scheme of an order

Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order over a smooth curve $C$. With $(C, \mathscr{A})$ we associate a functor

$$\mathcal{BS}(\mathscr{A})\colon \mathbf{Sch}_C^{\mathrm{op}} \to \mathbf{Sets}, \tag{2.9}$$

which assigns to a $C$-scheme $h\colon T \to C$ the set of left ideals $L \trianglelefteq h^*\mathscr{A}$ of rank $n$ such that the quotient $h^*\mathscr{A}/L$ is locally free as an $\mathcal{O}_T$-module.

In the context of central simple algebras over a field [72, §5], and Azumaya algebras over a scheme [74], it is well-known that this functor admits a fine moduli space. In those cases, the moduli space is a projective bundle with respect to the étale topology of the base scheme.

For an arbitrary $\mathcal{O}_C$-algebra $\mathscr{A}$, the representability of $\mathcal{BS}(\mathscr{A})$ was shown by Van den Bergh [137, Proposition 2]. We denote by $\underline{\mathsf{BS}}(\mathscr{A}) \to C$ the fine moduli space of $\mathcal{BS}(\mathscr{A})$. It is called the *Brauer–Severi scheme* associated to $(C, \mathscr{A})$. The Brauer–Severi scheme is well-behaved under base change. This means that for a morphism $g\colon T \to C$, we have

$$\underline{\mathsf{BS}}(g^*\mathscr{A}) \cong \underline{\mathsf{BS}}(\mathscr{A}) \times_C T. \tag{2.10}$$

by [137, Propositions 2 and 5].

**Remark 2.2.4.** In contrast to Azumaya algebras, the pullback $g^*\mathscr{A}$ of an $\mathcal{O}_C$-order $\mathscr{A}$ along a morphism $g\colon T \to C$ is not necessarily an $\mathcal{O}_T$-order. This leads to a more involved behavior of the Brauer–Severi scheme over $\Delta_{\mathscr{A}}$. For example, Artin [5, Example 3.4] observed that the Brauer–Severi scheme is not connected in general, cf. Example 2.3.35.

**The Artin model.** For orders, we can circumvent the problem of non-connectedness of the Brauer–Severi scheme using that they are generically Azumaya algebras. As before, let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order.

**Definition 2.2.5.** The *Artin model* of $(C, \mathscr{A})$ is the connected component $\mathsf{BS}(\mathscr{A}) \subseteq \underline{\mathsf{BS}}(\mathscr{A})$ containing the generic fiber $\underline{\mathsf{BS}}(\mathscr{A}) \times_C \operatorname{Spec} \mathbf{k}(C) = \underline{\mathsf{BS}}(A)$.

The morphism $f\colon \mathsf{BS}(\mathscr{A}) \to C$ is the composition of the inclusion $\iota\colon \mathsf{BS}(\mathscr{A}) \hookrightarrow \underline{\mathsf{BS}}(\mathscr{A})$ with the structure morphism $\underline{\mathsf{BS}}(\mathscr{A}) \to C$.

$$\begin{array}{ccc} \mathsf{BS}(\mathscr{A}) & \xrightarrow{\ \iota\ } & \underline{\mathsf{BS}}(\mathscr{A}) \\ & {}_{f}\searrow & \downarrow{\scriptstyle \pi} \\ & & C \end{array} \tag{2.11}$$

Moreover, $f$ is flat and projective, [5, Theorem 1.4]. A base change formula as (2.10) for the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$ does not make sense for $\mathsf{BS}(\mathscr{A})$. Hence, we proceed in two steps for the description of $\mathsf{BS}(\mathscr{A})_p := \mathsf{BS}(\mathscr{A}) \times \operatorname{Spec} \mathbf{k}(p)$.

i) First we show that $\mathsf{BS}(\mathscr{A}_p) \cong \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec}(\mathcal{O}_{C,p})$ for each $p \in C$.

ii) Then, we appeal to [5, Lemma 3.3] for the description of the fiber $\mathsf{BS}(\mathscr{A})_p$.

**Lemma 2.2.6.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order and $g\colon T \to C$ be an integral $C$-scheme such that $g^*\mathscr{A}$ is generically Azumaya. Then*

$$\mathsf{BS}(g^*\mathscr{A}) \cong \mathsf{BS}(\mathscr{A}) \times_C T. \tag{2.12}$$

*Proof.* Since $T$ is integral and the $\mathbf{k}(T)$-algebra $g^*\mathscr{A} \otimes_T \mathbf{k}(T)$ is central simple, the fiber $\underline{\mathsf{BS}}(g^*\mathscr{A}) \times_T \operatorname{Spec}(\mathbf{k}(T)) = \mathbb{P}^{n-1}_{\mathbf{k}(T)}$ over the generic point is connected, and $\mathsf{BS}(g^*\mathscr{A}) \subset \underline{\mathsf{BS}}(g^*\mathscr{A})$ is well-defined. Using (2.10), the embedding

$$\mathsf{BS}(\mathscr{A}) \times_C T \subseteq \underline{\mathsf{BS}}(\mathscr{A}) \times_C T \cong \underline{\mathsf{BS}}(g^*\mathscr{A}) \tag{2.13}$$

realizes $\mathsf{BS}(\mathscr{A}) \times_C T$ as a closed and open connected subscheme of $\underline{\mathsf{BS}}(g^*\mathscr{A})$ which is non-empty over the generic point $\operatorname{Spec} \mathbf{k}(T)$. Hence $\mathsf{BS}(g^*\mathscr{A}) \cong \mathsf{BS}(\mathscr{A}) \times_C T$. □

Lemma 2.2.6 implies that we can restrict to $\mathsf{BS}(\mathscr{A}_p)$ over the spectrum of a discrete valuation ring $\mathcal{O}_{C,p}$, which was the starting point in [5, 67]. We recall a criterion [5, Lemma 3.3] for an ideal $L \subseteq \mathscr{A}_p$ to be in $\mathsf{BS}(\mathscr{A}_p)$.

**Lemma 2.2.7.** *Let $C = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation ring $R$ with residue field $\mathbf{k}(p)$. Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ with the free generating set $E^{(p)}_{\alpha\beta}$ from* (2.6) *and induced $\mathbf{k}(p)$-basis $e_{\alpha\beta}$. Let $L \in \underline{\mathsf{BS}}(\mathscr{A}(p))$ be a left ideal in $\mathscr{A}(p)$. Then $L \in \mathsf{BS}(\mathscr{A})_p = \mathsf{BS}(\mathscr{A}) \times_C \mathrm{Spec}\,\mathbf{k}(p)$ if and only if $\dim_{\mathbf{k}(p)}(e_{\alpha\alpha} \cdot L) = 1$ for all $\alpha = 1, \ldots, n$.*

*Proof.* Since $R$ is a discrete valuation ring, we can assume that $\mathscr{A} = \mathscr{A}_{(\mathbf{n},e)}(R)$ from (2.3) for some ramification data $(\mathbf{n}, e)$. Let $\pi\colon \underline{\mathsf{BS}}(\mathscr{A}) \to C$ be the Brauer–Severi scheme with universal ideal sheaf $\mathcal{L} \trianglelefteq \pi^*\mathscr{A}$. The idempotents $E^{(p)}_{\alpha\alpha}$ pull back to a complete set of orthogonal idempotents of $\pi^*\mathscr{A}$. Hence, we have a decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_n$ as an $\mathcal{O}_{\mathsf{BS}(\mathscr{A})}$-module with respect to left multiplication by the $n$ idempotents.

For $x \in \underline{\mathsf{BS}}(\mathscr{A})$, the function $\dim_{\mathbf{k}(x)} \mathcal{L}_\alpha(x)$ is locally constant on $\underline{\mathsf{BS}}(\mathscr{A})$ as $\mathcal{L}$ is locally free of rank $n$. If $x \in \underline{\mathsf{BS}}(\mathscr{A})_\eta$ lies in the fiber over the generic point $\eta \in C$, the universal ideal sheaf restricts to an ideal $\mathcal{L}(x) \trianglelefteq \mathscr{A} \otimes_C \mathbf{k}(C)$ in a matrix algebra. Therefore, we find $\dim \mathcal{L}_\alpha(x) = 1$ for all $\alpha = 1, \ldots, n$. Since $\dim_{\mathbf{k}(x)} \mathcal{L}_\alpha(x)$ is constant on connected components of $\underline{\mathsf{BS}}(\mathscr{A})$, it follows for $x_L \in \underline{\mathsf{BS}}(\mathscr{A})_p = \underline{\mathsf{BS}}(\mathscr{A}) \times_C \mathrm{Spec}\,\mathbf{k}(p)$, corresponding to the ideal $L \trianglelefteq \mathscr{A}(p)$, that $x_L \in \mathsf{BS}(\mathscr{A})_p$ if and only if $\dim_{\mathbf{k}(p)} e_{\alpha\alpha}L = 1$ for all $\alpha = 1, \ldots, n$. □

## 2.3 The Artin model over ramified points

This section is devoted to the study of the fibers of the Artin model $\mathsf{BS}(\mathscr{A}) \to C$, which was initiated by Artin [5, Theorem 1.4] and continued by Frossard [67, Proposition 2.3], cf. Section 2.3.3. We extend their results by describing the intersection of an arbitrary number of irreducible components of the fiber $\mathsf{BS}(\mathscr{A})_p := \mathsf{BS}(\mathscr{A}) \times_C \mathrm{Spec}\,\mathbf{k}(p)$ in Section 2.3.5.

After recording some combinatorial data in Section 2.3.1, which will be needed later, we proceed to the construction of the Artin auxiliary varieties $V_{i,\mathbf{n},j}$ in Section 2.3.2. The Artin auxiliary varieties will be used for the description of the fibers $\mathsf{BS}(\mathscr{A})_p$, which is due to Frossard [67, Proposition 2.3] generalizing work of Artin [5, Theorem 1.4].

Section 2.3.4 presents a method to encode the Artin auxiliary varieties as matrices. It eases the computation of intersections of Artin auxiliary varieties. Then, we come to the main result, Theorem 2.3.29, the description of the intersection of irreducible components of $\mathsf{BS}(\mathscr{A})_p$ in terms of products of Artin auxiliary varieties.

### 2.3.1 Combinatorial data associated with a hereditary order

For the next section it is necessary to fix some data which can be associated with a hereditary $\mathcal{O}_C$-order $\mathscr{A}$. We denote by $n$ the degree of $\mathscr{A}$. Recall

- the *ramification index* $e \in \mathbb{Z}_{>0}$,
- the *ramification data* $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{Z}^e_{>0}$, satisfying $n = |\mathbf{n}| := \sum_{i=1}^{e} n_i$,

- the *tuple* $\mathbf{r} := (r_0, \ldots, r_e) \in \mathbb{Z}_{\geq 0}^{e+1}$ *of partial sums of* $\mathbf{n}$ defined by $r_i := \sum_{j=1}^{i} n_j$, and
- the *index function*

$$\text{ind}\colon \{1, \ldots, n\} \to \{1, \ldots, e\}, \tag{2.14}$$

  which sends $\alpha \in \{1, \ldots, n\}$ to $\text{ind}(\alpha)$ such that $r_{\text{ind}(\alpha)-1} + 1 \leq \alpha \leq r_{\text{ind}(\alpha)}$.

**Remark 2.3.1.** If the ramification data $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{N}^e$ satisfies $d = n_1 = \ldots = n_e$ for some $d > 0$, we write $\mathbf{d}_e = \mathbf{n}$ for short.

In addition to the above data, for each $i = 1, \ldots, e$, we set $\mathbf{s}_i(\mathbf{n}) := (s_{i,1}, \ldots, s_{i,e}) \in \mathbb{Z}_{>0}^e$, where

$$s_{i,j} := \sum_{k=1}^{\min\{j,i-1\}} n_{i-k} + \sum_{k=0}^{j-i} n_{e-k}. \tag{2.15}$$

If there is no ambiguity, we write $\mathbf{s}_i = \mathbf{s}_i(\mathbf{n})$. Note that $\mathbf{s}_i$ can be seen as a tuple of partial sums of $\mathbf{n}$ by starting with $s_{i,1} = n_{i-1}$ (or $s_{i,1} = n_e$ if $i = 1$) and going in the reversed direction.

Let $\mathbf{n}, \mathbf{s}_i(\mathbf{n}) = (s_{i,1}, \ldots, s_{i,e}) \in \mathbb{Z}_{>0}^e$ and $k \in \{1, \ldots, e\}$. We define the *$k$-th cut of* $\mathbf{n}$ *with respect to* $\mathbf{s}_i(\mathbf{n})$ as

$$\mathbf{m}(i, \mathbf{n}, k) := (s_{i,k} - s_{i,k-1}, \ldots, s_{i,2} - s_{i,1}, s_{i,1}) \in \mathbb{Z}_{>0}^k. \tag{2.16}$$

Hence, the $k$-tuple $\mathbf{m}(i, \mathbf{n}, k)$ has the entries $n_{i-1}, n_{i-2}, \ldots, n_{i-k}$ (counted modulo $e$).

Given an $e$-tuple $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{N}^e$ and $i \in \{1, \ldots, e\}$ we denote by

$$\mathbf{n}[i] = (n[i]_1, \ldots, n[i]_e) := (n_{i+1}, n_{i+2}, \ldots, n_e, n_1, \ldots, n_i) \tag{2.17}$$

the *cyclic shift*. For the $\mathbf{s}$-tuple, one observes that $\mathbf{s}_i(\mathbf{n}) = \mathbf{s}_1(\mathbf{n}[i-1])$.

**Example 2.3.2.** If $\mathbf{n} = \mathbf{1}_n$, as defined in Remark 2.3.1, with ramification index $e = n$. Then $\mathbf{r} = (0, 1, 2, \ldots, n) \in \mathbb{Z}^{n+1}$, and $\mathbf{s}_i = (1, 2, \ldots, n) \in \mathbb{N}^n$. Moreover, the index function is the identity on $\{1, \ldots, n\}$. This data is associated with the orders treated in Proposition 2.4.4 and Corollary 2.4.7. In particular, with $\mathbf{1}_2$ it covers conic bundles over curves.

The next example serves to illustrate the results in a more complicated situation.

**Example 2.3.3.** Let $\mathbf{n} = (3, 1, 4, 2) \in \mathbb{N}^4$. Then

$$\mathbf{s}_1 = (2, 6, 7, 10),\ \mathbf{s}_2 = (3, 5, 9, 10),\ \mathbf{s}_3 = (1, 4, 6, 10),\ \mathbf{s}_4 = (4, 5, 8, 10). \tag{2.18}$$

The tuple of partial sums is given by $\mathbf{r} = (0, 3, 4, 8, 10)$. Therefore for $\alpha \in \{1, 2, 3\}$, we have $\text{ind}(\alpha) = 1$, $\text{ind}(4) = 2$, for $\alpha \in \{5, 6, 7, 8\}$ the index is $\text{ind}(\alpha) = 3$, and $\text{ind}(9) = \text{ind}(10) = 4$.

### 2.3.2 Artin's auxiliary varieties

Let $\mathscr{A}$ be an $\mathcal{O}_C$-order. Assume that $\mathscr{A}$ is ramified at $p \in C$ with ramification data $(\mathbf{n}, e)$. In Theorem 2.3.15 we recall from [5, 67] that the fiber

$$\mathsf{BS}(\mathscr{A})_p = \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec} \mathbf{k}(p) = V_{1,\mathbf{n},e} \cup \ldots \cup V_{e,\mathbf{n},e} \tag{2.19}$$

is the union of $e$ irreducible components $V_{i,\mathbf{n},e}$, each a smooth projective variety birationally equivalent to $\mathbb{P}^{|\mathbf{n}|-1}$. To do so, we construct the irreducible components in this section.

Artin [5] already observed that it is convenient to describe the irreducible components $V_{i,\mathbf{n},e}$ inside the product of $e$ projective spaces of dimension $|\mathbf{n}| - 1$. This is why we refer to $V_{i,\mathbf{n},j}$ as *Artin auxiliary variety.*

As mentioned in Remark 2.3.13, our construction is close to [67, Définition 2.2], recording the successive blow-ups in the construction. The main difference to the above mentioned sources is that we do not describe the irreducible components up to isomorphism, but consider them up to equality in $\mathbb{P}^{|\mathbf{n}|-1} \times \ldots \times \mathbb{P}^{|\mathbf{n}|-1}$. This allows us to better understand the intersection of the irreducible components, cf. Theorem 2.3.29.

Fix $e \in \mathbb{N}$, $\mathbf{n} = (n_1, \ldots, n_e) \in \mathbb{N}^e$. For each $i \in \{1, \ldots, e\}$ we pick an $n$-dimensional $\mathbf{k}$-vector space $W_i$. with a fixed basis $e_{i,1}, \ldots, e_{i,n} \in W_i$, Recall the associated tuple $\mathbf{s}_i = \mathbf{s}_i(\mathbf{n}) \in \mathbb{N}^e$ from Section 2.3.1. We start by defining a flag of linear subspaces $W_i$ depending on the chosen $i$. Let

$$W_{i,k} = \langle e_{i,\alpha} \mid \operatorname{ind}(\alpha) \in \{i - \min(k, i-1), \ldots, i-1\} \cup \{e - (k-i), \ldots, e\} \rangle \tag{2.20}$$

for $k = 1, \ldots, e$. Then $W_{i,1} \subset W_{i,2} \subset \ldots \subset W_{i,e} = W_i$ is a flag of subspaces of $W_i$ such that

$$\mathbf{s}_i(\mathbf{n}) = (\dim_{\mathbf{k}} W_{i,1}, \ldots, \dim_{\mathbf{k}} W_{i,e}). \tag{2.21}$$

This follows from $s_{i,k} = \#\{\alpha \mid \operatorname{ind}(\alpha) \in \{i - \min(k, i-1), \ldots, i-1\} \cup \{e - (k-i), \ldots, e\}\}$.

**Example 2.3.4.** Let $\mathbf{n} = \mathbf{1}_n$ as in Example 2.3.2. Then $\mathbf{s}_i = (1, 2, \ldots, n)$ and the index function is the identity on $\{1, \ldots, n\}$. Hence, each $n$-dimensional $\mathbf{k}$-vector space $W_i$ comes with a complete flag

$$0 \subset W_{i,1} \subset W_{i,2} \subset \ldots \subset W_{i,n} = W_i \tag{2.22}$$

such that $W_{i,j} = \langle e_{i,i-1}, e_{i,i-2}, \ldots, e_{i,i-j} \rangle$, where the second index of $e_{i,k}$ is counted modulo $n$.

**Example 2.3.5.** Continuing Example 2.3.3, we consider $\mathbf{n} = (3, 1, 4, 2)$. If $\mathbf{s}_3 = (1, 4, 6, 10)$, the corresponding flag of subspaces of $W_3 \cong \mathbf{k}^{10}$ is given by

$$\begin{aligned}
W_{3,1} &= \langle e_{3,\alpha} \mid \operatorname{ind}(\alpha) = 2 \rangle = \langle e_{3,4} \rangle \cong \mathbf{k}, \\
W_{3,2} &= \langle e_{3,\alpha} \mid \operatorname{ind}(\alpha) = 1, 2 \rangle = \langle e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4} \rangle \cong \mathbf{k}^4, \\
W_{3,3} &= \langle e_{3,\alpha} \mid \operatorname{ind}(\alpha) = 1, 2, 4 \rangle = \langle e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}, e_{3,9}, e_{3,10} \rangle \cong \mathbf{k}^6, \\
W_{3,4} &= \langle e_{3,\alpha} \mid \operatorname{ind}(\alpha) = 1, 2, 3, 4 \rangle = W_3 \cong \mathbf{k}^{10}.
\end{aligned} \tag{2.23}$$

The other flags of subspaces are listed in Table 2.1.

| | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|
| $\mathbf{s}_i$ | $(2,6,7,10)$ | $(3,5,9,10)$ | $(1,4,6,10)$ | $(4,5,8,10)$ |
| $W_{i,1}$ | $e_{1,9}, e_{1,10}$ | $e_{2,1}, e_{2,2}, e_{2,3}$ | $e_{3,4}$ | $e_{4,5}, e_{4,6}, e_{4,7}, e_{4,8}$ |
| $W_{i,2}$ | $e_{1,5},e_{1,6},e_{1,7}, e_{1,8},e_{1,9},e_{1,10}$ | $e_{2,1},e_{2,2},e_{2,3}$ $e_{2,9},e_{2,10}$ | $e_{3,1}, e_{3,2}, e_{3,3}, e_{3,4}$ | $e_{4,4},e_{4,5}, e_{4,6},e_{4,7},e_{4,8}$ |
| $W_{i,3}$ | $e_{1,4},e_{1,5},e_{1,6},e_{1,7}, e_{1,8},e_{1,9},e_{1,10}$ | $e_{2,1},e_{2,2},e_{2,3},e_{2,5}, e_{2,6},e_{2,7},e_{2,8},e_{2,9},e_{2,10}$ | $e_{3,1},e_{3,2},e_{3,3},e_{3,4}, e_{3,9},e_{3,10}$ | $e_{4,1},e_{4,2},e_{4,3},e_{4,4},e_{4,5}, e_{4,6},e_{4,7},e_{4,8}$ |
| $W_{i,4}$ | $e_{1,1},e_{1,2},e_{1,3},e_{1,4},e_{1,5}, e_{1,6},e_{1,7},e_{1,8},e_{1,9},e_{1,10}$ | $e_{2,1},e_{2,2},e_{2,3},e_{2,4},e_{2,5}, e_{2,6},e_{2,7},e_{2,8},e_{2,9},e_{2,10}$ | $e_{3,1},e_{3,2},e_{3,3},e_{3,4},e_{3,5}, e_{3,6},e_{3,7},e_{3,8},e_{3,9},e_{3,10}$ | $e_{4,1},e_{4,2},e_{4,3},e_{4,4},e_{4,5}, e_{4,6},e_{4,7},e_{4,8},e_{4,9},e_{4,10}$ |

Table 2.1: Basis vectors for the four flags of subspaces associated with $\mathbf{n} = (3, 1, 4, 2)$.

The $\mathbf{k}$-vector spaces $W_1, \dots, W_e$ are used to define the product $\mathbb{P}(W_1) \times \dots \times \mathbb{P}(W_e)$ of $e$ projective spaces, each of dimension $n-1$. We proceed with the construction of the Artin auxiliary varieties as closed subvarieties

$$V_{i,\mathbf{n},j} \subset \mathbb{P}(W_1) \times \dots \times \mathbb{P}(W_e), \tag{2.24}$$

for $i, j \in \{1, \dots, e\}$. It will sometimes be necessary to use the coordinates $a_{i,1}, \dots, a_{i,n}$ for $\mathbb{P}(W_i)$. Note that the inclusion $W_{i,j} \subset W_i$ naturally identifies $\mathbb{P}(W_{i,j})$ with the linear subspace

$$\mathbb{V}(a_{i,\alpha} \mid \operatorname{ind}(\alpha) \notin \{i - \min(j, i-1), \dots, i-1\} \cup \{e - (j-i), \dots, e\}) \subseteq \mathbb{P}(W_i). \tag{2.25}$$

**Definition 2.3.6.** Let $(\mathbf{n}, e)$ be ramification data as defined in Section 2.3.1, and fix $i \in \{1, \dots, e\}$. With the notation from above we define the *Artin auxiliary variety* $V_{i,\mathbf{n},k}$ *of type* $(i, \mathbf{n}, k)$ recursively for $k \in \{1, \dots, e\}$ as follows:

- For $k = 1$ set $V_{i,\mathbf{n},1} = \mathbb{P}(W_{i,1})$.
- For $k \in \{2, \dots, e\}$ assume that for all $j < k$ the varieties

$$V_{i,\mathbf{n},j} \subseteq \mathbb{P}(W_{i,j}) \times \mathbb{P}(W_{i-1,j-1}) \times \dots \times \mathbb{P}(W_{i-j+1,1}) \tag{2.26}$$

  are constructed, where the first index $\ell$ of $W_{\ell,m}$ is counted modulo $e$.
- For the construction of $V_{i,\mathbf{n},k}$ start with $\mathbb{P}(W_{i,k})$. Let $1 \leq j < k$. Assume that we did $j-1$ successive blow-ups

$$\begin{aligned} &\mathrm{Bl}_{V_{i,\mathbf{n},j-1}}\left(\mathrm{Bl}_{V_{i,\mathbf{n},j-2}}\left(\dots \mathrm{Bl}_{V_{i,\mathbf{n},2}}(\mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}(W_{i,k}))) \dots\right)\right) \\ &\subset \mathbb{P}(W_{i,k}) \times \dots \times \mathbb{P}(W_{i-j+1,k-j+1}) \end{aligned} \tag{2.27}$$

  with center in the already constructed varieties $V_{i,\mathbf{n},\ell}$ for $\ell = 1, \dots, j-1$.

  The Artin auxiliary variety $V_{i,\mathbf{n},j}$ is a closed subvariety in (2.27) defined by the vanishing of $a_{i-j+1,\alpha}$ such that $\operatorname{ind}(\alpha) \neq i - j$ (counted modulo $e$). We perform the blow-up of (2.27) with center $V_{i,\mathbf{n},j}$.

After $k-1$ blow-ups of subvarieties we set

$$V_{i,\mathbf{n},k} := \mathrm{Bl}_{V_{i,\mathbf{n},k-1}}\left(\mathrm{Bl}_{V_{i,\mathbf{n},k-2}}\left(\ldots \mathrm{Bl}_{V_{i,\mathbf{n},2}}(\mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}(W_{i,k}))) \ldots\right)\right). \tag{2.28}$$

In this way, we obtain $V_{i,\mathbf{n},k}$ as a subvariety of

$$\mathbb{P}(W_{i,k}) \times \ldots \times \mathbb{P}(W_{i-k+1,1}) \subset \mathbb{P}(W_i) \times \ldots \times \mathbb{P}(W_{i-k+1}). \tag{2.29}$$

**Remark 2.3.7.** Alternatively, in the construction (2.27), we can view $V_{i,\mathbf{n},j}$ as the intersection

$$\begin{aligned} V_{i,\mathbf{n},j} = \mathrm{Bl}_{V_{i,\mathbf{n},j-1}}&\left(\mathrm{Bl}_{V_{i,\mathbf{n},j-2}}\left(\ldots \mathrm{Bl}_{V_{i,\mathbf{n},2}}(\mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}(W_{i,k}))) \ldots\right)\right) \\ &\cap \mathbb{P}(W_{i,k}) \times \ldots \times \mathbb{P}(W_{i-j+2,k-j+2}) \times \mathbb{P}(W_{i-j+1,1}) \end{aligned} \tag{2.30}$$

and then form the blow-up with center $V_{i,\mathbf{n},j}$. This follows from the construction (2.20) of the first component $W_{i-j+1,1} = \langle e_{i-j+1,\alpha} \mid \mathrm{ind}(\alpha) = i-j \rangle$ of the flag for $W_{i-j+1}$.

Lemma 2.3.12 records the most important geometric properties of the Artin auxiliary varieties. First, we continue by explaining the construction in concrete low-dimensional examples. Later on, Example 2.3.14 will illustrate a more complicated situation.

**Example 2.3.8.** Let $\mathbf{n} = \mathbf{1}_2 = (1,1) \in \mathbb{N}^2$. This is a special case of Example 2.3.4, and has already appeared in [5, Example 1.5]. Fix $i \in \{1,2\}$. Since $0 \subset W_{i,1} \subset W_{i,2} = W_i$ is a complete flag, the first step of Definition 2.3.6 yields $V_{i,\mathbf{1}_2,1} = \mathbb{P}(W_{i,1}) = \operatorname{Spec} \mathbf{k}$ a point.

In the second step, we embed the point into $\mathbb{P}(W_{i,2}) = \mathbb{P}^1_{a_{i,1},a_{i,2}}$ as the vanishing of $a_{i,\alpha}$ such that $\alpha \neq i$. Then we form the blow-up $\mathrm{Bl}_{V_{i,\mathbf{1}_2,1}}(\mathbb{P}(W_{i,2}))$. Since $\operatorname{codim}_{\mathbb{P}(W_{i,2})} V_{i,\mathbf{n},1} = 1$, this is a trivial operation. Hence $V_{i,\mathbf{1}_2,1}$ is a line. More precisely, using the linear subspaces defined by the flags (2.20), we have

$$V_{1,\mathbf{1}_2,2} = \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{2,1}), \quad V_{2,\mathbf{1}_2,2} = \mathbb{P}(W_{2,2}) \times \mathbb{P}(W_{1,1}). \tag{2.31}$$

We will explain in Example 2.3.11 how $V_{i,\mathbf{1}_2,2}$ encode ideals of a hereditary order $\mathscr{A}$ with ramification data $\mathbf{1}_2$.

**Example 2.3.9.** We consider another, more involved case of Example 2.3.4. Let $\mathbf{n} = \mathbf{1}_4 \in \mathbb{N}^4$. Recall that the index function is the identity on $\{1,2,3,4\}$. For simplicity, we fix $i = 1$.

**Step 1.** As we have seen in the example before $V_{1,\mathbf{1}_4,1} = \mathbb{P}(W_{1,1})$ is a point.

**Step 2.** For the second step, we start with the line $\mathbb{P}(W_{1,2})$. As in the example before, $V_{1,\mathbf{1}_4,1}$ can be considered as a point in $\mathbb{P}(W_{1,2}) = \mathbb{P}^1_{a_{1,3},a_{1,4}}$ defined by the vanishing of $a_{1,3}$. The blow-up is trivial. Therefore,

$$V_{1,\mathbf{1}_4,2} = \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{4,1}) \cong \mathbb{P}(W_{1,2}) \tag{2.32}$$

is a line as in Example 2.3.8.

**Step 3.** For the construction of $V_{1,\mathbf{1}_4,3}$, we start with the projective plane $\mathbb{P}^2_{a_{1,2},a_{1,3},a_{1,4}} = \mathbb{P}(W_{1,3})$. The singleton $V_{1,\mathbf{1}_4,1}$ embeds into $\mathbb{P}(W_{1,3})$ by the condition that $a_{1,2} = a_{1,3} = 0$. Hence, the first step to constructing $V_{1,\mathbf{1}_4,3}$ is the blow-up

$$\mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,3})) = \mathrm{Bl}_{[0:0:1]}(\mathbb{P}^2_{a_{1,2},a_{1,3},a_{1,4}}) \subset \mathbb{P}(W_{1,3}) \times \mathbb{P}(W_{4,2}) = \mathbb{P}^2_{a_{1,\alpha}} \times \mathbb{P}^1_{a_{4,2},a_{4,3}}. \quad (2.33)$$

The exceptional divisor is given by $V_{1,\mathbf{1}_4,1} \times \mathbb{P}(W_{4,2})$. The second step is the blow-up of (2.33) in $V_{1,\mathbf{1}_4,2}$. For this, we view $V_{1,\mathbf{1}_4,2}$ inside $\mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,3}))$ by the vanishing of $a_{4,2}$. Indeed, using the blow-up relations this realizes

$$V_{1,\mathbf{1}_4,2} = \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{4,1}) \subset \mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,3})) \quad (2.34)$$

as a closed subvariety of codimension one. The blow-up with center $V_{1,\mathbf{1}_4,2}$ is therefore trivial, and we obtain

$$V_{1,\mathbf{1}_4,3} = \mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,3})) \times \mathbb{P}(W_{3,1}) \subset \mathbb{P}(W_{1,3}) \times \mathbb{P}(W_{4,2}) \times \mathbb{P}(W_{3,1}). \quad (2.35)$$

**Step 4.** The final step begins with $V_{1,\mathbf{1}_4,1}$ which encodes the point $[0:0:0:1]$ inside the projective 3-space $\mathbb{P}(W_{1,4}) = \mathbb{P}^3_{a_{1,1},a_{1,2},a_{1,3},a_{1,4}}$. The blow-up in the point is given by

$$\mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,4})) =\subset \mathbb{P}(W_{1,4}) \times \mathbb{P}(W_{4,3}) = \mathbb{P}^3_{a_{1,\alpha}} \times \mathbb{P}^2_{a_{4,1},a_{4,2},a_{4,3}}. \quad (2.36)$$

The exceptional divisor is the plane $V_{1,\mathbf{1}_4,1} \times \mathbb{P}(W_{4,3})$. For the next step, $V_{1,\mathbf{1}_4,2}$ is the closed subvariety of (2.36) defined by the vanishing of $a_{4,1}$ and $a_{4,2}$. This realizes $V_{1,\mathbf{1}_4,2}$ as the line $\mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{4,1})$ inside (2.36). We blow-up (2.36) with center $V_{1,\mathbf{1}_4,2}$ and obtain

$$\mathrm{Bl}_{V_{1,\mathbf{1}_4,2}}(\mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,4}))) \subset \mathbb{P}(W_{1,4}) \times \mathbb{P}(W_{4,3}) \times \mathbb{P}(W_{3,2}) = \mathbb{P}^3_{a_{1,\alpha}} \times \mathbb{P}^2_{a_{4,\beta}} \times \mathbb{P}^1_{a_{3,1},a_{3,2}}. \quad (2.37)$$

The next step is needless as $V_{1,\mathbf{1}_4,3}$, embedded via the condition that $a_{3,1} = 0$, defines a closed subvariety of codimension one. Therefore, we obtain

$$\begin{aligned} V_{1,\mathbf{1}_4,4} &= \mathrm{Bl}_{V_{1,\mathbf{1}_4,2}}(\mathrm{Bl}_{V_{1,\mathbf{1}_4,2}}(\mathrm{Bl}_{V_{1,\mathbf{1}_4,1}}(\mathbb{P}(W_{1,4})))) \\ &\subset \mathbb{P}(W_{1,4}) \times \mathbb{P}(W_{4,3}) \times \mathbb{P}(W_{3,2}) \times \mathbb{P}(W_{2,1}). \end{aligned} \quad (2.38)$$

We can express coordinates of $V_{1,\mathbf{1}_4,4}$ as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 \end{pmatrix} \in \mathbb{P}(W_{1,4}) \times \mathbb{P}(W_{2,1}) \times \mathbb{P}(W_{3,2}) \times \mathbb{P}(W_{4,3}), \quad (2.39)$$

where the $i$-th row are the coordinates of $\mathbb{P}(W_i)$. This will be set on a formal footing in Section 2.3.4. Afterwards, we will revisit the above situation in Example 2.3.37.

We briefly mention the relation to the fiber $\mathsf{BS}(\mathscr{A})_p := \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec} \mathbf{k}(p)$.

**Remark 2.3.10.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order with ramification data $(\mathbf{n}, e)$. The explicit choice of the flags (2.20) for $W_i$ allows us to use the $\mathbf{k}(p)$-basis $e_{i,\beta}$ of $\mathscr{A}(p) = \mathscr{A} \otimes_C \mathbf{k}(p)$ to describe points in $\mathbb{P}(W_i)$. The criterion (2.2.7) says that every ideal $L \trianglelefteq \mathscr{A}(p)$ defines a point $e_{\alpha_i,\alpha_i} L \in \mathbb{P}(W_i)$, where $\mathrm{ind}(\alpha_i) = i$.

We use Examples 2.3.8 and 2.3.9 to illustrate the previous remark.

**Example 2.3.11.** First, let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order with ramification data $(\mathbf{1}_2, 2)$ at $p \in C$. A point $[a_{1,1} : a_{1,2}] \in V_{1,\mathbf{1}_2,2} \cong \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{2,1})$ encodes the left ideal

$$L = \langle a_{1,1}e_{1,1} + a_{1,2}e_{1,2}, e_{2,1} \rangle \subset \mathscr{A}(p). \tag{2.40}$$

Now, let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order with ramification data $(\mathbf{1}_4, 4)$ at $p \in C$. With the notation of Example 2.3.9, a point

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 \end{pmatrix} \in V_{1,\mathbf{1}_4,4} \tag{2.41}$$

encodes the left ideal

$$L = \left\langle \begin{matrix} a_{1,1}e_{1,1}a_{1,2}e_{1,2}+a_{1,3}e_{1,3}+a_{1,4}e_{1,4}, e_{2,1}, \\ a_{3,1}e_{3,1}+a_{3,2}e_{3,2}, a_{4,1}e_{4,1}+a_{4,2}e_{4,2}+a_{4,3}e_{4,3} \end{matrix} \right\rangle \subset \mathscr{A}(p). \tag{2.42}$$

In both cases, the ideals satisfy the condition of Lemma 2.2.7 for belonging to the Artin model.

With the goal of describing the fiber $\mathsf{BS}(\mathscr{A})_p$, we now come back to general properties of the Artin auxiliary varieties. The following lemma is due to [67, §2], as we explain in Remark 2.3.13.

**Lemma 2.3.12.** *We keep the notation from Definition 2.3.6. For $k \in \{1, \ldots, e\}$ the Artin auxiliary variety $V_{i,\mathbf{n},k}$ is a smooth projective variety birationally equivalent to $\mathbb{P}^{s_{i,k}-1}$.*

*Proof.* The lemma is shown by complete induction on $k$, the induction start for $k = 1$ being evident. Let $1 < k \leq e$. Assume that $V_{i,\mathbf{n},j}$ is smooth and projective, birationally equivalent to $\mathbb{P}^{s_{i,j}-1}$ for all $j < k$. Now, one constructs iteratively $V_{i,n,\mathbf{k}}$. From (2.30) we obtain a closed immersion of $V_{i,\mathbf{n},j}$ into

$$\mathrm{Bl}_{V_{i,\mathbf{n},j-1}} \left( \ldots \mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}(W_{i,k})) \ldots \right) \subset \mathbb{P}(W_{i,k}) \times \ldots \times \mathbb{P}(W_{i-j+1,k-j+1}). \tag{2.43}$$

Therefore, the blow-up of the left hand side of (2.43) along $V_{i,\mathbf{n},j}$ yields a smooth projective variety birationally equivalent to $\mathbb{P}(W_{i,k}) \cong \mathbb{P}^{s_{i,k}-1}$. □

**Remark 2.3.13.** The construction is similar to the one in [67, Définition 2.2] in the sense that $V_{1,\mathbf{n},e} \cong V^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{N}^e$ and $e > 1$ where $V^{\mathbf{n}}$ is the notation from *op. cit.* More generally for arbitrary $i \in \{1, \ldots, e\}$, it will follow from Lemma 2.3.28 that $V_{i,\mathbf{n},e} \cong V^{\mathbf{n}[i-1]}$, where $\mathbf{n}[i-1]$ is the shifting defined in (2.17). We decide to keep the additional two indices $i, k \in \{1, \ldots, e\}$ for $V_{i,\mathbf{n},k}$ as they provide additional information about the structure of the fiber $\mathsf{BS}(\mathscr{A})_p$.

- The index $i$ distinguishes isomorphic, but non-equal irreducible components of $\mathsf{BS}(\mathscr{A})_p$. If $\mathbf{n} = \mathbf{d}_e$, it follows from Lemma 2.3.28 that all irreducible components are isomorphic.

- The index $k$ keeps track of subspaces $V_{i,\mathbf{n},k} \subseteq V_{i,\mathbf{n},e}$, which reappear for example in Theorem 2.3.29, and are isomorphic to irreducible components of another $\mathsf{BS}(\mathscr{A}')_p$.

In Examples 2.3.8 and 2.3.9 it was convenient that the index function is the identity. We continue now with the setup of Examples 2.3.3 and 2.3.5 illustrating the construction of the Artin auxiliary variety in a more involved example.

**Example 2.3.14.** Let $\mathbf{n} = (3, 1, 4, 2)$ and $\mathbf{s_3} = (1, 4, 6, 10)$. We explain the construction of the Artin auxiliary variety $V_{3,\mathbf{n},4}$ in detail.

**Step 1.** The first Artin auxiliary variety is $V_{3,\mathbf{n},1} = \mathbb{P}(W_{3,1})$. Since $W_{3,1} = \langle e_{3,4} \rangle$ is a one-dimensional $\mathbf{k}$-vector space, $V_{3,\mathbf{n},1}$ is a point.

**Step 2.** The embedding of $V_{3,\mathbf{n},1}$ into $\mathbb{P}(W_{3,2}) = \mathbb{P}^3_{a_{3,1},\dots,a_{3,4}}$ is given by the vanishing of all $a_{3,\alpha}$ such that $\operatorname{ind}(\alpha) = 1$. Then by Definition 2.3.6 we have

$$V_{3,\mathbf{n},2} = \mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,2})) \subset \mathbb{P}(W_{3,2}) \times \mathbb{P}(W_{2,1}). \tag{2.44}$$

This is the blow-up of a point in $\mathbb{P}^3$. We denote the points of $V_{3,\mathbf{n},2}$ by

$$\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} \in V_{3,\mathbf{n},2}. \tag{2.45}$$

The first line describes the points of the two-dimensional linear subspace $\mathbb{P}(W_{2,1}) \subset \mathbb{P}(W_2)$. The second line describes the points of the three-dimensional linear subspace $\mathbb{P}(W_{3,2}) \subset \mathbb{P}(W_3)$. Note that for every $2 \times 2$-submatrix of the matrix without a '0' as a coordinate, we have a relation by taking its determinant $a_{2,\alpha}a_{3,\beta} - a_{2,\beta}a_{3,\alpha}$.

**Step 3.** We have to do successive blow-ups of $\mathbb{P}(W_{3,3}) = \mathbb{P}^5_{a_{3,1},\dots,a_{3,4},a_{3,9},a_{3,10}}$. First of all, the point $V_{3,\mathbf{n},1} = \mathbb{P}(W_{3,1}) \subset \mathbb{P}(W_{3,3})$ embeds as the vanishing of $a_{3,1}, a_{3,2}, a_{3,3}, a_{3,9}, a_{3,10}$. We form the blow-up $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3})) \subset \mathbb{P}(W_{3,3}) \times \mathbb{P}(W_{2,2})$. Since $\mathbb{P}(W_{2,2})$ is a four-dimensional space, a point in the blow-up will be denoted by

$$\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & 0 & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,9} & a_{3,10} \end{pmatrix} \in \mathbb{P}(W_{2,2}) \times \mathbb{P}(W_{3,3}). \tag{2.46}$$

The Artin auxiliary variety $V_{3,\mathbf{n},2}$ is a subvariety of $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3}))$ defined by the vanishing of $a_{2,9}, a_{2,10}$. The blow-up relations $a_{2,\alpha}a_{3,\beta} - a_{2,\beta}a_{3,\alpha}$ impose that $a_{3,9} = a_{3,10} = 0$ for points in $V_{3,\mathbf{n},2}$. Hence points of $V_{3,\mathbf{n},2}$ in $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3}))$ are given by

$$\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & 0 & 0 \end{pmatrix} \in \mathbb{P}(W_{2,2}) \times \mathbb{P}(W_{3,3}), \tag{2.47}$$

subject to the relations $a_{2,\alpha}a_{3,\beta} - a_{2,\beta}a_{3,\alpha}$ for every $2 \times 2$-submatrix without '0'.

We can describe $\mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3})))$ as the strict transform of $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3}))$ under the blow-up of $\mathbb{P}(W_{3,3}) \times \mathbb{P}(W_{2,2})$ along $Z_2 = \mathbb{P}(W_{3,3}) \times \mathbb{P}(W_{2,1})$. Now, the ambient space is blown up to

$$\begin{aligned} \mathrm{Bl}_{Z_2}(\mathbb{P}(W_{3,3}) \times \mathbb{P}(W_{2,2})) &\cong \mathbb{P}(W_{3,3}) \times \mathrm{Bl}_{\mathbb{P}(W_{2,1})}(\mathbb{P}(W_{2,2})) \\ &\subset \mathbb{P}(W_{3,3}) \times \mathbb{P}(W_{2,2}) \times \mathbb{P}(W_{1,1}). \end{aligned} \tag{2.48}$$

The second factor $\mathrm{Bl}_{\mathbb{P}(W_{2,1})}(\mathbb{P}(W_{2,2}))$ is the blow-up of $\mathbb{P}^4_{a_{2,1},a_{2,2},a_{2,3},a_{2,9},a_{2,10}}$ in a plane defined by $a_{2,9}=a_{2,10}=0$. It follows that

$$V_{3,\mathbf{n},3}=\mathrm{Bl}_{V_{3,\mathbf{n},2}}\left(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,3}))\right)\subset \mathbb{P}^5_{a_{3,\alpha}}\times\mathbb{P}^4_{a_{2,\beta}}\times\mathbb{P}^1_{a_{1,9},a_{1,10}}. \tag{2.49}$$

We describe the points of $V_{3,\mathbf{n},3}$ by a matrix with three rows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,9} & a_{1,10} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,9} & a_{3,10} \end{pmatrix}\in V_{3,\mathbf{n},3}. \tag{2.50}$$

The first line encodes the 1-dimensional liner subspace $\mathbb{P}(W_{1,1})\subset\mathbb{P}(W_1)$, the second line describes $\mathbb{P}(W_{2,2})$, and the third line describes $\mathbb{P}(W_{3,3})$. A point lies in $V_{3,\mathbf{n},3}$ if its coordinates satisfy the relations given by the vanishing of the $2\times 2$-minors of the matrices without a '0' as a coordinate.

**Step 4.** This is the step leading to the irreducible component $V_{3,\mathbf{n},4}$ starting from projective 9-space $\mathbb{P}(W_{3,4})=\mathbb{P}(W_3)$. The first step is the construction of $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3))$, the blow-up of $\mathbb{P}^9$ in a point. In analogy to the previous steps, its points are given by

$$\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,2} & 0 & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,2} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} \end{pmatrix}\in\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3}). \tag{2.51}$$

As in the previous step, we see that $V_{3,\mathbf{n},2}=\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_{3,4}))\cap(\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,1}))$ is the vanishing of $a_{2,\alpha}$ such that $\mathrm{ind}(\alpha)\neq 1$. For simplicity, we color the (possibly) non-zero coordinates in (2.51) describing $V_{3,\mathbf{n},2}$ inside $\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3))$ in blue.

The points of the blow-up $\mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3)))\subset\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3})\times\mathbb{P}(W_{1,2})$ are given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} \end{pmatrix}\in\mathrm{Bl}_{V_{3,\mathbf{n},2}}\left(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3))\right). \tag{2.52}$$

For the final step, we need to blow up $\mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3)))$ along $V_{3,\mathbf{n},3}$ constructed in (2.49). By Definition 2.3.6, $V_{3,\mathbf{n},3}$ is a subvariety of $\mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3)))$ cut out by $a_{1,5},\ldots,a_{1,8}$. We depict the (possibly) non-zero coordinates of $V_{3,\mathbf{n},3}$ in (2.52) in blue.

Now, we blow up the space $\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3})\times\mathbb{P}(W_{1,2})$ along the subvariety $Z_3=\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3})\times\mathbb{P}(W_{1,1})$. Then $V_{3,\mathbf{n},4}$ lies inside

$$\begin{aligned}&\mathrm{Bl}_{Z_3}(\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3})\times\mathbb{P}(W_{1,2}))\\ &\cong\mathbb{P}(W_{3,4})\times\mathbb{P}(W_{2,3})\times\mathrm{Bl}_{\mathbb{P}(W_{1,1})}(\mathbb{P}(W_{1,2})),\end{aligned} \tag{2.53}$$

where the last factor is the blow-up of a $\mathbb{P}^5$ along a line. Then $V_{3,\mathbf{n},4}$ is the strict transform of $\mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{V_{3,\mathbf{n},1}}(\mathbb{P}(W_3)))$ along the blow-up (2.53). We can describe its points by a matrix with four rows

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} \\ 0 & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & 0 & 0 \end{pmatrix}\in V_{3,\mathbf{n},4}. \tag{2.54}$$

The last row encodes a subspace of $\mathbb{P}(W_{4,1})$ and, analogously to (2.45) and (2.49), $V_{3,\mathbf{n},4}$ is the vanishing locus of every $2 \times 2$-minor of the matrix having no '0' as a coordinate. Using the notation of Section 2.3.4 the presentation of $V_{3,\mathbf{n},4}$ simplifies to

$$V_{3,\mathbf{n},4} \leftrightarrow \begin{pmatrix} 0 & 0 & * & * \\ * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \end{pmatrix}^{(3,1,4,2)} . \tag{2.55}$$

### 2.3.3 The fibers of the Artin model

Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ over a smooth separated curve $C$ over $\mathbf{k}$. Assume that $\mathscr{A}$ has ramification data $(\mathbf{n}, e)$ at $p \in C$. The goal of this section is an explicit description of the fiber $\mathsf{BS}(\mathscr{A})_p := \mathsf{BS}(\mathscr{A}) \times_C \operatorname{Spec}(\mathbf{k}(p))$ for $p \in C$ in terms of the Artin auxiliary varieties $V_{i,\mathbf{n},e}$. By Lemma 2.2.7 this means describing all left ideals $L \trianglelefteq \mathscr{A}(p) = \mathscr{A} \otimes_{\mathcal{O}_{C,p}} \mathbf{k}(p)$ satisfying $\dim_{\mathbf{k}(p)} e_{\alpha,\alpha} \cdot L = 1$ for all $\alpha = 1, \ldots, n$.

**Theorem 2.3.15** (Artin, Frossard)**.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$. Then the morphism $f \colon \mathsf{BS}(\mathscr{A}) \to C$ is a flat projective morphism. Let $p \in C$ be a point such that $\mathscr{A}$ has ramification data $(\mathbf{n}, e)$ at $p$ and denote by $\mathsf{BS}(\mathscr{A})_p$ the fiber at $p$.*

*i) The fiber $\mathsf{BS}(\mathscr{A})_p$ is a reduced projective variety of pure dimension $n-1$.*

*ii) The Artin auxiliary varieties $V_{1,\mathbf{n},e}, \ldots V_{e,\mathbf{n},e}$ from Definition 2.3.6 form the irreducible components of $\mathsf{BS}(\mathscr{A})_p$ and intersect transversely.*

Artin [5, Theorem 1.4] proved the theorem for restrictions of maximal orders along étale coverings. These comprise all orders with ramification data $\mathbf{n} = \mathbf{d}_e$ at a point $p \in C$, where $\mathscr{A}$ has ramification index $e$. Frossard [67, Proposition 2.3] extended the setting for the theorem to arbitrary hereditary orders. She defined the Artin auxiliary varieties for general $\mathbf{n}$ and showed that they form the irreducible components of the fiber $\mathsf{BS}(\mathscr{A})_p$. The transversality statement from [5, Theorem 1.4] carries over to the general case as well.

Since the understanding of how the varieties $V_{1,\mathbf{n},1}, \ldots, V_{e,\mathbf{n},e}$ form the fiber $\mathsf{BS}(\mathscr{A})_p$ is important for the discussion in Section 2.3, we recall the description of the fiber $\mathsf{BS}(\mathscr{A})_p$ using the setup introduced in Section 2.3.2.

Fix $p \in C$. By Lemma 2.2.6 we can restrict to an open neighborhood around $p$ such that $\mathscr{A}_p = \mathscr{A}_{(\mathbf{n},e)}(\mathcal{O}_{C,p})$ is of the form (2.3) with ramification data $(\mathbf{n}, e)$ at $p$. We fix

- the $\mathcal{O}_{C,p}$-generating set $\{E_{\alpha,\beta} \mid 1 \leq \alpha, \beta \leq n\}$ for the matrix algebra $\operatorname{Mat}_n(\mathcal{O}_{C,p})$,
- the induced generating set $E^{(p)}_{\alpha,\beta}$ for $\mathscr{A}_p$ from (2.6),
- and the associated $\mathbf{k}(p)$-basis $e_{\alpha,\beta}$ for $\mathscr{A}(p)$.
- Moreover, denote by $\mathbf{r}$ the tuple of partial sums of $\mathbf{n}$, and by $\mathbf{s}_i = \mathbf{s}_i(\mathbf{n})$ the $\mathbf{s}$-tuples from Section 2.3.1.

As already mentioned in Remark 2.3.10, the description of left ideals in $\mathscr{A}(p)$ is linked to the flags of subspaces from (2.20). First of all we note that a row $e_{\alpha,\alpha}\mathscr{A}(p)$ determines each other row $e_{\beta,\beta}\mathscr{A}(p)$ with $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$ by multiplication with an element from $\mathscr{A}(p)$.

**Lemma 2.3.16.** *Multiplication by $e_{\beta,\alpha}$ (from the left) defines an isomorphism $e_{\alpha,\alpha}\mathscr{A}(p) \to e_{\beta,\beta}\mathscr{A}(p)$ if and only if $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$.*

*Proof.* The lemma follows from the rule (2.8) for $\tau_{\alpha\beta\gamma}$. Spelling it out, we have $e_{\alpha,\beta}e_{\beta,\alpha} = e_{\alpha,\alpha}$ and $e_{\beta,\alpha}e_{\alpha,\beta} = e_{\beta,\beta}$ if and only if $\tau_{\alpha,\beta,\alpha} = \tau_{\beta,\alpha,\beta} = 1$ if and only if $\operatorname{ind}(\alpha) = \operatorname{ind}(\beta)$. □

In light of the preceding lemma, we fix a subset

$$\{\alpha_i \mid \operatorname{ind}(\alpha_i) = i,\ i = 1, \ldots, e\} \subseteq \{1, \ldots, n\} \tag{2.56}$$

of cardinality $e$. For example, one can choose $\alpha_i = r_{i-1} + 1$.

**Corollary 2.3.17.** *Let $L, L' \in \mathsf{BS}(\mathscr{A})_p$ be two left ideals in $\mathscr{A}(p)$ belonging to the Artin model. Then $L = L'$ if and only if $e_{\alpha_i,\alpha_i} \cdot L = e_{\alpha_i,\alpha_i} \cdot L'$ for all $i = 1, \ldots, e$.*

*Proof.* Since $L$ and $L'$ define elements in the Artin model, we know that for each $\alpha = 1, \ldots, n$ the $\mathbf{k}(p)$-vector space $e_{\alpha,\alpha} \cdot L$ (resp. $e_{\alpha,\alpha} \cdot L'$) is one-dimensional. Now the claim follows from Lemma 2.3.16. □

**Remark 2.3.18.** Keeping Corollary 2.3.17 in mind, we will abuse notation and simply write $e_{i,\beta} = e_{\alpha_i,\beta}$. In this way, the indexing $i \in \{1, \ldots, e\}$ and $\beta \in \{1, \ldots, n\}$ aligns well with the indexing of the basis vectors for the spaces $W_i$ from (2.20).

We define now $n$-dimensional $\mathbf{k}(p)$-vector spaces

$$W_i = e_{i,\alpha_i} \cdot \mathscr{A}(p) \quad \text{for } i = 1, \ldots, e. \tag{2.57}$$

For each $W_i$ we consider the flag $0 \subset W_{i,1} \subset W_{i,2} \subset \ldots \subset W_{i,e-1} \subset W_{i,e}$ as in (2.20) associated to the $\mathbf{s}$-tuple $\mathbf{s}_i$.

The remainder of this section is essentially devoted to showing that every ideal $L \in \mathsf{BS}(\mathscr{A})_p$ is determined by an $e$-tuple of non-zero vectors

$$(\zeta_i, \zeta_{i-1}, \ldots, \zeta_{i+1}) \in W_{i,e} \times W_{i-1,e-1} \times \ldots \times W_{i+1,1}, \tag{2.58}$$

where the first index is counted modulo $e$.

**Definition 2.3.19.** A non-zero element $\eta \in \mathscr{A}(p)$ is called *essential* if there exists $i \in \{1, \ldots, e\}$ such that $\eta \in W_{i,1}$.

**Example 2.3.20.** We continue the discussion of Examples 2.3.3, 2.3.5 and 2.3.14 with ramification data $\mathbf{n} = (3, 1, 4, 2)$ and flags of subspaces for $W_1, W_2, W_3, W_4$ given in Table 2.1. There are four distinct families of essential elements, belonging to one of the subspaces

$$\begin{aligned} W_{2,1} &= \langle e_{4,1}, e_{4,2}, e_{4,3} \rangle \subset W_2, \\ W_{3,1} &= \langle e_{5,4} \rangle \subset W_3, \\ W_{4,1} &= \langle e_{9,5}, e_{9,6}, e_{9,7}, e_{9,8} \rangle \subset W_4, \\ W_{1,1} &= \langle e_{1,9}, e_{1,10} \rangle \subset W_1. \end{aligned} \tag{2.59}$$

It will follow from the proof of Proposition 2.3.23 below that an ideal $L \in \mathsf{BS}(\mathscr{A})_p$ contains an essential element in $W_{i+1,1}$ if and only if it lies in $V_{i,\mathbf{n},e}$.

**Lemma 2.3.21.** *Let $L \in \mathsf{BS}(\mathscr{A})_p$ be an ideal.*

*i) If $L$ contains two essential elements belonging to the same $W_{i,1}$, then they are linearly dependent.*

*ii) Every ideal $L$ contains an essential element.*

*Proof.* By Lemma 2.2.7 and Corollary 2.3.17 the ideal $L$ is determined by $e$ non-zero vectors $\zeta_i = \sum_{\alpha=1}^{n} a_{i,\alpha} e_{i,\alpha} \in W_i$. Since the subspace $e_{i,\alpha_i} L \subset W_i$ is one-dimensional (spanned by $\zeta_i$), the first part is clear.

For the second part, assume that none of $\zeta_1, \ldots, \zeta_e$ is essential. Using the rule (2.8), one calculates that

$$e_{i,\alpha_e} \cdot \zeta_e = \sum_{\mathrm{ind}(\alpha)<i} a_{e,\alpha} e_{i,\alpha} + \sum_{\mathrm{ind}(\alpha)=e} a_{e,\alpha} e_{i,\alpha}. \tag{2.60}$$

For $i = 1$ the calculation implies $e_{1,\alpha_e} \cdot \zeta_e \in W_{1,1} = \langle e_{1,\alpha} \mid \mathrm{ind}(\alpha) = e \rangle$, hence $e_{1,\alpha_e} \cdot \zeta_e = 0$, because we assumed that there is no essential element in $L$. By applying the same reasoning recursively for $i = 2, \ldots, e$, one concludes that $\zeta_e = 0$ which is a contradiction to $L \in \mathsf{BS}(\mathscr{A})_p$. □

An essential element of an ideal $L \in \mathsf{BS}(\mathscr{A})_p$ determines the structure of the ideal with respect to the flag of subspaces of $W_j$.

**Lemma 2.3.22.** *Assume that $L \in \mathsf{BS}(\mathscr{A})_p$. Then there exists $i \in \{1, \ldots, e\}$ such that*

$$\bigoplus_{j=1}^{e} e_{j,\alpha_j} \cdot L \subset W_{1,e-i+2} \times W_{2,e-i+3} \times \ldots \times W_{e-1,e-i} \times W_{e,e-i+1}, \tag{2.61}$$

*where the second index of each subspace has to be read modulo $e$.*

*Proof.* Let $L \in \mathsf{BS}(\mathscr{A})_p$. We know from Corollary 2.3.17 that the ideal $L$ is uniquely determined by $e$ non-zero vectors $\zeta_j = \sum_{\alpha=1}^{n} a_{j,\alpha} e_{j,\alpha} \in e_{j,\alpha_j} \cdot L \subset W_j$, $j = 1, \ldots, e$. By Lemma 2.3.21, there exists $i = 1, \ldots, e$ such that $\zeta_i \in W_{i,1}$ is essential, which means that $a_{i,\alpha} = 0$ unless $\mathrm{ind}(\alpha) = i - 1$ (where the index has to be read modulo $e$).

Since $e_{i,\alpha_i} \cdot L$ is a one-dimensional vector space with basis $\zeta_i$, a necessary condition for $L$ being an ideal is that $\zeta_1, \ldots, \zeta_e$ satisfy $e_{i,\alpha_j}\zeta_j \in W_{i,1}$. From (2.8) it follows that

$$e_{i,\alpha_j}\zeta_j = \begin{cases} \sum\limits_{\mathrm{ind}(\alpha)<i} a_{j,\alpha} e_{i,\alpha} + \sum\limits_{\mathrm{ind}(\alpha)\geq j} a_{j,\alpha} e_{i,\alpha} & \text{if } i < j, \\ \sum\limits_{j \leq \mathrm{ind}(\alpha) < i} a_{j,\alpha} e_{i,\alpha} & \text{if } i > j. \end{cases} \tag{2.62}$$

In the case $i < j$, one has $e_{i,\alpha_j}\zeta_j \in W_{i,1}$ if and only if $a_{j,\alpha} = 0$, unless $i-1 \leq \mathrm{ind}(\alpha) \leq j-1$. Now, one compares to the construction of the flag (2.20) of subspaces for $W_j$. The condition on the coefficients is satisfied if and only if $\zeta_j \in W_{j,j-i+1}$.

In order to avoid confusion, let us mention that if $i = 1$, the condition $a_{j,\alpha} = 0$ unless $i-1 \leq \mathrm{ind}(\alpha) \leq j-1$ has to be read modulo $e$. More precisely, this means that $a_{j,\alpha} = 0$ unless $\mathrm{ind}(\alpha) \leq j-1$ or $\mathrm{ind}(\alpha) = e$. Because $W_{1,1} = \langle e_{1,\alpha} \mid \mathrm{ind}(\alpha) = e \rangle$, the coefficients $a_{j,\alpha}$ are allowed to be non-zero for $\mathrm{ind}(\alpha) = e$. The conclusion is that $\zeta_j \in W_{j,j}$.

The case $i > j$ is treated similarly. The calculation (2.62) imposes $a_{j,\alpha} = 0$ unless $1 \leq \mathrm{ind}(\alpha) \leq j-1$ or $i-1 \leq \mathrm{ind}(\alpha) \leq e$. Comparing to the definition of the flag (2.20) of subspaces for $W_j$, we see that $\zeta_j \in W_{j,e+j-i+1}$. □

We have now all the ingredients at hand, to describe the **k**-points of $\mathsf{BS}(\mathscr{A})_p$.

**Proposition 2.3.23.** *The fiber* $\mathsf{BS}(\mathscr{A})_p$ *is a closed subvariety of* $\mathbb{P}(W_1) \times \ldots \mathbb{P}(W_e)$ *such that*

$$\mathsf{BS}(\mathscr{A})_p = V_{1,\mathbf{n},e} \cup \ldots V_{e,\mathbf{n},e} \subset \mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e). \tag{2.63}$$

*Proof.* By Lemma 2.2.7 and Corollary 2.3.17 a left ideal $L \in \mathsf{BS}(\mathscr{A})_p$ is determined by $e$ one-dimensional subspaces $e_{i,\alpha_i}L$. Therefore, the assignment $L \mapsto (e_{1,\alpha_1}L, \ldots, e_{e,\alpha_e}L)$ embeds $\mathsf{BS}(\mathscr{A})_p$ into $\mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e)$.

Thus, it remains to identify all points $([\zeta_1], \ldots, [\zeta_e]) \in \mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e)$ such that

$$L = \bigoplus_{i=1}^{e} \bigoplus_{\mathrm{ind}(\alpha)=i} \langle e_{\alpha,\alpha_i}\zeta_i \rangle \subset \mathscr{A}(p) \tag{2.64}$$

is an ideal.

First let $L \in \mathsf{BS}(\mathscr{A})_p$ be an ideal. We show that $L \in V_{k,\mathbf{n},e}$ for some $k$. Denote by $\zeta_j = \sum_{\alpha=1}^{n} a_{j,\alpha} e_{j,\alpha} \in e_{j,\alpha_j} \cdot L$ a non-zero vector for each $j = 1, \ldots, n$. By Lemma 2.3.22, there exists an $i \in \{1, \ldots, e\}$ such that

$$(\zeta_1, \ldots, \zeta_e) \in W_{1,e-i+2} \times W_{2,e-i+3} \times \ldots \times W_{e-1,e-i} \times W_{e,e-i+1}. \tag{2.65}$$

because $L$ is an ideal. It is a lengthy but straightforward calculation that $(\zeta_1, \ldots, \zeta_e)$ satisfies $e_{k,\alpha_j}\zeta_j \in W_{k,k+e-i+1}$. More precisely, excluding the trivial case $k = j$ and using the constraints from (2.65), we have

$$e_{k,\alpha_j}\zeta_j = \begin{cases} \sum\limits_{i-1\le \mathrm{ind}(\alpha)\le k-1} a_{j,\alpha}e_{k,\alpha} \in W_{k,k-i+1} & \text{if } i\le k<j,\\ \sum\limits_{\mathrm{ind}(\alpha)\le k-1} a_{j,\alpha}e_{k,\alpha} + \sum\limits_{\mathrm{ind}(\alpha)\ge i-1} a_{j,\alpha}e_{k,\alpha} \in W_{k,e+k-i+1} & \text{if } k<j<i,\\ \sum\limits_{i-1\le \mathrm{ind}(\alpha)\le k-1} a_{j,\alpha}e_{k,\alpha} \in W_{k,k-i+1} & \text{if } k\ge i>j,\\ 0 & \text{otherwise.}\end{cases} \tag{2.66}$$

Since $L$ is an ideal, we have in addition that $e_{k,\alpha_j}\zeta_j \in \langle \zeta_k\rangle$ for all $j,k$ by Lemma 2.2.7. This imposes

$$a_{j,\alpha}\cdot a_{k,\beta} = a_{j,\beta}\cdot a_{k,\alpha} \quad \text{if } i\le k<j, k<j<i, \text{ or } j<i\le k, \tag{2.67}$$

and $\mathrm{ind}(\alpha),\mathrm{ind}(\beta)$ such that $e_{k,\alpha}, e_{k,\beta}\in W_{k,k-i+1}$ (counted modulo $e$). The relations (2.67) are those realizing the embedding

$$V_{i-1,\mathbf{n},e} \subset \mathbb{P}(W_{i-1,e})\times\mathbb{P}(W_{i-2,e-1})\times\ldots\mathbb{P}(W_{i+1,2})\times\mathbb{P}(W_{i,1}) \tag{2.68}$$

from Definition 2.3.6. Hence, it follows that $([\zeta_1],\ldots,[\zeta_e]) = (e_{1,\alpha_1}L,\ldots,e_{e,\alpha_e}L)\in V_{i-1,\mathbf{n},e}$.

Vice versa, a point $V_{i-1,\mathbf{n},e}$ is given by a collection of $e$ non-zero vectors $(\zeta_1,\ldots,\zeta_e)\in W_{1,e-i+2}\times\ldots\times W_{e,e-i+1}$ as in (2.65), because of the embedding (2.68). The blow-up relations yield $e_{k,\alpha_j}\zeta_j\in\langle\zeta_k\rangle$ for all $j,k=1,\ldots,e$. Here we use the left action of $\mathscr{A}(p)$ on $W_i = e_{i,\alpha_i}\mathscr{A}(p)$. Hence $(\zeta_1,\ldots,\zeta_e)$ defines an ideal as in (2.64). ☐

### 2.3.4 Indicator matrices

In order to make the calculations with the irreducible components $V_{1,\mathbf{n},e},\ldots,V_{e,\mathbf{n},e}$ from Definition 2.3.6 more comprehensible, we introduce the following notation.

**Definition 2.3.24.** Fix ramification data $(\mathbf{n},e)$. An *indicator matrix of shape* $(\mathbf{n},e)$ is an $e\times e$ matrix $M^{\mathbf{n}} = (m_{ij})^{\mathbf{n}}_{1\le i,j\le e}$ with entries $m_{ij}\in\{0,*\}$. We denote by $\mathcal{M}^{\mathbf{n},e}$ the set of indicator matrices of shape $(\mathbf{n},e)$.

Recall the index function associated with $(\mathbf{n},e)$ from Section 2.3.1. Moreover, let $W_1,\ldots,W_e$ be the $|\mathbf{n}|$-dimensional $\mathbf{k}(p)$-vector spaces from (2.57) and recall their flags of subspaces (2.20).

We use $M^{\mathbf{n}}$ to encode certain subvarieties of $\mathbb{P}(W_1)\times\ldots\times\mathbb{P}(W_e)$. Denote by $(a_{i,\alpha})_{\alpha=1,\ldots,n}$ the coordinates of $\mathbb{P}(W_i)$. Then $M^{\mathbf{n}} = (m_{ij})^{\mathbf{n}}_{1\le i,j\le e}$ encodes the reduced subvariety in $\mathbb{P}(W_1)\times\ldots\times\mathbb{P}(W_e)$ given by the vanishing of the following functions

- if $m_{ij}=0$, we require $a_{i,\alpha}=0$ for all $\alpha$ with $\mathrm{ind}(\alpha)=j$,
- if $\begin{pmatrix} m_{i_1,j_1} & m_{i_1,j_2}\\ m_{i_2,j_1} & m_{i_2,j_2}\end{pmatrix} = \begin{pmatrix} * & *\\ * & *\end{pmatrix}$, where $i_1\ne i_2$, then

$$a_{i_1,\alpha}\cdot a_{i_2,\beta} - a_{i_1,\beta}\cdot a_{i_2,\alpha} = 0 \quad \text{for all } \alpha,\beta \text{ with } \mathrm{ind}(\alpha),\mathrm{ind}(\beta)\in\{j_1,j_2\}. \tag{2.69}$$

If $M^{\mathbf{n}}$ encodes a subvariety $V \subset \mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e)$, we will sometimes write $M(V)$ for $M^{\mathbf{n}}$, or indicate the correspondence by $V \leftrightarrow M^{\mathbf{n}}$.

**Remark 2.3.25.** The construction implies that a '$*$' in the $(i,j)$-th entry of an indicator matrix $M^{\mathbf{n}}$ encodes the coordinates of an $(n_j - 1)$-dimensional linear subspace of $\mathbb{P}(W_i)$.

Indicator matrices encode the irreducible components of the Artin model $\mathsf{BS}(\mathscr{A})_p$ where the order $\mathscr{A}$ is ramified over $p$ with ramification data $(\mathbf{n}, e)$.

**Proposition 2.3.26.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order and $p \in C$ such that $\mathscr{A}$ has ramification data $(\mathbf{n}, e)$ at $p$.*

*i) For all $i, j = 1, \ldots, e$, there is a unique indicator matrix $M^{\mathbf{n}}_{i,j} = M(V_{i,\mathbf{n},j}) \in \mathcal{M}^{\mathbf{n},e}$ encoding the Artin auxiliary variety $V_{i,\mathbf{n},j} \subset \mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e)$.*

*ii) In particular, the irreducible components of $\mathsf{BS}(\mathscr{A})_p$ satisfy*

$$V_{1,\mathbf{n},e} \leftrightarrow \begin{pmatrix} * & * & \dots & * & * \\ * & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ * & * & & 0 & 0 \\ * & * & \dots & * & 0 \end{pmatrix}^{\mathbf{n}}, \ldots, \; V_{e-1,\mathbf{n},e} \leftrightarrow \begin{pmatrix} 0 & 0 & \dots & * & * \\ * & 0 & \dots & * & * \\ \vdots & & & & \vdots \\ * & * & \dots & * & * \\ 0 & 0 & \dots & * & 0 \end{pmatrix}^{\mathbf{n}}, \; V_{e,\mathbf{n},e} \leftrightarrow \begin{pmatrix} 0 & 0 & \dots & 0 & * \\ * & 0 & \dots & 0 & * \\ \vdots & \ddots & \ddots & & \vdots \\ * & * & & 0 & * \\ * & * & \dots & * & * \end{pmatrix}^{\mathbf{n}}.$$

*Proof.* The linear subspace $\mathbb{P}(W_{i,j}) \subset \mathbb{P}(W_i)$ from (2.20) corresponds to an indicator matrix $M^{\mathbf{n}}$ with $m_{ij} = 0$ for all $j \neq i$. To encode $\mathbb{P}(W_{i,j})$, one fills in '$*$'s in the $i$-th row of $M^{\mathbf{n}}$ starting in the column $k = i - 1$ decreasing $k$ until $k = i - j$ (counted modulo $e$). It follows from the construction of the $W_{i,j} \subset W_i$ in (2.20) that the obtained $M^{\mathbf{n}}$ encodes $\mathbb{P}(W_{i,j})$ inside $\mathbb{P}(W_1) \times \ldots \times \mathbb{P}(W_e)$. The discussion implies that $V_{i,\mathbf{n},1}$ is encoded by the indicator matrix $M^{\mathbf{n}}_{i,1} = (m_{s,t})^{\mathbf{n}}_{1 \leq s,t \leq e}$ with

$$m_{s,t} = \begin{cases} * & \text{if } (s,t) = (i, i-1) \text{ or } (s,t) = (1, e) \\ 0 & \text{otherwise.} \end{cases} \tag{2.70}$$

Using the construction of the Artin auxiliary varieties from Definition 2.3.6 with respect to the flags of subspaces of $W_i$, we obtain the indicator matrices for the $V_{i,\mathbf{n},j}$ as follows. Fix $i \in \{1, \ldots, e\}$. It follows from the embedding (2.26) that the '$*$'s for $V_{i,\mathbf{n},j}$ are concentrated in the rows starting from $i$ to $i - j + 1$ in decreasing order (counted modulo $e$).

Then, for $k$ between $i$ and $i - j + 1$ (counted modulo $e$ in decreasing order) the nonzero entries sit precisely in $\mathbb{P}(W_{k,j-(i-k)})$, whose indicator matrix we have described above. Therefore, we require that the $k$-th row of the indicator matrix $M^{\mathbf{n}}_{i,j}$ encoding $V_{i,\mathbf{n},j}$ equals the $k$-th row of the indicator matrix encoding $\mathbb{P}(W_{k,j-(i-k)})$.

Since $V_{i,\mathbf{n},j}$ is constructed by successive blow-ups we obtain the relations (2.69) whenever there occurs a $2 \times 2$-submatrix of '$*$'s or two '$*$'s lie in the same column of $M^{\mathbf{n}}_{i,j}$.

The uniqueness follows from the fact that in the embedding

$$V_{i,\mathbf{n},j} \subset \mathbb{P}(W_{i,j}) \times \ldots \times \mathbb{P}(W_{i-j+1,1}) \tag{2.71}$$

from (2.29), we cannot replace any of the linear subspaces on the right by a smaller one. This settles part (i). The description of the $V_{i,\mathbf{n},e}$ by an indicator matrix follows from the above discussion for $j = e$. □

**Example 2.3.27.** Let $\mathbf{n} = (3, 1, 4, 2)$ as in Examples 2.3.3, 2.3.5, 2.3.14 and 2.3.20. Then the indicator matrices associated with the components are given by:

$$
\begin{aligned}
&V_{1,\mathbf{n},1} \leftrightarrow \begin{pmatrix} 0&0&0&*\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{1,\mathbf{n},2} \leftrightarrow \begin{pmatrix} 0&0&*&*\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{1,\mathbf{n},3} \leftrightarrow \begin{pmatrix} 0&*&*&*\\ 0&0&0&0\\ 0&*&0&0\\ 0&*&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{1,\mathbf{n},4} \leftrightarrow \begin{pmatrix} *&*&*&*\\ *&0&0&0\\ *&*&0&0\\ *&*&*&0 \end{pmatrix}^{\mathbf{n}}, \\
&V_{2,\mathbf{n},1} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ *&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{2,\mathbf{n},2} \leftrightarrow \begin{pmatrix} 0&0&0&*\\ *&0&0&*\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{2,\mathbf{n},3} \leftrightarrow \begin{pmatrix} 0&0&*&*\\ *&0&*&*\\ 0&0&0&0\\ 0&0&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{2,\mathbf{n},4} \leftrightarrow \begin{pmatrix} 0&*&*&*\\ *&*&*&*\\ 0&*&0&0\\ 0&*&*&0 \end{pmatrix}^{\mathbf{n}}, \\
&V_{3,\mathbf{n},1} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&*&0&0\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{3,\mathbf{n},2} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ *&0&0&0\\ *&*&0&0\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{3,\mathbf{n},3} \leftrightarrow \begin{pmatrix} 0&0&0&*\\ *&0&0&*\\ *&*&0&*\\ 0&0&0&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{3,\mathbf{n},4} \leftrightarrow \begin{pmatrix} 0&0&*&*\\ *&0&*&*\\ *&*&*&*\\ 0&0&*&0 \end{pmatrix}^{\mathbf{n}}, \\
&V_{4,\mathbf{n},1} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{4,\mathbf{n},2} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&*&0&0\\ 0&*&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{4,\mathbf{n},3} \leftrightarrow \begin{pmatrix} 0&0&0&0\\ *&0&0&0\\ *&*&0&0\\ *&*&*&0 \end{pmatrix}^{\mathbf{n}}, \quad
V_{4,\mathbf{n},4} \leftrightarrow \begin{pmatrix} 0&0&0&*\\ *&0&0&*\\ *&*&0&*\\ *&*&*&* \end{pmatrix}^{\mathbf{n}}.
\end{aligned}
$$

### 2.3.5 Intersections of irreducible components

We start by recording some isomorphisms between the Artin auxiliary varieties as a preparation for Theorem 2.3.29.

**Lemma 2.3.28.** *Denote by $(\mathbf{n}, e)$ ramification data and let $\mathbf{s}_i(\mathbf{n}) \in \mathbb{N}^e$ be the $\mathbf{s}$-tuple for some $i = 1, \ldots, e$ as defined in Section 2.3.1.*

*i) Let $i, k \in \{1, \ldots, e\}$ Then*

$$V_{i,\mathbf{n},k} \cong V_{1,\mathbf{m}(i,\mathbf{n},k),k}, \tag{2.72}$$

*where $\mathbf{m}(i, \mathbf{n}, k) \in \mathbb{Z}^k_{>0}$ is the $k$-th cut of $\mathbf{n}$ with respect to $\mathbf{s}_i$.*

*ii) For each $i, k \in \{1, \ldots, e\}$*

$$V_{i,\mathbf{n},k} \cong V_{1,\mathbf{n}[i-1],k}. \tag{2.73}$$

*iii) If $\mathbf{n} = \mathbf{d}_e := (d, \ldots, d) \in \mathbb{N}^e$, then for all $i, j, k \in \{1, \ldots, e\}$*

$$V_{i,\mathbf{n},k} \cong V_{j,\mathbf{n},k}. \tag{2.74}$$

*Proof.* Let $\mathbf{m} = \mathbf{m}(i, \mathbf{n}, k) \in \mathbb{N}^k$. Its $\mathbf{s}$-tuple $\mathbf{s}_1(\mathbf{m}) \in \mathbb{N}^k$ coincides with the first $k$ entries of $\mathbf{s}_i(\mathbf{n})$. Therefore the flag (2.20) associated with $\mathbf{s}_1(\mathbf{m})$ is isomorphic to the first $k$ subspaces of the flag (2.20) associated with $\mathbf{s}_i(\mathbf{n})$. Now, the isomorphism (i) follows directly from the construction of $V_{i,\mathbf{n},k}$.

The second isomorphism follows similarly from the observation that $\mathbf{s}_i(\mathbf{n}) = \mathbf{s}_1(\mathbf{n}[i-1])$. Using that $\mathbf{d}_e[i-1] = \mathbf{d}_e$, it follows from (ii) that $V_{i,\mathbf{d}_e,k} \cong V_{1,\mathbf{d}_k,k} \cong V_{j,\mathbf{d}_e,k}$. This shows the third statement. □

We come now to the main result of this section. This extends parts of the results from [67, Proposition 2.5], where it was shown that the intersection of all $e$ irreducible components of $\mathsf{BS}(\mathscr{A})_p$ is a product of projective spaces, see Corollary 2.3.31. It also extends Artin's description [5, Proposition 3.10] of the non-zero coordinates of points in the intersection, in the restricted setting where $\mathbf{n} = \mathbf{d}_e$.

**Theorem 2.3.29.** *Let $\mathscr{A}$ be a hereditary order over a smooth curve $C$, and $p \in C$ be a point such that $\mathscr{A}$ has ramification data $(\mathbf{n}, e)$ at $p$. Let $k \in \{1, \ldots, e\}$ and $(i_1 < \ldots < i_k)$ be a $k$-tuple of increasing integers with entries in $\{1, \ldots, e\}$. Then the intersection of the $k$ irreducible components $V_{i_1,\mathbf{n},e}, \ldots V_{i_k,\mathbf{n},e} \subset \mathsf{BS}(\mathscr{A})_p$ can be encoded by an indicator matrix.*

*The intersection is given by*

$$\bigcap_{j=1}^{k} V_{i_j,\mathbf{n},e} = V_{i_2,\mathbf{n},i_2-i_1} \times V_{i_3,\mathbf{n},i_3-i_2} \times \ldots \times V_{i_1,\mathbf{n},e-i_k+i_1}. \tag{2.75}$$

*Proof.* From the calculations of [5, §4], one knows that the irreducible components intersect transversely. Therefore, the intersection is a reduced closed subvariety of $\mathbb{P}(W_1)\times\ldots\times\mathbb{P}(W_e)$.

Consider the component $V_{i_j,\mathbf{n},e}$. Among all $k$ irreducible components $V_{i_1,\mathbf{n},e}, \ldots, V_{i_k,\mathbf{n},e}$, the linear subspaces for $V_{i_j,\mathbf{n},e}$ in the inclusion (2.26) are of the lowest dimension inside $\mathbb{P}(W_{i_j+1}), \mathbb{P}(W_{i_j+2}), \ldots, \mathbb{P}(W_{i_{j+1}})$. This imposes that the intersection lies in the linear subspaces

$$\mathbb{P}(W_{i_j+1,1}) \times \mathbb{P}(W_{i_j+2,2}) \times \ldots \times \mathbb{P}(W_{i_{j+1},e-i_j+i_{j+1}}). \tag{2.76}$$

The subspace inside $V_{i_{j+1},\mathbf{n},e}$ intersected with the product (2.76) is $V_{i_{j+1},\mathbf{n},i_{j+1}-i_j}$ as one can see from (2.30) (modulo $e$ for the last index).

It remains to check that the space $V_{i_{j+1},\mathbf{n},i_{j+1}-i_j}$ occurs as a proper factor on the right hand side of (2.75). This follows from the fact that $V_{i_{j-1},\mathbf{n},e}$ and $V_{i_{j+1},\mathbf{n},e}$ impose that there are no non-zero entries in the rows describing the coordinates of $\mathbb{P}(W_\ell)$ if $\ell \notin \{i_j+1, \ldots, i_{j+1}\}$ and the index of the second coordinate agrees with the second one of the coordinates describing non-zero entries of $V_{i_{j+1},\mathbf{n},i_{j+1}-i_j}$. Hence, we obtain the decomposition into factors of Artin auxiliary varieties.

Now, we can construct the indicator matrix $M^{\mathbf{n}}_{(i_1,\ldots,i_k)} = (m_{s,t})^{\mathbf{n}}_{1\leq i,j\leq n}$ of the intersection as follows. One sets $m_{s,t} = 0$ if there exists an indicator matrix $M^{\mathbf{n}}_{i_j,e}$ with a 0 at $(s,t)$ and $m_{s,t} = *$ otherwise. □

We refer to Section 2.3.6 for applications of the above proposition to low-dimensional examples.

Note that (2.75) is an equality in $\mathsf{BS}(\mathscr{A})_p$. We can use Lemma 2.3.28 to get an isomorphism to a product of Artin auxiliary varieties with strictly smaller ramification index $e$. We demonstrate this in the special case, where the ramification data is $\mathbf{n} = \mathbf{d}_e$ with $e > 0$.

**Corollary 2.3.30.** *Let $\mathscr{A}$ be a hereditary order of degree $n$ over a smooth curve $C$ with associated ramification data $(\mathbf{d}_e, e)$ at $p \in C$. Let $k \in \{1, \dots, e\}$. Then the intersection of $k$ distinct irreducible components is given by*

$$\bigcap_{j=1}^{k} V_{i_j,\mathbf{d}_e,e} \cong V_{1,\mathbf{d}_{i_2-i_1},i_2-i_1} \times V_{1,\mathbf{d}_{i_3-i_2},i_3-i_2} \times \ldots \times V_{1,\mathbf{d}_{e-i_k+i_1},e-i_k+i_1}. \tag{2.77}$$

*Proof.* Starting from Theorem 2.3.29, we obtain

$$\begin{aligned} \bigcap_{j=1}^{k} V_{i_j,\mathbf{d}_e,e} &= V_{i_2,\mathbf{d}_e,i_2-i_1} \times V_{i_3,\mathbf{d}_e,i_3-i_2} \times \ldots \times V_{i_1,\mathbf{d}_e,e-i_k+i_1} \\ &\cong V_{1,\mathbf{d}_{i_2-i_1},i_2-i_1} \times V_{1,\mathbf{d}_{i_3-i_2},i_3-i_2} \times \ldots \times V_{1,\mathbf{d}_{e-i_k+i_1},e-i_k+i_1}. \end{aligned} \tag{2.78}$$

The isomorphism follows from Lemma 2.3.28 as $(s_{i,i_j-i_{j-1}} - s_{i,i_j-i_{j-1}-1}, \dots, s_{i,1}) = \mathbf{d}_{i_j-i_{j-1}}$. □

A corollary of Theorem 2.3.29 is the result [67, Proposition 2.4] that the intersection of all irreducible components is a product of projective spaces.

**Corollary 2.3.31.** *With the notation from Theorem 2.3.29 we have*

$$\bigcap_{i=1}^{e} V_{i,\mathbf{n},e} = \mathbb{P}(W_{1,1}) \times \mathbb{P}(W_{2,1}) \times \ldots \times \mathbb{P}(W_{e,1}). \tag{2.79}$$

*Proof.* Apply Theorem 2.3.29 with $k = e$ and observe that $i_2 - i_1 = \ldots = e - i_e + i_1 = 1$. By Definition 2.3.6, we have $V_{i,\mathbf{n},1} = \mathbb{P}(W_{i,1})$. □

**Remark 2.3.32.** We see that the intersection of all irreducible components of $\mathsf{BS}(\mathscr{A})_p$ is the product of the linear subspaces spanned by the essential elements. As a result, the indicator matrix encoding the intersection (2.79) is given by

$$M\left(\bigcap_{i=1}^{e} V_{i,\mathbf{n},e}\right) = \begin{pmatrix} 0 & \cdots & 0 & * \\ * & & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & * & 0 \end{pmatrix}^{\mathbf{n}}. \tag{2.80}$$

### 2.3.6 Examples

**Unramified points.** Let $\mathscr{A}$ be a hereditary order of degree $n$. Assume that $p \in C$ is an unramified point, i.e. $\mathscr{A}_p \cong \mathrm{Mat}_n(\mathcal{O}_{C,p})$. In this case, the fiber $\mathsf{BS}(\mathscr{A})_p$ consists of a single irreducible component $V_{1,n,1} = \mathbb{P}^{n-1}$. Its indicator matrix is given by $M(V_{1,n,1}) = (*)^{(n)}$. Moreover, $\underline{\mathsf{BS}}(\mathscr{A}(p))$ is connected and hence $\underline{\mathsf{BS}}(\mathscr{A}(p)) = \mathsf{BS}(\mathscr{A})_p$

**Totally ramified orders.** In this paragraph we introduce the orders which are of interest for Proposition 2.4.4. They have already appeared in Examples 2.3.2, 2.3.4, 2.3.9 and 2.3.34.

**Definition 2.3.33.** A hereditary $\mathcal{O}_C$-order $\mathscr{A}$ is *totally ramified* at $p \in C$ if its ramification index $e$ is equal to the degree $n = \deg \mathscr{A}$.

If $\mathscr{A}$ is totally ramified at $p \in C$, then its ramification data at $p$ are necessarily $\mathbf{n} = \mathbf{1}_n \in \mathbb{N}^n$. Totally ramified orders play a special role among hereditary $\mathcal{O}_C$-orders. This is because at every point $p \in C$ the stalk of a hereditary order $\mathscr{A}_p$ Morita equivalent to a hereditary $\mathcal{O}_{C,p}$-order which is totally ramified at $p$, see [51, Proposition 7.2].

By Proposition 2.3.23 the fiber $\mathsf{BS}(\mathscr{A})_p$ of the Artin model over $p \in C$ consists of $n$ irreducible components of the form

$$V_{i,\mathbf{1}_n,n} = \{L \trianglelefteq A_p \mid e_{i+1,i} \in L\}, \tag{2.81}$$

where $i = 1, \ldots, n$ and the first index of $e_{i+1,i}$ is counted modulo $e = n$. By Lemma 2.3.28 the irreducible components are all isomorphic to each other, as long as $n$ is fixed.

In the following we present some low-dimensional examples of hereditary orders with totally ramified points.

**Example 2.3.34.** We start with a continuation of Example 2.3.8. If $n = 2$, then $\mathsf{BS}(\mathscr{A}) \to C$ is a conic bundle over the curve $C$ and $\mathscr{A}$ is totally ramified over finitely many points. The fibers were already described in [5, Examples 1.5(i)]. If $\mathscr{A}$ ramifies at $p \in C$ the fiber $\mathsf{BS}(\mathscr{A})_p$ consists of two lines. Using indicator matrices, they are encoded as

$$M(V_{1,\mathbf{1}_2,2}) = \left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)^{\mathbf{1}_2}, \quad M(V_{2,\mathbf{1}_2,2}) = \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)^{\mathbf{1}_2}. \tag{2.82}$$

Their intersection is given by a point, encoded as

$$\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)^{\mathbf{1}_2} \leftrightarrow V_{1,\mathbf{1}_2,2} \cap V_{2,\mathbf{1}_2,2} \cong V_{1,1,1} \times V_{1,1,1} \tag{2.83}$$

The point is given by the ideal $L = \langle e_{12}, e_{21} \rangle$. Note that in the particular case of $n = 2$, the Brauer–Severi scheme is connected, i.e. $\mathsf{BS}(\mathscr{A}) = \underline{\mathsf{BS}}(\mathscr{A})$.

The distinction between the Artin model $\mathsf{BS}(\mathscr{A})$ and the Brauer–Severi scheme $\underline{\mathsf{BS}}(\mathscr{A})$ becomes necessary for higher degree $n \geq 3$. For example if $\mathscr{A}$ is a hereditary order of degree 3 with ramification index $e = 3$, Artin [5, Exampe 1.3(ii)] showed that $\underline{\mathsf{BS}}(\mathscr{A})$ consists of more than one connected component. We now present the example in detail.

**Example 2.3.35.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree 3 and $p \in C$ such that $\mathscr{A}$ is totally ramified at $p \in C$. Then $\mathsf{BS}(\mathscr{A})_p$ consists of three irreducible components, each the blow-up of the plane in a point. The three indicator matrices of the three irreducible components are given by

$$M(V_{1,\mathbf{1}_3,3}) = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}^{\mathbf{1}_3}, \quad M(V_{2,\mathbf{1}_3,3}) = \begin{pmatrix} 0 & * & * \\ * & * & * \\ 0 & * & 0 \end{pmatrix}^{\mathbf{1}_3}, \quad M(V_{3,\mathbf{1}_3,3}) = \begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ * & * & * \end{pmatrix}^{\mathbf{1}_3}. \tag{2.84}$$

We can read off the intersections of the irreducible components from their indicator matrices:

$$\begin{aligned} \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}^{\mathbf{1}_3} &\leftrightarrow V_{1,\mathbf{1}_3,3} \cap V_{2,\mathbf{1}_3,3} = V_{2,\mathbf{1}_3,1} \times V_{1,\mathbf{1}_3,2} \cong V_{1,\mathbf{1}_3,2} \\ \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix}^{\mathbf{1}_3} &\leftrightarrow V_{1,\mathbf{1}_3,3} \cap V_{3,\mathbf{1}_3,3} = V_{3,\mathbf{1}_3,2} \times V_{1,\mathbf{1}_3,1} \cong V_{3,\mathbf{1}_3,2} \\ \begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix}^{\mathbf{1}_3} &\leftrightarrow V_{2,\mathbf{1}_3,3} \cap V_{3,\mathbf{1}_3,3} = V_{3,\mathbf{1}_3,1} \times V_{2,\mathbf{1}_3,2} \cong V_{2,\mathbf{1}_3,2}. \end{aligned} \tag{2.85}$$

The Artin auxiliary variety $V_{i,\mathbf{1}_3,2}$ is the exceptional divisor of $V_{i,\mathbf{1}_3,3} \cong \mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2)$. They define a ruling for the $\mathbb{P}^1$-bundle structure in the blow-up $V_{i+1,\mathbf{1}_3,3}$. Thus, we have three lines in $\mathsf{BS}(\mathscr{A})_p$, where the Artin auxiliary varieties intersect. The three lines intersect in a single point

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}^{\mathbf{1}_3} \leftrightarrow V_{1,\mathbf{1}_3,3} \cap V_{2,\mathbf{1}_3,3} \cap V_{3,\mathbf{1}_3,3} = \{L\}, \tag{2.86}$$

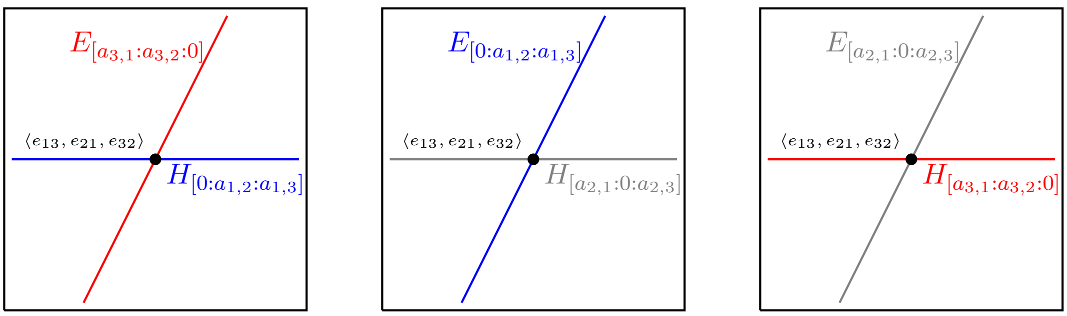
where $L = \langle e_{1,3}, e_{2,1}, e_{3,2} \rangle$.

We summarize the geometry in Figure 2.1 using coordinates $(a_{i,\alpha})_{\alpha=1,2,3}$ for $\mathbb{P}(W_i)$. The lines with the same color (in different irreducible components) are identified in $\mathsf{BS}(\mathscr{A}_p)$. Moreover we have added the three disconnected points of $L_1, L_2, L_3 \in \underline{\mathsf{BS}}(\mathscr{A}(p))$ which do not belong to $\mathsf{BS}(\mathscr{A})_p$.

$\bullet\, L_3 = \langle e_{13}, e_{21}, e_{23} \rangle$

$L_2 = \langle e_{12}, e_{13}, e_{32} \rangle \,\bullet$

$\bullet\, L_1 = \langle e_{21}, e_{31}, e_{32} \rangle$




$V_{1,\mathbf{1}_3,3}$ $\qquad$ $V_{2,\mathbf{1}_3,3}$ $\qquad$ $V_{3,\mathbf{1}_3,3}$

Figure 2.1: The fiber $\underline{\mathsf{BS}}(\mathscr{A}(p))$ of a totally ramified order at $p$ with ramification index $e = 3$.

**Remark 2.3.36.** More generally, assume that $\mathscr{A}$ is a hereditary order of degree $n \geq 3$ which is totally ramified at $p \in C$. Using Lemma 2.2.7, it is straightforward that $\underline{\mathsf{BS}}(\mathscr{A})$ is not connected. The ideal $L = \langle e_{2,1}, \dots e_{n,n-1}, e_{n,n-2} \rangle \trianglelefteq \mathscr{A}(p)$ has dimension $n$, but $e_{1,1}L = 0$. Hence, $L \notin \mathsf{BS}(\mathscr{A})_p$.

From what we have seen so far for totally ramified orders, one can ask whether the intersections of $k$ components $V_{i_1,\mathbf{1}_n,n} \cap \ldots \cap V_{i_k,\mathbf{1}_n,n}$ are not only birationally equivalent but actually isomorphic. Extending Example 2.3.9, the next example shows that this is not true.

**Example 2.3.37.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $\deg \mathscr{A} = 4$. Assume that $\mathscr{A}$ is totally ramified at $p \in C$. The four irreducible components of $\mathsf{BS}(\mathscr{A})_p$ are given by

$$V_{1,\mathbf{1}_4,4} \leftrightarrow \left(\begin{smallmatrix} * & * & * & * \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4}, \; V_{2,\mathbf{1}_4,4} \leftrightarrow \left(\begin{smallmatrix} 0 & * & * & * \\ * & * & * & * \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4}, \; V_{3,\mathbf{1}_4,4} \leftrightarrow \left(\begin{smallmatrix} 0 & 0 & * & * \\ * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4}, \; V_{4,\mathbf{1}_4,4} \leftrightarrow \left(\begin{smallmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & * \\ * & * & 0 & * \\ * & * & * & * \end{smallmatrix}\right)^{\mathbf{1}_4}.$$

Each of these varieties is the blow-up of $\mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^3)$ in a line intersecting the exceptional divisor in a point. Therefore, it is a Fano variety of type 3-30 in the notation of [23]. Using coordinates this becomes clear. We can write $V_{1,\mathbf{1}_4,4}$ as the blow-up of $V(a_{4,1}, a_{4,2})$ in

$$\mathrm{Bl}_{[0:0:0:1]}(\mathbb{P}^3_{[a_{1,1}:a_{1,2}:a_{1,3}:a_{1,4}]}) \subset \mathbb{P}^3_{[a_{1,1}:a_{1,2}:a_{1,3}:a_{1,4}]} \times \mathbb{P}^2_{[a_{4,1}:a_{4,2}:a_{4,3}]}. \tag{2.87}$$

The intersections can be encoded via indicator matrices as follows:

$$\begin{aligned}
\left(\begin{smallmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{1,\mathbf{1}_4,4} \cap V_{2,\mathbf{1}_4,4} \cong V_{1,1,1} \times V_{1,\mathbf{1}_3,3} \cong \mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2), \\
\left(\begin{smallmatrix} 0 & 0 & * & * \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{1,\mathbf{1}_4,4} \cap V_{3,\mathbf{1}_4,4} \cong V_{1,\mathbf{1}_2,2} \times V_{1,\mathbf{1}_2,2} \cong \mathbb{P}^1 \times \mathbb{P}^1, \\
\left(\begin{smallmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{1,\mathbf{1}_4,4} \cap V_{4,\mathbf{1}_4,4} \cong V_{1,\mathbf{1}_3,3} \times V_{1,1,1} \cong \mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2), \\
\left(\begin{smallmatrix} 0 & 0 & * & * \\ * & 0 & * & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{2,\mathbf{1}_4,4} \cap V_{3,\mathbf{1}_4,4} \cong V_{1,1,1} \times V_{1,\mathbf{1}_3,3} \cong \mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2), \\
\left(\begin{smallmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{2,\mathbf{1}_4,4} \cap V_{4,\mathbf{1}_4,4} \cong V_{1,\mathbf{1}_2,2} \times V_{1,\mathbf{1}_2,2} \cong \mathbb{P}^1 \times \mathbb{P}^1, \\
\left(\begin{smallmatrix} 0 & 0 & 0 & * \\ * & 0 & 0 & * \\ * & * & 0 & * \\ 0 & 0 & * & 0 \end{smallmatrix}\right)^{\mathbf{1}_4} &\leftrightarrow V_{3,\mathbf{1}_4,4} \cap V_{4,\mathbf{1}_4,4} \cong V_{1,\mathbf{1}_3,3} \times V_{1,1,1} \cong \mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2).
\end{aligned} \tag{2.88}$$

The triple intersections are each a line, intersecting in one common point given by the ideal $L = \langle e_{2,1}, e_{3,2}, e_{4,3}, e_{1,4} \rangle$.

**Low-dimensional examples.** In this paragraph we collect all the other possible local forms of hereditary orders of degree 3 and 4, which were not covered by the unramified and totally ramified case, respectively.

**Example 2.3.38.** Let $\mathscr{A}$ be a hereditary order of degree $\deg \mathscr{A} = 3$. Let $p \in C$ be a point such that $\mathscr{A}$ is ramified at $p$ with ramification index $e = 2$. Therefore the ramification data is $\mathbf{n} = \{(1,2),(2,1)\}$. We treat the case $\mathbf{n} = (1,2)$. Using the associated flags (2.57), the fiber $\mathsf{BS}(\mathscr{A})_p$ consists of two irreducible components

$$\begin{aligned}
\left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)^{(1,2)} &\leftrightarrow V_{1,(1,2),2} = \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{2,1}) \cong \mathbb{P}^2_{a_{1,1},a_{1,2},a_{1,3}} \\
\left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)^{(1,2)} &\leftrightarrow V_{2,(1,2),2} = \mathrm{Bl}_{\mathbb{P}(W_{2,1})}(\mathbb{P}(W_{2,2})) \subset \mathbb{P}^2_{a_{2,1},a_{2,2},a_{2,3}} \times \mathbb{P}^1_{a_{1,2},a_{1,3}}.
\end{aligned} \tag{2.89}$$

Hence, the second component is the blow-up of $\mathbb{P}^2_{a_{2,1},a_{2,2},a_{2,3}}$ in the point $[1:0:0]$. The two components intersect in $\mathbb{P}(W_{1,1}) \times \mathbb{P}(W_{2,1}) \cong \mathbb{P}^1_{a_{1,2},a_{1,3}}$. It is the line which is the exceptional divisor of $V_{2,(1,2),2}$.

Now, we come to orders of degree 4, which are ramified at $p \in C$ with ramification data $\mathbf{n} \neq \mathbf{1}_4$. The first example is due to [5, Example 1.5].

**Example 2.3.39.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree 4 ramified over $p \in C$ with ramification data $(\mathbf{n}, e) = ((2,2),2)$. In what follows, we use again the flag of subspaces (2.57) with respect to the ramification data $(2,2)$. The fiber

$$\mathsf{BS}(\mathscr{A})_p = V_{1,\mathbf{2}_2,2} \cup V_{2,\mathbf{2}_2,2} \tag{2.90}$$

consists of two irreducible components $V_{1,\mathbf{2}_2,2}, V_{2,\mathbf{2}_2,2}$. Both are given by the blow-up of $\mathbb{P}^3$ in a line. More precisely, we have

$$\begin{aligned} \left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)^{\mathbf{2}_2} &\leftrightarrow V_{1,\mathbf{2}_2,2} = \mathrm{Bl}_{\mathbb{P}(W_{1,1})}(\mathbb{P}(W_{1,2})) \subset \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{2,1}), \\ \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)^{\mathbf{2}_2} &\leftrightarrow V_{2,\mathbf{2}_2,2} = \mathrm{Bl}_{\mathbb{P}(W_{2,1})}(\mathbb{P}(W_{2,2})) \subset \mathbb{P}(W_{2,2}) \times \mathbb{P}(W_{1,1}). \end{aligned} \tag{2.91}$$

The two components intersect in

$$\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)^{\mathbf{2}_2} \leftrightarrow V_{1,\mathbf{2}_2,2} \cap V_{2,\mathbf{2}_2,2} = \mathbb{P}(W_{1,1}) \times \mathbb{P}(W_{2,1}) \cong \mathbb{P}^1 \times \mathbb{P}^1, \tag{2.92}$$

the exceptional divisor of both components.

The next two examples are hereditary orders, which cannot appear as the pullback of a maximal order along an étale covering, see [67, Proposition 1.7].

**Example 2.3.40.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree 4. Assume that $\mathscr{A}$ is ramified at $p \in C$. We have already treated the cases $\mathbf{n} = \mathbf{1}_4$ (Example 2.3.37), and $\mathbf{n} = \mathbf{2}_2$ (Example 2.3.39). The other ramification data are $\mathbf{n} \in \{(1,3),(3,1),(1,1,2),(1,2,1),(2,1,1)\}$.

i) *Ramification index 2.* Let $\mathbf{n} = (1,3)$. The fiber $\mathsf{BS}(\mathscr{A})_p$ consists of two irreducible components given by

$$\begin{aligned} \left(\begin{smallmatrix} * & * \\ * & 0 \end{smallmatrix}\right)^{(1,3)} &\leftrightarrow V_{1,(1,3),2} = \mathbb{P}(W_{1,2}) \times \mathbb{P}(W_{2,1}) \cong \mathbb{P}^3_{a_{1,1},\dots,a_{1,4}}, \\ \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix}\right)^{(1,3)} &\leftrightarrow V_{2,(1,3),2} = \mathrm{Bl}_{\mathbb{P}(W_{2,1})}(\mathbb{P}(W_{2,2})) \subset \mathbb{P}^3_{a_{2,1},\dots,a_{2,4}} \times \mathbb{P}^2_{a_{1,2},a_{1,3},a_{1,4}}. \end{aligned} \tag{2.93}$$

The second component is the blow-up of $\mathbb{P}^3_{a_{2,1},\dots,a_{2,4}}$ in the point $[1:0:0:0]$. The two components intersect in $\mathbb{P}(W_{1,1}) \times \mathbb{P}(W_{2,1})$, the exceptional divisor of $V_{2,(1,3),2}$. The case $\mathbf{n} = (3,1)$ is done similarly.

ii) *Ramification index 3.* Let $\mathbf{n} = (1,1,2)$. Then, the fiber $\mathsf{BS}(\mathscr{A})_p$ consists of three irreducible components. Using the coordinates $(a_{i,\alpha})_{\alpha=1,2,3,4}$ for $\mathbb{P}(W_i)$, the irreducible components are given by

$$\begin{aligned}
\left(\begin{smallmatrix} * & * & * \\ * & 0 & 0 \\ * & * & 0 \end{smallmatrix}\right)^{(1,1,2)} &\leftrightarrow V_{1,(1,1,2),3} \cong \mathrm{Bl}_{\mathbb{P}^1_{a_{1,3},a_{1,4}}}(\mathbb{P}^3_{a_{1,\alpha}}), \\
\left(\begin{smallmatrix} 0 & * & * \\ * & * & * \\ 0 & * & 0 \end{smallmatrix}\right)^{(1,1,2)} &\leftrightarrow V_{2,(1,1,2),3} \cong \mathrm{Bl}_{[1:0:0:0]}(\mathbb{P}^3_{a_{2,\alpha}}), \\
\left(\begin{smallmatrix} 0 & 0 & * \\ * & 0 & * \\ * & * & * \end{smallmatrix}\right)^{(1,1,2)} &\leftrightarrow V_{3,(1,1,2),3} = \mathrm{Bl}_{V_{3,\mathbf{n},2}}(\mathrm{Bl}_{[0:1:0:0]}(\mathbb{P}^3_{a_{3,\alpha}})).
\end{aligned} \tag{2.94}$$

The component $V_{3,(1,1,2),3}$ is the blow-up of $\mathrm{Bl}_{[0:1:0:0]}(\mathbb{P}^3_{a_{3,\alpha}})$ in a line intersecting the exceptional divisor in the point $[1:0:0] \in \mathbb{P}^2_{a_{2,1},a_{2,3},a_{2,4}}$.

Using Theorem 2.3.29 and Lemma 2.3.28 we find that

$$\left(\begin{smallmatrix} 0 & * & * \\ * & 0 & 0 \\ 0 & * & 0 \end{smallmatrix}\right)^{(1,1,2)} \leftrightarrow V_{1,\mathbf{n},3} \cap V_{2,\mathbf{n},3} \cong V_{1,1,1} \times V_{1,(1,2),2} \cong \mathbb{P}^2_{a_{1,2},a_{1,3},a_{1,4}}, \tag{2.95}$$

$$\left(\begin{smallmatrix} 0 & 0 & * \\ * & 0 & 0 \\ * & * & 0 \end{smallmatrix}\right)^{(1,1,2)} \leftrightarrow V_{1,\mathbf{n},3} \cap V_{3,\mathbf{n},3} \cong V_{1,\mathbf{1}_2,2} \times V_{1,2,1} \cong \mathbb{P}^1_{a_{3,1},a_{3,2}} \times \mathbb{P}^1_{a_{1,3},a_{1,4}}, \tag{2.96}$$

$$\left(\begin{smallmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & * & 0 \end{smallmatrix}\right)^{(1,1,2)} \leftrightarrow V_{2,\mathbf{n},3} \cap V_{3,\mathbf{n},3} \cong V_{1,1,1} \times V_{1,(2,1),2} \cong \mathrm{Bl}_{[1:0:0]}(\mathbb{P}^2_{a_{2,1},a_{2,3},a_{2,4}}). \tag{2.97}$$

The intersection (2.95) is the exceptional divisor of $V_{1,\mathbf{n},2}$. Note that the exceptional divisors of $V_{1,\mathbf{n},3}$ and $V_{1,\mathbf{n},3}$ are both isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, described by their intersection (2.96). The triple intersection is the projective line

$$\left(\begin{smallmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{smallmatrix}\right)^{(1,1,2)} \leftrightarrow V_{1,\mathbf{n},3} \cap V_{2,\mathbf{n},3} \cap V_{3,\mathbf{n},3} \cong V_{1,1,1} \times V_{1,1,1} \times V_{1,2,1} \cong \mathbb{P}^1_{a_{1,3},a_{1,4}}. \tag{2.98}$$

## 2.4 A motivic description of the Artin model

In this section we assume $\mathbf{k}$ to be algebraically closed of characteristic zero. We start by recalling the definition of the Grothendieck ring of varieties $\mathsf{K}_0(\mathsf{Var}_\mathbf{k})$ and some of its properties in Section 2.4.1. Then, we proceed by giving a recursive formula for the computation of the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_\mathbf{k})$. In the special case, where $\mathscr{A}$ has ramification data $\mathbf{n} = \mathbf{1}_n$ over each point $p \in \Delta_\mathscr{A}$, this leads to a closed formula for the class $[\mathsf{BS}(\mathscr{A})]$ in terms of $[C]$ and $\mathbb{L} = [\mathbb{A}^1]$, see Corollary 2.4.7.

In the final section, we give a motivic description of the bounded derived category $\mathsf{D}^\mathsf{b}(\mathsf{BS}(\mathscr{A}))$ of the Artin model $\mathsf{BS}(\mathscr{A})$. We will use this to present the conjectural shape of a semi-orthogonal decomposition of $\mathsf{D}^\mathsf{b}(\mathsf{BS}(\mathscr{A}))$.

### 2.4.1 The Grothendieck ring of varieties

The Grothendieck ring of varieties $\mathsf{K}_0(\mathsf{Var}_\mathbf{k})$ is generated, as an abelian group, by isomorphism classes $[X]$ of algebraic varieties over $\mathbf{k}$ subject to the cut-and-paste relation

$$[X] = [U] + [Z], \tag{2.99}$$

for every closed subvariety $Z \subseteq X$ with complement $U = X \setminus Z$. The multiplication is given by the fiber product

$$[X] \cdot [Y] := [X \times Y], \tag{2.100}$$

and has the unit $1 = [\mathrm{pt}]$. The *Lefschetz class* $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line.

**Properties of the Grothendieck ring of varieties.** Using the cut-and-paste relation we can decompose many algebro-geometric constructions, such as fibrations and blow-ups, in $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$. The following list of formulas will be used later on.

- *Projective space.* From the decomposition $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$ it follows inductively that

$$[\mathbb{P}^n] = \sum_{i=0}^{n} \mathbb{L}^i \quad \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}). \tag{2.101}$$

- *Vector bundle formula*, [100, Lemma 2.1]. Let $f\colon Y \to X$ be a morphism of varieties such that there is a (Zariski) open cover $X = \bigcup_{i=1}^n U_i$ such that $f^{-1}(U_i) \cong Z \times U_i$ for each $i \in \{1, \ldots, n\}$. Then

$$[Y] = [X] \cdot [Z] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}). \tag{2.102}$$

  In particular, for a locally free sheaf $\mathcal{V}$ of rank $r$ on a variety $X$, the class of the associated projective bundle $\mathbb{P}_X(\mathcal{V})$ is $[\mathbb{P}_X(\mathcal{V})] = [X] \cdot [\mathbb{P}^{r-1}] = [X] \cdot \sum_{i=0}^{r-1} \mathbb{L}^i$.

- *Blow-up formula*, [32, Lemma 3.5]. Let $X$ be a smooth connected variety and $Z \subset X$ a smooth closed connected subvariety of codimension $c = \operatorname{codim}_X(Z) \geq 2$. Then

$$[\mathrm{Bl}_Z(X)] = [X] + [Z] \cdot \sum_{i=1}^{c-1} \mathbb{L}^i. \tag{2.103}$$

Let $E \subset \mathrm{Bl}_Z(X)$ be the exceptional divisor of the blow-up $\mathrm{Bl}_Z(X)$ of $X$ in $Z$. Using only classes of smooth and proper varieties, the blow-up formula (2.103) can be rewritten as

$$[\mathrm{Bl}_Z(X)] - [E] = [X] - [Z]. \tag{2.104}$$

This formula for the blow-up can be used to give a different presentation due to Bittner [32, Theorem 3.1] if $\mathbf{k}$ has characteristic zero. The Bittner presentation states that the Grothendieck ring of varieties $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ is isomorphic to the ring generated by isomorphism classes of smooth and proper varieties over $\mathbf{k}$, subject to the relation (2.104) for each smooth closed subvariety $Z \subset X$. It is useful for motivic invariants of a variety which require the variety to be smooth or proper, cf. Section 2.4.4.

We will show now that we can decompose the class of a reducible variety into classes of intersections of its irreducible components. This will be useful in Proposition 2.4.4 for the computation of the class of the reducible fibers $\mathsf{BS}(\mathscr{A})_p$.

**Lemma 2.4.1.** *Let $X = \bigcup_{i=1}^n V_i$ be the decomposition of a variety $X$ into its irreducible components $V_1, \dots, V_n$. Then*

$$[X] = \sum_{1\le i\le n} [V_i] - \sum_{1\le i_1<i_2\le n} [V_{i_1} \cap V_{i_2}] + \sum_{1\le i_1<i_2<i_3\le n} [V_{i_1} \cap V_{i_2} \cap V_{i_3}] - \dots + (-1)^{n+1}[V_1 \cap \dots \cap V_n]. \quad (2.105)$$

*Proof.* The formula is proven by induction on the number $n$ of irreducible components. For $i = 1$, the statement is clear. Assume that the formula holds for every union of $n-1$ subvarieties and let $X = \bigcup_{i=1}^n V_i$. We can consider $X \setminus V_n = \bigcup_{i=1}^{n-1} W_i$, where $W_i = V_i \setminus (V_i \cap V_n)$. Then by the induction hypothesis

$$[X] - [V_n] = \sum_{i=1}^{n-1} [W_i] - \sum_{1\le i_1<i_2\le n-1} [W_{i_1} \cap W_{i_2}] \pm \dots + (-1)^n [W_1 \cap \dots \cap W_{n-1}]. \quad (2.106)$$

For the $k$-th summand on the right-hand side one has

$$\begin{aligned} \sum_{1\le i_1<\dots<i_k\le n-1} [W_{i_1} \cap \dots \cap W_{i_k}] &= \sum_{1\le i_1<\dots<i_k\le n-1} [(V_{i_1} \cap \dots \cap V_{i_k}) \setminus (V_{i_1} \cap \dots \cap V_{i_k} \cap V_n)] \\ &= \sum_{1\le i_1<\dots<i_k\le n-1} [V_{i_1} \cap \dots \cap V_{i_k}] - \sum_{1\le i_1<\dots<i_k\le n-1} [V_{i_1} \cap \dots \cap V_{i_k} \cap V_n] \end{aligned} \quad (2.107)$$

The second part on the right-hand side contributes to the $(k+1)$-st summand yielding all the terms in the formula with $[\dots \cap V_n]$ in the intersection. After rearranging the summands this leads to the formula for $[X]$. □

**Motivic measures.** Assume that $\mathbf{k} = \mathbb{C}$. For a smooth proper algebraic variety $X$, we can compute the singular cohomology $\mathrm{H}^i(X, \mathbb{C})$ and the Hodge numbers $\mathrm{h}^{p,q}(X) = \dim_{\mathbb{C}} \mathrm{H}^q(X, \Omega_X^p)$. Using the Bittner presentation for $\mathsf{K}_0(\mathsf{Var}_{\mathbb{C}})$ we can extend these invariants to a function on $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ as follows.

Let $S$ be a ring. A *motivic measure* is a ring homomorphism

$$\mu\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to S. \quad (2.108)$$

Now, as explained in [108, §4.1],

- the Poincaré polynomial $\mathrm{P}([X]) = \sum_{i\in\mathbb{N}_0} \dim_{\mathbb{C}} \mathrm{H}^i(X, \mathbb{C}) t^i \in \mathbb{Z}[t]$, and
- the Hodge polynomial $\mathrm{h}([X]) = \sum_{p,q\in\mathbb{N}_0} \mathrm{h}^{p,q}(X) u^p v^q \in \mathbb{Z}[u, v]$

are motivic measures. Therefore, we can calculate the Hodge diamond from the presentation of $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbb{C}})$. The computation of $[\mathsf{BS}(\mathscr{A})]$ is the content of the next section.

### 2.4.2 A motivic formula for the Artin model

We start with a recursive method for computing the class $[\mathsf{BS}(\mathscr{A})] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ of the Artin model of an arbitrary hereditary $\mathcal{O}_C$-order $\mathscr{A}$. The class $[\mathsf{BS}(\mathscr{A})]$ can be represented as a polynomial in $\mathbb{L}$ and $[C]$. For this, we need a presentation of the Artin auxiliary variety $V_{i,\mathbf{n},e}$ from Definition 2.3.6.

**Proposition 2.4.2.** *Let $n, e \in \mathbb{Z}_{>0}$ and $\mathbf{n} \in \mathbb{N}^e$. Denote by $\mathbf{s}_i = (s_{i,1}, \dots, s_{i,e})$ the associated $\mathbf{s}$-tuple from Section 2.3.1. Then, with the notation from Section 2.3.2, for each $i \in \{1, \dots, n\}$ the class of the Artin auxiliary variety $V_{i,\mathbf{n},e}$ is given by*

$$\begin{aligned}
[V_{i,\mathbf{n},e}] &= [\mathbb{P}^{n-1}] + \sum_{k=1}^{e-1} [V_{i,\mathbf{n},k}] \cdot (\mathbb{L}^{n-s_{i,k}-1} + \dots + \mathbb{L}) \\
&= [\mathbb{P}^{n-1}] + \sum_{k=1}^{e-1} [V_{1,\mathbf{m}(i,\mathbf{n},k),k}] \cdot (\mathbb{L}^{n-s_{i,k}-1} + \dots + \mathbb{L}).
\end{aligned} \tag{2.109}$$

The recursive nature of the formula relies on the presentation of $[V_{i,\mathbf{n},e}]$ by classes of Artin auxiliary varieties with smaller ramification index $k < e$.

*Proof.* From Definition 2.3.6 and the blow-up formula (2.103) we see that

$$\begin{aligned}
[V_{i,\mathbf{n},e}] &= \left[\mathrm{Bl}_{V_{i,\mathbf{n},e-1}} \left(\mathrm{Bl}_{V_{i,\mathbf{n},e-2}} \left(\dots \mathrm{Bl}_{V_{i,\mathbf{n},2}}(\mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}^{n-1}))\dots\right)\right)\right] \\
&= [\mathbb{P}^{n-1}] + \sum_{k=1}^{e-1} [V_{i,\mathbf{n},k}] \cdot \left(\mathbb{L}^{n-s_{i,k}-1} + \dots + \mathbb{L}\right).
\end{aligned} \tag{2.110}$$

Note that $V_{i,\mathbf{n},k}$ has codimension $n - s_{i,k}$ in $\mathrm{Bl}_{V_{i,\mathbf{n},k-1}}\left(\dots \mathrm{Bl}_{V_{i,\mathbf{n},2}}(\mathrm{Bl}_{V_{i,\mathbf{n},1}}(\mathbb{P}^{n-1}))\dots\right)$. The remaining reduction follows from the isomorphism $V_{i,\mathbf{n},k} \cong V_{1,\mathbf{m}(i,\mathbf{n},k),k}$ as established in Lemma 2.3.28. ☐

The next paragraph is devoted to a closed formula for $V_{i,\mathbf{1}_n,n}$ for points $p \in \Delta_{\mathscr{A}}$, where $\mathscr{A}$ is totally ramified. We also present low-degree examples in Section 2.4.3. Here, we continue with the discussion of Examples 2.3.3, 2.3.5, 2.3.14, 2.3.20 and 2.3.27.

**Example 2.4.3.** Let $\mathbf{n} = (3, 1, 4, 2)$ and $\mathbf{s}_3 = (1, 4, 6, 10)$. Then Proposition 2.4.2 reduces the computation of $[V_{3,\mathbf{n},4}] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ to

$$[V_{3,\mathbf{n},4}] = [\mathbb{P}^9] + [V_{1,1,1}] \cdot \sum_{j=1}^{8} \mathbb{L}^j + [V_{1,(3,1),2}] \cdot \sum_{j=1}^{5} \mathbb{L}^j + [V_{1,(2,3,1),3}] \cdot \sum_{j=1}^{3} \mathbb{L}^j. \tag{2.111}$$

Recursively, we can compute all four classes of the auxiliary varieties. They are given by

$$\begin{aligned}
[V_{1,\mathbf{n},4}] &= \mathbb{L}^9 + 4 \cdot \mathbb{L}^8 + 9 \cdot \mathbb{L}^7 + 14 \cdot \mathbb{L}^6 + 17 \cdot \mathbb{L}^5 + 17 \cdot \mathbb{L}^4 + 14 \cdot \mathbb{L}^3 + 9 \cdot \mathbb{L}^2 + 4 \cdot \mathbb{L} + 1, \\
[V_{2,\mathbf{n},4}] &= \mathbb{L}^9 + 3 \cdot \mathbb{L}^8 + 6 \cdot \mathbb{L}^7 + 9 \cdot \mathbb{L}^6 + 11 \cdot \mathbb{L}^5 + 11 \cdot \mathbb{L}^4 + 9 \cdot \mathbb{L}^3 + 6 \cdot \mathbb{L}^2 + 3 \cdot \mathbb{L} + 1, \\
[V_{3,\mathbf{n},4}] &= \mathbb{L}^9 + 4 \cdot \mathbb{L}^8 + 9 \cdot \mathbb{L}^7 + 15 \cdot \mathbb{L}^6 + 19 \cdot \mathbb{L}^5 + 19 \cdot \mathbb{L}^4 + 15 \cdot \mathbb{L}^3 + 9 \cdot \mathbb{L}^2 + 4 \cdot \mathbb{L} + 1, \\
[V_{4,\mathbf{n},4}] &= \mathbb{L}^9 + 4 \cdot \mathbb{L}^8 + 8 \cdot \mathbb{L}^7 + 12 \cdot \mathbb{L}^6 + 15 \cdot \mathbb{L}^5 + 15 \cdot \mathbb{L}^4 + 12 \cdot \mathbb{L}^3 + 8 \cdot \mathbb{L}^2 + 4 \cdot \mathbb{L} + 1.
\end{aligned}$$

Applying Theorem 2.3.29 and Proposition 2.4.2, we can also compute the class of intersections of the irreducible components. Using Lemma 2.4.1, we obtain a formula for $[\mathsf{BS}(\mathscr{A})_p] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ for a hereditary order with ramification data $\mathbf{n} = (3,1,4,2)$ at $p \in C$. It is given by

$$[\mathsf{BS}(\mathscr{A})_p] = 4\cdot\mathbb{L}^9+9\cdot\mathbb{L}^8+15\cdot\mathbb{L}^7+20\cdot\mathbb{L}^6+22\cdot\mathbb{L}^5+20\cdot\mathbb{L}^4+15\cdot\mathbb{L}^3+9\cdot\mathbb{L}^2+4\cdot\mathbb{L}+1. \quad (2.112)$$

As a conclusion, let $\mathscr{A}$ be for a hereditary $\mathcal{O}_C$-order with a single ramified point $p \in C$ such that the ramification data at $p$ is $\mathbf{n} = (3,1,4,2)$. The class of its Artin model is given by

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})] &= ([C]-1)\cdot[\mathbb{P}^9] + [\mathsf{BS}(\mathscr{A})_p] \\ &= [C]\cdot[\mathbb{P}^9] + 3\mathbb{L}^9 + 8\mathbb{L}^8 + 14\mathbb{L}^7 + 19\mathbb{L}^6 + 21\mathbb{L}^5 + 19\mathbb{L}^4 + 14\mathbb{L}^3 + 8\mathbb{L}^2 + 3\mathbb{L}.\end{aligned}$$

**Orders with totally ramified points.** Recall that a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ of degree $n$ is totally ramified at $p \in \Delta_{\mathscr{A}}$ if $e = n$. Using Corollary 2.3.30, we show below how to write the class $[\mathsf{BS}(\mathscr{A})_p] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ as a polynomial in $\mathbb{L}$.

**Proposition 2.4.4.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ such that $\mathscr{A}$ is totally ramified at $p \in \Delta_{\mathscr{A}}$. Then*

$$[\mathsf{BS}(\mathscr{A})_p] = \sum_{i=0}^{n-1} \binom{n}{i} \mathbb{L}^i. \quad (2.113)$$

We split the proof into two lemmas, the first one being the computation of $[V_{i,\mathbf{1}_n,n}] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ of the irreducible components of $\mathsf{BS}(\mathscr{A})_p$.

**Lemma 2.4.5.** *Let $p \in \Delta_{\mathscr{A}}$ such that $\mathscr{A}$ is totally ramified at $p$. Then for every irreducible component $V_{i,\mathbf{1}_n,n} \subset \mathsf{BS}(\mathscr{A})_p$ one has*

$$[V_{i,\mathbf{1}_n,n}] = \sum_{j=0}^{n-1} \binom{n-1}{j} \mathbb{L}^j. \quad (2.114)$$

*Proof.* Since $V_{i,\mathbf{1}_n,n} \cong V_{j,\mathbf{1}_n,n}$ for all $1 \le i,j \le n$, it suffices to prove the statement for $V_{1,\mathbf{1}_n,n}$ which is done by induction on $n \ge 1$. The base cases for $V_{1,1,1} = \mathrm{pt}$ and $V_{1,\mathbf{1}_2,2} = \mathbb{P}^1$ are straightforward.

Assume that the claim is holds for all $V_{1,\mathbf{1}_m,m}$, $1 \le m \le n$ for a fixed $n \ge 2$. Then, by Proposition 2.4.2, the class of the irreducible component $V_{1,\mathbf{1}_{n+1},n+1}$ of a totally ramified order of degree $n+1$ over a point $p \in \Delta_{\mathscr{A}}$ can be written as

$$[V_{1,\mathbf{1}_{n+1},n+1}] = [\mathbb{P}^n] + \sum_{k=1}^{n} \left( [V_{1,\mathbf{1}_k,k}] \cdot \sum_{j=1}^{n-k} \mathbb{L}^j \right) = [\mathbb{P}^n] + \sum_{j=1}^{n-1} \mathbb{L}^j \cdot \sum_{k=1}^{n-j} [V_{1,\mathbf{1}_k,k}]. \quad (2.115)$$

Applying the induction hypothesis for $\ell \le n-1$, we obtain

$$\sum_{k=1}^{n-j} [V_{1,\mathbf{1}_k,k}] = \sum_{k=1}^{n-j} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \mathbb{L}^\ell = \sum_{\ell=0}^{n-j-1} \left( \sum_{k=\ell}^{n-j-1} \binom{k}{\ell} \right) \mathbb{L}^\ell = \sum_{\ell=0}^{n-j-1} \binom{n-j}{\ell+1} \mathbb{L}^\ell. \quad (2.116)$$

For the last equality we used the hockey-stick identity $\sum_{k=\alpha}^{\beta}\binom{k}{\alpha}=\binom{\beta+1}{\alpha+1}$ for $\beta=n-j-1$ and $\alpha=\ell$. After plugging this into the formula for $[V_{1,\mathbf{1}_{n+1},n+1}]$ and rearranging the summands we obtain

$$\begin{aligned}
[V_{1,\mathbf{1}_{n+1},n+1}] &= [\mathbb{P}^n]+\sum_{j=1}^{n-1}\mathbb{L}^j\sum_{\ell=0}^{n-j-1}\binom{n-j}{\ell+1}\mathbb{L}^\ell \\
&= 1+\sum_{j=1}^{n}\mathbb{L}^j\left(\binom{n-j+1}{1}+\sum_{\ell=1}^{n-j-1}\binom{n-j}{\ell+1}\mathbb{L}^\ell\right) \qquad (2.117)\\
&= 1+\sum_{k=1}^{n}\left(\binom{n-k+1}{1}+\sum_{\ell=1}^{k-1}\binom{n+\ell-k}{\ell+1}\right)\mathbb{L}^k.
\end{aligned}$$

By repeatedly applying the Pascal identity $\binom{\alpha-1}{\beta}+\binom{\alpha-1}{\beta-1}=\binom{\alpha}{\beta}$ starting with $\alpha=n-k+2$, $\beta=2$, the coefficient in front of $\mathbb{L}^k$, $0\le k\le n$, results in $\binom{n}{k}$. This proves the statement. □

Even though Example 2.3.37 shows that the intersections of $k$ irreducible components of the fiber $\mathsf{BS}(\mathscr{A})_p$ need not be isomorphic, we show that the classes $[V_{i_1,\mathbf{1}_n,n}\cap\ldots\cap V_{i_k,\mathbf{1}_n,n}]$ are the same in $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ for every $k$-tuple $(i_1,\ldots,i_k)$.

**Lemma 2.4.6.** *Let $p\in\Delta_{\mathscr{A}}$ such that $\mathscr{A}$ is totally ramified at $p$. For each (ordered) subset $\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$ we have*

$$[V_{i_1,\mathbf{1}_n,n}\cap\ldots\cap V_{i_k,\mathbf{1}_n,n}]=[V_{1,\mathbf{1}_{n-k+1},n-k+1}]. \qquad (2.118)$$

*Proof.* By Corollary 2.3.30 the intersection of $k$ components is given by

$$V_{i_1,\mathbf{1}_n,n}\cap\ldots\cap V_{i_k,\mathbf{1}_n,n}\cong V_{1,\mathbf{1}_{i_2-i_1},i_2-i_1}\times V_{1,\mathbf{1}_{i_3-i_2},i_3-i_2}\times\ldots\times V_{1,\mathbf{1}_{n-i_k+i_1},n-i_k+i_1}. \qquad (2.119)$$

If $\{i_1,\ldots,i_k\}=\{1,\ldots,k\}$ we have

$$V_{1,\mathbf{1}_n,n}\cap\ldots\cap V_{k,\mathbf{1}_n,n}\cong V_{1,1,1}\times\ldots\times V_{1,1,1}\times V_{1,\mathbf{1}_{n-k+1},n-k+1}\cong V_{1,\mathbf{1}_{n-k+1},n-k+1}, \qquad (2.120)$$

as the auxiliary varieties $V_{1,1,1}$ are just points. By applying Lemma 2.4.5 to (2.119) and (2.120), we are left with showing

$$\prod_{\ell=1}^{k}\sum_{j_\ell=0}^{n_\ell-1}\binom{n_\ell-1}{j_\ell}\mathbb{L}^{j_\ell}=\sum_{i=0}^{n-k}\binom{n-k}{i}\mathbb{L}^i, \qquad (2.121)$$

where $n_\ell=i_{\ell+1}-i_\ell$ for $\ell=1,\ldots,k-1$ and $n_k=n-i_k+i_1$. Using the Vandermonde identity on the left hand side for the sum over all coefficients of $\mathbb{L}^i$, we obtain

$$\sum_{j_1+\ldots+j_k=i}\binom{n_1-1}{j_1}\cdot\ldots\cdot\binom{n_k-1}{j_k}=\binom{n_1-1+\ldots+n_k-1}{i}=\binom{n-k}{i}. \qquad (2.122)$$

This proves the equality (2.121) and hence the lemma. □

The remaining part for the proof of Proposition 2.4.4 is a calculation

*Proof of Proposition 2.4.4.* Collecting the two results Lemma 2.4.5 and Lemma 2.4.6, we arrive at

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})_p] &\overset{2.4.1,2.4.6}{=} \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i-1}[V_{i,\mathbf{1}_n,n}] \\ &\overset{2.4.5}{=} \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i-1}\sum_{j=0}^{i-1}\binom{i-1}{j}\mathbb{L}^j \\ &= \sum_{k=0}^{n-1}\left(\sum_{j=k}^{n-1}(-1)^{n-j-1}\binom{n}{j}\binom{j}{k}\right)\mathbb{L}^k. \end{aligned} \tag{2.123}$$

For the coefficient of $\mathbb{L}^k$ we observe that $\binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k}$. Thus we obtain

$$\begin{aligned}\sum_{j=k}^{n-1}(-1)^{n-j-1}\binom{n}{j}\binom{j}{k} &= (-1)^{n-k-1}\binom{n}{k}\sum_{j=0}^{n-k-1}(-1)^j\binom{n-k}{j} \\ &= (-1)^{n-k-1}\binom{n}{k}\cdot(-1)^{n-k-1} = \binom{n}{k}. \end{aligned} \tag{2.124}$$

Plugging this into (2.123), we arrive at the claimed equality. □

**Corollary 2.4.7.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ which is totally ramified over $r$ points and unramified everywhere else. Denote by $f\colon \mathsf{BS}(\mathscr{A}) \to C$ the Artin model of $\mathscr{A}$. Then*

$$[\mathsf{BS}(\mathscr{A})] = \sum_{i=0}^{n-1}\left([C] + r\cdot\binom{n}{i} - r\right)\mathbb{L}^i = [C]\cdot[\mathbb{P}^{n-1}] + \sum_{i=1}^{n-1} r\cdot\left(\binom{n}{i} - 1\right)\mathbb{L}^i. \tag{2.125}$$

*Proof.* Outside the ramification locus $\Delta_{\mathscr{A}}$, the Artin model $\mathsf{BS}(\mathscr{A})$ is a $\mathbb{P}^{n-1}$-bundle. By the vector bundle formula (2.102), we have $[\mathsf{BS}(\mathscr{A})] = r\cdot[\mathsf{BS}(\mathscr{A})_p] + ([C]-r)\cdot[\mathbb{P}^{n-1}]$. Now the corollary follows from the presentation (2.101) of projective space and Proposition 2.4.4. □

For the computation of the Hodge polynomial $\mathrm{h}(\mathsf{BS}(\mathscr{A}))$, we restrict ourselves to a smooth projective curve $C$ of genus $g$. Then $\mathrm{h}(C) = 1 + g\cdot u + g\cdot v + uv \in \mathbb{Z}[u,v]$. As an application we can compute the Hodge polynomial to $\mathsf{BS}(\mathscr{A})$. Note that $\mathsf{BS}(\mathscr{A})$ is a smooth projective variety by [5, Theorem 1.4].

**Corollary 2.4.8.** *Let $\mathscr{A}$ be a hereditary order of degree $n$ over a smooth projective curve $C$ of genus $g$. If $\mathscr{A}$ is totally ramified over $r$ points and unramified everywhere else, the Hodge polynomial of $\mathsf{BS}(\mathscr{A})$ is*

$$\mathrm{h}(\mathsf{BS}(\mathscr{A})) = \sum_{i=0}^{n} u^i v^i + \sum_{i=1}^{n-1}\left(1 + r\cdot\binom{n}{i} - r\right)u^i v^i + \sum_{i=0}^{n-1} g(u^{i+1}v^i + u^i v^{i+1}). \tag{2.126}$$

### 2.4.3 Examples

First, we apply Corollary 2.4.7 to low-dimensional examples, then we draw conclusions for the presentation of $[\mathsf{BS}(\mathscr{A})_p] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ from Section 2.3.6.

**Conic bundles over a curve.** If $f\colon \mathsf{BS}(\mathscr{A}) \to C$ is a conic bundle associated to a hereditary order of degree 2, the class of $\mathsf{BS}(\mathscr{A})$ is given by

$$[\mathsf{BS}(\mathscr{A})] = [C] \cdot [\mathbb{P}^1] + r\mathbb{L}, \tag{2.127}$$

where $r = \#\Delta_{\mathscr{A}}$. We will explain in Section 2.4.4 how this relates to the quadric fibration formula [94, Theorem 4.2] for $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$.

**Hereditary orders of degree 3.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree 3 with ramification locus $\Delta_{\mathscr{A}}$ of cardinality $r$. Assume that $\mathscr{A}$ is totally ramified at each $p \in \Delta_{\mathscr{A}}$. Applying Corollary 2.4.7 to $\mathscr{A}$ leads to

$$[\mathsf{BS}(\mathscr{A})] = [C] \cdot [\mathbb{P}^2] + 2r \cdot \mathbb{L} + 2r \cdot \mathbb{L}^2. \tag{2.128}$$

**Hereditary orders of degree 3, 4 and 5.** In the next lemma we collect some results for low-dimensional Artin models which follow from the explicit descriptions in Section 2.3.6 and Proposition 2.4.4.

**Lemma 2.4.9.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order and $p \in C$ a point with ramification index $e > 1$.*

*i) If $\deg(\mathscr{A}) = 2$ and $e = 2$, then $[\mathsf{BS}(\mathscr{A})_p] = 2\mathbb{L} + 1$.*

*ii) If $\deg(\mathscr{A}) = 3$ and*

*a) $e = 2$, then $[\mathsf{BS}(\mathscr{A})_p] = 2\mathbb{L}^2 + 2\mathbb{L} + 1$;*

*b) $e = 3$, then $[\mathsf{BS}(\mathscr{A})_p] = 3\mathbb{L}^2 + 3\mathbb{L} + 1$.*

*iii) If $\deg(\mathscr{A}) = 4$, and*

*a) $e = 2$, then* $[\mathsf{BS}(\mathscr{A})_p] = \begin{cases} 2\mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} + 1 & \text{if } \mathbf{n} = (1,3), \\ 2\mathbb{L}^3 + 3\mathbb{L}^2 + 2\mathbb{L} + 1 & \text{if } \mathbf{n} = \mathbf{2}_2. \end{cases}$

*b) $e = 3$, then $[\mathsf{BS}(\mathscr{A})_p] = 3\mathbb{L}^3 + 4\mathbb{L}^2 + 3\mathbb{L} + 1$;*

*c) $e = 4$, then $[\mathsf{BS}(\mathscr{A})_p] = 4\mathbb{L}^3 + 6\mathbb{L}^2 + 4\mathbb{L} + 1$.*

*iv) If $\deg(\mathscr{A}) = 5$ and $e = 5$, then $[\mathsf{BS}(\mathscr{A})_p] = 5\mathbb{L}^4 + 10\mathbb{L}^3 + 10\mathbb{L}^2 + 5\mathbb{L} + 1$.*

*Proof.* The parts (i), (ii.b), (iii.c) and (iv) follow from Proposition 2.4.4. Each of the remaining formulas is proven using Lemma 2.4.1 together with the examples studied in Section 2.3.6.

For (ii.a) we refer to Example 2.3.38 yielding

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})_p] &= [V_{1,(1,2),2}] + [V_{2,(1,2),2}] - [V_{1,(1,2),2} \cap V_{2,(1,2),2}] = [\mathbb{P}^2] + [\mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2)] - [\mathbb{P}^1] \\ &= 2\mathbb{L}^2 + 2\mathbb{L} + 1.\end{aligned} \tag{2.129}$$

For (iii) we come back to Examples 2.3.39 and 2.3.40. Assume first that $e = 2$. If $\mathbf{n} = \mathbf{2}_2$, then we use the decomposition of Example 2.3.39 to calculate

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})_p] &= [V_{1,\mathbf{2}_2,2}] + [V_{2,\mathbf{2}_2,2}] - [V_{1,\mathbf{2}_2,2} \cap V_{2,\mathbf{2}_2,2}] = [\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^3)] + [\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^3)] - [\mathbb{P}^1 \times \mathbb{P}^1] \\ &= 2\mathbb{L}^3 + 3\mathbb{L}^2 + 2\mathbb{L} + 1.\end{aligned}$$

If instead $\mathbf{n} = (1, 3)$, it follows from Example 2.3.40(i) that

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})_p] &= [V_{1,(1,3),2}] + [V_{2,(1,3),2}] - [V_{1,(1,3),2} \cap V_{2,(1,3),2}] = [\mathbb{P}^3] + [\mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^3)] - [\mathbb{P}^2] \\ &= 2\mathbb{L}^3 + 2\mathbb{L}^2 + 2\mathbb{L} + 1.\end{aligned} \tag{2.130}$$

If $e = 3$, set $\mathbf{n} = (1, 1, 2)$ and consider the decomposition of Example 2.3.40(ii). Then

$$\begin{aligned}[\mathsf{BS}(\mathscr{A})_p] &= [V_{1,\mathbf{n},3}] + [V_{2,\mathbf{n},3}] + [V_{3,\mathbf{n},3}] - [V_{1,\mathbf{n},3} \cap V_{2,\mathbf{n},3}] - [V_{1,\mathbf{n},3} \cap V_{3,\mathbf{n},3}] \\ &\quad - [V_{2,\mathbf{n},3} \cap V_{3,\mathbf{n},3}] + [V_{1,\mathbf{n},3} \cap V_{2,\mathbf{n},3} \cap V_{3,\mathbf{n},3}] \\ &= 2 \cdot [\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^3)] + [\mathrm{Bl}_{\mathbb{P}^1}(\mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^3))] - [\mathbb{P}^2] - [\mathbb{P}^1 \times \mathbb{P}^1] - [\mathrm{Bl}_{\mathrm{pt}}(\mathbb{P}^2)] + [\mathbb{P}^1] \\ &= 3\mathbb{L}^3 + 4\mathbb{L}^2 + 3\mathbb{L} + 1.\end{aligned}$$

This proves all the formulas from the lemma. □

### 2.4.4 Predictions for the derived category

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Recall that a motivic measure (2.108) is a ring homomorphism with $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ as a domain. We present another motivic measure from [38, §7] which allows us to make predictions about $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ using the results of Section 2.4.2.

To construct the motivic measure, we define the Grothendieck ring $\mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}})$ of smooth and proper dg categories. As an abelian group it is generated by quasi-equivalence classes $[\mathsf{D}]$ of pretriangulated smooth and proper $\mathbf{k}$-linear dg categories. The relations are given by

$$[\mathsf{D}] = [\mathsf{A}] + [\mathsf{B}] \quad \in \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}}) \tag{2.131}$$

for each semiorthogonal decomposition $\mathsf{Ho}(\mathsf{D}) = \langle \mathsf{Ho}(\mathsf{A}), \mathsf{Ho}(\mathsf{B}) \rangle$ of the homotopy categories. By [38, Corollary 4.7], $\mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}})$ is a commutative ring. The multiplication is given by $[\mathsf{A}_1] \cdot [\mathsf{A}_2] := [\mathrm{Perf}(\mathsf{A}_1 \otimes_{\mathbf{k}} \mathsf{A}_2)]$, the class of perfect dg modules over the dg tensor product $\mathsf{A}_1 \otimes_{\mathbf{k}} \mathsf{A}_2$. The unit is given by $[\mathsf{mod}^{\mathbf{dg}}(\mathbf{k})]$, the class of the dg category of bounded cochain complexes of finite-dimensional $\mathbf{k}$-vector spaces.

Using the Bittner presentation, we obtain a motivic measure

$$\mathrm{d}\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}}), \tag{2.132}$$

which sends $[X]$ to the class of the unique dg enhancement of $\mathsf{D}^{\mathsf{b}}(X)$. We refer to [47, Theorem 4.9] for a discussion of (the uniqueness of) dg enhancements in this context. We will denote the image by $[\mathsf{D}^{\mathsf{b}}(X)] = \mathrm{d}([X])$. Since $\mathrm{d}(\mathbb{L}) = 1$, we have that the ideal $(\mathbb{L} - 1)$ lies in $\mathrm{Ker}(\mathrm{d})$.

**Relation between the motivic measures.** In Section 2.4.1 we have seen that the Hodge polynomial h and the Poincaré polynomial P are motivic measures $\mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$.

Given a smooth and proper $\mathbf{k}$-linear dg category $\mathsf{D}$ we can define its Hochschild homology $\mathrm{HH}_\bullet(\mathsf{D}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{HH}_i(\mathsf{D})$. If we denote $\mathrm{hh}_i(\mathsf{D}) := \dim_{\mathbf{k}} \mathrm{HH}_i(\mathsf{D})$, then, by [96, Theorem 7.3],

$$\mathrm{hh}_\bullet\colon \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}}) \to \mathbb{Z}[t, t^{-1}], \quad [\mathsf{D}] \mapsto \sum_{i\in\mathbb{Z}} \mathrm{hh}_i(\mathsf{D}) t^i \tag{2.133}$$

is a motivic measure.

We would like to compare the four motivic measures $\mathrm{h}, \mathrm{P}, \mathrm{d}$ and $\mathrm{hh}_\bullet$. To do so, we define the following maps between polynomial rings.

- For $f = f(u, v) \in \mathbb{Z}[u, v]$, we set $\mathrm{b}(f) := f(t, t)$.
- For $f = f(u, v) \in \mathbb{Z}[u, v]$, we set $\mathrm{hkr}(f) := f(t, t^{-1})$.
- For $g \in \mathbb{Z}[t]$, we set $\chi_{\mathrm{top}}(g) := g(-1)$.

Then the diagram of solid arrows

$$\begin{array}{ccccccc} \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) & \xrightarrow{\mathrm{h}} & \mathbb{Z}[u,v] & \xrightarrow{\mathrm{b}} & \mathbb{Z}[t] & \xrightarrow{\chi_{\mathrm{top}}} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \mathrm{hkr}} & & & \nearrow{\scriptstyle (\text{dashed})} & \\ \mathsf{K}_0(\mathsf{dgcat}_{\mathbf{k}}) & \xrightarrow{\mathrm{hh}_\bullet} & \mathbb{Z}[t,t^{-1}] & & & & \end{array} \quad (\mathrm{P}\colon \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}}) \to \mathbb{Z}[t]) \tag{2.134}$$

is commutative.

We have $\mathrm{P} = \mathrm{b} \circ \mathrm{h}$, because $\dim \mathrm{H}^i(X) = \sum_{p+q=i} \mathrm{h}^{p,q}(X)$ for every smooth projective variety $X$. Regarding the square, the commutativity follows from the Hochschild–Kostant–Rosenberg isomorphism

$$\mathrm{HH}_i(X) \cong \bigoplus_{p-q=i} \mathrm{H}^q(X, \Omega_X^p) \tag{2.135}$$

for a smooth projective variety $X$.

The dashed arrow is defined on the subring $\mathbb{Z}[t] \subset \mathbb{Z}[t, t^{-1}]$ by $g \mapsto g(0)$. It makes the whole diagram commutative when we restrict to the ideal $(\mathbb{L}) \subset \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$. We can think of the subring generated by $\mathrm{hkr}(\mathrm{h}(\mathbb{L}))$ as the image of $[X] \in \mathsf{K}_0(\mathsf{Var}_{\mathbf{k}})$ such that $\mathsf{D}^{\mathsf{b}}(X)$ admits a full exceptional collection.

**The derived category of the Artin model.** Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order over a smooth projective curve $C$. The results of Section 2.4.2 allow us to give heuristics about a possible decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$. We restrict ourselves to the situation of Corollary 2.4.7, where $\mathscr{A}$ is a hereditary order of degree $n$, totally ramified over $r$ points. In this case it follows from [5, Theorem 1.4] that $\mathsf{BS}(\mathscr{A})$ is a smooth projective variety, since $C$ is smooth and projective. First we give a motivic decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$.

**Proposition 2.4.10.** *Let $C$ be a smooth projective curve over $\mathbf{k}$. Moreover, let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ with $r$ totally ramified points. Then*

$$[\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))] = [C] + [\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})] + \sum_{i=2}^{n-1} [\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_i)], \tag{2.136}$$

*where $\mathscr{B}_i$ is a hereditary order of degree $\deg \mathscr{B}_i = \binom{n}{i}$ totally ramified over $\Delta_{\mathscr{B}_i} = \Delta_{\mathscr{A}}$.*

*Proof.* Let $\mathscr{B}_i$ be a totally ramified hereditary $\mathcal{O}_C$-order of degree $\binom{n}{i}$ with $r$ ramified points. By Theorem 3.B, there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_i) = \langle E_1^{(1)}, \dots, E^{(1)}_{\binom{n}{i}-1}, \dots, E_1^{(r)}, \dots, E^{(r)}_{\binom{n}{i}-1} \mathsf{D}^{\mathsf{b}}(C) \rangle, \tag{2.137}$$

where all $E_j^{(k)}$ are exceptional objects. Hence the right hand side of (2.136) consists of summands of the form

$$[\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_i)] = [\mathsf{D}^{\mathsf{b}}(C)] + r \cdot \left( \binom{n}{i} - 1 \right). \tag{2.138}$$

Substituting this equality on the right hand side of (2.136) we obtain

$$n \cdot [\mathsf{D}^{\mathsf{b}}(C)] + \sum_{i=1}^{n-1} r \cdot \left( \binom{n}{i} - 1 \right) = [\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))] \tag{2.139}$$

by applying the motivic measure d to Corollary 2.4.7. ☐

In light of the motivic decomposition we expect $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ to have a semiorthogonal decomposition of the following shape.

**Conjecture 2.4.11.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order of degree $n$ such that $\Delta_{\mathscr{A}} = \{o\}$ and $\mathscr{A}$ is totally ramified over $o$. Then $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ admits a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_{n-1}), \dots, \mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_2), \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}), \mathsf{D}^{\mathsf{b}}(C) \rangle, \tag{2.140}$$

*where $\mathscr{B}_i$ is a hereditary order of degree $\deg \mathscr{B}_i = \binom{n}{i}$ with ramification divisor $\Delta_{\mathscr{B}_i} = \{o\}$ and ramification index $e_i = \binom{n}{i}$ at $o$.*

We give some evidences to the conjecture. First, consider the situation of Azumaya algebras (i.e. $\Delta_{\mathscr{A}} = \emptyset$). Bernardara [30, Theorem 4.1] constructed a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ for an Azumaya algebra $\mathscr{A}$ of degree $n$ on a smooth projective variety $X$. The semiorthogonal decomposition is $X$-linear and consists of $n$ components. It is given by

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}^{\otimes(n-1)}), \dots, \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}), \mathsf{D}^{\mathsf{b}}(X) \rangle. \tag{2.141}$$

If $\mathscr{A}$ is a hereditary $\mathcal{O}_C$-order of degree 2, then $\mathsf{BS}(\mathscr{A}) \to C$ is a conic bundle. Therefore we can appeal to Kuznetsov's result [94, Theorem 4.2] for quadric fibrations. It states that there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A})) = \langle \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}), \mathsf{D}^{\mathsf{b}}(C) \rangle. \tag{2.142}$$

Finally, if $\mathscr{A}$ is hereditary of higher degree, we can look at what happens under base change of semiorthogonal decompositions [97, Theorem 5.6]. A semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ will base change to a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A} \otimes \mathbf{k}(C)))$. Then, by (2.141), it may come from $n$ (hereditary) orders in Azumaya algebras.

Despite the above discussion, it is already an open problem whether $\mathsf{D}^{\mathsf{b}}(C)$ and $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ embed as admissible components into $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ as the 'start' of a semiorthogonal decomposition. By extending cohomological criteria from [53, §6], it is work-in-progress to show that $\mathsf{D}^{\mathsf{b}}(C)$ and $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ are admissible components of $\mathsf{D}^{\mathsf{b}}(\mathsf{BS}(\mathscr{A}))$ such that $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ lies in the right-orthogonal of $\mathsf{D}^{\mathsf{b}}(C)$.

It is completely open how to construct $\mathscr{B}_i$. As the tensor product of hereditary orders with non-empty ramification locus is not hereditary again [77], mimicking the case of Azumaya algebras does not work.

# Chapter 3

# Categorical absorption for hereditary orders

In this chapter, we present our work [16]. We provide more details to the base change formula for coherent ringed schemes.

## 3.1 Introduction

Hereditary orders over a curve form a well-studied class of sheaves of (noncommutative) algebras. Their classification up to étale-local isomorphism [119, Theorem 39.14], and up to Morita equivalence [44, Proposition 7.7] is well-established.

From the geometric point of view, [51, Corollary 7.8] provides a dictionary between

a) hereditary orders $\mathscr{A}$ on a smooth curve $C$, and

b) smooth root stacks $\mathcal{C}$ with coarse moduli space $C$,

in the sense that there is an equivalence $\mathsf{coh}(C, \mathscr{A}) \simeq \mathsf{coh}(\mathcal{C})$ of abelian categories. The dictionary identifies ramification points of the order $\mathscr{A}$ with the points with non-trivial stabilizer of the root stack $\mathcal{C}$.

The bounded derived category $\mathsf{D^b}(C, \mathscr{A}) \simeq \mathsf{D^b}(\mathcal{C})$ has been examined on both sides of the dictionary with various objectives in mind. In particular, when trying to decompose these categories, one has the following results if $C$ is projective.

- If $C = \mathbb{P}^1$, the stacky curve $\mathcal{C}$ is a weighted projective line, and $\mathsf{D^b}(\mathcal{C})$ admits a tilting bundle [70, Proposition 4.1]. This was recently generalized to a certain class of hereditary orders over non-algebraically closed fields in [39, Theorem 3.12].
- Among other things, [81, Theorem 1.2] construct a semiorthogonal decomposition of $\mathsf{D^b}(\mathcal{C})$.

- By [29, Theorem 6.4] $\mathsf{D}^{\mathsf{b}}(\mathcal{C})$ is geometric, i.e. it is an admissible subcategory inside $\mathsf{D}^{\mathsf{b}}(X)$, for some smooth projective scheme $X$.

We propose a novel approach, using the framework of Kuznetsov–Shinder [101] to decompose $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$, by viewing $(C, \mathscr{A})$ as a family of finite-dimensional algebras over the curve $C$. The idea fits into the general perspective of studying the bounded derived category of families of varieties over a base scheme, see [99] for a survey. It provides the first example of a deformation absorption of singularities [101] in a noncommutative setting.

To state the first main result of the paper, let $C$ be a smooth curve over an algebraically closed field $\mathbf{k}$ of characteristic zero. Consider a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ with ramification locus $\Delta_{\mathscr{A}} = \{o\}$ and ramification index $r \in \mathbb{Z}_{\geq 1}$. Hence, the algebra $\mathscr{A}$ is Azumaya on $C \setminus \{o\}$. The algebra $\mathscr{A}(o) := \mathscr{A} \otimes_C \operatorname{Spec} \mathbf{k}(o)$ is described in Lemma 3.2.9.

The first main result is the existence of a triangulated subcategory in $\mathsf{D}^{\mathsf{b}}(\mathscr{A}(o))$ which absorbs singularities.

**Theorem 3.A** (Theorem 3.4.7)**.** *The sequence $(S_1, \ldots, S_{r-1})$ of simple $\mathscr{A}(o)$-modules is semiorthogonal in $\mathsf{D}^{\mathsf{b}}(\mathscr{A}(o))$ and absorbs singularities, i.e. the triangulated subcategory*

$$\mathsf{S} = \langle S_1, \ldots, S_{r-1} \rangle \subset \mathsf{D}^{\mathsf{b}}(\mathscr{A}(o)) \tag{3.1}$$

*is admissible and both of its complements ${}^{\perp}\mathsf{S}$ and $\mathsf{S}^{\perp}$ are smooth and proper.*

More precisely, we show in Lemma 3.4.8 that the sequence (3.1) forms a semiorthogonal sequence of $\mathbb{P}^{\infty,2}$-objects. The proof uses the representation theory of finite-dimensional algebras. Appealing to a noncommutative version of [101, Theorem 1.8] we use $\mathsf{S}$ to provide a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

**Theorem 3.B** (Theorem 3.4.11)**.** *Let $\mathrm{i}_o \colon (\operatorname{Spec} \mathbf{k}(o), \mathscr{A}(o)) \to (C, \mathscr{A})$ be the inclusion of $o \in C$. There is a strong $C$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathrm{i}_{o,*}S_1, \ldots, \mathrm{i}_{o,*}S_{r-1}, \mathsf{D} \rangle, \tag{3.2}$$

*such that*

*i) the sequence $(\mathrm{i}_{o,*}S_1, \ldots, \mathrm{i}_{o,*}S_{r-1})$ is exceptional,*

*ii) the admissible subcategory $\mathsf{D}$ is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$.*

In fact, we will show in Lemma 3.4.15 that $\mathsf{D}$ is equivalent to $\mathsf{D}^{\mathsf{b}}(C)$.

The existence of a semiorthogonal decomposition (3.2) can also be deduced in an indirect way from the stacks–orders dictionary [51, Corollary 7.8] and the semiorthogonal decomposition of [29, Theorem 4.7] for root stacks. We explain in Section 3.4.4 how our results translate to the stacky side in [29, Theorem 4.7].

**Categorical absorption and deformation absorption of singularities.** We outline the idea of categorical absorption and deformation absorption of singularities from [101].

Given a flat family $\mathcal{X} \to (C, o)$ over a smooth pointed curve with a single singular fiber $\mathcal{X}_o = \mathcal{X} \times_C \operatorname{Spec} \mathbf{k}(o)$ and smooth total space $\mathcal{X}$, there is sometimes an interplay of a certain semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathcal{X}_o) = \langle \mathsf{S}, {}^{\perp}\mathsf{S} \rangle$ of the fiber and a $C$-linear semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathcal{X}) = \langle \mathsf{A}_{\mathsf{S}}, \mathsf{B} \rangle$ of the total space in the following sense.

a) As in Theorem 3.A, the category $\mathsf{S} \subset \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ is admissible and both of its complements ${}^{\perp}\mathsf{S}$ and $\mathsf{S}^{\perp}$ are smooth and proper. In this case [101, Definition 1.1] says that $\mathsf{S}$ *absorbs singularities of* $\mathcal{X}_o$.

b) The category $\mathsf{A}_{\mathsf{S}}$ is the (thick) closure of the pushforward of $\mathsf{S}$ to $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$. If it is admissible in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$, [101, Theorem 1.5] show that its complement $\mathsf{B}$ is $C$-linear, as well as smooth and proper such that

$$\mathsf{B}_p \simeq \begin{cases} {}^{\perp}\mathsf{S} & \text{if } p = o, \\ \mathsf{D}^{\mathsf{b}}(\mathcal{X}_p) & \text{otherwise.} \end{cases} \tag{3.3}$$

If additionally, (b) is satisfied for $\mathsf{S}$, then Definition 1.4 of *op. cit.* says that $\mathsf{S}$ *provides a deformation absorption of singularities of* $\mathcal{X}_o$. Theorem 3.B shows that (b) holds in the noncommutative setting for $\mathsf{S}$ defined in Theorem 3.A.

Finding $\mathsf{S} \subsetneq \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ which satisfies (a) and (b) is a non-trivial task. Among other things, Kuznetsov–Shinder thus introduce so-called $\mathbb{P}^{\infty,2}$-objects $S \in \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$. This is an object $S$ such that its ring of self-extensions is $\operatorname{Ext}^{\bullet}_{\mathcal{X}_o}(S, S) \cong \mathbf{k}[\theta]$ with $\deg \theta = 2$, see Definition 3.4.1. By [101, Theorem 1.8] a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ forms a triangulated subcategory satisfying (a) and (b). Notably, (b) is satisfied independently of the chosen smoothing $\mathcal{X}$ of the singular fiber $\mathcal{X}_o$.

Examples of this phenomenon are odd-dimensional varieties with isolated nodal singularities [101, Theorem 6.1]. Generalizing to the notion of compound $\mathbb{P}^{\infty}$-objects, [140] observed a similar phenomenon for a projective threefold with non-isolated singularity. An application of categorical absorption in the context of mirror symmetry is given by [106]. Interpreting a hereditary order $\mathscr{A}$ as a flat family over its central curve $C$, we present an example $(C, \mathscr{A}) \to C$ where the role of the singular fiber is played by the restriction $\mathscr{A}(o)$ of $\mathscr{A}$ to $o$ in Section 3.4.

**Noncommutative base change.** In Section 3.3.4 we will make sense of the fiber $\mathsf{B}_p$ from (3.3) in the derived category of a coherent ringed scheme $(C, \mathscr{A})$ by providing a base change formula which follows from some modifications of the results in [97].

Following [147], a coherent ringed scheme $(X, \mathscr{A})$ is a quasiprojective integral scheme $X$ over $\mathbf{k}$ with a coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{A}$ on it. A morphism $\mathfrak{f} \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ of coherent ringed schemes consists of a morphism of schemes $f \colon X \to Y$ and an $\mathcal{O}_X$-algebra morphism $f_{\mathsf{alg}} \colon f^*\mathscr{B} \to \mathscr{A}$.

We show in Lemma 3.2.5 that, under mild conditions on the morphisms, one can form the fiber product of morphisms of coherent ringed schemes. This was already shown in [93, Lemma 10.37] for Azumaya varieties $(X, \mathscr{A})$, i.e. when $\mathscr{A}$ is an Azumaya algebra over $X$.

With the construction of the fiber product, we can come to a key ingredient for the study of derived categories of families of varieties over a base scheme, a base change formula à la [97, Theorem 5.6]. This is fully developed in the noncommutative setting for the category of perfect complexes and the unbounded derived category [116, §2] using $\infty$-categories, and has a specific version for Azumaya varieties in [93, Theorem 2.46].

Given $\mathfrak{f} = (f, f_{\mathsf{alg}})\colon (X, \mathscr{A}) \to S$ such that $\mathscr{A}$ is a flat $\mathcal{O}_X$-module, $f\colon X \to S$ is flat, and $h\colon T \to S$, we set $X_T = X \times_S T$ and $\mathscr{A}_T = h_T^* \mathscr{A}$, where $h_T\colon X_T \to X$ is the induced morphism.

**Theorem 3.C** (Theorem 3.3.27)**.** *Assume that*

$$\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m \rangle \tag{3.4}$$

*is an $S$-linear strong semiorthogonal decomposition such that the projection functors have finite cohomological amplitude. If $\mathscr{A}$ has finite global dimension, then there is a $T$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(X_T, \mathscr{A}_T) = \langle \mathsf{A}_{1,T}, \dots, \mathsf{A}_{m,T} \rangle \tag{3.5}$$

*compatible with pullback and pushforward.*

It should be mentioned that Theorem 3.A and Theorem 3.B are independent from the base change formula in Theorem 3.C. The reader interested in the deformation absorption result can therefore skip Section 3.3.

The organization of the paper is as follows. We fix the setup and recall properties of coherent ringed schemes in Section 3.2.1. Moreover, we provide the construction of a fiber product in Lemma 3.2.5. We then apply this point of view to hereditary orders in Section 3.2.2. Section 3.3 focuses on the derived category of coherent ringed schemes. Most of the results are well-known or only slight modifications of the commutative case. The goal of this section is to explain how to obtain Theorem 3.C from its commutative version. In Section 3.4, we prove Theorem 3.A and Theorem 3.B. Moreover, we show in Section 3.4.4 how our results relate to smooth stacky curves using the dictionary [51].

**Notations and conventions.** Throughout the paper $\mathbf{k}$ denotes an algebraically closed field of characteristic zero. Every scheme is assumed to be integral and quasi-projective over $\mathbf{k}$. If $X$ is a scheme, we denote by $\mathbf{k}(X)$ its function field.

## 3.2 Preliminaries

Let $X$ be an (integral) scheme. An *$\mathcal{O}_X$-order* is a coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ which is torsion-free as an $\mathcal{O}_X$-module such that $\mathscr{A} \otimes_X \mathbf{k}(X)$ is a central simple $\mathbf{k}(X)$-algebra.

This means that there is a maximal open dense subset $U \subset X$ such that the restriction $\mathscr{A}|_U$ is Azumaya. The complement $\Delta_{\mathscr{A}} = X \setminus U$ is the *ramification locus of $\mathscr{A}$*.

We say that an $\mathcal{O}_X$-order $\mathscr{B}$ is an *overorder of* $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{A} \otimes_X \mathbf{k}(X)$. The order $\mathscr{A}$ is *maximal* if there are no proper overorders of $\mathscr{A}$.

Orders form an important example of a coherent ringed scheme $(X, \mathscr{A})$. We start by recalling the general setup of coherent ringed schemes and define the category of coherent ringed schemes $\mathbf{ncSch_k}$ over $\mathbf{k}$. This allows us to construct a fiber product in $\mathbf{ncSch_k}$ for a large class of pairs of morphisms. In Section 3.2.2 we recall the theory of hereditary orders, their fibers and their overorders, in the framework of coherent ringed schemes. A more extensive treatment of hereditary orders can be found in Chapter 1.

### 3.2.1 Coherent ringed schemes

In this section we define coherent ringed schemes $(X, \mathscr{A})$. Sometimes they are also referred to as 'mild noncommutative schemes', see [58]. Their derived category $\mathsf{D}(X, \mathscr{A})$ (and bounded versions of it) were for example studied in [93, Appendix D], [44], [146, Appendix A], [57, 58]. Since we are eventually interested in $\mathcal{O}_X$-orders, we restrict our attention to coherent sheaves of algebras.

**Definition 3.2.1.** A *coherent ringed scheme* is a pair $(X, \mathscr{A})$ of a scheme $X$ over $\mathbf{k}$ and a coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{A}$.

A *morphism of coherent ringed schemes* $\mathfrak{f} = (f, f_{\mathsf{alg}}) \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ consists of

- a morphism of schemes $f \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.
- an $\mathcal{O}_X$-algebra morphism $f_{\mathsf{alg}} \colon f^*\mathscr{B} \to \mathscr{A}$.

Given $\mathfrak{f} = (f, f_{\mathsf{alg}}) \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $\mathfrak{g} = (g, g_{\mathsf{alg}}) \colon (Y, \mathscr{B}) \to (Z, \mathscr{C})$ we define their *composition* $\mathfrak{h} = (h, h_{\mathsf{alg}}) = \mathfrak{g} \circ \mathfrak{f} \colon (X, \mathscr{A}) \to (Z, \mathscr{C})$ as follows:

- on the underlying schemes $h := g \circ f$, and
- on algebras $h_{\mathsf{alg}} := f_{\mathsf{alg}} \circ f^*(g_{\mathsf{alg}}) \colon h^*\mathscr{C} \to f^*\mathscr{B} \to \mathscr{A}$.

This allows us to form the category of coherent ringed schemes $\mathbf{ncSch_k}$. Note that every scheme $X$ is an object in $\mathbf{ncSch_k}$ by setting $\mathscr{A} = \mathcal{O}_X$. In other words, we have a fully faithful functor form the category of $\mathbf{k}$-schemes to $\mathbf{ncSch_k}$.

Following [93, Definition 10.3], we call a morphism $\mathfrak{f} \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ *strict* if $f_{\mathsf{alg}} = \mathrm{id}_{f^*\mathscr{B}}$. A morphism $\mathfrak{f}$ is called an *extension* if $X = Y$ and $f$ is the identity.

**Remark 3.2.2.** Every coherent ringed scheme $(X, \mathscr{A})$ comes with a *structure morphism* $\mathfrak{f} \colon (X, \mathscr{A}) \to X$. This is an extension, where $f_{\mathsf{alg}} \colon \mathcal{O}_X \to \mathscr{A}$ induces the $\mathcal{O}_X$-algebra structure on $\mathscr{A}$. Since $\mathscr{A}$ is a coherent $\mathcal{O}_X$-algebra, we can view the structure morphism as a *finite noncommutative covering* in light of the equivalence $\mathsf{coh}(\underline{\mathrm{Spec}}_X(\mathscr{A})) \cong \mathsf{coh}(X, \mathscr{A})$ for commutative $\mathscr{A}$ by allowing noncommutative finite-dimensional $\mathbf{k}$-algebras as fibers, see [105].

**Modules on coherent ringed schemes.** Given a coherent ringed scheme $(X, \mathscr{A})$, we denote

- by $\mathsf{QCoh}(X, \mathscr{A})$ the category of right $\mathscr{A}$-modules which are quasicoherent, and
- by $\mathsf{coh}(X, \mathscr{A})$ the subcategory of right $\mathscr{A}$-modules which are coherent as $\mathcal{O}_X$-modules.

Then we can define the following functors on $\mathscr{A}$-modules.

- Let $\mathsf{Ab}(X)$ be the category of sheaves of abelian groups in $X$. The *sheaf hom-functor* is the functor $\mathcal{H}om_{\mathscr{A}}(-,-)$ given by
$$\mathsf{QCoh}(X, \mathscr{A}) \times \mathsf{QCoh}(X, \mathscr{A}) \to \mathsf{Ab}(X), \quad (M, N) \mapsto \mathcal{H}om_{\mathscr{A}}(M, N). \tag{3.6}$$
Note that $\mathcal{H}om_{\mathscr{A}}(M, N)$ is quasi-coherent if $M \in \mathsf{coh}(X, \mathscr{A})$, see [15, Lemma 0GMV].
- Denote by $\mathscr{A}^{\mathrm{op}}$ the opposite algebra. The *tensor product of $\mathscr{A}$-modules* is given by
$$\mathsf{QCoh}(X, \mathscr{A}) \times \mathsf{QCoh}(X, \mathscr{A}^{\mathrm{op}}) \to \mathsf{QCoh}(X), \quad (M, N) \mapsto M \otimes_{\mathscr{A}} N. \tag{3.7}$$
- Given a morphism $\mathfrak{f} \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ of coherent ringed schemes, we let
$$\mathfrak{f}_* \colon \mathsf{QCoh}(X, \mathscr{A}) \to \mathsf{QCoh}(Y, \mathscr{B}), \quad M \mapsto f_* M \tag{3.8}$$
be the *pushforward*. Note that $f_* M$ carries an induced $\mathscr{B}$-module structure via
$$f_* M \otimes_Y \mathscr{B} \cong f_*(M \otimes_X f^* \mathscr{B}) \xrightarrow{f_*(f_{\mathsf{alg}})} f_*(M \otimes \mathscr{A}) \xrightarrow{f_*(\mu_M)} f_* M, \tag{3.9}$$
where $\mu_M \colon M \otimes_X \mathscr{A} \to M$ is the $\mathscr{A}$-module structure of $M$.
- The *pullback* along the morphism $\mathfrak{f} \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ is defined as
$$\mathfrak{f}^* \colon \mathsf{QCoh}(Y, \mathscr{B}) \to \mathsf{QCoh}(X, \mathscr{A}), \quad N \mapsto f^* N \otimes_{f^* \mathscr{B}} \mathscr{A}. \tag{3.10}$$
The left $f^* \mathscr{B}$-module structure on $\mathscr{A}$ is given by $f_{\mathsf{alg}} \colon f^* \mathscr{B} \to \mathscr{A}$. It is interesting that $\mathfrak{f}^* = f^*$ if $\mathfrak{f}$ is strict, and $\mathfrak{f}^* = - \otimes_{\mathscr{B}} \mathscr{A}$ if $\mathfrak{f}$ is an extension.

**Remark 3.2.3.** Since we assume that $\mathscr{A}$ is coherent, it follows that $\mathcal{H}om_{\mathscr{A}}$, $\otimes_{\mathscr{A}}$ and $\mathfrak{f}^*$ map coherent modules to coherent modules. For the pushforward (as in the commutative case), one has to additionally require properness of $f \colon X \to Y$, cf. [146, Lemma A.5].

**Remark 3.2.4.** We have the usual adjunctions for $\mathscr{A}$-modules.

i) The *push-pull* adjunction $\mathfrak{f}^* \dashv \mathfrak{f}_*$ carries over to our setting by [146, Lemma A.5].

ii) The *tensor-hom* adjunction can in general be formulated as follows. For two coherent ringed schemes $(X, \mathscr{A})$ and $(X, \mathscr{B})$ with the same underlying scheme, and $P \in \mathsf{QCoh}(X, \mathscr{A} \otimes_C \mathscr{B}^{\mathrm{op}})$ an $\mathscr{A}$-$\mathscr{B}$-bimodule there is an isomorphism
$$\mathcal{H}om_{\mathscr{A}}(N \otimes_{\mathscr{B}} P, M) \cong \mathcal{H}om_{\mathscr{B}}(N, \mathcal{H}om_{\mathscr{A}}(P, M)) \tag{3.11}$$
natural in $M \in \mathsf{QCoh}(X, \mathscr{A})$ and $N \in \mathsf{QCoh}(X, \mathscr{B})$.

**Fiber products of coherent ringed schemes.** We extend fiber products [93, Lemma 10.37] for Azumaya varieties to fiber products in $\mathbf{ncSch_k}$. Let $\mathfrak{f}_1\colon (X_1, \mathscr{A}_1) \to (X, \mathscr{A})$ and $\mathfrak{f}_2\colon (X_2, \mathscr{A}_2) \to (X, \mathscr{A})$ be morphisms of coherent ringed schemes. Moreover, denote by $p_i\colon X_1 \times_X X_2 \to X_i$ the two canonical projections.

$$\begin{array}{ccc} (X_1 \times_X X_2, p_j^*\mathscr{A}_j) & \xrightarrow{\mathfrak{p}_2} & (X_2, \mathscr{A}_2) \\ \downarrow{\scriptstyle \mathfrak{p}_1} & & \downarrow{\scriptstyle \mathfrak{f}_2} \\ (X_1, \mathscr{A}_1) & \xrightarrow{\mathfrak{f}_1} & (X, \mathscr{A}) \end{array} \tag{3.12}$$

**Lemma 3.2.5.** *Assume that there exists $i \in \{1, 2\}$ such that the morphism $\mathfrak{f}_i$ is strict and let $j \neq i$. The coherent ringed scheme $(X_1 \times_X X_2, p_j^*\mathscr{A}_j)$ is a fibre product of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ in $\mathbf{ncSch_k}$. It is unique up to unique isomorphism.*

*Proof.* By symmetry, we assume without loss of generality that $\mathfrak{f}_1$ is strict. Since $X, X_1, X_2$ are noetherian, the $\mathcal{O}_{X_1\times_X X_2}$-algebra $p_2^*\mathscr{A}_2$ is coherent. The morphism $\mathfrak{p}_2 = (p_2, \mathrm{id}_{p_2^*\mathscr{A}_2})$ is the strict morphism induced from the structure morphism $p_2\colon X_1 \times_X X_2 \to X_2$ of the fiber product of $X_1$ and $X_2$. The morphism $\mathfrak{p}_1 = (p_1, p_2^*(f_{2,\mathsf{alg}}))$ is given by the structure morphism $p_1\colon X_1 \times_X X_2 \to X_1$ for schemes and, using that $\mathfrak{f}_1$ is strict, by

$$p_{1,\mathsf{alg}}\colon p_1^*\mathscr{A}_1 \cong p_2^* f_2^* \mathscr{A} \xrightarrow{p_2^*(f_{2,\mathsf{alg}})} p_2^*\mathscr{A}_2. \tag{3.13}$$

Next, we outline the universal property of the fiber product. Let $\mathfrak{q}_i\colon (Y, \mathscr{B}) \to (X_i, \mathscr{A}_i)$, $i = 1, 2$, be morphisms of coherent ringed schemes such that they commute with the morphisms in the diagram (3.12). By the universal property of the fiber product in the category of schemes, there exists a morphism of schemes $h\colon Y \to X_1 \times_X X_2$ such that $p_i \circ h = q_i$ for $i = 1, 2$. As the morphism $\mathfrak{p}_2$ is strict, it follows that $h_{\mathsf{alg}} = q_{2,\mathsf{alg}}$ is uniquely determined by $\mathfrak{q}_2$. Since $\mathfrak{f}_1$ is strict and $\mathfrak{f}_1 \circ \mathfrak{q}_1 = \mathfrak{f}_2 \circ \mathfrak{q}_2$, it follows as well that $\mathfrak{p}_1 \circ \mathfrak{h} = \mathfrak{q}_1$. ☐

**Remark 3.2.6.** Note that a morphism $h\colon T \to S$ of schemes is always strict. Hence we can base change a coherent ringed scheme $\mathfrak{f} = (f, f_{\mathsf{alg}})\colon (X, \mathscr{A}) \to S$ over a commutative base $S$ along $h\colon T \to S$, to obtain a coherent ringed scheme $(X_T, \mathscr{A}_T)$ over $T$, where $X_T = X \times_S T$ and $\mathscr{A}_T = p_2^*\mathscr{A}$. This setup will be important in Section 3.3.4.

**K-injectives and locally projectives.** The category $\mathsf{QCoh}(X, \mathscr{A})$ is a Grothendieck abelian category, and therefore every cochain complex $M^\bullet$ of quasicoherent $\mathscr{A}$-modules admits a K-injective resolution, i.e. a quasi-isomorphism $M^\bullet \to I^\bullet$ such that $\mathrm{Hom}_{\mathscr{A}}(-, I^\bullet)$ maps acyclic cochain complexes of $\mathscr{A}$-modules to acyclic complexes in $\mathsf{Ab}(X)$. A proof of this can be found in [146, Lemma A.2].

The notion of locally free modules has to be replaced by locally projective modules.

**Definition 3.2.7.** A coherent $\mathscr{A}$-module $P \in \mathsf{coh}(X, \mathscr{A})$ is *locally projective* if there exists a (Zariski-)open covering $X = \bigcup_{i\in I} U_i$ such that $P|_{U_i}$ is a finitely generated projective $\mathscr{A}|_{U_i}$-module for all $i \in I$.

The next lemma shows that our definition agrees with [44, Definition 3.6] and that local projectivity is of complete local nature. For $x \in X$ denote by $\widehat{\mathscr{A}_p} = \mathscr{A}_p \otimes_X \widehat{\mathcal{O}}_{X,p}$ the completion of the stalk $\mathscr{A}_p$ at $p \in X$.

**Lemma 3.2.8.** *Let $(X, \mathscr{A})$ be a coherent ringed scheme and $P \in \mathsf{coh}(X, \mathscr{A})$ a coherent $\mathscr{A}$-module. Then the following are equivalent:*

*i) The $\mathscr{A}$-module $P$ is locally projective.*

*ii) For every $p \in X$, the $\mathscr{A}_p$-module $P_p$ is projective.*

*iii) For every $p \in X$, the $\widehat{\mathscr{A}_p}$-module $\widehat{P}_p$ is projective.*

*Proof.* The equivalence of (i) and (ii) can be found in [119, Corollary 3.23] after choice of an affine open covering.

Let $x \in X$ and denote by $R = \mathcal{O}_{X,p}$. Taking the completion commutes with direct sums, whence $\widehat{P}_p$ is projective if $P_p$ is projective. Conversely, since $R$ is noetherian, the completion $R \to \widehat{R}$ is faithfully flat. Since $\mathscr{A}_p$ is noetherian and $P_p$ is finitely generated, [119, Theorem 2.39] provides an isomorphism $\widehat{R} \otimes_R \operatorname{Hom}_{\mathscr{A}_p}(P_p, -) \cong \operatorname{Hom}_{\widehat{\mathscr{A}_p}}(\widehat{P}_p, - \otimes_R \widehat{R})$ of functors, which proves the equivalence between (ii) and (iii). □

Under the (standing) assumption that $X$ is quasi-projective, [44, Proposition 3.7] show that $\mathsf{coh}(X, \mathscr{A})$ *admits enough locally projectives*, i.e. for every coherent $\mathscr{A}$-module $M$ there exists a locally projective $\mathscr{A}$-module $P$ and an $\mathscr{A}$-module epimorphism $P \twoheadrightarrow M$.

Since every quasicoherent $\mathscr{A}$-module is the filtered colimit of its submodules, we can also achieve that every quasicoherent $\mathscr{A}$-module admits a surjection by a locally projective quasicoherent $\mathscr{A}$-module (taking infinite direct sums of locally projective $\mathscr{A}$-modules).

### 3.2.2 Hereditary orders

Throughout this section, let $\mathscr{A}$ be a hereditary order over $C$ with ramification locus $\Delta_{\mathscr{A}}$ and structure morphism $\mathfrak{f}\colon (C, \mathscr{A}) \to C$. It is shown in [119, Corollary 3.24, Theorem 40.5] that being a hereditary order is a (complete) local property. This means that the following are equivalent:

- The $\mathcal{O}_C$-order $\mathscr{A}$ is hereditary.
- The localization $\mathscr{A}_p$ is a hereditary $\mathcal{O}_{C,p}$-order for every point $p \in C$.
- The completion $\widehat{\mathscr{A}_p} = \mathscr{A}_p \otimes_C \widehat{\mathcal{O}}_{C,p}$ is a hereditary $\widehat{\mathcal{O}}_{C,p}$-order for all $p \in C$.

In the following, let $R = \widehat{\mathcal{O}}_{C,p}$ be the completion of $\mathcal{O}_{C,p}$ at $p \in C$ and $\mathfrak{m} \trianglelefteq R$ be the maximal ideal of $R$. It is a classical fact, see [119, Theorem 39.14], that there is an isomorphism of $R$-algebras

$$\widehat{\mathscr{A}_p} \cong \begin{pmatrix} R & R & \dots & R \\ \mathfrak{m} & R & \dots & R \\ \vdots & & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \dots & R \end{pmatrix}^{(n_1,\dots,n_r)} \subseteq \mathrm{Mat}_n(R), \tag{3.14}$$

where $n^2 = \operatorname{rk} \mathscr{A}$, $n_1 + \ldots + n_r = n$, and the superscript $(n_1, \ldots, n_r)$ indicates that the $(i,j)$-th coordinate of the matrix on the right-hand side is to be read as an $n_i \times n_j$-matrix with entries in $\mathfrak{m}$, or $R$ respectively. We call $(n_1, \ldots, n_r) \in \mathbb{N}^r$ the *ramification data* of $\mathscr{A}$ at $p \in C$. If $\mathscr{A}$ is an $\mathcal{O}_C$-order such that the ramification data at $p \in C$ satisfies $r > 1$, then $\mathscr{A}$ is *ramified at* $p$ with *ramification index* $r$.

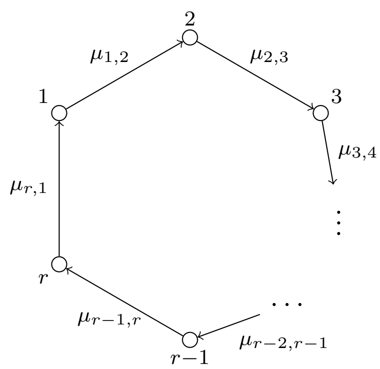
In addition, by [119, Theorem 39.23], the indecomposable projective $\widehat{\mathscr{A}_p}$-modules are given (up to isomorphism) by the rows

$$L_p^{(j)} = E_{\alpha_j,\alpha_j} \widehat{\mathscr{A}_p}, \quad j = 1, \ldots, r, \tag{3.15}$$

where $\alpha_j = n_1 + \ldots + n_j$, and $E_{\alpha,\beta} \in \mathrm{Mat}_n(R)$ denote the elementary matrices.

**The fibers of hereditary orders.** Let $p \in C$. We are interested in the description of the fiber $\mathscr{A}(p) := \widehat{\mathscr{A}_p} \otimes_R \mathbf{k}(p)$ over $p$. In the language of Remark 3.2.6 this is the base change of the structure morphism $\mathfrak{f}\colon (C, \mathscr{A}) \to C$ along the closed immersion $i_p\colon \operatorname{Spec} \mathbf{k}(p) \to C$.

Let $Q_r$ be the cyclic quiver with $r$ vertices $Q_0 = \{1, \ldots, r\}$ and $r$ arrows $Q_1 = \{\mu_{i,i+1}\colon i \to i+1\}$, where here and in the following the numbering has to be understood modulo $r$. The quiver $Q_r$ can be depicted as follows:




(3.16)

Denote by $\mu_{[i,j]}\colon i \to i+1 \to \ldots \to j-1 \to j$ the shortest path of positive length from $i$ to $j$ and let

$$I = (\mathsf{rad}\, \mathbf{k}Q_r)^{n-1} \lhd \mathbf{k}Q_r \tag{3.17}$$

be the admissible ideal generated by all cycles $\mu_{[i,i]}$. The quotient

$$\Lambda_r = \mathbf{k}Q_r / I \tag{3.18}$$

is an $r^2$-dimensional $\mathbf{k}$-algebra of infinite global dimension.

**Lemma 3.2.9.** *Let $(C, \mathscr{A})$ be a hereditary order which is ramified at $p \in C$ with ramification index $r$. Then the $\mathbf{k}$-algebra $\mathscr{A}(p) = \mathscr{A}_p \otimes_C \mathbf{k}(p)$ is Morita equivalent to $\Lambda_r$.*

*Proof.* By [51, Theorem 7.6], we find that the completion $\widehat{\mathscr{A}_p}$ is Morita equivalent to an $R$-algebra $\Gamma \subset \mathrm{Mat}_r(R)$ as in (3.14) with $n_1 = \ldots = n_r = 1$. Note that the residue field of a local ring does not change after completion. The algebra $\Gamma \otimes_R \mathbf{k}(p)$ is isomorphic to $\Lambda_r$ by [90, Theorem 3.1], and hence $\mathscr{A}(p)$ is Morita equivalent to $\Lambda_r$.

We sketch a different proof of why $\Gamma \otimes_R \mathbf{k}$ is isomorphic to $\Lambda_r$ using the algorithmic characterization of finite-dimensional basic $\mathbf{k}(p)$-algebras as quotients of path algebras of a quiver by an admissible ideal from [13, Section II.3]. By [5, Section 2], there is a $\mathbf{k}$-basis $\{e_{ij}\}_{1 \le i,j \le r}$ of $\Gamma \otimes_R \mathbf{k}(p)$ such that

$$e_{ij} \cdot e_{jk} = \begin{cases} e_{ik} & \text{if } i \le j \le k \text{ or } j \le k < i \text{ or } k < i \le j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.19}$$

Moreover, one has $e_{ij} \cdot e_{j'k} = 0$ if $j \neq j'$.

The algebra $\Gamma \otimes_R \mathbf{k}(p)$ is basic and connected, with $e_{11}, \ldots, e_{rr}$ a complete set of primitive orthogonal idempotents.

Define a $\mathbf{k}$-algebra homomorphism $\varphi \colon \mathbf{k}Q_r \to \Gamma \otimes_R \mathbf{k}(p)$ by sending the lazy path $e_i$ associated to the vertex $i \in Q_0$ to the idempotent $e_{ii}$, and the arrow $\mu_{i,i+1} \in Q_1$ to $e_{i,i+1}$. It follows from (3.19) that this is a surjective $\mathbf{k}$-algebra homomorphism. For dimension reasons and from the relations $e_{i,i+1} \cdot \ldots \cdot e_{i-1,i} = 0$ for each $i = 1, \ldots r$, it follows that $\varphi$ induces an isomorphism $\Lambda_r \cong \Gamma \otimes_R \mathbf{k}(p)$. □

We can use the representation theory of $\mathbf{k}Q_r/I$ to characterize simple $\Lambda_r$-modules. Since there are no non-zero oriented cycles, thanks to the relations, the simple $\Lambda_r$-modules up to isomorphism are given by $S_1, \ldots, S_r$ with

$$(S_i)_j = \begin{cases} \mathbf{k} & \text{if } j = i, \\ 0 & \text{if } j \in Q_0 \setminus \{i\}. \end{cases} \tag{3.20}$$

**Remark 3.2.10.** The number of isomorphism classes of simple $\mathscr{A}(p)$-modules does not change under the Morita equivalence from Lemma 3.2.9. It is straightforward that the simple $\mathscr{A}(p)$-module corresponding to $S_i$ consists of a single $n_i$-dimensional non-trivial $\mathbf{k}$-vector space at the vertex $i \in Q_0$. In the remainder we will only be interested in properties of the simple $\Lambda_r$-modules which are preserved under the Morita equivalence. In particular Theorem 3.4.7 and Theorem 3.4.11 do only depend on $\mathscr{A}(p)$ up to Morita equivalence.

**Maximal overorders.** Denote by $\Delta_{\mathscr{A}} = \{p_1, \ldots, p_m\} \subset C$ the ramification locus of $\mathscr{A}$ and let $r_i > 1$ be the corresponding ramification index of $\mathscr{A}$ at $p_i \in C$.

The inclusion of every maximal overorder $j_{\mathscr{B},\mathsf{alg}}\colon \mathscr{A} \hookrightarrow \mathscr{B}$ gives rise to an extension

$$\mathrm{j}_{\mathscr{B}} = (\mathrm{id}_{\mathscr{B}}, j_{\mathscr{B},\mathsf{alg}})\colon (C,\mathscr{B}) \to (C,\mathscr{A}). \tag{3.21}$$

It was shown in [59, Theorem 3.1] that the pushforward $\mathrm{j}_{\mathscr{B},*}\mathscr{B}$ is a locally projective (left and right) $\mathscr{A}$-module. We study the shape of the locally projective $\mathscr{A}$-modules more closely with respect to the indecomposable projective $\widehat{\mathscr{A}_p}$-modules $L_p^{(j)}$ for $j = 1,\dots,r$ from (3.15).

**Definition 3.2.11.** Let $p \in \Delta_{\mathscr{A}}$ such that $\mathscr{A}$ has ramification index $r$ at $p$. A nonzero locally projective $\mathscr{A}$-module $P$ is called *purely of type $j$ at $p$* if

$$\widehat{P}_p \cong {L_p^{(j)}}^{\oplus k} \quad \text{for some } k \in \mathbb{N}. \tag{3.22}$$

If the type $j$ is not specified, we say that $M$ is *purely of one type at $p$*.

As shown in [119, Theorem 40.10], hereditary orders can be characterized by the maximal orders containing them. Using the classification of the indecomposable projective $\widehat{\mathscr{A}_p}$-modules, one can construct all the maximal orders containing $\mathscr{A}$ explicitly.

**Proposition 3.2.12.** *Let $(C,\mathscr{A})$ be a hereditary order with ramification data as above.*

*i) There are precisely $r_1 \cdot \ldots \cdot r_m$ maximal overorders of $\mathscr{A}$.*

*ii) Every maximal overorder $\mathscr{B}$ of $\mathscr{A}$ is locally projective as a left and a right $\mathscr{A}$-module.*

*iii) Every maximal overorder $\mathscr{B} \supseteq \mathscr{A}$ is purely of one type at each ramification point. Moreover, a maximal overorder is uniquely determined by its types at the ramification points.*

*Proof.* The first two parts can be found in [119, Theorem 39.23 and Theorem 40.8]. Let us indicate how to prove these two statements for an algebraically closed field using the classification (3.15) of the indecomposable projective $\widehat{\mathscr{A}_p}$-modules for $p \in C$. By Tsen's theorem, $\mathscr{A} \otimes_C \mathbf{k}(C) \cong \mathrm{End}_{\mathbf{k}(C)}(V)$, for some $n$-dimensional vector space $V$.

Let $\mathscr{B} \supset \mathscr{A}$ be a maximal overorder. Then $\widehat{\mathscr{B}_p} \supset \widehat{\mathscr{A}_p}$ is a maximal overorder, because being maximal is preserved under completion by [119, Corollary 11.6]. From the classification of maximal orders in discrete valuation rings (see Corollary 17.4 of *op. cit.*), it follows that there exists an $\widehat{\mathcal{O}}_{C,p}$-lattice $L_p$ in $V$ such that $\widehat{\mathscr{B}_p} = \mathrm{End}_{\widehat{\mathcal{O}}_{C,p}}(L_p)$ and $L_p$ has the structure of a right $\widehat{\mathscr{A}_p}$-module from the induced $\widehat{\mathscr{B}_p}$-module structure on $L_p$. It must be indecomposable as $\widehat{\mathscr{A}_p}$-module, because its endomorphism ring is generically central simple. Therefore $L_p$ is an indecomposable projective $\widehat{\mathscr{A}_p}$-module, e.g. by [119, Theorem 10.6] and the fact that $\widehat{\mathscr{A}_p}$ is hereditary.

Now $\mathscr{B} \subset \mathrm{End}_{\mathbf{k}(C)}(V)$ is coherent and torsion-free, and therefore uniquely determined by the completion at every point $p \in C$. For a ramified point $p_i \in \Delta_i$ we have $r_i$ choices of $L_{p_i} = L_{p_i}^{(j)}$, $j = 1,\dots,r_i$, hence $r_i$ different possible specializations of $\mathscr{B}$ at $p_i$, and we conclude (i).

For the second part, it suffices to note that by Lemma 3.2.8, the question is complete local. From the construction of $\mathscr{B}$ above, it is clear that $\mathscr{B}$ is locally projective as a right $\mathscr{A}$-module. The isomorphism as left modules follows similarly using that $\mathrm{Hom}_{\widehat{\mathcal{O}}_{C,p}}(L_{p_i}^{(j)}, \widehat{\mathcal{O}}_{C,p})$ describes indecomposable projective left modules.

The third part can be calculated complete locally as well. We have at $p_i \in \Delta_{\mathscr{A}}$ that

$$\widehat{\mathscr{B}_{p_i}} \cong \mathrm{End}_{\widehat{\mathcal{O}}_{C,p_i}}(L_{p_i}^{(j)}) \cong \mathrm{Hom}_{\widehat{\mathcal{O}}_{C,p_i}}(\widehat{\mathcal{O}}_{C,p_i}^{\oplus n}, L_{p_i}^{(j)}) \cong L_{p_i}^{(j)\oplus n}. \tag{3.23}$$

The isomorphism as left $\widehat{\mathscr{A}_{p_i}}$-modules is given similarly. The uniqueness follows directly from the characterization of all maximal overorders of $\mathscr{A}$. □

The explicit description of $\mathscr{B}$ allows us to draw some powerful conclusions about $\mathsf{j}_{\mathscr{B}}$.

**Lemma 3.2.13.** *Let $\mathsf{j}_{\mathscr{B}}\colon (C, \mathscr{B}) \to (C, \mathscr{A})$ be the extension corresponding to a maximal overorder $\mathscr{B} \supset \mathscr{A}$. Then the following hold.*

*i) The pullback $\mathsf{j}_{\mathscr{B}}^*\colon \mathsf{QCoh}(C, \mathscr{A}) \to \mathsf{QCoh}(C, \mathscr{B})$ is exact.*

*ii) The pushforward $\mathsf{j}_{\mathscr{B},*}\colon \mathsf{QCoh}(C, \mathscr{B}) \to \mathsf{QCoh}(C, \mathscr{A})$ is exact and fully faithful.*

*Proof.* The pushforward $\mathsf{j}_{\mathscr{B},*}$ is the restriction of scalars to $\mathscr{A}$, forgetting the $\mathscr{B}$-module structure. Therefore, it does not affect exactness of a sequence, and hence $\mathsf{j}_{\mathscr{B},*}$ is exact. Since $\mathsf{j}_{\mathscr{B}}$ is an extension, the pullback is given by $\mathsf{j}_{\mathscr{B}}^* = - \otimes_{\mathscr{A}} \mathscr{B}$. By Proposition 3.2.12, $\mathscr{B}$ is locally projective, hence $\mathsf{j}_{\mathscr{B}}^*$ is exact.

It remains to show that $\mathsf{j}_{\mathscr{B},*}$ is fully faithful. Similarly to [120, Proposition 1.5], one has that the multiplication on $\mathscr{B}$ induces an isomorphism $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \cong \mathscr{B}$. Using that $\mathscr{B}$ is a locally projective $\mathscr{A}$-module, the isomorphism is a consequence of the inclusion $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \hookrightarrow A$ into the central simple $\mathbf{k}(C)$-algebra $A = \mathscr{A} \otimes_C \mathbf{k}(C)$. Now, given $M \in \mathsf{QCoh}(C, \mathscr{B})$, we find that

$$\mathsf{j}_{\mathscr{B}}^* \mathsf{j}_{\mathscr{B},*} M = M \otimes_{\mathscr{A}} \mathscr{B} \cong M \otimes_{\mathscr{B}} \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B} \cong M \otimes_{\mathscr{B}} \mathscr{B} \cong M. \tag{3.24}$$

Hence, the pushforward $\mathsf{j}_{\mathscr{B},*}$ is fully faithful. □

## 3.3 The derived category of a coherent ringed scheme

Mostly, we are interested in the bounded derived category $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ of a coherent ringed scheme. However, we will need to leave this world from time to time.

- We denote by $\mathsf{D}(X, \mathscr{A}) = \mathsf{D}(\mathsf{QCoh}(X, \mathscr{A}))$ the *unbounded derived category of quasicoherent $\mathscr{A}$-modules*, and by $\mathsf{D}^*(X, \mathscr{A})$, for $* \in \{+, -, \mathsf{b}\}$, the bounded below, bounded above, resp. bounded derived category of quasicoherent $\mathscr{A}$-modules.
- By $\mathsf{D}_{\mathsf{coh}}(X, \mathscr{A}) = \mathsf{D}_{\mathsf{coh}}(\mathsf{QCoh}(X, \mathscr{A}))$ we denote the *unbounded derived category of quasicoherent $\mathscr{A}$-modules with coherent cohomology.* In particular, we call $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) = \mathsf{D}^b_{\mathsf{coh}}(\mathsf{QCoh}(X, \mathscr{A}))$ the *bounded derived category* of $(X, \mathscr{A})$.

- The *category of perfect complexes* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ is the full triangulated subcategory of $\mathsf{D}(X, \mathscr{A})$ consisting of objects which are represented by complexes that are locally quasi-isomorphic to bounded complexes of locally projective $\mathscr{A}$-modules.
- Moreover, for $a \leq b \in \mathbb{Z}$, we denote by $\mathsf{D}^{[a,b]}(X, \mathscr{A}) \subset \mathsf{D}(X, \mathscr{A})$ the objects $M \in \mathsf{D}(X, \mathscr{A})$ such that the cohomology sheaf $\mathcal{H}^i(M)$ vanishes for $i \notin [a, b]$. If $a = -\infty$, we write $\mathsf{D}^{\leq b}(X, \mathscr{A})$, and if $b = \infty$, we write $\mathsf{D}^{\geq a}(X, \mathscr{A})$.

It is a straightforward, but important observation, cf. [57, Example 3.7], that the pushforward $\mathfrak{f}_*\colon \mathsf{D}(X, \mathscr{A}) \to \mathsf{D}(X)$ of the structure morphism $\mathfrak{f}\colon (X, \mathscr{A}) \to X$ is exact as exactness does not depend on the module structure.

Since $\mathscr{A}$ is coherent, [146, Lemma A.4] provides the useful equivalence

$$\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(X, \mathscr{A})). \tag{3.25}$$

### 3.3.1 Perfect complexes

Recall that an object $P \in \mathsf{D}(X, \mathscr{A})$ is *compact* if $\mathrm{Hom}_{\mathsf{D}(X,\mathscr{A})}(P, -)$ commutes with filtered colimits.

**Remark 3.3.1.** For $(X, \mathscr{A})$ a coherent ringed scheme, [44, Proposition 3.14] shows that $\mathsf{D}(X, \mathscr{A})$ is compactly generated, and the compact objects are $\mathsf{D}(X, \mathscr{A})^{\mathsf{c}} = \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$.

With the identification (3.25) it is straightforward to extend [134, Proposition 2.3.1] to coherent ringed schemes $(X, \mathscr{A})$, where $\mathscr{A}$ has finite global dimension. In the commutative case it says that a perfect complex on a quasi-projective scheme $X$ is quasi-isomorphic to a bounded cochain complex of locally free sheaves.

**Lemma 3.3.2.** *Let $(X, \mathscr{A})$ be a coherent ringed scheme.*

*i) Every perfect complex of $\mathscr{A}$-modules is quasi-isomorphic to a bounded cochain complex of coherent $\mathscr{A}$-modules.*

*ii) If $\mathscr{A}$ has finite global dimension or $X = \operatorname{Spec}\mathbf{k}$ is a point, each perfect complex is globally quasi-isomorphic to a bounded cochain complex of locally projective modules.*

*iii) If $\mathscr{A}$ has finite global dimension, then $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$.*

*Proof.* Let $M \in \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$. Since $X$ is quasi-compact and the cohomology sheaf $\mathcal{H}^\ell(M)$ does only depend on the $\mathcal{O}_X$-module structure, it follows that $\mathcal{H}^\ell(M)$ is coherent for all $\ell \in \mathbb{Z}$, and non-zero for only finitely many $\ell$. From (3.25) one sees that $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(X, \mathscr{A}))$ since $\mathscr{A}$ is coherent.

For $X = \operatorname{Spec}\mathbf{k}$ the second statement follows by definition of perfect complexes. In the other case (ii) follows from the first and the third statement.

Hence, we pass to (iii). Let $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$. By [44, Proposition 3.7], there is a locally projective resolution $P^\bullet \xrightarrow{\sim} M$, with $P^\bullet \in \mathsf{D}^-(\mathsf{coh}(X, \mathscr{A}))$ bounded above. Since $M$ is bounded on both sides, there exists $n \ll 0$ such that $\mathcal{H}^k(P^\bullet) = 0$ for all $k \leq n$. Then the canonical truncation $N^\bullet = \tau_{\geq n} P^\bullet$ (cf. [15, Section 0118]) is quasi-isomorphic to $P^\bullet$. The canonical truncation consists of locally projective $\mathscr{A}$-modules except at the $n$-th position, we have $N^n = \operatorname{Coker}(d^{n-1}\colon P^{n-1} \to P^n) \cong \operatorname{Im}(d^n)$, by exactness at $n$. Since $N^n$ is a coherent $\mathscr{A}$-module, it admits a locally projective resolution $\varepsilon\colon Q^\bullet \to N^n$, which is of finite length if $\mathscr{A}$ is of finite global dimension. Replacing $N^n$ by its locally projective resolution, we obtain a new bounded cochain complex $L^\bullet$ of locally projective $\mathscr{A}$-modules and a morphism of cochain complexes

$$\begin{array}{ccccccccccc} L^\bullet & & \cdots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \xrightarrow{\iota\varepsilon} & P^{n+1} & \longrightarrow & \cdots \\ \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ N^\bullet & & \cdots & \longrightarrow & 0 & \longrightarrow & N^n & \xrightarrow{\iota} & P^{n+1} & \longrightarrow & \cdots \end{array} \tag{3.26}$$

where $\iota\colon N^n \to P^{n+1}$ is the inclusion of the image. This is a quasi-isomorphism. All in all, we obtain that $M$ quasi-isomorphic to $L^\bullet \in \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$. □

Recall from [114, Definition 1.6] that an object $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ is *homologically finite* if for every $N \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ the $\mathbf{k}$-vector space $\bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}^i_{\mathscr{A}}(M, N)$ is finite-dimensional. From the perspective of induced semiorthogonal decompositions (see Section 3.3.3), homologically finite objects are better behaved than perfect complexes, as a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ induces one for homologically finite objects.

The following generalizes [114, Proposition 1.11] to the noncommutative setting.

**Lemma 3.3.3.** *Let $(X, \mathscr{A})$ be a coherent ringed scheme such that $\mathscr{A}$ is of finite global dimension or $X = \operatorname{Spec} \mathbf{k}$ is a point. An object $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ is perfect if and only if it is homologically finite.*

*Proof.* We explain how to modify the proof in [114, Proposition 1.11] so that it works in our setting. Since every homologically finite object is bounded, by [95, Proposition 2.9] it is perfect.

Vice versa, assume that $M$ is perfect. By Lemma 3.3.2 it is represented by a bounded cochain complex $P^\bullet$ of locally projective $\mathscr{A}$-modules. The spectral sequence

$$\mathrm{H}^p(X, \mathcal{H}^q(\mathcal{H}om_{\mathscr{A}}(P^\bullet, N))) \quad \Rightarrow \quad \operatorname{Ext}^{p+q}_{\mathscr{A}}(M, N) \tag{3.27}$$

is concentrated in $p \in [0, \dim X]$. It is zero in the $q$-direction outside the bounds of $P^\bullet$. Moreover, each $\mathbf{k}$-vector spaces on the $E_2$-page is finite-dimensional, because $\mathscr{A}$ is coherent. Therefore, $\operatorname{Ext}^i_{\mathscr{A}}(M, N)$ is finite-dimensional for all $i \in \mathbb{Z}$ and non-zero for only finitely many $i \in \mathbb{Z}$. □

Next, we come to a generalization of [95, Proposition 2.5] to coherent ringed schemes. We refer to Section 2.3 of *op. cit.* for a short introduction to (bounded) t-structures.

**Definition 3.3.4.** Let $\mathsf{T}$ be a triangulated category with a t-structure $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$, and $\mathsf{C} \subset \mathsf{D}(X, \mathscr{A})$ be a triangulated subcategory. A functor $\Phi\colon \mathsf{C} \to \mathsf{T}$ has *finite cohomological amplitude* if there are $a, b$ such that

$$\Phi(\mathsf{C} \cap \mathsf{D}^{\geq 0}(X, \mathscr{A})) \subset \mathsf{T}^{\geq a}, \quad \text{and} \quad \Phi(\mathsf{C} \cap \mathsf{D}^{\leq 0}(X, \mathscr{A})) \subset \mathsf{T}^{\leq b}. \tag{3.28}$$

**Lemma 3.3.5.** *Let $(X, \mathscr{A})$ be a coherent ringed scheme, where $X$ is a quasi-projective variety. Let $\mathsf{T}$ be a triangulated category which admits a bounded t-structure. Then every functor $\Phi\colon \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) \to \mathsf{T}$ has finite cohomological amplitude.*

*Proof.* The proof of [95, Proposition 2.5] needs only a small modification. Let $\mathcal{O}_X(1)$ be an ample line bundle on $X$, and denote by $\mathscr{A}(n) = \mathscr{A} \otimes_X \mathcal{O}_X(n)$. Since $\mathsf{T}$ has a bounded t-structure, and $X$ is quasi-projective, one finds $a, b \in \mathbb{Z}$ such that $\Phi(\mathscr{A}(n)) \in \mathsf{T}^{[a,b]}$ for all $n \in \mathbb{Z}$.

The remainder works as in [95, Proposition 2.5]. A perfect object $M \in \mathsf{D}^{[s,t]}(X, \mathscr{A})$ can be represented by bounded above cochain complex

$$P^\bullet = \ldots \to P^{t-1} \to P^t \tag{3.29}$$

such that $P^i = \bigoplus_{k_i} \mathscr{A}(n_{k_i})^{\oplus r_{k_i}}$ is locally projective (of finite rank). Let $\ell = s - \dim X$. Consider the stupid truncation $\sigma_{\geq \ell} P^\bullet$, see [15, Section 0118] for the definition. The morphism $\sigma_{\geq \ell} P^\bullet \to M$ fits into a distinguished triangle

$$\mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet)[-\ell] \to \sigma_{\geq \ell} P^\bullet \to M \to \mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet)[1-\ell]. \tag{3.30}$$

We want to show that $\mathrm{Hom}_{\mathsf{D}(S,\mathscr{A})}(M, \mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet)[1-\ell]) = 0$. Since $M \in \mathsf{D}^{\geq s}(X, \mathscr{A})$ and and $\mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet) \in \mathsf{coh}(S, \mathscr{A})$, it follows that $\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet))$ is computed by a cochain complex of $\mathcal{O}_X$-modules in $\mathsf{D}^{\leq -s}(X)$. Therefore

$$\mathrm{Hom}_{\mathsf{D}(S,\mathscr{A})}(M, \mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet)[1-\ell]) = \mathrm{Ext}^{1-\ell}_{\mathscr{A}}(M, \mathcal{H}^\ell(\sigma_{\geq \ell} P^\bullet)) = 0 \tag{3.31}$$

It follows that $M$ is quasi-isomorphic to a direct summand of the bounded cochain complex $\sigma_{\geq \ell} P^\bullet$, which satisfies $\Phi(\sigma_{\geq \ell} P^\bullet) \in \mathsf{T}^{[a+\ell, b+t]}$. Thus $\Phi(M)$ lies in the same bounds as well. Hence $\Phi$ has finite cohomological amplitude. □

### 3.3.2 Semiorthogonal decompositions linear over the base

Let $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$, and consider a morphism of schemes $\mathfrak{f}\colon (X, \mathscr{A}) \to S$. We start by recalling $S$-linearity of $\mathsf{T}$, see [93, Section 2.6], [116, Section 2 and 3].

**Definition 3.3.6.** A triangulated subcategory $\mathsf{A} \subset \mathsf{T}$ is called *$S$-linear* if

$$M \otimes^{\mathbf{L}}_X \mathbf{L} f^* \mathcal{F} \in \mathsf{A} \quad \text{for all } M \in \mathsf{A}, \mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S). \tag{3.32}$$

**Remark 3.3.7.** If $\mathsf{T} = \mathsf{D}^-(X, \mathscr{A})$ or $\mathsf{T} = \mathsf{D}(X, \mathscr{A})$, and $\mathsf{A} \subset \mathsf{T}$ is $S$-linear for $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S)$, then $S$-linearity automatically holds for $\mathsf{D}^-(S)$, resp. $\mathsf{D}(S)$. This follows from [97, Lemma 4.5].

Before we continue, we record some properties of the sheaf hom-functor, which will be needed afterwards. Let $F^\vee := \mathbf{R}\mathcal{H}om_X(F, \mathcal{O}_X)$ be the derived dual of $F \in \mathsf{D}^{\mathsf{perf}}(X)$.

**Lemma 3.3.8.** *Let $(X, \mathscr{A})$ be a coherent ringed scheme, $M, N \in \mathsf{D}(X, \mathscr{A})$ and $F \in \mathsf{D}^{\mathsf{perf}}(X)$. Then there are natural isomorphisms*

$$\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M \otimes^{\mathbf{L}}_X F, N) \cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, N \otimes^{\mathbf{L}}_X F^\vee), \tag{3.33}$$

$$\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, N \otimes^{\mathbf{L}}_X F) \cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, N) \otimes^{\mathbf{L}}_X F. \tag{3.34}$$

*Proof.* Let $\mathfrak{f}\colon (X, \mathscr{A}) \to X$ be the structure morphism. Then $\mathbf{L}\mathfrak{f}^* F = F \otimes^{\mathbf{L}}_X \mathscr{A}$ is a perfect complex of $\mathscr{A}$-bimodules. By applying the tensor-Hom adjunction twice, it follows that

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M \otimes^{\mathbf{L}}_X F, N) &\cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M \otimes^{\mathbf{L}}_{\mathscr{A}} \mathbf{L}\mathfrak{f}^* F, N) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(F \otimes^{\mathbf{L}}_X \mathscr{A}, N)) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathbf{R}\mathcal{H}om_X(F, N)) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, F^\vee \otimes^{\mathbf{L}}_X N). \end{aligned} \tag{3.35}$$

For the last identification, one uses the isomorphism $\mathbf{R}\mathcal{H}om_X(F, -) \cong F^\vee \otimes^{\mathbf{L}}_X -$.

The second isomorphism follows from the first and

$$F \otimes^{\mathbf{L}}_X \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, N) \cong \mathbf{R}\mathcal{H}om_X(F^\vee, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, N)) \cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(F^\vee \otimes^{\mathbf{L}}_X M, N), \tag{3.36}$$

where one uses again the tensor-Hom adjunction. □

**Lemma 3.3.9.** *Assume that $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ is a strict morphism. For all $M \in \mathsf{D}^-(X, \mathscr{A})$ and $N \in \mathsf{D}(Y, \mathscr{B})$ there is a natural isomorphism*

$$\mathbf{L}f^* \mathbf{R}\mathcal{H}om_{\mathscr{B}}(M, N) \cong \mathbf{R}\mathcal{H}om_{\mathscr{A}}(\mathbf{L}\mathfrak{f}^* M, \mathbf{L}\mathfrak{f}^* N). \tag{3.37}$$

*natural in $M$ and $N$.*

*Proof.* Since $\mathfrak{f}$ is strict, it follows that $\mathfrak{f}^* \mathscr{B} = \mathscr{A}$ and the existence of the morphism

$$\alpha\colon \mathbf{L}f^* \mathbf{R}\mathcal{H}om_{\mathscr{B}}(M, N) \to \mathbf{R}\mathcal{H}om_{\mathscr{A}}(\mathbf{L}\mathfrak{f}^* M, \mathbf{L}\mathfrak{f}^* N) \tag{3.38}$$

follows as in the commutative case, see [15, Remark 0GMX]. If $M$ is a cochain complex of locally projective $\mathscr{A}$-modules, $\alpha$ is an isomorphism. For general $M \in \mathsf{D}^-(X, \mathscr{A})$, one replaces $M$ by a projective resolution. □

Let $\mathsf{T}$ be an $S$-linear triangulated category. If $\mathsf{A} \subset \mathsf{T}$ is a full triangulated subcategory, the *right orthogonal* $\mathsf{A}^\perp$ (resp. *left orthogonal* ${}^\perp\mathsf{A}$) of $\mathsf{A}$ is defined as the full subcategory containing all objects $T \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(A, T[i]) = 0$ (resp. $\mathrm{Hom}_{\mathsf{T}}(T, A[i]) = 0$) for all $A \in \mathsf{A}$ and $i \in \mathbb{Z}$.

The following lemma, a straightforward generalization of [93, Lemma 2.36], shows that the orthogonal complement of an $S$-linear subcategory is again $S$-linear.

**Lemma 3.3.10.** *Let* $\mathfrak{f}\colon (X, \mathscr{A}) \to S$ *be a morphism of schemes, and* $\mathsf{A} \subset \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ *be an* $S$*-linear category such that* $\mathsf{A}$ *is* $S$*-linear. Then* ${}^{\perp}\mathsf{A}$ *and* $\mathsf{A}^{\perp}$ *are* $S$*-linear as well.*

*Proof.* Set $\mathsf{B} = {}^{\perp}\mathsf{A}$. Let $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}(S)$, $M \in \mathsf{B}$ and $N \in \mathsf{A}$. Then Lemma 3.3.8 implies

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(M \otimes_X^{\mathbf{L}} \mathbf{L}f^*\mathcal{F}, N) \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})}(M, \mathbf{L}f^*\mathcal{F}^{\vee} \otimes_X^{\mathbf{L}} N). \tag{3.39}$$

Since $\mathsf{A}$ is $S$-linear, it follows that $\mathbf{L}f^*\mathcal{F}^{\vee} \otimes_X^{\mathbf{L}} N \in \mathsf{A}$. Therefore the right-hand side is zero, because $\mathsf{B} = {}^{\perp}\mathsf{A}$. This eventually implies that $M \otimes_X^{\mathbf{L}} \mathbf{L}f^*\mathcal{F} \in \mathsf{B}$ as well. □

Recall that an ($S$-linear) triangulated subcategory $\mathsf{A}$ of $\mathsf{T}$ is called *right admissible* (resp. *left admissible*) if its embedding functor $\alpha\colon \mathsf{A} \to \mathsf{T}$ admits a right adjoint $\alpha^!$ (resp. a left adjoint $\alpha^*$). We say that $\mathsf{A}$ is *admissible* if it is both, left and right admissible.

A sequence of ($S$-linear) triangulated categories $\mathsf{A}_1, \ldots, \mathsf{A}_m$ is called an *($S$-linear) semiorthogonal decomposition* of $\mathsf{T}$ if

i) for every $i > j$ one has $\mathsf{A}_j \subset \mathsf{A}_i^{\perp}$,

ii) the smallest triangulated subcategory containing all $\mathsf{A}_i$ is $\mathsf{T}$.

In this case we write $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ and every object $T \in \mathsf{T}$ can be decomposed into distinguished triangles

$$T_\ell \to T_{\ell-1} \to A_\ell \to T_\ell[1] \quad \text{for } 1 \le \ell \le m, \tag{3.40}$$

such that $T_m = 0$, $T_0 = T$, and $A_\ell \in \mathsf{A}_\ell$.

**Remark 3.3.11.** We denote by $\mathrm{pr}_\ell\colon \mathsf{T} \to \mathsf{T}$, $T \mapsto A_\ell$ the *$\ell$-th projection functor* of the semiorthogonal decomposition. If the decomposition is $S$-linear, it follows from [97, Lemma 3.1] that $\mathrm{pr}_\ell(M \otimes_X \mathbf{L}f^*\mathcal{F}) = \mathrm{pr}_\ell(M) \otimes_X \mathbf{L}f^*\mathcal{F}$ for $M \in \mathsf{T}$, $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S)$. In other words, the projection functors are $S$-linear.

In order to compare semiorthogonal decompositions we need the notion fo compatibility from [97, Section 3] for exact functors between $S$-linear triangulated categories.

**Definition 3.3.12.** Let $\Phi\colon \mathsf{T} \to \mathsf{T}'$ be an exact functor between triangulated categories with semiorthogonal decompositions $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ and $\mathsf{T}' = \langle \mathsf{A}'_1, \ldots, \mathsf{A}'_m \rangle$. We say that $\Phi$ *is compatible with the semiorthogonal decompositions* if $\Phi(\mathsf{A}_k) \subset \mathsf{A}'_k$.

Admissibility of components of a semiorthogonal decomposition is not automatic. Therefore [97, Definition 2.6] introduced the notion of strong semiorthogonal decompositions.

**Definition 3.3.13.** We call a semiorthogonal decomposition $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ *strong* if the component $\mathsf{A}_k$ is admissible in $\langle \mathsf{A}_k, \ldots, \mathsf{A}_m \rangle$.

We finish this section with a relative criterion for $S$-linear semiorthogonality generalizing [97, Lemma 2.7] to coherent ringed schemes.

**Lemma 3.3.14.** *Let* $\mathfrak{f}\colon (X,\mathscr{A}) \to S$ *be a morphism and* $\mathsf{A},\mathsf{B} \subseteq \mathsf{D}(X,\mathscr{A})$ *be* $S$*-linear admissible subcategories. Then* $\mathsf{A} \subseteq \mathsf{B}^{\perp}$ *if and only if* $\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(\mathsf{B},\mathsf{A}) = 0$.

*Proof.* Assume that $\mathsf{A} \subset \mathsf{B}^{\perp}$. By the proof of [97, Lemma 2.7] it suffices to show that $\mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N,M)) = 0$ for all $P \in \mathsf{D}^{\mathsf{perf}}(S)$, $N \in \mathsf{B}$ and $M \in \mathsf{A}$. Using the adjunction $\mathbf{L}f^* \dashv \mathbf{R}f_*$ and the tensor-hom adjunction, one obtains

$$\mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N,M)) \cong \mathbf{R}\mathrm{Hom}_S(\mathbf{L}f^*P, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(N,M)) \tag{3.41}$$

$$\cong \mathbf{R}\mathrm{Hom}_{\mathscr{A}}(\mathbf{L}f^*P \otimes^{\mathbf{L}}_X N, M) = 0. \tag{3.42}$$

The vanishing in the last step follows from the $S$-linearity of $\mathsf{B}$.

For the converse, note that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_S, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N,M)) &\cong \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(N,M)) \\ &\cong \mathbf{R}\mathrm{Hom}_{\mathscr{A}}(N,M)\end{aligned} \tag{3.43}$$

by adjunction and the fact that the pushforward along the structure morphism $(X,\mathscr{A}) \to X$ is exact. □

### 3.3.3 Induced semiorthogonal decompositions

We treat the question when semiorthogonal decompositions on $\mathsf{D}(X,\mathscr{A})$ and its bounded versions are induced from each other. The exposition is close to [97, Chapter 4] and we only mention the necessary modifications for the noncommutative setting. The only additional assumption is that $\mathscr{A}$ is of finite global dimension (or $X = \operatorname{Spec}\mathbf{k}$) in Lemma 3.3.15 in order to obtain an induced semiorthogonal decomposition on $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$ from one on $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$.

Throughout the section we let $\mathfrak{f}\colon (X,\mathscr{A}) \to S$ be a morphism of coherent ringed schemes.

**Lemma 3.3.15.** *Let* $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m \rangle$ *be a strong* $S$*-linear semiorthogonal decomposition. If* $\mathscr{A}$ *has finite global dimension or* $X = \operatorname{Spec}\mathbf{k}$*, then there is a unique* $S$*-linear decomposition*

$$\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}} \rangle, \tag{3.44}$$

*which is compatible with the inclusion* $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) \subset \mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$*. The components are given by* $\mathsf{A}_i^{\mathsf{perf}} = \mathsf{A}_i \cap \mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$.

*Proof.* As in the commutative case, [114, Proposition 1.10] can be applied, because by Lemma 3.3.3 homologically finite objects coincide with perfect objects. The uniqueness follows from [97, Lemma 3.3]. Since the pullback of a perfect object is perfect, and for every perfect object $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(X)$, the tensor product $M \otimes_X \mathcal{F}$ for $M \in \mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$ is perfect, it follows from the $S$-linearity of $\mathsf{A}_i$ that also $\mathsf{A}_i^{\mathsf{perf}}$ is $S$-linear. □

The next step was already provided for $\infty$-enhanced categories by [116, Lemma 3.12]. We restrict ourselves to the derived category of coherent ringed schemes.

**Lemma 3.3.16.** *Assume that* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m \rangle$ *is an* $S$*-linear semiorthogonal decomposition.*

*i) There is a unique* $S$*-linear semiorthogonal decomposition* $\mathsf{D}(S, \mathscr{A}) = \langle \widehat{\mathsf{A}}_1, \dots, \widehat{\mathsf{A}}_m \rangle$ *compatible with the inclusion* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) \subset \mathsf{D}(S, \mathscr{A})$.

*ii) If the semiorthogonal decomposition was induced by one of* $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$*, where the projection functors* $\mathrm{pr}_i \colon \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ *have finite cohomological amplitude, then the semiorthogonal decomposition of* $\mathsf{D}(X, \mathscr{A})$ *is compatible with the inclusion* $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \subset \mathsf{D}(X, \mathscr{A})$ *as well.*

*Proof.* Part (i) carries over analogously from [97, Proposition 4.2] using Remark 3.3.1 that $\mathsf{D}(X, \mathscr{A})^{\mathsf{c}} = \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ and $\mathsf{D}(X, \mathscr{A})$ is compactly generated. Note that $\widehat{\mathsf{A}}_i$ is obtained from $\mathsf{A}_i^{\mathsf{perf}} \subset \mathsf{D}(S, \mathscr{A})$ as the smallest triangulated category closed under (arbitrary) direct sums and cones containing $\mathsf{A}_i^{\mathsf{perf}}$. The $S$-linearity follows from the fact that the functor $- \otimes_X \mathbf{L}f^*\mathcal{F} \colon \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ commutes with (arbitrary) direct sums.

Part (ii) also follows as in [97, Proposition 4.2], because every object in $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ admits a resolution by locally projective $\mathscr{A}$-modules, see [44, Proposition 3.7]. □

Without any further assumptions, we obtain an induced semiorthogonal decomposition on $\mathsf{D}^-(X, \mathscr{A})$.

**Lemma 3.3.17.** *Assume that* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}} \rangle$ *is an* $S$*-linear semiorthogonal decomposition. There is a unique* $S$*-linear semiorthogonal decomposition*

$$\mathsf{D}^-(S, \mathscr{A}) = \langle \mathsf{A}_1^-, \dots, \mathsf{A}_m^- \rangle, \tag{3.45}$$

*compatible with the inclusions* $\mathsf{D}^{\mathsf{perf}}(S, \mathscr{A}) \subset \mathsf{D}^-(S, \mathscr{A}) \subset \mathsf{D}(S, \mathscr{A})$*. The components are given by* $\mathsf{A}_i^- = \widehat{\mathsf{A}}_i \cap \mathsf{D}^-(S, \mathscr{A})$.

*Proof.* The projection functors $\mathrm{pr}_i \colon \mathsf{A}_i^{\mathsf{perf}} \to \mathsf{A}_i^{\mathsf{perf}}$ have finite cohomological amplitude by Lemma 3.3.5. One can therefore argue as in [97, Proposition 4.3]. □

### 3.3.4 Base change of semiorthogonal decompositions

We generalize the base change formulas [97, Section 5] for semiorthogonal decompositions to the noncommutative setting. For $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ and $\mathsf{D}(X, \mathscr{A})$ this can be seen as a special case of [116, Lemma 3.15]. There have been several generalizations of Kuznetsov's base change formula, notably [18, Theorem 3.17], and [27, Theorem 3.5] weakening the assumptions on the schemes. Besides the base change formula [93, Theorem 2.46] for certain Azumaya varieties, we are not aware of results for the bounded derived category $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$. However, most of the proofs can be adapted from the commutative case.

First we observe that if Lemma 3.2.5 applies, i.e. if $\mathfrak{f}_i \colon (X_i, \mathscr{A}_i) \to (X, \mathscr{A})$ are two morphisms of coherent ringed schemes such that one of them is strict, one obtains a natural transformation of functors from the diagram (3.12) in $\mathbf{ncSch}_{\mathbf{k}}$.

**Lemma 3.3.18.** *In the situation of* (3.12)*, there is a natural transformation of functors*

$$\mathbf{L}\mathfrak{f}_1^* \circ \mathbf{R}\mathfrak{f}_{2,*} \Rightarrow \mathbf{R}\mathfrak{p}_{1,*} \circ \mathbf{L}\mathfrak{p}_2^*. \tag{3.46}$$

*from* $\mathsf{D}(X_2, \mathscr{A}_2)$ *to* $\mathsf{D}(X_1, \mathscr{A}_1)$*.*

*Proof.* Using the push-pull adjunction [146, Lemma A.6] the construction of the natural transformation translates verbatim from [15, Section 02N6 and Remark 08HY]. □

We specialize now to the situation mentioned in Remark 3.2.6 Assume that $h\colon T \to S$ is a morphism of schemes and $\mathfrak{f}\colon (X, \mathscr{A}) \to S$ is a morphism of coherent ringed schemes. Consider the base change

$$\begin{array}{ccc} (X_T, \mathscr{A}_T) & \xrightarrow{\mathfrak{h}_T} & (X, \mathscr{A}) \\ \downarrow{\scriptstyle \mathfrak{f}_T} & & \downarrow{\scriptstyle \mathfrak{f}} \\ T & \xrightarrow{h} & S \end{array} \tag{3.47}$$

of $\mathfrak{f}$ along $h\colon T \to S$. Lemma 3.2.5 implies that $\mathscr{A}_T = h_T^* \mathscr{A}$.

**Definition 3.3.19.** A morphism $h\colon T \to S$ is *faithful* for $\mathfrak{f}$ if the natural transformation of functors $\mathbf{L}h^* \circ \mathbf{R}\mathfrak{f}_* \Rightarrow \mathbf{R}\mathfrak{f}_{T,*} \circ \mathbf{L}\mathfrak{h}_T^*$ from $\mathsf{D}(X, \mathscr{A})$ to $\mathsf{D}(T)$ is an equivalence.

**Lemma 3.3.20.** *If $\mathscr{A}$ is a flat $\mathcal{O}_X$-algebra and $f\colon X \to S$ is flat, then every morphism $h\colon T \to S$ is faithful for $\mathfrak{f}$.*

*Proof.* If $\mathscr{A}$ is flat over $X$, a K-flat resolution $F^\bullet$ of an $\mathscr{A}$-module $M \in \mathsf{D}(X, \mathscr{A})$ is also flat over $X$. Hence $\mathbf{L}\mathfrak{h}_T^* = \mathbf{L}h_T^*$. Thus the commutative base change isomorphism applies. □

In the following we will always assume that $h\colon T \to S$ is faithful for $\mathfrak{f}\colon (X, \mathscr{A}) \to S$.

**Perfect complexes.** For base changing perfect complexes, one uses that $\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ is generated by 'box tensors' of the components $\mathsf{D}^{\mathsf{perf}}(T)$ and $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ inside the unbounded derived category $\mathsf{D}(X_T, \mathscr{A}_T)$. This holds in the full generality of $\infty$-categories by [116, Lemma 2.7]. We provide the small modifications to apply the proof of [97, Lemma 5.2].

**Lemma 3.3.21.** *The category $\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ is the minimal triangulated subcategory of $\mathsf{D}(X, \mathscr{A}_T)$ closed under taking direct summands which is generated by the objects*

$$\mathbf{L}\mathfrak{h}_T^* M \otimes_T^{\mathbf{L}} \mathbf{L}f_T^* \mathcal{F}, \quad \text{where } M \in \mathsf{D}^{\mathsf{perf}}(S, \mathscr{A}) \text{ and } \mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T). \tag{3.48}$$

*Proof.* Since $h\colon T \to S$ is a quasi-projective morphism, every coherent $\mathcal{O}_{X_T}$-module $\mathcal{F}$ admits a surjection $f_T^* \mathcal{O}_h(n)^{\oplus k} \twoheadrightarrow \mathcal{F}$ for some $n, k \in \mathbb{Z}$, where $\mathcal{O}_h(1)$ is an $h$-ample line bundle on $T$. Note that by [15, Lemma 0893], the line bundle $f_T^* \mathcal{O}_h(1)$ is $h_T$-ample.

Therefore, every coherent $\mathscr{A}_T$-module $M$ admits a surjection $f_T^* \mathcal{O}_h(n)^{\oplus k} \otimes_T \mathscr{A}_T \twoheadrightarrow M$ of $\mathscr{A}_T$-modules, where we use the right $\mathscr{A}_T$-module structure on $M$. Since $\mathfrak{h}_T$ is strict, $\mathscr{A}_T$

is the pullback of (the locally projective $\mathscr{A}$-module) $\mathscr{A}$. For this reason we find for every $M \in \mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ a bounded above locally projective $\mathscr{A}_T$-resolution $P^\bullet \to M$ such that for each $P^i \cong \mathfrak{h}_T^* Q^i \otimes_T f_T^* \mathcal{E}^i$, where $Q^i$ is locally projective, and $\mathcal{E}^i$ is a locally free $\mathcal{O}_T$-module. This allows us to proceed as in [97, Lemma 5.2]. □

With this lemma, we are ready to define the base change of admissible subcategories in the category of perfect complexes. Given an $S$-linear semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}} \rangle, \tag{3.49}$$

we define $\mathsf{A}_{iT}^{\mathsf{perf}}$ to be the smallest triangulated subcategory, closed under direct summands, which contains all objects of the form

$$\mathbf{L}\mathfrak{h}_T^* M \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{F} \quad \text{for } M \in \mathsf{A}_i^{\mathsf{perf}}, \mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T). \tag{3.50}$$

**Proposition 3.3.22.** *From the $S$-linear semiorthogonal decomposition* (3.49) *one obtains a $T$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T) = \langle \mathsf{A}_{1T}^{\mathsf{perf}}, \dots, \mathsf{A}_{mT}^{\mathsf{perf}} \rangle \tag{3.51}$$

*compatible with* $\mathbf{L}\mathfrak{h}_T^*$.

*Proof.* Given $\mathbf{L}\mathfrak{h}_T^* M_i \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{F} \in \mathsf{A}_{iT}^{\mathsf{perf}}$ and $\mathbf{L}\mathfrak{h}_T^* M_j \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{G} \in \mathsf{A}_{jT}^{\mathsf{perf}}$, we assume that $i > j$. Then

$$\begin{aligned}
&\mathbf{R} f_{T,*} \mathbf{R}\mathcal{H}om_{\mathscr{A}_T}(\mathbf{L}\mathfrak{h}_T^* M_i \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{F}, \mathbf{L}\mathfrak{h}_T^* M_j \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{G}) \\
&\cong \mathbf{R} f_{T,*} \left( \mathbf{R}\mathcal{H}om_{\mathscr{A}_T}(\mathbf{L}\mathfrak{h}_T^* M_i, \mathbf{L}\mathfrak{h}_T^* M_j) \otimes_{X_T}^{\mathbf{L}} (\mathbf{L} f_T^* \mathcal{F})^\vee \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{G} \right) && \text{(Lemma 3.3.8)} \\
&\cong \mathbf{R} f_{T,*} \left( \mathbf{L} h_T^* \mathbf{R}\mathcal{H}om_{\mathscr{A}_T}(M_i, M_j) \otimes_{X_T}^{\mathbf{L}} \mathbf{L} f_T^* (\mathcal{F}^\vee \otimes_X^{\mathbf{L}} \mathcal{G}) \right) && \text{(Lemma 3.3.8)} \\
&\cong \mathbf{R} f_{T,*} \mathbf{L} h_T^* \mathbf{R}\mathcal{H}om_{\mathscr{A}_T}(M_i, M_j) \otimes_X^{\mathbf{L}} \mathcal{F}^\vee \otimes_X^{\mathbf{L}} \mathcal{G} && \text{(projection formula)} \\
&\cong \mathbf{L} h^* \mathbf{R} f_* \mathbf{R}\mathcal{H}om_{\mathscr{A}_T}(M_i, M_j) \otimes_X^{\mathbf{L}} \mathcal{F}^\vee \otimes_X^{\mathbf{L}} \mathcal{G} && \text{(base change)} \\
&= 0 && \text{(Lemma 3.3.14).}
\end{aligned}$$

By Lemma 3.3.14 this is sufficient in order to show $\mathsf{A}_{jT}^{\mathsf{perf}} \subset \mathsf{A}_{iT}^{\mathsf{perf}\perp}$.

The construction (3.50) of $\mathsf{A}_{iT}^{\mathsf{perf}}$ implies $T$-linearity and $\mathbf{L}\mathfrak{h}_T^*(\mathsf{A}_i^{\mathsf{perf}}) \subseteq \mathsf{A}_{iT}^{\mathsf{perf}}$. Generation follows Lemma 3.3.21. □

**The unbounded derived category.** We extend [97, Proposition 5.3] to the noncommutative setting.

**Proposition 3.3.23.** *Let $\mathsf{D}(X, \mathscr{A}) = \langle \widehat{\mathsf{A}}_1, \dots, \widehat{\mathsf{A}}_m \rangle$ be an $S$-linear semiorthogonal decomposition. There is a $T$-linear semiorthogonal decomposition*

$$\mathsf{D}(X_T, \mathscr{A}_T) = \langle \widehat{\mathsf{A}}_{1,T}, \dots, \widehat{\mathsf{A}}_{m,T} \rangle \tag{3.52}$$

*compatible with* $\mathbf{L}\mathfrak{h}_{T,*}$ *and* $\mathbf{L}\mathfrak{h}_T^*$.

*Moreover, if the decomposition is induced from a semiorthogonal decomposition* (3.49) *on* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$*, the new semiorthogonal decomposition* (3.52) *is compatible with the one constructed in Proposition 3.3.22.*

*Proof.* Recall from the proof of Lemma 3.3.16 that by construction $\widehat{A}_{iT}$ is obtained as the closure of $\mathsf{A}_{iT}^{\mathsf{perf}}$ in $\mathsf{D}(X_T, \mathscr{A}_T)$ under direct sums and iterated cones. In particular, it contains all homotopy colimits of perfect complexes in $\mathsf{A}_{iT}^{\mathsf{perf}}$.

The compatibility with the pullback follows now directly from $\mathbf{L}\mathfrak{h}_T^*\mathsf{A}_i^{\mathsf{perf}} \subset \mathsf{A}_{iT}^{\mathsf{perf}} \subset \widehat{\mathsf{A}}_{iT}$, where the first inclusion follows from the compatible base change in Proposition 3.3.22.

Next consider the pushforward. By Lemma 3.3.21, we can consider $M \in \mathsf{A}_i^{\mathsf{perf}}$ and $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T)$. It is to show that $\mathbf{R}\mathfrak{h}_{T,*}(\mathbf{L}\mathfrak{h}_T^*M \otimes^{\mathbf{L}}_{X_T} \mathbf{L}f_T^*\mathcal{F}) \in \widehat{\mathsf{A}}_i$. Because $\mathbf{L}\mathfrak{h}_T^*M \otimes^{\mathbf{L}}_{X_T} \mathbf{L}f_T^*\mathcal{F} \cong \mathbf{L}\mathfrak{h}_T^*M \otimes^{\mathbf{L}}_{\mathscr{A}_T} \mathbf{L}\mathfrak{f}_T^*\mathcal{F}$ in $\mathsf{D}^{\mathsf{b}}(X_T, \mathscr{A}_T)$, we can use the noncommutative projection formula [146, Proposition A.6] to obtain

$$\mathbf{R}\mathfrak{h}_{T,*}(\mathbf{L}\mathfrak{h}_T^*M \otimes^{\mathbf{L}}_{\mathscr{A}_T} \mathbf{L}\mathfrak{f}_T^*\mathcal{F}) = M \otimes^{\mathbf{L}}_{\mathscr{A}} (\mathbf{R}\mathfrak{h}_{T,*}\mathbf{L}f_T^*\mathcal{F}). \tag{3.53}$$

The right hand side lies in $\widehat{A}_i$ as $\mathbf{R}h_{T,*}\mathbf{L}f_T^*\mathcal{F} \cong \mathbf{L}f^*\mathbf{R}h_*\mathcal{F} \in \mathsf{D}(X)$, because $h$ is faithful for $\mathfrak{f}$, and $\widehat{\mathsf{A}}_i$ is $S$-linear. Note that by Remark 3.3.7, each component is $S$-linear for pullbacks of all quasi-coherent sheaves $\mathcal{F} \in \mathsf{D}(S)$. □

To give a more precise description we need the following lemma.

**Lemma 3.3.24.** *Let* $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ *be a morphism of coherent ringed schemes, such that* $f$ *is quasi-projective* $M \in \mathsf{D}(X, \mathscr{A})$ *and* $\mathcal{O}_f(1)$ *an* $f$*-ample line bundle on* $X$.

*i) Then* $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$ *if and only if there is a sequence* $\mathcal{O}_f(k_1) \to \mathcal{O}_f(k_2) \to \ldots$ *with* $k_i \to \infty$ *such that* $\operatorname{hocolim} \mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y, \mathscr{B})$.

*ii) Moreover,* $M = 0$ *if and only if there is a sequence* $\mathcal{O}_f(k_1) \to \mathcal{O}_f(k_2) \to \ldots$ *with* $k_i \to \infty$ *such that* $\operatorname{hocolim} \mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) = 0$.

*Proof.* Given a quasi-coherent $\mathscr{A}$-module $M$, we have $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$ if and only if $M \in \mathsf{D}^{[p,q]}(X)$, because the cohomology sheaf of $M$ does not depend on its $\mathscr{A}$-module structure. By [97, Lemma 5.4], this holds if and only if $\operatorname{hocolim} \mathbf{R}f_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y)$. Since $\mathbf{R}\mathfrak{f}_* = \mathbf{R}f_*$, we find in particular that $\operatorname{hocolim} \mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y, \mathscr{B})$.

The second claim follows analogously with Lemma 5.4 of *op. cit.* □

We can give a more precise description of the components $\widehat{\mathsf{A}}_{iT}$ in the semiorthogonal decomposition of Proposition 3.3.23.

**Lemma 3.3.25.** *With the setup as in Proposition 3.3.23 the components $\widehat{\mathsf{A}}_{iT}$ are given by objects $M \in \mathsf{D}(X_T, \mathscr{A}_T)$ such that*

$$\mathbf{R}\mathfrak{h}_{T,*}(M \otimes_T^{\mathbf{L}} \mathbf{L} f_T^* \mathcal{F}) \in \widehat{\mathsf{A}}_i \tag{3.54}$$

*for all $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T)$.*

With statement (ii) of Lemma 3.3.24 the proof is the same as in [97, Proposition 5.3].

**Remark 3.3.26.** With the assumption that the projection functors in the semiorthogonal decompositions of Proposition 3.3.22 have finite cohomological amplitude one obtains a semiorthogonal decomposition of the bounded above derived category

$$\mathsf{D}^-(T, \mathscr{A}_T) = \langle \mathsf{A}_{1T}^-, \ldots, \mathsf{A}_{mT}^- \rangle \tag{3.55}$$

compatible with derived pushforward and pullback. The $T$-linear components are obtained as $\mathsf{A}_{iT}^- = \widehat{\mathsf{A}}_{iT} \cap \mathsf{D}^-(T, \mathscr{A}_T)$.

**The bounded derived category.** We are now ready to formulate the main statement of this section, the generalization of [97, Theorem 5.6] to coherent ringed schemes.

**Theorem 3.3.27.** *Let $\mathfrak{f}\colon (X, \mathscr{A}) \to S$ and $h\colon T \to S$ be morphisms of coherent ringed schemes. Assume that*

$$\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle \tag{3.56}$$

*is an $S$-linear strong semiorthogonal decomposition such that the projection functors have finite cohomological amplitude and $h$ is faithful for $\mathfrak{f}$. If $\mathscr{A}$ has finite global dimension, there is a $T$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(X_T, \mathscr{A}_T) = \langle \mathsf{A}_{1,T}, \ldots, \mathsf{A}_{m,T} \rangle \tag{3.57}$$

*compatible with*

- *the induced semiorthogonal decomposition on $\mathsf{D}(X_T, \mathscr{A}_T)$ and $\mathsf{D}^-(X_T, \mathscr{A}_T)$,*
- *the pullback $\mathbf{L}\mathfrak{h}_T^*\colon \mathsf{D}^{\mathsf{b}}(S, \mathscr{A}) \to \mathsf{D}^-(X_T, \mathscr{A}_T)$, and*
- *the pushforward $\mathbf{R}\mathfrak{h}_{T,*}\colon \mathsf{D}^{\mathsf{b}}(X_T, \mathscr{A}_T) \to \mathsf{D}(S, \mathscr{A})$.*

The proof follows as in [97, Theorem 5.6] with the following lemma for the approximation of the pushforward of bounded quasi-coherent complexes $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$.

**Lemma 3.3.28.** *Let $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of coherent ringed schemes, such that $f$ is quasi-projective $M \in \mathsf{D}(X, \mathscr{A})$ and $\mathcal{O}_f(1)$ an $f$-ample line bundle on $X$. If $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$ and $k \gg 0$, then there is a direct system $\{N_m\}$ in $\mathsf{D}^{[p,q]}(Y, \mathscr{B})$ such that $\mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k)) \cong \mathsf{hocolim}\, N_m$.*

Since every quasi-coherent $\mathscr{A}$-module is a colimit of its coherent submodules, one can use the same argument as in [97, Lemma 2.20] to prove this lemma.

**Remark 3.3.29.** Note that the admissible component $\mathsf{A}_{iT} = \widehat{\mathsf{A}}_{iT}^- \cap \mathsf{D}^-(X_T, \mathscr{A}_T) \subset \mathsf{D}^\mathsf{b}(X_T, \mathscr{A}_T)$ is not the same as the (derived) pullback of $\mathsf{A}_i$. For example, for $\mathsf{D}^\mathsf{b}(X)$ of a smooth projective variety $X$, this would only produce a description of $\mathsf{D}^{\mathsf{perf}}(X_T)$.

## 3.4 Deformation absorption applied to hereditary orders

We start by recalling in Section 3.4.1 the notion of absorption of singularities and deformation absorption of singularities from [101] for a flat family of varieties over a curve. A crucial role is played by $\mathbb{P}^{\infty,2}$-objects, which can only exist in the bounded derived category of a singular variety $X$.

In Section 3.4.2 and Section 3.4.3 we apply this theory to hereditary orders, where the role of the singular fiber is played by the restriction of the order to a ramified point. This yields a semiorthogonal decomposition of the bounded derived category of a hereditary order. In Section 3.4.4 we compare this result to the semiorthogonal decomposition [29, Theorem 4.7] for stacky curves.

### 3.4.1 Background

**$\mathbb{P}^\infty$-objects.** Let $\mathsf{T}$ be a triangulated category and $q \in \mathbb{Z}_{>0}$. We recall [101, Definition 2.6].

**Definition 3.4.1.** An object $S \in \mathsf{T}$ is a $\mathbb{P}^{\infty,q}$*-object* if

i) there is a $\mathbf{k}$-algebra isomorphism $\mathrm{Ext}^\bullet_\mathsf{T}(S,S) \cong \mathbf{k}[\theta]$ with $\deg \theta = q$, and

ii) the induced map $\theta \colon S \to S[q]$ satisfies $\mathsf{hocolim}(S \to S[q] \to S[2q] \to \dots) = 0$ in a cocomplete category $\widehat{\mathsf{T}}$ containing $\mathsf{T}$.

Similarly to [101, Remark 2.7] for the bounded derived category of a projective variety the definition simplifies for finite-dimensional $\mathbf{k}$-algebras.

**Remark 3.4.2.** Let $A$ be a finite-dimensional $\mathbf{k}$-algebra. An object $S \in \mathsf{D}^\mathsf{b}(A)$ is a $\mathbb{P}^{\infty,q}$-object if and only if $\mathrm{Ext}^\bullet_\mathsf{T}(S,S) \cong \mathbf{k}[\theta]$.

Indeed, since $\mathsf{D}(A) \supset \mathsf{D}^\mathsf{b}(A)$ is cocomplete with a compact generator $A$ which satisfies for every $M \in \mathsf{D}^\mathsf{b}(A)$ that $\mathrm{Ext}^\bullet_A(A,M) \cong \mathrm{H}^\bullet(M)$ is bounded above, we can apply [101, Lemma 2.3].

Every $\mathbb{P}^{\infty,q}$-object comes with the distinguished triangle

$$M \to S \xrightarrow{\theta} S[q] \to M[1] \tag{3.58}$$

induced by the non-trivial morphism $\theta \in \mathrm{Hom}_\mathsf{T}(S, S[q]) = \mathrm{Ext}^q_\mathsf{T}(S,S)$. Following [101, Definition 2.8], we call this triangle the *canonical self-extension* of $S$.

**Remark 3.4.3.** The object $M$ can be used to detect whether the triangulated subcategory $\langle S \rangle \subset T$ generated by $S$ is admissible by [101, Lemma 2.10]. For right (resp. left) admissibility one needs $M$ to be *homologically left* (resp. *right*) *finite-dimensional*, that is, $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(M,N)$, (resp. $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(N,M)$) is finite-dimensional for all $N \in \mathsf{T}$, see [102, §4.1] for a definition.

**Overview and definitions.** A (sequence of) $\mathbb{P}^{\infty,q}$-object(s) detects the difference between smooth and singular varieties resp. categories of finite and infinite global dimension. In special cases, they generate a proper subcategory $\mathsf{S} \subset \mathsf{T}$ which absorbs the singularities in the sense of [101, Definition 1.1] for projective varieties. We present an analogous definition for $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(\Lambda)$, where $\Lambda$ is a finite-dimensional $\mathbf{k}$-algebra.

**Definition 3.4.4.** A triangulated subcategory $\mathsf{S} \subset \mathsf{T}$ absorbs singularities of $\mathsf{T}$ if $\mathsf{S}$ is admissible and its complements $\mathsf{S}^{\perp}$ as well as ${}^{\perp}\mathsf{S}$ are smooth and proper.

**Remark 3.4.5.** Note that by [61, Theorem A] $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(\Lambda)$ is always smooth, but not proper unless $\Lambda$ has finite global dimension. This is the same phenomenon as for $\mathsf{D}^{\mathsf{b}}(X)$, where $X$ is a projective variety.

We propose the following definition for a noncommutative smoothing.

**Definition 3.4.6.** Assume that $\mathfrak{f}\colon (\mathcal{X},\mathscr{B}) \to C$ is a morphism of coherent ringed schemes such that $C$ is a smooth pointed curve with fixed (closed) point $o \in C$. If the fiber over $o \in C$ is

$$(\mathcal{X} \times_C \operatorname{Spec}\mathbf{k}(o), \mathscr{B}|_{\operatorname{Spec}\mathbf{k}(o)}) = (X, \mathscr{A}) \tag{3.59}$$

we say that $\mathfrak{f}\colon (\mathcal{X},\mathscr{B}) \to C$ is a *smoothing for* $(X,\mathscr{A})$ if the following conditions are satisfied:

- for each $p \in C \setminus \{o\}$ the fiber $(\mathcal{X} \times \operatorname{Spec}\mathbf{k}(p), \mathscr{B}|_{\operatorname{Spec}\mathbf{k}(p)})$ is a smooth scheme with an Azumaya algebra $\mathscr{B}|_{\operatorname{Spec}\mathbf{k}(p)}$,
- the total space $\mathcal{X}$ is smooth and $f$ is flat, and
- the $\mathcal{O}_X$-algebra $\mathscr{B}$ is flat as an $\mathcal{O}_X$-module, and $\operatorname{gldim}\mathscr{B} = \dim\mathcal{X}$.

Kuznetsov–Shinder [101] provided powerful results for the derived category of such a smoothing, when $\mathscr{A} = \mathcal{O}_X$ and $\mathscr{B} = \mathcal{O}_{\mathcal{X}}$. We recall some of their results here, stressing the link to $\mathbb{P}^{\infty,2}$-objects.

First, they showed that if $X$ admits a smoothing and $S \in \mathsf{D}^{\mathsf{b}}(X)$ is a $\mathbb{P}^{\infty,q}$-object, then $q \in \{1,2\}$, cf. [101, Corollary 4.6].

Second, if a singular projective variety $X$ admits a semiorthogonal collection $S_1,\dots,S_r$ of $\mathbb{P}^{\infty,2}$-objects such that the triangulated subcategory $\mathsf{S} \subset \mathsf{D}^{\mathsf{b}}(X)$ generated by these objects absorbs singularities of $X$, then by [101, Theorem 1.8]:

- the pushforward of $S_1,\dots,S_r$ to any smoothing $\mathcal{X} \to B$ defines a collection of exceptional objects in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$, and

- the triangulated subcategory in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$ *provides a deformation absorption of singularities of $X$ with respect to any smoothing $\mathcal{X} \to B$*, i.e. it is admissible in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$.

In particular the second point implies by Theorem 1.5 of *op. cit.* that there is a $B$-linear semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathcal{X}) = \langle \mathsf{A}_S, \mathsf{B} \rangle \tag{3.60}$$

where $\mathsf{A}_{\mathsf{S}}$ is the triangulated subcategory generated by the pushforward of $\mathsf{S}$, and $\mathsf{B}$ is smooth and proper such that the base change satisfies

$$\mathsf{B}_p \simeq \begin{cases} {}^{\perp}\mathsf{S} & \text{if } p = o, \\ \mathsf{D}^{\mathsf{b}}(\mathcal{X}_p) & \text{else.} \end{cases} \tag{3.61}$$

In Theorem 3.4.11 we provide the last two results in the special case where a hereditary order is viewed as the smoothing of the finite-dimensional algebra $\Lambda_r$ defined in (3.18).

### 3.4.2 Absorption of singularities for the fiber over a ramified point

Let $\mathscr{A}$ be a hereditary order over a curve $C$. Let $o \in C$ be a ramified point with ramification index $r$. From Lemma 3.2.9 we know that the fiber $\mathscr{A}(o)$ is Morita equivalent to the algebra $\Lambda_r$ defined in (3.18).

For each $i \in Q_0 = \{1, \dots, r\}$, denote by $S_i$ the simple $\Lambda_r$-module as defined in (3.20), and by $P_i$ the unique indecomposable projective $\Lambda_r$-module such that $P_i/\mathsf{rad}\, P_i \cong S_i$.

**Theorem 3.4.7.** *With the notation as above let $i \in Q_0$.*

*i) There is a semiorthogonal collection $(S_{i+1}, \dots, S_{i-1})$ of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$.*

*ii) Let*

$$\mathsf{S}_i = \langle S_{i+1}, \dots, S_{i-1} \rangle \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r) \tag{3.62}$$

*be the triangulated subcategory generated by $S_{i+1}, \dots, S_{i-1}$. Then there is a semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle \mathsf{S}_i, P_i \rangle. \tag{3.63}$$

*iii) The triangulated subcategory $\mathsf{S}_i$ absorbs singularities of $\Lambda_r$.*

Note that every $S_i$ has infinite projective dimension and admits a 2-periodic projective resolution of the form

$$C_i^\bullet := (\dots \xrightarrow{\mu_{[i+1,i]}} P_{i+1} \xrightarrow{\mu_{i,i+1}} P_i \xrightarrow{\mu_{[i+1,i]}} P_{i+1} \xrightarrow{\mu_{i,i+1}} P_i), \tag{3.64}$$

where the maps in the cochain complex are given by left multiplication with the elements written above the arrows. See (3.16) for their definition.

**Lemma 3.4.8.** *Let $i \in Q_0$.*

*i) The simple $\Lambda_r$-module $S_i$ is a $\mathbb{P}^{\infty,2}$-object.*

*ii) The triangulated subcategory $\langle S_i \rangle \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ generated by $S_i$ is admissible.*

*Proof.* Let $i \in Q_0$. By Remark 3.4.2, it suffices to show that $\operatorname{Ext}^{\bullet}_{\Lambda_r}(S_i, S_i) \cong \mathbf{k}[\theta]$ with $\deg \theta = 2$. Since $\operatorname{Hom}_{\Lambda_r}(P_{i+1}, S_i) = 0$, it follows from (3.64) that

$$\operatorname{Ext}^k_{\Lambda_r}(S_i, S_i) \cong \begin{cases} \mathbf{k}\theta^k & \text{if } k \in 2\mathbb{Z}_{\geq 0}, \\ 0 & \text{else.} \end{cases} \tag{3.65}$$

The map $\theta \colon S_i \to S_i[2]$, can be explicitly described, using the projective resolution $C_i^{\bullet}$, as the morphism of cochain complexes which is the identity map in each degree below $-1$.

For the admissibility of $\langle S_i \rangle$, we use that the cochain complex $M_i = (P_{i+1} \to P_i) \in \mathsf{D}^{\mathsf{perf}}(\Lambda_r)$ of projective $\Lambda_r$-modules, which is concentrated in degrees $\{-1, 0\}$, represents the third object in the canonical self-extension (3.58) of $S_i$. The algebra $\Lambda_r$ is Gorenstein in the sense of [83, Assumption 0.1], because every injective $\Lambda_r$-module is projective as well. Hence by [102, Proposition 6.9] the cochain complex $M_i$ is homologically left and right finite-dimensional. It follows from [101, Lemma 2.10] that the triangulated category $\langle S_i \rangle$ generated by $S_i$ is admissible. ☐

**Lemma 3.4.9.** *For all $1 \leq i, j \leq r$, one has*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\Lambda_r)}(S_j, S_i[\ell]) = 0 \quad \textit{for all } \ell \in \mathbb{Z}, \tag{3.66}$$

*if and only if $j \notin \{i, i+1\}$.*

*Proof.* From (3.64) and $\operatorname{Hom}_{\Lambda_r}(P_j, S_k) = 0$ unless $j = k$, one obtains that $\operatorname{Ext}^{\bullet}_{\Lambda_r}(S_j, S_k) = 0$ unless $k \in \{j, j+1\}$. Therefore, the collection is semiorthogonal. ☐

**Lemma 3.4.10.** *With the notation from Theorem 3.4.7, let $P_i$ be the indecomposable projective and $I_i$ the indecomposable injective $\Lambda_r$-module associated with the vertex $i \in Q_0$. Then*

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle \mathsf{S}_i, P_i \rangle, \tag{3.67}$$

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle I_i, \mathsf{S}_i \rangle. \tag{3.68}$$

*are semiorthogonal decompositions of $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$.*

*Proof.* By rotational symmetry, it suffices to consider $i = r$. Both collections are semiorthogonal by Lemma 3.4.9, and because

$$\operatorname{Hom}_{\Lambda_r}(P_r, S_j) = \operatorname{Hom}_{\Lambda_r}(S_j, I_r) = 0 \quad \text{for all } 1 \leq j < r. \tag{3.69}$$

It remains to show that $S_1, \ldots, S_{r-1}, P_r$ generate $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$. Since $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$ is generated by $\Lambda_r$, it suffices to show that all indecomposable projective $\Lambda_r$-modules $P_1, \ldots, P_r$ are in the smallest triangulated subcategory $\mathsf{T}$ generated by $S_1, \ldots, S_{r-1}, P_r$.

From the canonical self-extension for $S_i$, we obtain that $M_1 = (P_2 \to P_1), \dots, M_{r-1} = (P_r \to P_{r-1})$ belong to $\mathsf{T}$. For every $i \in 1, \dots, r-1$, one obtains a distinguished triangle

$$P_i \to M_i \to P_{i+1} \to P_i[1]. \tag{3.70}$$

Since $\mathsf{T}$ is closed under taking cones and $P_r \in \mathsf{T}$, the claim follows. ☐

We are now ready to prove the main assertion of this section.

*Proof of Theorem 3.4.7.* Part (i) is a consequence of Lemma 3.4.8 and Lemma 3.4.9. Statement (ii) is part of Lemma 3.4.10.

Lemma 3.4.10 implies in particular that $\mathsf{S}_i$ is admissible, and both ${}^{\perp}\mathsf{S}_i$ as well as $\mathsf{S}_i^{\perp}$ are generated by an exceptional object, hence smooth and proper. Therefore, $\mathsf{S}_i$ absorbs singularities. ☐

### 3.4.3 Deformation absorption of singularities

Throughout this section let $(C, \mathscr{A})$ be a pair of a curve $C$, and a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ with ramification locus $\Delta_{\mathscr{A}} = \{o\}$ a single closed point of ramification index $r \geq 1$. Since orders are generically Azumaya, this can always be achieved for every hereditary $\mathcal{O}_C$-order $\mathscr{A}$ by shrinking $C$ to a Zariski open neighborhood around a chosen ramification point.

Given a closed point $p \in C$, we denote by $i_p\colon \operatorname{Spec} \mathbf{k}(p) \to C$ the inclusion. Consider the base change diagram

$$\begin{array}{ccc} (\operatorname{Spec} \mathbf{k}(p), \mathscr{A}(p)) & \xrightarrow{\mathfrak{i}_p = (i_p, \mathrm{id}_{\mathscr{A}(p)})} & (C, \mathscr{A}) \\ \downarrow{\scriptstyle \mathfrak{f}_p} & & \downarrow{\scriptstyle \mathfrak{f}} \\ \operatorname{Spec} \mathbf{k}(p) & \xrightarrow{\quad i_p \quad} & C \end{array} \tag{3.71}$$

of the structure morphism $\mathfrak{f}\colon (C, \mathscr{A}) \to C$ (see Remark 3.2.2) along $i_p$. Since $\mathfrak{f}$ is an extension and every order over a curve is flat, the morphism $i_p$ is faithful for $\mathfrak{f}$ by Lemma 3.3.20.

If $p \neq o$, the $\mathbf{k}(p)$-algebra $\mathscr{A}(p)$ is a matrix algebra and therefore Morita equivalent to $\mathbf{k}(p)$. From Lemma 3.2.9, one knows that $\mathscr{A}(o)$ is Morita equivalent to the algebra $\Lambda_r$ from (3.18).

Base changing $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ along a given morphism $h\colon T \to C$ makes therefore sense by Theorem 3.3.27, and we can formulate the main result.

**Theorem 3.4.11.** *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order ramified over $\Delta_{\mathscr{A}} = \{o\}$ with ramification index $r \in \mathbb{Z}_{\geq 1}$. For each $i \in \{1, \dots, r\}$ there is a strong $C$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathfrak{i}_{o,*} S_{i+1}, \dots, \mathfrak{i}_{o,*} S_{i-1}, \mathsf{D} \rangle \tag{3.72}$$

*such that*

*i) the sequence* $\mathsf{i}_{o,*}S_{i+1},\dots,\mathsf{i}_{o,*}S_{i-1}$ *is exceptional,*

*ii) the admissible subcategory* $\mathsf{D}$ *is smooth and proper over* $\mathsf{D}^{\mathsf{b}}(C)$*,*

*iii) the fibers of* $\mathsf{D}$ *over* $p \in C$ *are equivalent to* $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathbf{k}(p))$*.*

In light of [101, Theorem 1.5] we say that $\mathsf{S}_i = \langle S_{i+1},\dots,S_{i-1}\rangle$ (from Theorem 3.4.7) *provides a deformation absorption of singularities of* $\Lambda_r$ *with respect to* $(C,\mathscr{A})$, i.e. it is admissible in $\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$.

We split the proof of Theorem 3.4.11 into three steps. In the first step, we show that, as in the commutative case of [101, Theorem 1.8], a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects on the singular fiber pushes forward to an exceptional collection along the noncommutative smoothing. The idea of the proof is similar, but requires the extension to $\mathbf{ncSch_k}$. Let us spell out the details.

**Lemma 3.4.12.** *The semiorthogonal collection of* $\mathbb{P}^{\infty,2}$*-objects* $(S_{i+1},\dots,S_{i-1})$ *of* $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$ *pushes forward to an exceptional collection*

$$(\mathsf{i}_{o,*}S_{i+1},\dots,\mathsf{i}_{o,*}S_{i-1}) \quad \textit{in } \mathsf{D}^{\mathsf{b}}(C,\mathscr{A}). \tag{3.73}$$

*Proof.* By the rotational symmetry of the question, it suffices to consider the case $i = r$.

Since $\mathsf{i}_o$ is the base change of $(C,\mathscr{A}) \to C$ along $i_o\colon \operatorname{Spec}\mathbf{k}(o) \to C$, the object $\mathbf{L}\mathsf{i}_o^*\mathsf{i}_{o,*}S_k$ fits into the distinguished triangle

$$S_k[1] \to \mathbf{L}\mathsf{i}_o^*\mathsf{i}_{o,*}S_k \to S_k \xrightarrow{\theta} S_k[2] \tag{3.74}$$

for Cartier divisors with trivial normal bundle [101, Section 4.2]. The map $S_k \to S_k[2]$ is non-zero, because the pullback $\mathbf{L}\mathsf{i}^*$ preserves perfect complexes and $S_k \notin \mathsf{D}^{\mathsf{perf}}(\Lambda_r)$. It follows from the comparison of this triangle to the canonical self-extension (3.58) that

$$\operatorname{Ext}^\bullet_{\mathscr{A}}(\mathsf{i}_{o,*}S_k,\mathsf{i}_{o,*}S_k) \cong \operatorname{Ext}^\bullet_{\Lambda_r}(\mathbf{L}\mathsf{i}^*\mathsf{i}_{o,*}S_k,S_k) \cong \operatorname{Ext}^\bullet_{\Lambda_r}(M_k,S_k) \cong \mathbf{k}[0]. \tag{3.75}$$

Therefore, each object is exceptional.

Similarly, the distinguished triangle (3.74) leads to the vanishing of $\operatorname{Ext}^\bullet_{(C,\mathscr{A})}(\mathsf{i}_{o,*}S_k,\mathsf{i}_{o,*}S_j)$ for $1 \le j < k \le r-1$, because $M_k = (P_{k+1} \to P_k)$ maps only trivially to $S_j$. □

Together with Theorem 3.4.7 this lemma already shows that $\mathsf{S}_i \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ provides a deformation absorption of singularities with respect to $(C,\mathscr{A})$.

**Remark 3.4.13.** The exceptional sequence (3.73) is not a strong exceptional sequence on the nose, because

$$\operatorname{Ext}^1_{\mathscr{A}}(\mathsf{i}_{o,*}S_k,\mathsf{i}_{o,*}S_{k+1}) \cong \mathbf{k}. \tag{3.76}$$

However, this is the only non-trivial extension group between different exceptional objects. Therefore $(\mathsf{i}_{o,*}S_{i+1},\mathsf{i}_{o,*}S_{i+2}[1],\dots,\mathsf{i}_{o,*}S_{i-1}[r-2])$ is an exceptional sequence such that all extension groups vanish, except in degree zero. Let

$$T_{S,i} = \mathsf{i}_{o,*}S_{i+1} \bigoplus \mathsf{i}_{o,*}S_{i+2}[1] \bigoplus \dots \bigoplus \mathsf{i}_{o,*}S_{i-1}[r-2] \tag{3.77}$$

be the direct sum over these modules. Then

$$\mathbf{R}\mathrm{End}_{\mathscr{A}}(T_{S,i}) \cong \mathbf{k}\mathbf{A}_{r-1}/(\nu_j \nu_{j+1})_{j=i+1,\dots,i-3}, \tag{3.78}$$

where $\mathbf{A}_{r-1}$ is given by the following quiver:

$$\underset{i+1}{\circ} \xrightarrow{\nu_{i+1}} \underset{i+2}{\circ} \xrightarrow{\nu_{i+2}} \underset{i+3}{\circ} \xrightarrow{\nu_{i+3}} \cdots \xrightarrow{\nu_{i-3}} \underset{i-2}{\circ} \xrightarrow{\nu_{i-2}} \underset{i-1}{\circ} \tag{3.79}$$

In the next step we prove that (3.72) is a $C$-linear semiorthogonal decomposition

**Lemma 3.4.14.** *There is a $C$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathsf{i}_{o,*}S_{i+1}, \dots, \mathsf{i}_{o,*}S_{i-1}, \mathsf{D} \rangle. \tag{3.80}$$

*Proof.* A sequence of exceptional objects generates an admissible subcategory by [35, Theorem 3.2]. Hence we obtain a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ with $\mathsf{D} = {}^{\perp}\langle \mathsf{i}_{o,*}S_{i+1}, \dots, \mathsf{i}_{o,*}S_{i-1} \rangle$.

As before we consider only the case $i = r$. For the $C$-linearity, we use that each of the admissible subcategories $\langle \mathsf{i}_{o,*}S_k \rangle$ is $C$-linear. More precisely, for each $S_k \in \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ and every $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(C)$, the projection formula implies

$$(\mathsf{i}_{o,*}S_k) \otimes^{\mathbf{L}}_C f^*\mathcal{F} \cong \mathsf{i}_{o,*}\left(S_k \otimes_{\mathbf{k}} \mathbf{L}(f_o \circ i_o)^*\mathcal{F}\right). \tag{3.81}$$

Since $\mathbf{L}(f_o \circ i_o)^*\mathcal{F}$ can be represented by a bounded cochain complex of $\mathbf{k}$-vector spaces, one obtains $S_k \otimes_{\mathbf{k}} \mathbf{L}(f_o \circ i_o)^*\mathcal{F} \in \langle S_k \rangle$. Since $\mathsf{i}_{o,*}$ is exact and commutes with direct sums, the $C$-linearity of $\langle \mathsf{i}_{o,*}S_k \rangle$ follows.

The $C$-linearity of the complement $\mathsf{D}$ follows from Lemma 3.3.10. □

For the last lemma recall the notion of locally projective $\mathscr{A}$-modules purely of one type from Definition 3.2.11 and the characterization of maximal overorders of $\mathscr{A}$ from Proposition 3.2.12. For $i \in Q_0 = \{1, \dots, r\}$ denote by $\mathscr{B}_i$ the (unique) maximal overorder of $\mathscr{A}$ which is purely of type $i$ at $o$.

**Lemma 3.4.15.** *The component $\mathsf{D}$ in the semiorthogonal decomposition from Lemma 3.4.14 is equivalent to $\mathsf{D}^{\mathsf{b}}(C)$ given by the thick closure of the embedding*

$$\mathsf{j}_{\mathscr{B}_i,*} \colon \mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_i) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}), \tag{3.82}$$

*where $\mathscr{B}_i$ is purely of type $i$ at $o$. In particular, $\mathsf{D}$ is an admissible subcategory.*

*Proof.* Each of the maximal orders $\mathscr{B}_i$ is in fact Azumaya. We consider the case $i = r$. Since $\mathbf{k}$ is algebraically closed, we have $\mathsf{coh}(C, \mathscr{B}_r) \simeq \mathsf{coh}(C)$.

By Lemma 3.2.13, the pushforward and the pullback are both exact, and the pushforward is fully faithful. This implies that

$$\mathsf{j}_{\mathscr{B}_r,*}\colon \mathsf{D}^{\mathsf{b}}(C,\mathscr{B}_r) \to \mathsf{D}^{\mathsf{b}}(C,\mathscr{A}) \tag{3.83}$$

is fully faithful on the bounded derived category as well.

We need to show that $\mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^{\mathsf{b}}(C,\mathscr{B}_r) = \mathsf{D}$. Using Lemma 3.3.14, one has to show that $M \in \mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^{\mathsf{b}}(C,\mathscr{B}_r)$ if and only if $\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathsf{i}_{o,*}S_k) = 0$ for every $k = 1,\dots,r-1$. Note that $\mathfrak{f}$ is an extension. Therefore $\mathbf{R}f_*$ is the identity.

Since $M \in \mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$, we can calculate $\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathsf{i}_{o,*}S_k)$ by replacing $M$ by a bounded cochain complex of projective $\mathscr{A}$-modules. Moreover $\mathsf{i}_{o,*}S_k \in \mathsf{coh}(C,\mathscr{A})$ and it is supported at $o \in C$. Thus, it suffices to show

$$\mathbf{R}\mathrm{Hom}_{\widehat{\mathscr{A}_o}}(\widehat{M}_o, \mathsf{i}_{o,*}S_k) = 0. \tag{3.84}$$

Assume that $M \in \mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^b(C,\mathscr{B}_r)$. As $\mathscr{B}_r$ is purely of type $r$ at $o$, it follows from Proposition 3.2.12 and its proof that $\widehat{\mathscr{B}_r} \cong \mathrm{End}_{\widehat{\mathcal{O}}_{C,o}}(L_o^{(r)})$, where $L_o^{(r)}$ is the indecomposable projective $\widehat{\mathscr{A}_{C,o}}$-module defined in (3.15). Hence $\widehat{M}_o$ can be expressed as an iterated cone of direct sums of $L_o^{(r)}$. Since $k \neq r$, it follows that $M \in \mathsf{D}$ as

$$\mathbf{R}\mathrm{Hom}_{\widehat{\mathscr{A}_o}}(L_o^{(r)}, \mathsf{i}_{o,*}S_k) = \mathrm{Hom}_{\widehat{\mathscr{A}_o}}(L_o^{(r)}, \mathsf{i}_{o,*}S_k) = 0. \tag{3.85}$$

Vice versa, assume that $M \in \mathsf{D}$. Restricting to $(\mathrm{Spec}(\widehat{\mathcal{O}}_{C,o}), \widehat{\mathscr{A}_o})$, we can represent $\widehat{M}_o$ by a bounded cochain complex $Q^\bullet$ of projective $\widehat{\mathscr{A}_o}$-modules. Since we have by assumption that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\widehat{\mathscr{A}_o})}(Q^\bullet, \mathsf{i}_{o,*}S_k) = 0$ for every $i = 1,\dots,r-1$, the cochain complex $Q^\bullet$ belongs to the subcategory generated by $L_o^{(r)}$. It follows that $M$ must belong to the $C$-linear subcategory of $\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$ generated by $\mathscr{B}_r$, i.e. $M \in \mathsf{j}_{r,*}\mathsf{D}^{\mathsf{b}}(C,\mathscr{B}_r)$.

The right admissibility of $\mathsf{D}$ is automatic and left admissibility follows from the adjunction $\mathsf{j}^*_{\mathscr{B}_r} \dashv \mathsf{j}_{\mathscr{B}_r,*}$. ☐

*Proof of Theorem 3.4.11.* The fact that the semiorthogonal decomposition (3.72) is strong follows from Lemma 3.4.15. We have shown in Lemma 3.4.12 that $(\mathsf{i}_{o,*}S_{i+1},\dots,\mathsf{i}_{o,*}S_{i-1})$ is an exceptional collection in $\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$. From the equivalence $\mathsf{D} \simeq \mathsf{D}^{\mathsf{b}}(C)$ of Lemma 3.4.15, it follows that $\mathsf{D}$ is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$. For the third point, note that over each point $p \neq o$, the restriction $\mathscr{A}(p)$ is Azumaya, and hence Morita equivalent to a point. Moreover, $\mathsf{D}_o$ is the admissible subcategory generated by the $r$-th indecomposable projective $\Lambda_r$-module, which is exceptional. ☐

**Remark 3.4.16.** There is a two-dimensional analogue given by *tame* orders of global dimension 2. By [120, Theorem 1.1] a tame order on a surface is uniquely determined by its overorders. Moreover Theorem 1.14 of *op. cit.* hints to a similar decomposition of the derived category of a tame order as in Lemma 3.4.15 using a maximal overorder. The precise shape of such a decomposition must take into account that maximal orders on surfaces are not necessarily Azumaya. In light of the recently developed stacks–orders dictionary [62] in dimension two, such a decomposition would be interesting. For example, it may extend the semiorthogonal decomposition for root stacks to more complicated stacky surfaces.

**Periodicity of the semiorthogonal decomposition.** As an application we show that the semiorthogonal decomposition (3.72) of $\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$ is $2r$-periodic. Let

$$\mathsf{T} = \langle \mathsf{A}, \mathsf{B} \rangle \tag{3.86}$$

be a semiorthogonal decomposition. We denote the *right dual semiorthogonal decomposition* by $\mathsf{T} = \langle \mathsf{B}, \mathbb{R}_{\mathsf{B}}\mathsf{A} \rangle$, where $\mathbb{R}_{\mathsf{B}}$ is the right mutation functor as defined in [35, §2]. By [33, Definition 4.2] a semiorthogonal decomposition $\mathsf{T} = \langle \mathsf{A}, \mathsf{B} \rangle$ is *$N$-periodic* if the $N$th right dual is again the original decomposition.

In [33, Section 4] the periodicity of a semiorthogonal decomposition for the derived category of a root stack is studied. We provide the same result for the semiorthogonal decomposition (3.72) using the representation theory of orders. Moreover, this explains the connection between the $r$ different version of (3.72).

**Theorem 3.4.17.** *The semiorthogonal decomposition* (3.72) *is* $2r$*-periodic.*

*Proof.* Start with the semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(C,\mathscr{A}) = \langle \mathsf{i}_{o,*}S_{i+1}, \ldots, \mathsf{i}_{o,*}S_{i-1}, \mathsf{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C,\mathscr{A}) \rangle. \tag{3.87}$$

By the proof of Lemma 3.4.15, the category $\mathsf{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$ is generated by $\{P_i \otimes_C \mathcal{L}_\alpha\}$, where $\{\mathcal{L}_\alpha\}_\alpha$ is a generating set of $\mathsf{D}^{\mathsf{b}}(C)$, and $P_i$ is the locally projective $\mathscr{A}$-module purely of type $i$ at $o$ such that $\widehat{P}_{i,o} \cong L_o^{(i)}$ is indecomposable. See (3.15) for a definition.

By [136, Theorem 1] $\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$ possesses a Serre functor

$$\mathbb{S}_{\mathscr{A}} \colon \mathsf{D}^{\mathsf{b}}(C,\mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(C,\mathscr{A}), \quad M \mapsto M \otimes_{\mathscr{A}} \omega_{\mathscr{A}}[1], \tag{3.88}$$

where the dualizing bimodule is $\omega_{\mathscr{A}} = \mathcal{H}om_X(\mathscr{A},\omega_X)$. We have that $\mathbb{S}_{\mathscr{A}}(P_i) = P_{i+1}[1]$, and $\mathbb{S}_{\mathscr{A}}(\mathsf{i}_{o,*}S_i) = \mathsf{i}_{o,*}S_{i+1}$.

If we denote by $\mathsf{A} = \langle \mathsf{i}_{o,*}S_{i+1}, \ldots, \mathsf{i}_{o,*}S_{i-1} \rangle$, and by $\mathsf{B} = \mathsf{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C,\mathscr{A})$, it follows from [36, Proposition 3.6] that $\mathbb{R}_{\mathsf{B}}(\mathsf{A}) = \mathbb{S}(\mathsf{A})$. Hence, after $2r$ times taking the right dual of $\mathsf{B}$, the category $\mathsf{B}$ is replaced by $\mathbb{S}^r(\mathsf{B})$. Since $\mathbb{S}^r(\mathsf{B})$ is generated by $\{\mathbb{S}^r(P_i) \otimes_C \mathcal{L}_\alpha\}_\alpha$, and $\mathsf{B}$ is triangulated, it follows that $\mathbb{S}^r(\mathsf{B}) = \mathsf{B}$. Similarly, we have that $\mathbb{S}^r(\mathsf{A}) = \mathsf{A}$. □

### 3.4.4 The dictionary between hereditary orders and smooth root stacks

There is a stacks–orders dictionary in dimension one and two [51, 17, 62].

Let $C$ be a quasi-projective curve. By [51, Corollary 7.8], resp. Section 4.2.2 in the language of root stacks, the dictionary relates the following two objects.

i) Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order with ramification divisor $\Delta = \{p_1, \ldots, p_m\}$ and ramification indices $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, and

ii) denote by $\pi\colon \sqrt[\mathbf{r}]{C;\Delta} \to C$ the iterated root stack over $C$ by doing the $r_i$-th root construction at $p_i \in \Delta_{\mathscr{A}}$.

**Theorem 3.4.18** (Chan–Ingalls)**.** *With the notation as above, there is an equivalence of categories*

$$\mathsf{coh}(C, \mathscr{A}) \simeq \mathsf{coh}(\sqrt[\mathbf{r}]{C;\Delta_{\mathscr{A}}}). \tag{3.89}$$

Therefore, there is a stacky version of Theorem 3.4.11. Assume that $\Delta = \{o\}$ and do the $r$-th root construction at $o$. Then the singular fiber is described by

$$\operatorname{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta} \cong \left[\operatorname{Spec}\left(\frac{\mathbf{k}[t]}{(t^r)}\right)/\mu_r\right], \tag{3.90}$$

where $\mu_r$ is the group scheme of $r$-th roots of unity, and the $r$-th primitive root acts by multiplication on $t$. Coherent sheaves on the singular fiber are given by

$$\mathsf{coh}([\operatorname{Spec}(\mathbf{k}[t]/(t^r))/\mu_r]) \simeq \mathsf{coh}^{\mu_r}(\mathbf{k}[t]/(t^r)) \simeq \mathsf{coh}(\mathbf{k}[t]/(t^r) * \mu_r). \tag{3.91}$$

Note that the multiplication on the skew group algebra $\mathbf{k}[t]/(t^r) * \mu_r$ is given by $f\zeta \cdot g\xi = f\zeta(g)\zeta\xi$ for $f, g \in \mathbf{k}[t]/(t^r)$ and $\zeta, \xi \in \mu_r$.

On the other hand each closed point in $\sqrt[r]{C;\Delta}$ (i.e. a morphism $\operatorname{Spec}\mathbf{k} \to \sqrt[r]{C;\Delta}$) factors through its residual gerbe $\mathrm{BAut}(p)$, where $p \in C$ is obtained by postcomposing with the map to the coarse moduli space $C$.

The residual gerbe is the reduced substack of (3.90). Hence, we have $\mathrm{B}\mu_r = [\operatorname{Spec}\mathbf{k}(o)/\mu_r]$, and

$$\mathsf{coh}(\mathrm{B}\mu_r) \simeq \mathsf{mod}(\mathbf{k}[\mu_r]). \tag{3.92}$$

The group algebra $\mathbf{k}[\mu_r]$ is semisimple with $r$ simple modules Thus, we obtain $r$ generalized points $(o, \zeta_1), \ldots, (o, \zeta_r)$, each corresponding to one character of $\mathbf{k}[\mu_r]$, following the terminology of [107, Section 2.2]. Denote by $\mathcal{O}_{o,\zeta_i}$ the irreducible $\mu_r$-representation corresponding to $(o, \zeta_i)$.

By [45, Example 2.4.3], the residual gerbe over $o$ embeds as a closed substack into the fiber $\operatorname{Spec}\mathbf{k}(o) \times_C \sqrt[r]{C;\Delta}$ so that we obtain a commutative diagram

$$\begin{array}{ccccc} [\operatorname{Spec}\mathbf{k}(o)/\mu_r] & \xrightarrow{j} & [\operatorname{Spec}(\mathbf{k}[t]/(t^r))/\mu_r] & \xrightarrow{\iota_o} & \sqrt[r]{C;\Delta} \\ & \searrow & \downarrow & & \downarrow \pi \\ & & \operatorname{Spec}\mathbf{k}(o) & \xrightarrow{i_o} & C \end{array} \tag{3.93}$$

On the level of modules the inclusion $j$ corresponds to the surjective $\mathbf{k}$-algebra homomorphism $\mathbf{k}[t]/(t^r) * \mu_r \to \mathbf{k}[\mu_r]$, which identifies the irreducible module $\mathcal{O}_{o,\zeta_i}$ with a simple $\mathbf{k}[t]/(t^r) * \mu_r$-module, denoted by $\widetilde{\mathcal{O}}_{o,\zeta_i}$.

We can use the modules $\widetilde{\mathcal{O}}_{o,\zeta_i}$ for the version of Theorem 3.4.7 for stacky curves.

**Corollary 3.4.19.** *Let $i \in \{1, \dots, r\}$. The collection $\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \dots, \widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ (counted modulo $r$) is a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\mathrm{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta})$.*

*Moreover, the smallest triangulated category $\mathsf{O}_i \subset \mathsf{D}^{\mathsf{b}}(\mathrm{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta})$ containing $\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \dots, \widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ absorbs singularities of $\mathrm{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta}$.*

*Proof.* This follows from Theorem 3.4.7 and the $\mathbf{k}$-algebra isomorphism $\mathbf{k}[t]/(t^r) * \mu_r \cong \Lambda_r$, which identifies the simple $\mathbf{k}[t]/(t^r) * \mu_r$-module $\widetilde{\mathcal{O}}_{o,\zeta_i}$ with the simple $\Lambda_r$-module $S_i$. □

Using [29, Theorem 4.7] we obtain the deformation absorption result for $\sqrt[r]{C;\Delta}$ as well.

**Theorem 3.4.20.** *Let $\sqrt[r]{C;o} \to C$ be smooth stacky curve with nontrivial stabilizer $\mu_r$ over the closed point $o \in C$. For each $i \in \{1, \dots, r\}$ there is a strong $C$-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\sqrt[r]{C;o}) = \langle \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \dots, \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i-1}}, \mathsf{D}\rangle \tag{3.94}$$

*such that*

*i) the sequence $\iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \dots, \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ is exceptional,*

*ii) the admissible subcategory $\mathsf{D}$ is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$,*

*iii) the fibers of $\mathsf{D}$ over $p \in C$ are equivalent to $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathbf{k}(p))$.*

*In other words $\mathsf{O}_i$ provides a deformation absorption of singularities of $[\mathrm{Spec}(\mathbf{k}[t]/(t^r))/\mu_r]$ with respect to the smoothing $\sqrt[r]{C;o} \to C$.*

*Proof.* We have that $\iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_k} = \mathbf{R}(\iota_{o,*} \circ j)\mathcal{O}_{o,\zeta_k}$. Therefore, [29, Theorem 4.7] provides the semiorthogonal decomposition (3.94) for $i = 0$, where $\mathsf{D} = \mathbf{L}\pi^*\mathsf{D}^{\mathsf{b}}(C)$. For $i > 0$ the semiorthogonal decomposition follows by [33, Theorem 4.3]. □

# Chapter 4

# Central curves on noncommutative surfaces

In this chapter, we present our joint work [17] with Pieter Belmans and Okke van Garderen, which will be published in Transactions of the American Mathematical Society.

## 4.1 Introduction

There are rich interactions between noncommutative algebraic geometry and algebraic stacks. First of all, there exists a one-to-one correspondence between

1. hereditary orders on smooth (and separated) curves;
2. smooth (and separated) Deligne–Mumford stacks which are generically curves

by [51], giving rise to an equivalence of categories between the category of coherent sheaves of modules for a hereditary order and the category of coherent sheaves on the associated stacky curve (and vice versa). The curve underlying the hereditary order is the coarse moduli space of the Deligne–Mumford stack, and the Deligne–Mumford stack admits a bottom-up description in terms of root stacks [71], and thus we completely understand their (derived) categories of sheaves.

Recently, this correspondence was extended to a dictionary between

1. tame orders of global dimension 2 on normal surfaces, and
2. Azumaya algebras on certain smooth stacky surfaces.

Again it takes the form of an equivalence between the category of coherent sheaves of modules for the tame order $\mathscr{A}$ on the surface $S$ and the category of coherent sheaves of modules for an Azumaya algebra $\mathscr{A}_{\mathsf{can}}$ on an associated stacky surface $\mathcal{S} = \mathcal{S}_{\mathsf{can}}$ [62], with notation as given in Section 4.2.3. The surface $S$ is the coarse moduli space of $\mathcal{S}$.

**Restricting the dictionary to central curves** In this paper we explain what happens if we take a curve $C$ on the underlying surface $S$, the titular *central curve*, and restrict the tame order $\mathscr{A}$ on $S$ to a sheaf of algebras $\mathscr{A}|_C$ on $C$.

On the other side of the dictionary, we can take the fiber product

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ C & \hookrightarrow & S \end{array} \tag{4.1}$$

and restrict the Azumaya algebra $\mathscr{A}_{\mathsf{can}}$ on $\mathcal{S}$ to an Azumaya algebra on $\mathcal{C}$.

Our first result is that this restriction is well-behaved with respect to the equivalence from [62].

**Theorem 4.A.** *Let $\mathscr{A}$ be a tame order of global dimension 2 on a normal quasiprojective surface $S$, and let $\mathscr{A}_{\mathsf{can}}$ be the Azumaya algebra on the Deligne–Mumford stack $\mathcal{S}$ for which $\mathsf{coh}(S,\mathscr{A}) \simeq \mathsf{coh}(\mathcal{S},\mathscr{A}_{\mathsf{can}})$. Given a curve $C \subset S$ with $\mathcal{C}$ the pullback to the stack as in* (4.1)*, there exists an equivalence*

$$\mathsf{coh}(C,\mathscr{A}|_C) \simeq \mathsf{coh}(\mathcal{C},\mathscr{A}_{\mathsf{can}}|_{\mathcal{C}}). \tag{4.2}$$

The more precise version is given in Theorem 4.3.1. We in fact show that the equivalence from [62] induces equivalences of subcategories of $\mathcal{I}$-torsion objects, for any ideal sheaf $\mathcal{I} \lhd \mathcal{O}_S$, and thus for any subscheme of $S$, not just a curve.

Next, we want a more precise understanding of the sheaf of algebras $\mathscr{A}|_C$. Denoting $\Delta \subset S$ the discriminant of $\mathscr{A}$, we have the following result, confirming the expected behavior of the restriction.

**Proposition 4.B.** *Let $S, \mathcal{A}, C$ be as in Theorem 4.A, and assume moreover that $C$ is integral. Then:*

1. *The sheaf of algebras $\mathscr{A}|_C$ is an order if and only if $C$ is not contained in $\Delta$.*

*Assume that $\mathscr{A}|_C$ is an order. Then its discriminant is $C \cap \Delta$, and in particular:*

2. *$\mathscr{A}|_C$ is Azumaya if and only if $C \cap \Delta = \emptyset$.*
3. *If $C$ is smooth and intersects the $\Delta$ transversely in its smooth locus, then $\mathscr{A}|_C$ is hereditary.*

The proof, and the more precise versions of this result, is given in Section 4.3.2.

These results point towards the extension of the dictionary between hereditary orders and smooth Deligne–Mumford curves from [51], by including also non-hereditary orders, and thus singular Deligne–Mumford curves. At least for orders on curves obtained by restricting tame orders on surfaces, one can explicitly describe the resulting stacky curve. We will apply this principle in two interesting classes of examples, bringing the extended dictionary into view. It is not clear exactly how far the dictionary can be extended, and, moreover, it should not be restricted only to orders on curves restricted from tame orders on surfaces. We leave this for future work.

**Application: noncommutative plane curves** The examples of the restriction results in Theorem 4.A and Proposition 4.B that we will study originate in a construction which a priori does not involve orders on surfaces. Namely, we will first translate a class of objects that can reasonably be called "noncommutative plane curves" into this setup of central curves on orders on surfaces. Subsequently, the perspective of central curves and the dictionary provide new geometric proofs and significant extensions of several results in the literature on noncommutative plane curves, obtained by vastly different methods.

Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra, so that $\mathsf{qgr}\,A$ is a noncommutative projective plane. Let $f \in \mathrm{Z}(A)_d$ be a central element of degree $d$. The noncommutative projective variety $\mathsf{qgr}\,A/(f)$ can be considered as a noncommutative plane curve. This is a two-sided notion of noncommutative curve, different from, e.g., the line modules of [9, §6], which are quotients by a left ideal generated by a non-zero element of degree 1.

Noncommutative plane curves defined by degree-2 elements are studied in [79, 135], and in a more specialized situation for degree-3 elements in [84]. The methods in [84, 79, 135] do not generalize to higher degrees.

The Artin–Schelter regular algebras $A$ in all of the cited papers are in fact finite over their centers[1]. Thus, to such an algebra $A$ we can apply the central Proj construction of [104] to obtain an order $\mathscr{A}$ on an associated $\mathbb{P}^2$, as recalled in Theorem 4.4.3. The element $f$ defines a central curve $C \subset \mathbb{P}^2$, and Proposition 4.4.5 describes an equivalence

$$\mathsf{qgr}\,A/(f) \simeq \mathsf{coh}(C, \mathscr{A}|_C). \tag{4.3}$$

This allows us to apply the dictionary, to give a unified treatment of the noncommutative conics and cubics, which we describe below. This approach moreover generalizes easily to other settings, we will discuss two interesting phenomena in Examples 4.6.7 and 4.6.8.

It is classical that any 3-dimensional quadratic Artin–Schelter regular algebra admits a normal element $g \in A_3$, which is very often (e.g., when $A$ is Sklyanin) in fact central [8]. The noncommutative plane curve $\mathsf{qgr}\,A/(g)$ is the plane cubic curve $C$ appearing in the classification of Artin–Schelter regular algebras, with $A/(g)$ being the twisted homogeneous coordinate ring of $C$ [10]. Generically, the center of $A$ is generated by $g$, meaning that no other interesting noncommutative plane curves exist. Asking that $A$ is finite over its center is therefore a natural condition to ensure the existence of noncommutative plane curves.

[1]Except for one exceptional case: by [79, Theorem 3.6], the only 3-dimensional Calabi–Yau Artin–Schelter regular algebra with a central element of degree 2 which is not a graded Clifford algebra or the commutative polynomial algebra, is $\mathbf{k}\langle x,y,z\rangle/(yz-zy+x^2, zx-xz, xy-yx)$, the quantization of the Weyl algebra, which only has the central element $x^2$. We will ignore this case.

**Noncommutative conics in graded Clifford algebras** We will revisit the degree-2 case, studied and classified in [79, 135], from the perspective of central curves. The ambient Artin–Schelter regular algebras we will consider are graded Clifford algebras, see Section 4.5.1 for more details, and these admit a 3-dimensional linear system of noncommutative conics. A central element $f \in \mathrm{Z}(A)_2$ defines a plane curve of degree 1 in the associated $\mathbb{P}^2$ coming from the central Proj construction.

We obtain the following classification, which was obtained by very different methods in [79, Theorem 5.11], and which did not include the local descriptions in Tables 4.1 and 4.2.

**Theorem 4.C.** *There are 6 isomorphism classes of noncommutative conics within 3-dimensional graded Clifford algebras, i.e., up to equivalence there are 6 abelian categories of the form* (4.3)*.*

*Their algebraic resp. stacky properties are summarized in Tables 4.1 and 4.2.*

We refer to Theorems 4.5.15, 4.5.21 and 4.5.24 for the precise version of this theorem:

- Theorem 4.5.15 gives the order-theoretic description and classification up to Morita equivalence within a fixed graded Clifford algebra;
- Theorem 4.5.21 gives the stacky description and classification up to isomorphism within a fixed graded Clifford algebra;
- Theorem 4.5.24 extends this to a global classification.

Tables 4.1 and 4.2 suggest how to extend the dictionary for hereditary orders and root stacks beyond the smooth case. For 5 out of 6 isomorphism classes in Theorem 4.C, the abelian category $\mathsf{qgr}\, A/(f)$ has infinite global dimension, so that the sheaf of algebras $\mathscr{A}|_{\mathbb{P}^1}$ is no longer a hereditary order, resp. the Deligne–Mumford stack $\mathcal{C}$ from Theorem 4.A is no longer smooth.

E.g., the property of being a *tiled* order is seen to correspond to being a *root stack*, possibly in a non-reduced divisor, at least in the setting of noncommutative conics. Understanding the precise shape of the dictionary, relating order-theoretic properties to stacky properties, is beyond the scope of this article, and left for future work.

**Noncommutative (Fermat) cubics in skew polynomial algebras** For a second application we will consider the noncommutative Fermat cubics in 3-dimensional skew polynomial algebras studied by Kanazawa [84]. Op. cit. considers the skew polynomial algebra $A = \mathbf{k}_q[x, y, z]$, where

$$q = \begin{pmatrix} 1 & q_{1,2} & q_{1,3} \\ q_{1,2}^{-1} & 1 & q_{2,3} \\ q_{1,3}^{-1} & q_{2,3}^{-1} & 1 \end{pmatrix} \tag{4.4}$$

and the $q_{i,j}$ are cube roots of unity; with central element $f = x^3 + y^3 + z^3$ the Fermat cubic, and $B := A/(f)$ the homogeneous coordinate ring of the noncommutative plane curve. Then [84, Theorem 1.1] (for $n = 3$) states that the category $\mathsf{qgr}\, B$ is 1-Calabi–Yau if and only if

$$q_{1,2}^{-1} q_{1,3}^{-1} = q_{1,2} q_{2,3}^{-1} = q_{1,3} q_{2,3}. \tag{4.5}$$

We will re-examine this case in Section 4.6.1 using our machinery, and explain how we actually obtain an equivalence $\mathsf{qgr}\, B \simeq \mathsf{coh}\, E$ for the Fermat elliptic curve $E$, which is not mentioned explicitly in [84].

The benefit of our approach is that our methods in fact work for:

1. *every* element in the 3-dimensional linear system (referred to as net) of central cubics $\langle x^3, y^3, z^3 \rangle$;
2. the case where (4.5) *does not hold*.

In the latter case, by [84, Theorem 1.1] $\mathsf{qgr}\, B$ is *not* 1-Calabi–Yau. However, op. cit. shows that it has global dimension 1 for the Fermat cubic, and one can moreover deduce from [84, Proposition 2.4] that its Serre functor is 3-torsion. In Section 4.6.2 we explain how $\mathsf{qgr}\, B$ is the (up to isomorphism unique) tubular weighted projective line of type $(3, 3, 3)$, which is indeed fractional Calabi–Yau of dimension $3/3$, and we extend the result to any central cubic in the linear system $\langle x^3, y^3, z^3 \rangle$. We refer to Proposition 4.6.1 (resp. Proposition 4.6.2) for the precise statements in case (4.5) holds (resp. does not hold).

**Notation and conventions** Throughout we work over an algebraically closed field $\mathbf{k}$ of characteristic 0.

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## 4.2 Dictionaries between orders and stacks

In this section we survey the dictionary between orders and stacks. For hereditary orders on curves this is done in Section 4.2.2, for tame orders on surfaces this is done in Section 4.2.3. We also recall some of the details of the proofs in [62] as they are necessary for the proofs in this paper. To highlight the role that generalized Rees algebras (implicitly) play in [62], we start by recalling their definition in Section 4.2.1, and use them to give an alternative proof of the dictionary in dimension 1.

### 4.2.1 Generalized Rees algebras

Generalized Rees algebras are a construction due to Van Oystaeyen [139] and studied by Reiten–Van den Bergh [121], associating a type of noncommutative Rees algebra to an order. In the rest of the subsection $R$ denotes a noetherian integrally closed domain containing the field $\mathbf{k}$ and $\Lambda$ is a tame $R$-order: an order which is reflexive as a sheaf, and for which the localization $\Lambda_\mathfrak{p}$ is hereditary for each prime $\mathfrak{p} \lhd R$ of height one.

Recall that a *fractional ideal* of $\Lambda$ is a $\Lambda$-submodule $I \subset \mathrm{Frac}(R)\Lambda$ which is reflexive as an $R$-module and satisfies $\mathrm{Frac}(R)I = \mathrm{Frac}(R)\Lambda$, and that $I$ is called *divisorial* if $I_\mathfrak{p}$ is invertible for every prime $\mathfrak{p} \in \operatorname{Spec} R$ of height 1. The operation $I \cdot J = (I \otimes J)^{\vee\vee}$, with $(-)^\vee$ denoting the $R$-dual, endows the set of divisorial ideals with the structure of a group. In particular, for any $n \in \mathbb{Z}$ there is a well-defined symbolic power $I^{(n)}$ given by the $n$th power under this operation.

Reiten–Van den Bergh [121, Chapter 5] define the ramification ideal of the order $\Lambda$ over $R$ as follows. For each height-one prime $\mathfrak{p} \lhd R$ the submodule $\mathsf{rad}\,\Lambda_\mathfrak{p} \cap \Lambda \subset \Lambda$ is a divisorial ideal of $\Lambda$ which contains $\mathfrak{p}\Lambda$. There exist a finite number of such primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \lhd R$ for which this divisorial ideal is not equal to $\mathfrak{p}\Lambda$, which we will denote

$$P_i := \mathsf{rad}\,\Lambda_{\mathfrak{p}_i} \cap \Lambda \subset \Lambda. \tag{4.6}$$

For each of these there exists a minimal integer $e_i \geq 2$ such that $P_i^{(e_i)} \cong (\mathfrak{p}_i\Lambda)^{(1)}$, called the *ramification index* of $\Lambda$ over $\mathfrak{p}_i$.

**Definition 4.2.1.** The *ramification ideal* of $\Lambda$ is the product

$$\mathrm{D}(\Lambda/R) = P_1^{(e_1-1)} \cdots P_n^{(e_n-1)}. \tag{4.7}$$

We obtain the following construction.

**Definition 4.2.2.** The ramification ideal defines the *ramification Rees algebra*

$$\widetilde{\Lambda} := \bigoplus_{n\in\mathbb{Z}} \mathrm{D}(\Lambda/R)^{(n)} T^n, \tag{4.8}$$

By [103, Theorem II.4.38 and Remark II.4.40] this is a maximal $\mathbb{Z}$-graded order over its center, and it moreover follows by the proof of [121, Proposition 5.1(c)] that the order $\widetilde{\Lambda}$ is *reflexive Azumaya*, i.e., it is not just a hereditary order at codimension-one points, but in fact Azumaya. An explicit description of the center is also given in loc. cit. as the so-called *scaled Rees ring*

$$\widetilde{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_n; e_1, \ldots, e_n) := \bigoplus_{k\in\mathbb{Z}} \mathfrak{p}_1^{\lceil k(e_1-1)/e_1 \rceil} \cdots \mathfrak{p}_n^{\lceil k(e_n-1)/e_n \rceil} T^k. \tag{4.9}$$

Because [103, Theorem II.4.38] does not give a complete proof of this fact, we include one for completeness' sake, under the assumption that each prime $\mathfrak{p}_\mathfrak{i}$ is a principal ideal.

**Lemma 4.2.3.** *Let $\Lambda$ be a tame order over a noetherian integrally closed domain $R$, ramified over primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. If each $\mathfrak{p}_i$ is principal, then $\mathrm{Z}(\widetilde{\Lambda}) = \widetilde{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_n; e_1, \ldots, e_n)$.*

*Proof.* We note that the center $\mathrm{Z}(\widetilde{\Lambda})$ is again graded, and its graded summands can be obtained by intersecting the graded summands of $\widetilde{\Lambda}$ with $R$ (in positive degrees) or with a suitable localization of $R$ in negative degrees; we consider only positive degrees for simplicity.

For any $k \geq 1$ the localization of $\mathrm{D}(\Lambda/R)^{(k)}$ at a prime $\mathfrak{p}$ is given by $\mathrm{D}(\Lambda/R)^k_{\mathfrak{p}} = (P_i)^{k(e_i-1)}_{\mathfrak{p}_i}$ if $\mathfrak{p} = \mathfrak{p}_i$ for some $i$ or $\mathrm{D}(\Lambda/R)^k_{\mathfrak{p}} = \Lambda_{\mathfrak{p}}$ otherwise. Note that after localization at $\mathfrak{p}$, taking the reflexive hull of the tensor product of fractional ideals is the identity, which is why we have $\mathrm{D}(\Lambda/R)^k_{\mathfrak{p}} = \mathrm{D}(\Lambda/R)^{(k)}_{\mathfrak{p}}$ in this case. Now since $\mathrm{D}(\Lambda/R)$ is reflexive, it follows that

$$R \cap \mathrm{D}(\Lambda/R)^{(k)} = R \cap \bigcap_{\mathfrak{p} \text{ height } 1} \mathrm{D}(\Lambda/R)^k_{\mathfrak{p}} = R \cap \bigcap_{i=1}^{n} (R_{\mathfrak{p}_i} \cap (P_i)^{k(e_i-1)}_{\mathfrak{p}_i}). \tag{4.10}$$

Letting $k_i, r_i \in \mathbb{N}$ be integers such that $k(e_i - 1) = k_i e_i + r_i$ and $0 < r_i \leq e_i$. It follows from the definition of the ramification index that $P_i^{(k(e_i-1))} = \mathfrak{p}_i^{k_i} P_i^{(r_i)}$, and so in particular

$$R_{\mathfrak{p}_i} \cap (P_i)^{k(e_i-1)}_{\mathfrak{p}_i} = \mathfrak{p}_i^{k_i}(R_{\mathfrak{p}_i} \cap (P_i)^{r_i}_{\mathfrak{p}_i}). \tag{4.11}$$

Now $R_{\mathfrak{p}_i} \cap (P_i)^{r_i}_{\mathfrak{p}_i}$ is a proper ideal of $R_{\mathfrak{p}_i}$ which includes the maximal ideal $R_{\mathfrak{p}_i} \cap (P_i)^{e_i}_{\mathfrak{p}_i} = \mathfrak{p}_i R_{\mathfrak{p}_i}$, hence must be equal to it. It follows that $R_{\mathfrak{p}_i} \cap (P_i)^{k(e_i-1)}_{\mathfrak{p}_i} = \mathfrak{p}_i^{k_i+1} R_{\mathfrak{p}_i}$, and therefore

$$R \cap \mathrm{D}(\Lambda/R)^{(k)} = R \cap \bigcap_{i=1}^{n} (\mathfrak{p}_i R_{\mathfrak{p}_i})^{\lceil k(e_i-1)/e_i \rceil} = \mathfrak{p}_1^{\lceil k(e_1-1)/e_1 \rceil} \cap \cdots \cap \mathfrak{p}_n^{\lceil k(e_n-1)/e_n \rceil}, \tag{4.12}$$

where we note that $k_i + 1 = \lceil k(e_i - 1)/e_i \rceil$. By assumption $\mathfrak{p}_i = (a_i)$ for distinct prime elements $a_i \in R$. Hence the intersection is of the claimed form

$$\begin{aligned} \mathfrak{p}_1^{\lceil k(e_1-1)/e_1 \rceil} \cap \cdots \cap \mathfrak{p}_n^{\lceil k(e_n-1)/e_n \rceil} &= (a_1^{\lceil k(e_1-1)/e_1 \rceil} \cdots a_n^{\lceil k(e_n-1)/e_n \rceil}) \\ &= \mathfrak{p}_1^{\lceil k(e_1-1)/e_1 \rceil} \cdots \mathfrak{p}_n^{\lceil k(e_n-1)/e_n \rceil}. \end{aligned}$$ □

In what follows we will write the scaled Rees ring simply as $\widetilde{R}$, leaving the data implicit.

It is often convenient to work with a module-finite version of the Rees algebra, which is constructed as follows. Under the assumption that $\mathfrak{p}_i = (a_i)$ for some $a_i \in R$, we can take $e = \operatorname{lcm}(e_1, \ldots, e_n)$ to find

$$\mathrm{D}(\Lambda/R)^{(e)} = P_1^{(e(e_1-1))} \cdots P_n^{(e(e_n-1))} = \Lambda a \Lambda, \tag{4.13}$$

where $a = a_1^{e(e_1-1)/e_1} \cdots a_n^{e(e_n-1)/e_n} \in R$ is central. We then consider the $\mathbb{Z}/e\mathbb{Z}$-graded algebra

$$\Lambda_e := \frac{\widetilde{\Lambda}}{(1 - aT^e)} = \Lambda \oplus \mathrm{D}(\Lambda/R)T \oplus \cdots \oplus \mathrm{D}(\Lambda/R)^{(e-1)}T^{e-1}. \tag{4.14}$$

It follows by [121, Proposition 5.1] that $\Lambda_e$ is a tame order over its center, and it is again reflexive Azumaya. Applying Lemma 4.2.3 gives a description of the center $R_e := \widetilde{R}/(1-aT^e)$ via the scaled Rees ring. Under additional assumptions, which will hold for the cases we consider in Sections 4.5 and 4.6, this has a particularly nice description.

**Corollary 4.2.4.** *With notation as above suppose that $e_1 = \ldots = e_n = e$. Then the center of $\Lambda_e$ is*

$$R_e = \frac{\widetilde{R}}{(1 - aT^e)} \cong \frac{R[t]}{(t^e - a_1 \cdots a_n)}. \tag{4.15}$$

*Proof.* Write $\mathfrak{p}_i = (a_i)$ as above. If the $e_i$ are all equal to $e$, then the graded summands of the scaled Rees ring are all of the form $(\mathfrak{p}_1 \cdots \mathfrak{p}_n)^{\lceil k(e-1)/e \rceil} T^k = (a_1 \cdots a_n)^{\lceil k(e-1)/e \rceil} T^k$. In particular, $t = a_1 \cdots a_n T$ generates the positive part, hence the quotient $R_e$. This generator satisfies $t^e = (a_1 \cdots a_n)^e T^e = a_1 \cdots a_n$ as claimed. □

Geometrically $\operatorname{Spec} R_e \to \operatorname{Spec} R$ is an $e$-fold covering ramified in each component $\{a_i = 0\}$ with ramification index $e$.

### 4.2.2 Hereditary orders and stacky curves

The structure of hereditary orders on smooth curves is well-understood, since they have an étale-local normal form described as follows. Let $C$ be a smooth curve and $\mathscr{A}$ a hereditary order on $C$. Given a closed point $p$, let $R$ denote the henselisation of $\mathcal{O}_{C,p}$ and $\Lambda := \mathscr{A}_p \otimes_C R$ the henselisation of the stalk. Then [119, Theorem 39.14] shows that $\Lambda$ is isomorphic to an order of the form

$$\Lambda \cong \begin{pmatrix} R & R & \ldots & R \\ \mathfrak{m} & R & \ldots & R \\ \vdots & & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \ldots & R \end{pmatrix}^{(n_1,\ldots,n_r)} \subset \operatorname{Mat}_n(R), \tag{4.16}$$

where the superscript $(n_1, \ldots, n_r)$ refers to a block decomposition of the indicated sizes, as in [119, Definition 39.2]. The number $r$ is called the *ramification index* of $\Lambda$ at $p$.

Every hereditary order on $C$ is ramified at a finite number of points, and the ramification indices of these points determine the order up to Morita equivalence.

Another object which is determined uniquely by the same numerical data are *smooth* stacky curves, by the bottom-up characterization of [71].

**Definition 4.2.5.** A *stacky curve* is a separated Deligne–Mumford stack of dimension 1 with trivial generic stabilizer.

In [51, Corollary 7.8] a dictionary is given which relates hereditary orders to stacky curves.

**Theorem 4.2.6** (Chan–Ingalls)**.** *Let $\mathscr{A}$ be a hereditary order on a smooth curve $C$. Then there exists a unique smooth stacky curve $\mathcal{C}$ whose coarse moduli space is $C$, together with an equivalence*

$$\mathsf{coh}\,\mathcal{C} \simeq \mathsf{coh}(C, \mathscr{A}). \tag{4.17}$$

*Conversely, for every smooth stacky curve $\mathcal{C}$ there exists a hereditary order $\mathscr{A}$ on the coarse moduli space $C$, unique up to Morita equivalence, together with an equivalence* (4.17)*.*

The stack $\mathcal{C}$ is constructed in [51, Theorem 7.7] using an iterative procedure, predating the introduction of root stacks in [45].

For the benefit of the reader we give a more direct construction of this dictionary using the generalized Rees algebras. This is the 1-dimensional version of the construction in [62], and it highlights the implicit role of root stacks in [51].

**Using generalized Rees algebras** Let $C$ again be a smooth curve, and $\mathscr{A}$ a hereditary order on $C$. The canonical bimodule $\omega_{\mathscr{A}}$ is an invertible sheaf, which defines a sheaf of graded algebras

$$\widetilde{\mathscr{A}} := \bigoplus_{n\in\mathbb{Z}} \omega_{\mathscr{A}}^{\otimes -n}. \tag{4.18}$$

The center $\mathrm{Z}(\widetilde{\mathscr{A}})$ is a sheaf of graded commutative algebras on $C$, and $\widetilde{\mathscr{A}}$ can therefore be viewed as a sheaf of graded algebras on the relative spectrum

$$r'\colon \widetilde{C} := \underline{\mathrm{Spec}}_C\Big(\mathrm{Z}(\widetilde{\mathscr{A}})\Big) \longrightarrow C. \tag{4.19}$$

The following lemma explains how $\widetilde{\mathscr{A}}$ globalizes the generalized Rees algebra construction from Section 4.2.1.

**Lemma 4.2.7.** *Over every point $p \in C$ the stalk $\widetilde{\mathscr{A}_p}$ is the generalized Rees algebra of $\mathscr{A}_p$.*

*Proof.* For every $p \in C$ the stalk $\Lambda := \mathscr{A}_p$ is a hereditary order over $R = \mathcal{O}_{C,p}$, and it follows by [130, Proposition 2.7] that

$$\omega_{\mathscr{A},p}^{-1} \cong \omega_R^{-1} \otimes_R \mathrm{D}(\Lambda/R) \cong \mathrm{D}(\Lambda/R). \tag{4.20}$$

It is then immediate that $(\widetilde{\mathscr{A}})_p = \widetilde{\Lambda}$. □

The grading on $\mathrm{Z}(\widetilde{\mathscr{A}})$ induces an action of $\mathbb{G}_{\mathrm{m}}$ on $\widetilde{C}$, and therefore defines a quotient stack

$$r\colon \mathcal{C}_{\mathsf{root}} := [\widetilde{C}/\mathbb{G}_{\mathrm{m}}] \longrightarrow C, \tag{4.21}$$

which is equipped with a reflexive Azumaya algebra $\mathscr{A}_{\mathsf{root}}$ corresponding to the graded reflexive Azumaya $\widetilde{\mathscr{A}}$ under the equivalence between coherent sheaves on $\mathcal{C}_{\mathsf{root}}$ and graded coherent sheaves on $\widetilde{C}$. We have the following description of $(\mathcal{C}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}})$.

**Proposition 4.2.8.** *The stack $\mathcal{C}_{\mathsf{root}}$ is the root stack associated to the ramification data of $\mathscr{A}$, and $\mathscr{A}_{\mathsf{root}}$ is an Azumaya algebra.*

*Proof.* Given any point $p \in C$ let again $\Lambda = \mathscr{A}_p$ denote the order over $R = \mathcal{O}_{C,p}$, so that $\widetilde{\Lambda} = \widetilde{\mathscr{A}_p}$ as in Lemma 4.2.7. Since $R$ is regular it follows by Lemma 4.2.3 that $\mathrm{Z}(\widetilde{\mathscr{A}})_p = \widetilde{R}$. By Corollary 4.2.4 we obtain

$$[\mathrm{Spec}\,\widetilde{R}/\mathbb{G}_{\mathrm{m}}] \cong [\mathrm{Spec}\,R_e/\mu_e] = \left[\mathrm{Spec}\,\frac{R[t]}{(t^e - s)}/\mu_e\right], \tag{4.22}$$

where $s \in R$ is any generator of the maximal ideal of $R$, and $e$ is the ramification index of $\mathscr{A}_p$ at $p$. It follows that $\mathcal{C}_{\mathsf{root}}$ is the root stack over $C$ associated to the ramification data. The stack $\mathcal{C}_{\mathsf{root}}$ is smooth and 1-dimensional, so any reflexive sheaf is automatically locally free, and thus any reflexive Azumaya algebra is automatically Azumaya. □

We aim to show that there is an equivalence as in Theorem 4.2.6.

**Lemma 4.2.9.** *There exists an adjoint pair of equivalences of categories*

$$\mathsf{coh}^{\mathbb{Z}}(\widetilde{C}, \widetilde{\mathscr{A}}) \underset{-\otimes_{\mathscr{A}}\widetilde{\mathscr{A}}}{\overset{(-)_0}{\rightleftarrows}} \mathsf{coh}(C, \mathscr{A}). \tag{4.23}$$

*Proof.* It suffices to show that the adjunction between the two functors induce isomorphisms on objects. For each module $\mathcal{N} \in \mathsf{coh}(C, \mathscr{A})$ the adjunction gives rise to the natural unit map

$$\mathcal{N} \to (\mathcal{N} \otimes_{\mathscr{A}} \widetilde{\mathscr{A}})_0, \quad n \mapsto n \otimes 1. \tag{4.24}$$

It is clear that this is an isomorphism, since $\widetilde{\mathscr{A}_0} = \mathscr{A}$ by construction. Conversely, given a module $\mathcal{M} \in \mathsf{coh}^{\mathbb{Z}}(\widetilde{C}, \widetilde{\mathscr{A}})$, the adjunction gives rise to the natural counit map

$$\mathcal{M}_0 \otimes_{\mathscr{A}} \widetilde{\mathscr{A}} \to \mathcal{M}, \tag{4.25}$$

given by the restriction of the module action on $\mathcal{M}$. It can be checked étale-locally that this is an isomorphism: picking a closed point $p \in C$ and writing $R = \mathcal{O}^{\mathrm{sh}}_{C,p} \cong \mathbf{k}\{u\}$, $\Lambda = \mathscr{A}_p \otimes_C R$, and $M = \mathcal{M}_p \otimes_C R$, it suffices to consider

$$M_0 \otimes_{\Lambda} \widetilde{\Lambda} \to M, \quad m_0 \otimes a \mapsto m_0 a. \tag{4.26}$$

Note that $\Lambda$ has a normal form (4.16), and that we can consider (4.26) after a Morita equivalence ensuring that $n_1 = \ldots = n_r = 1$. To see (4.26) is an isomorphism, we note, using the normal form (4.16), that the generalized Rees algebra is of the form

$$\widetilde{\Lambda} = \bigoplus_{n \in \mathbb{Z}} \Lambda \zeta^n, \tag{4.27}$$

where $\zeta \in \Lambda$ satisfies $\zeta^r = u$, and $\zeta^{-1} = \zeta^{r-1} u^{-1}$ is its inverse in $\widetilde{\Lambda}$. Since every homogeneous $m_n \in M_n$ can be written as $m_n = (m_n \zeta^{-n}) \cdot \zeta^n$, we moreover have $M_n = M_0 \zeta^n$ for each $n \in \mathbb{Z}$. The composition

$$M_0 \otimes_{\mathbf{k}} \mathbf{k}\zeta^n \xrightarrow{\sim} (M_0 \otimes_{\Lambda} \widetilde{\Lambda})_n \xrightarrow{m_0 \otimes a \mapsto m_0 a} M_n, \tag{4.28}$$

is therefore an isomorphism for every $n \in \mathbb{Z}$, and it follows that (4.26) is also an isomorphism. □

It follows by Tsen's theorem and our standing assumption that $\mathbf{k}$ is algebraically closed that the Brauer group of a smooth (stacky) curve is trivial, so that $\mathscr{A}_{\mathsf{root}}$ is necessarily a *split* Azumaya algebra on $\mathcal{C}_{\mathsf{root}}$. Hence, we obtain the following corollary.

**Corollary 4.2.10.** *There are equivalences* $\mathsf{coh}(\mathcal{C}_{\mathsf{root}}) \simeq \mathsf{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}) \simeq \mathsf{coh}(C, \mathscr{A})$.

### 4.2.3 Tame orders of global dimension 2 and Azumaya algebras on stacky surfaces

The 1-dimensional dictionary from [51] was extended to a 2-dimensional version in [62]. We will now recall its statement and some of the details of the proof, as they will be needed for what follows.

**Theorem 4.2.11** (Faber–Ingalls–Okawa–Satriano)**.** *Let $S$ be a normal quasiprojective surface. Let $\mathscr{A}$ be a tame order of global dimension 2. Then there exists a diagram*

$$\begin{array}{ccccc} \mathscr{A}_{\mathsf{can}} & & \mathscr{A}_{\mathsf{root}} & & \mathscr{A} \\ \vdots & & \vdots & & \vdots \\ \mathcal{S}_{\mathsf{can}} & \xrightarrow{c} & \mathcal{S}_{\mathsf{root}} & \xrightarrow{r} & S \end{array} \tag{4.29}$$

*where*

- $r\colon \mathcal{S}_{\mathsf{root}} \to S$ *is a root stack construction;*
- $c\colon \mathcal{S}_{\mathsf{can}} \to \mathcal{S}_{\mathsf{root}}$ *is the canonical stack associated to* $\mathcal{S}_{\mathsf{root}}$*;*
- $\mathscr{A}_{\mathsf{root}}$ *is a reflexive Azumaya algebra on* $\mathcal{S}_{\mathsf{root}}$ *such that* $r_*\mathscr{A}_{\mathsf{root}} \cong \mathscr{A}$*;*
- $\mathscr{A}_{\mathsf{can}}$ *is an Azumaya algebra on* $\mathcal{S}_{\mathsf{can}}$ *such that* $c_*\mathscr{A}_{\mathsf{can}} \cong \mathscr{A}_{\mathsf{root}}$*;*

*such that there exist equivalences*

$$\mathsf{coh}(\mathcal{S}_{\mathsf{can}}, \mathscr{A}_{\mathsf{can}}) \simeq \mathsf{coh}(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}) \simeq \mathsf{coh}(S, \mathscr{A}). \tag{4.30}$$

**The root stack** Consider as in Theorem 4.2.11 a normal connected quasi-projective surface $S$ with a tame $\mathcal{O}_S$-order $\mathscr{A}$ of global dimension 2. Then the order $\mathscr{A}_{\mathsf{root}}$ on the root stack $r\colon \mathcal{S}_{\mathsf{root}} \to S$ is defined via a global version of the generalized Rees algebra construction, see [62, §3] (although the terminology does not appear as such in op. cit.): replacing the ramification ideal by the reflexive dual $\omega_{\mathscr{A}}^{-1}$ of the dualizing bimodule $\omega_{\mathscr{A}}$, one obtains a sheaf of $\mathbb{Z}$-graded algebras

$$\widetilde{\mathscr{A}} = \bigoplus_{n\in\mathbb{Z}} \omega_{\mathscr{A}}^{(-n)}, \tag{4.31}$$

which we can interpret as a $\mathbb{Z}$-graded reflexive Azumaya algebra over the relative affine scheme

$$r'\colon \widetilde{S} := \underline{\mathrm{Spec}}_S\left(\mathrm{Z}(\widetilde{\mathscr{A}})\right) \longrightarrow S. \tag{4.32}$$

The grading on the center yields a $\mathbb{G}_{\mathrm{m}}$-action on $\widetilde{S}$. The map $r'$ is invariant for the action and therefore factors over the quotient to a map

$$r\colon \mathcal{S}_{\mathsf{root}} = [\widetilde{S}/\mathbb{G}_{\mathrm{m}}] \to S. \tag{4.33}$$

The order $\mathscr{A}_{\mathsf{root}}$ of [62, §3] is the sheaf of algebras corresponding to the sheafification $\widetilde{\mathscr{A}}$ along the equivalence $\operatorname{coh} \mathcal{S}_{\mathsf{root}} \simeq \operatorname{coh}^{\mathbb{Z}} \widetilde{S}$. This order satisfies $r_* \mathscr{A}_{\mathsf{root}} \cong \mathscr{A}$ by [62, Lemma 4.4] and furthermore there are equivalences of categories

$$\operatorname{coh}(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}) \underset{r^*(-)\otimes_{r^*\mathscr{A}} \mathscr{A}_{\mathsf{root}}}{\overset{r_*}{\rightleftarrows}} \operatorname{coh}(S, \mathscr{A}). \tag{4.34}$$

We remark that $r$ is an isomorphism over $S \setminus \Delta$, where $\Delta$ denotes the ramification locus of $\mathscr{A}_{\mathsf{root}}$, and that $\mathscr{A}_{\mathsf{root}}$ is isomorphic to $\mathscr{A}$ when restricted to $S \setminus \Delta$.

It will be useful for the rest of the paper to detail the relation between [62] and [121]. Since $S$ is normal, every point in $S$ admits a neighborhood $U \subset S$ such that $R = \Gamma(U, \mathcal{O}_S)$ is an integrally closed domain, equipped with a tame order $\Lambda = \Gamma(U, \mathscr{A})$. Writing $P_1, \ldots, P_n \subset \Lambda$ for the ramified primes over $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \lhd R$ as before, the formula in [130, Proposition 2.7] shows that the dualizing bimodule of $\Lambda$ is given by

$$\omega_\Lambda = \omega_R \otimes_R \left( \prod_{i=1}^{n} P_i \mathfrak{p}_i^{-1} \right). \tag{4.35}$$

As explained in [62, Remark 2.12], the reflexive dual $\omega_\Lambda^{-1}$ is given (as a bimodule) by $\mathrm{D}(\Lambda/R)$ from (4.7). In particular, there is an isomorphism of $\mathbb{Z}$-graded algebras

$$\Gamma(U, \widetilde{\mathscr{A}}) \cong \widetilde{\Lambda}, \tag{4.36}$$

and the root stack is locally given by the map $\widetilde{S} \times_S U \cong \operatorname{Spec} \widetilde{R} \to \operatorname{Spec} R$. If the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are principal, which can be assumed if for example $S$ is locally factorial, then one can again define the finite quotients $\Lambda_e$ and $R_e$. It follows from [62, Corollary 2.16] that there is an isomorphism of stacks

$$\mathcal{S}_{\mathsf{root}} \times_S U \cong [\operatorname{Spec} \widetilde{R}/\mathbb{G}_{\mathrm{m}}] \cong [\operatorname{Spec} R_e/\mu_e], \tag{4.37}$$

which identifies the sheaves of orders associated to $\widetilde{\Lambda}$ and $\Lambda_e$ with $\mathscr{A}_{\mathsf{root}}|_{\mathcal{S}_{\mathsf{root}} \times_S U}$.

**Remark 4.2.12.** By inspecting the construction of the root stack using generalized Rees algebras in [62] for orders on surfaces, as just recalled, or Section 4.2.2 for orders on curves, and the papers upon which this construction builds in the affine case, namely [121, §5] (see also [130, §2]) one notices that the construction should also work in a higher-dimensional setting. It would then give rise to an equivalence between

- the category of left modules over a tame order of finite global dimension over a normal variety;
- the category of left modules over a reflexive Azumaya algebra (which is Azumaya if the ramification divisor is smooth) on the associated root stack.

We do not address this generalization, as we are only concerned with 1- and 2-dimensional objects.

**The canonical stack** It is shown in [62] that $\mathcal{S}_{\mathsf{root}}$ is a Deligne–Mumford stack with linearly reductive quotient singularities. Such a stack has a natural resolution given by the canonical stack $c\colon \mathcal{S}_{\mathsf{can}} \to \mathcal{S}_{\mathsf{root}}$ of [141]. Let us recall the complete-local description of the canonical stack given in [62, §4]. Around each point $p \in S$ we can consider a complete-local chart $\operatorname{Spec} R \subset S$ such that $\mathcal{S}_{\mathsf{root}} \to S$ can be locally presented as

$$[\operatorname{Spec} R_e/\mu_e] \to \operatorname{Spec} R. \tag{4.38}$$

By [62, Proposition 2.24] the ring $R_e$ can be presented as an invariant ring $T^H \subset T$ of a finite subgroup $H < \mathrm{GL}_2(k)$ acting on $T = k[\![x, y]\!]$. The canonical stack is constructed such that $c\colon \mathcal{S}_{\mathsf{can}} \to \mathcal{S}_{\mathsf{root}}$ pulls back along the étale cover $\operatorname{Spec} R_e \to [\operatorname{Spec} R_e/\mu_e]$ to the map

$$[\operatorname{Spec} T/H] \to \operatorname{Spec} R_e. \tag{4.39}$$

This local description shows that this is a birational modification in codimension 2. In [62] the authors construct the $\mathcal{O}_{\mathcal{S}_{\mathsf{can}}}$-order $\mathscr{A}_{\mathsf{can}}$ as the reflexive hull

$$\mathscr{A}_{\mathsf{can}} := (c^*\mathscr{A}_{\mathsf{root}})^{\vee\vee}, \tag{4.40}$$

of the pullback of $\mathscr{A}_{\mathsf{root}}$, and show that this is a sheaf of Azumaya algebras satisfying $c_*\mathscr{A}_{\mathsf{can}} \cong \mathscr{A}_{\mathsf{root}}$.

The order can be locally described as follows. Writing $\Lambda_e$ for the pullback of $\mathscr{A}_{\mathsf{root}}$ to $\operatorname{Spec} R_e$, the pullback of $\mathscr{A}_{\mathsf{can}}$ to $\operatorname{Spec} T$ is given by the reflexive closure

$$\Gamma = (\Lambda_e \otimes_{R_e} T)^{\vee\vee}. \tag{4.41}$$

There is an induced action of $H$ on $\Gamma$ and it is shown in [62] that there is an equivalence

$$\mathsf{mod}^H \Gamma \underset{-\otimes_{\Lambda_e}\Gamma}{\overset{\operatorname{Hom}^H_\Gamma(\Gamma,-)}{\rightleftarrows}} \mathsf{mod}\,\Lambda_e. \tag{4.42}$$

It is shown in [62] that this induces a global equivalence

$$\mathsf{coh}(\mathcal{S}_{\mathsf{can}}, \mathscr{A}_{\mathsf{can}}) \underset{c^*(-)\otimes_{c^*\mathscr{A}_{\mathsf{root}}}\mathscr{A}_{\mathsf{can}}}{\overset{c_*}{\rightleftarrows}} \mathsf{coh}(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}). \tag{4.43}$$

We end this section by explicitly spelling out some properties of the bottom-up construction.

**Proposition 4.2.13.** *Let $p \in S$ and consider the scheme $S_p = \operatorname{Spec} \mathcal{O}_{S,p} \subset S$ associated to the local ring $\mathcal{O}_{S,p}$ at $p$. Then:*

1. *$p \notin \Delta$ if and only if $\mathcal{S}_{\mathsf{root}} \times_S S_p \to S_p$ is an isomorphism,*
2. *$p \notin (\Delta^{\mathrm{sing}} \cup S^{\mathrm{sing}})$ if and only if $\mathcal{S}_{\mathsf{can}} \times_S S_p \to \mathcal{S}_{\mathsf{root}} \times_S S_p$ is an isomorphism.*

*Proof.* (1) If $p \notin \Delta$ then it follows that the algebra $A = \mathscr{A} \otimes_S \mathcal{O}_{S,p}$ is unramified on $R = \Gamma(S_p, \mathcal{O}_{S,p})$, hence $\omega_A^{-1} = A$ is trivial. The Rees algebra is therefore of the form $\widetilde{A} = \bigoplus_{k\in\mathbb{Z}} A$, and its center is simply $\widetilde{R} = R[T^{\pm}]$. Therefore the $\mathbb{G}_{\mathrm{m}}$-action on $\operatorname{Spec}\widetilde{R} = \operatorname{Spec} R \times \mathbb{G}_{\mathrm{m}}$ is free and induces an isomorphism

$$\mathcal{S}_{\mathsf{root}} \times_S S_p = [\operatorname{Spec}\widetilde{R}/\mathbb{G}_{\mathrm{m}}] \cong \operatorname{Spec} R = S_p. \tag{4.44}$$

(2) The map $\mathcal{S}_{\mathsf{can}} \to \mathcal{S}_{\mathsf{root}}$ is by construction an isomorphism outside of the singularities of $\mathcal{S}_{\mathsf{root}}$. Hence it suffices to prove that $\mathcal{S}_{\mathsf{root}} \times_S S_p$ is smooth if and only if $p$ does not hit the singular loci of $S$ and $\Delta \subset S$. If $p \in S$ is smooth with $p \notin \Delta$, then it follows from (1) that $\mathcal{S}_{\mathsf{root}} \times_S S_p \cong S_p$ is again smooth, and conversely, so we are done. Hence we can suppose that $p \in S$ is a smooth point with $p \in \Delta$ and $p \notin \Delta^{\mathrm{sing}}$.

Writing again $A = \mathscr{A} \otimes_S \mathcal{O}_{S,p}$ for the order on $R = \Gamma(S_p, \mathcal{O}_{S,p})$, we note that the condition $p \notin \Delta^{\mathrm{sing}}$ is equivalent to the fact that $A$ is ramified at exactly one prime $\mathfrak{p}_1$ with some ramification index $e \in \mathbb{N}$. Since $p$ is smooth, $R$ is a UFD and hence $\mathfrak{p}_1 = (a_1)$ for some irreducible element $a_1$. It follows from Corollary 4.2.4 that

$$\mathcal{S}_{\mathsf{root}} \times_S S_p = [\operatorname{Spec} R_e/\mu_e] \cong \left[\operatorname{Spec} \frac{R[t]}{(t^e - a_1)}/\mu_e\right]. \tag{4.45}$$

Now because $p \notin \Delta^{\mathrm{sing}}$ it follows that $R/(a_1)$ is smooth, and hence $R[t]/(t^e - a_1)$ is also smooth (for example by the Jacobian criterion for smoothness). It follows that $\mathcal{S}_{\mathsf{root}} \times_S S_p$ is smooth and hence $\mathcal{S}_{\mathsf{can}} \times_S S_p \to \mathcal{S}_{\mathsf{root}} \times_S S_p$ is an isomorphism. This also proves the converse. □

## 4.3 Restricting to central curves

We now come to the first main result of the paper, which addresses what happens when we take an order $\mathscr{A}$ on a surface $S$, together with a curve $C \subset S$ (or more generally, closed subscheme) and consider the restriction $(C, \mathscr{A}|_C)$. This will usually (but not always) be an order on $C$, and we will explain how the dictionary from Section 4.2.3 can be used to understand $(C, \mathscr{A}|_C)$.

### 4.3.1 Restricting the equivalence

We work in the setting of Section 4.2.3, with $\mathscr{A}$ a tame order of global dimension 2, on a normal quasiprojective surface. Let $C \subset S$ be a closed subscheme, cut out by the ideal sheaf $\mathcal{I} \lhd \mathcal{O}_S$. We will use the notation $C$ even when $\dim C = 0$, because the curve case is the one of interest to us.

We will consider the diagram of fiber products

$$\begin{array}{ccccc} \mathcal{C}_{\mathsf{can}} & \xrightarrow{c_C} & \mathcal{C}_{\mathsf{root}} & \xrightarrow{r_C} & C \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{S}_{\mathsf{can}} & \xrightarrow{c} & \mathcal{S}_{\mathsf{root}} & \xrightarrow{r} & S \end{array} \tag{4.46}$$

induced from the bottom row in (4.29). Restricting the orders on the (stacky) surfaces to each of these closed subschemes (resp. substacks) yields the sheaves of algebras

$$\mathscr{B} := \mathscr{A}|_C, \quad \mathscr{B}_{\mathsf{root}} := \mathscr{A}_{\mathsf{root}}|_{\mathcal{C}_{\mathsf{root}}}, \quad \mathscr{B}_{\mathsf{can}} := \mathscr{A}_{\mathsf{can}}|_{\mathcal{C}_{\mathsf{can}}}. \tag{4.47}$$

on $C$, $\mathcal{C}_{\mathsf{root}}$ and $\mathcal{C}_{\mathsf{can}}$ respectively. We will show that these sheaves of algebras are related via the maps $c_C$ and $r_C$, and that the associated categories of modules are again equivalent. Namely, we have the following precise incarnation of Theorem 4.A.

**Theorem 4.3.1.** *Let $S$ be a normal quasiprojective surface, and let $\mathscr{A}$ be a tame order of global dimension 2. Let $C \subset S$ be a closed subscheme. With the above notation, there exist isomorphisms $c_{C,*}\mathscr{B}_{\mathsf{can}} \cong \mathscr{B}_{\mathsf{root}}$ and $r_{C,*}\mathscr{B}_{\mathsf{root}} \cong \mathscr{B}$, and equivalences of categories*

$$\mathsf{coh}(C, \mathscr{B}) \simeq \mathsf{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{B}_{\mathsf{root}}) \simeq \mathsf{coh}(\mathcal{C}_{\mathsf{can}}, \mathscr{B}_{\mathsf{can}}). \tag{4.48}$$

The precise shape of the functors is given in the proof. We first consider the restriction of the root stack construction.

**Proposition 4.3.2.** *There is an isomorphism $\mathscr{B} \cong r_{C,*}\mathscr{B}_{\mathsf{root}}$ and an equivalence*

$$\mathsf{coh}(C, \mathscr{B}) \simeq \mathsf{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{B}_{\mathsf{root}}). \tag{4.49}$$

*Proof.* Recall that the root stack $\mathcal{S}_{\mathsf{root}}$ is defined as the $\mathbb{G}_{\mathrm{m}}$-quotient of the affine map $r'$ in (4.32), defined by the $\mathbb{Z}$-graded $\mathcal{O}_S$-algebra $\mathcal{O}_{\widetilde{S}} = \mathrm{Z}(\widetilde{\mathscr{A}})$. Writing $\mathcal{I} \lhd \mathcal{O}_S$ for the sheaf of ideals associated to $C$, the base change of $r'$ along $C \hookrightarrow S$ is again an affine map

$$r'_C \colon \widetilde{C} = \underline{\mathrm{Spec}}_C\Big(\mathcal{O}_{\widetilde{S}}/\mathcal{I}\mathcal{O}_{\widetilde{S}}\Big) \to C, \tag{4.50}$$

and the fibre product $\mathcal{S}_{\mathsf{root}} \times_S C$ is given by the quotient $r_C \colon \mathcal{C}_{\mathsf{root}} = [\widetilde{C}/\mathbb{G}_{\mathrm{m}}] \to C$. The pushforward along $r_{C,*}$ is given by the composition

$$\mathsf{coh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{C}) \xrightarrow[\sim]{r'_{C,*}} \mathsf{coh}^{\mathbb{Z}}(C, \mathcal{O}_{\widetilde{S}}/\mathcal{I}\mathcal{O}_{\widetilde{S}}) \xrightarrow{(-)_0} \mathsf{coh}(C) \tag{4.51}$$

sending an equivariant sheaf on $\widetilde{C}$ to the degree-0 part of the corresponding $\mathcal{O}_{\widetilde{S}}/\mathcal{I}\mathcal{O}_{\widetilde{S}}$-module on $C$. Along this pushforward, the restricted algebra $\mathscr{B}_{\mathsf{root}} \in \mathsf{coh}\,\mathcal{C}_{\mathsf{root}} = \mathsf{coh}^{\mathbb{G}_{\mathrm{m}}}(\widetilde{C})$ maps to

$$r_{C,*}\mathscr{B}_{\mathsf{root}} = (r'_{C,*}\mathscr{B}_{\mathsf{root}})_0 = (\widetilde{\mathscr{A}}/\mathcal{I}\widetilde{\mathscr{A}})_0 = \widetilde{\mathscr{A}_0}/\mathcal{I}\widetilde{\mathscr{A}_0} \cong \mathscr{A}/\mathcal{I}\mathscr{A} = \mathscr{B}. \tag{4.52}$$

Hence, if we denote $\widetilde{\mathscr{B}} = \widetilde{\mathscr{A}}/\mathcal{I}\widetilde{\mathscr{A}}$ we obtain two adjoint functors as in [62, §4.2]:

$$\mathsf{coh}(C, \mathscr{B}) \underset{F=(-)_0}{\overset{G=(-)\otimes_{\mathscr{B}}\widetilde{\mathscr{B}}}{\rightleftarrows}} \mathsf{coh}^{\mathbb{Z}}(\widetilde{C}, \widetilde{\mathscr{B}}) \simeq \mathsf{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{B}_{\mathsf{root}}). \tag{4.53}$$

Similar to [62, Theorem 4.5], we already know that for a module $\mathcal{M} \in \mathsf{coh}(C, \mathscr{B})$ the natural map $\mathcal{M} \to F(G(\mathcal{M}))$ is an isomorphism. Conversely, identifying an object $\mathcal{N} \in$

$\mathsf{coh}^{\mathbb{Z}}(\widetilde{C},\widetilde{\mathscr{B}})$ with the corresponding $\mathcal{I}$-torsion object of $\mathsf{coh}^{\mathbb{Z}}(S,\widetilde{\mathscr{A}})$ (by the graded version of Lemma 4.4.7) the natural map $G(F(\mathcal{N}))\to\mathcal{N}$ fits into a commutative diagram

$$\begin{array}{ccc} G(F(\mathcal{N}))=\mathcal{N}_0\otimes_{\mathscr{B}}\widetilde{\mathscr{B}} & \longrightarrow & \mathcal{N} \\ \downarrow\wr & & \downarrow\wr \\ (\mathcal{N}\otimes_{\widetilde{\mathscr{A}}}\widetilde{\mathscr{B}})_0\otimes_{\mathscr{A}}\widetilde{\mathscr{A}} & \longrightarrow & \mathcal{N}\otimes_{\widetilde{\mathscr{A}}}\widetilde{\mathscr{B}} \end{array} \tag{4.54}$$

It is shown in [62, Theorem 4.5] that the bottom map is an isomorphism, hence so is the top. □

For the canonical stack we first consider the complete-local description in Section 4.2.3 where $\Gamma$ is an $H$-equivariant order on $\operatorname{Spec} T$ for $T=\mathbf{k}[\![x,y]\!]$ obtained from an order $\Lambda_e$ on a complete-local chart $\operatorname{Spec} R_e$ of the root stack. We have the following.

**Lemma 4.3.3.** *If $J\subset R_e=T^H$ is an ideal, then* (4.42) *induces an equivalence*

$$\mathsf{mod}^H\,\Gamma/J\Gamma \underset{-\otimes_{\Lambda_e/J\Lambda_e}\Gamma/J\Gamma}{\overset{\operatorname{Hom}^H_{\Gamma/J}(\Gamma/J\Gamma,-)}{\rightleftarrows}} \mathsf{mod}\,\Lambda_e/J\Lambda_e. \tag{4.55}$$

*Proof.* Using [62, Proposition 2.26] the inclusion $\Lambda_e\to(\Lambda_e\otimes_{R_e}T)^{\vee\vee}=\Gamma$ induces an isomorphism $\Lambda_e\cong\Gamma^H$, and via this identification there are isomorphisms

$$\Gamma * H\cong\operatorname{End}_{\Gamma^H}(\Gamma)\cong\operatorname{End}_{\Lambda_e}(\Gamma). \tag{4.56}$$

These isomorphisms are $R_e$-linear along the isomorphism $R_e\cong T^H$, and therefore they induce $R_e$-linear equivalences

$$\mathsf{mod}^H\,\Gamma\xrightarrow{\operatorname{Hom}^H_\Gamma(\Gamma,-)}\mathsf{mod}\operatorname{End}_{\Lambda_e}(\Gamma)\xrightarrow{\ \sim\ }\mathsf{mod}\,\Lambda_e, \tag{4.57}$$

where the second equivalence is the Morita equivalence, using the fact that $\Gamma$ is a projective $\Lambda_e$-module which contains $\Lambda_e$ as a direct summand. It follows from $R_e$-linearity that these equivalences restrict to the subcategories of $J$-torsion objects

$$\mathsf{mod}^H\,\Gamma/J\Gamma\simeq\{M\in\mathsf{mod}^H\,\Gamma\mid JM=0\},\quad \mathsf{mod}\,\Lambda_e/J\Lambda_e\simeq\{M\in\mathsf{mod}\,\Lambda_e\mid JM=0\} \tag{4.58}$$

for any ideal $J\triangleleft R_e\cong T^H$. □

This allows us to discuss the restriction of the canonical stack construction.

**Proposition 4.3.4.** *There is an isomorphism $\mathscr{B}_{\mathsf{root}}\cong c_{C,*}\mathscr{B}_{\mathsf{can}}$ which induces an equivalence*

$$\mathsf{coh}(\mathcal{C}_{\mathsf{root}},\mathscr{B}_{\mathsf{root}})\simeq\mathsf{coh}(\mathcal{C}_{\mathsf{can}},\mathscr{B}_{\mathsf{can}}). \tag{4.59}$$

*Proof.* By the construction in (4.46), the stacky subscheme in the canonical stack is given by the fiber product

$$\begin{array}{ccc} \mathcal{C}_{\mathsf{can}} & \xrightarrow{i} & \mathcal{S}_{\mathsf{can}} \\ \downarrow{\scriptstyle c_C} & & \downarrow{\scriptstyle c} \\ \mathcal{C}_{\mathsf{root}} & \xrightarrow{j} & \mathcal{S}_{\mathsf{root}} \end{array} \tag{4.60}$$

The usual adjunctions yield a canonical base-change map $j^*c_*\mathcal{F} \to c_{C,*}i^*\mathcal{F}$ for any sheaf $\mathcal{F}$ on $\mathcal{S}_{\mathsf{can}}$. In particular, since $c_*\mathscr{A}_{\mathsf{can}} = \mathscr{A}_{\mathsf{root}}$ by Theorem 4.2.11, we find a canonical morphism

$$\mathscr{B}_{\mathsf{root}} = j^*c_*\mathscr{A}_{\mathsf{can}} \to c_{C,*}i^*\mathscr{A}_{\mathsf{can}} = c_{C,*}\mathscr{B}_{\mathsf{can}}. \tag{4.61}$$

To check that this is an isomorphism, it suffices to check it is an isomorphism on complete-local neighborhoods of points $p \in C \subset S$ in $S$. For any such point we can consider the root stack over a complete-local chart

$$[\operatorname{Spec} R_e/\mu_e] \to \operatorname{Spec} R, \tag{4.62}$$

and let $J \subset R$ be the ideal cutting out $C$. Then the stacky subscheme $\mathcal{C}_{\mathsf{root}}$ restricts to the quotient stack

$$[\operatorname{Spec} R_e/\mu_e] \times_{\operatorname{Spec} R} \operatorname{Spec}(R/JR) \cong [\operatorname{Spec}(R_e/JR_e)/\mu_e]\,. \tag{4.63}$$

Base changing along the étale cover $\operatorname{Spec} R_e$, the map $c_C\colon \mathcal{C}_{\mathsf{can}} \to \mathcal{C}_{\mathsf{root}}$ restricts to

$$[\operatorname{Spec}(T/JT)/H] \to \operatorname{Spec}(R_e/JR_e). \tag{4.64}$$

In particular, the sheaves of algebras $\mathscr{B}_{\mathsf{can}} = \mathscr{A}_{\mathsf{can}}|_{\mathcal{C}_{\mathsf{can}}}$ and $\mathscr{B}_{\mathsf{root}} = \mathscr{A}_{\mathsf{root}}|_{\mathcal{C}_{\mathsf{root}}}$ restrict on $\operatorname{Spec} T/JT$ resp. $\operatorname{Spec} R_e/JR_e$ to the quotients $\Gamma/J\Gamma$ and $\Lambda_e/J\Lambda_e$. To describe the map (4.61), we use the short exact sequence

$$0 \to J\Gamma \to \Gamma \to \Gamma/J\Gamma \to 0. \tag{4.65}$$

On this local chart the direct image is given by the functor $\operatorname{Hom}_{T*H}(T,-)\colon \mathsf{mod}\,T*H \to \mathsf{mod}\,T^H = \mathsf{mod}\,R_e$, and applying it to (4.65) yields the long exact sequence

$$0 \to \operatorname{Hom}_{T*H}(T, J\Gamma) \to \operatorname{Hom}_{T*H}(T,\Gamma) \to \operatorname{Hom}_{T*H}(T,\Gamma/J\Gamma) \to \operatorname{Ext}^1_{T*H}(T, J\Gamma) \to \dots \tag{4.66}$$

Because $H$ is a finite group, $T$ is projective as a $T*H$-module and therefore $\operatorname{Ext}^1_{T*H}(T, J\Gamma) = 0$. By [62, Lemma 4.4] there is moreover a canonical isomorphism $\operatorname{Hom}_{T*H}(T,\Gamma) \cong \Lambda_e$, and by $R_e$-linearity of the functor a canonical isomorphism $\operatorname{Hom}_{T*H}(T, J\Gamma) \cong J\Lambda_e$ and we therefore obtain an exact sequence

$$0 \to J\Lambda_e \to \Lambda_e \to \operatorname{Hom}_{T*H}(T,\Gamma/J\Gamma) \to 0, \tag{4.67}$$

which implies that $\operatorname{Hom}^H_T(T,\Gamma/J\Gamma) \cong \Lambda_e/J\Lambda_e$. We conclude that $\mathscr{B}_{\mathsf{root}} \cong c_{C,*}\mathscr{B}_{\mathsf{can}}$.

For the second statement, using Lemma A.1.2 the full subcategories $\mathsf{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{B}_{\mathsf{root}}) \subset \mathsf{coh}(\mathcal{S}_{\mathsf{root}}, \mathcal{A}_{\mathsf{root}})$ and $\mathsf{coh}(\mathcal{C}_{\mathsf{can}}, \mathscr{B}_{\mathsf{can}}) \subset \mathsf{coh}(\mathcal{S}_{\mathsf{can}}, \mathcal{A}_{\mathsf{can}})$ are identified with the subcategories of $r^*\mathcal{I}$-torsion and $c^*r^*\mathcal{I}$-torsion objects, where $\mathcal{I}$ is the defining sheaf of ideals of $C$ in $S$. It then suffices to show that the inverse pair of functors

$$G\colon \mathsf{coh}(\mathcal{S}_{\mathsf{root}}, \mathscr{A}_{\mathsf{root}}) \rightleftarrows \mathsf{coh}(\mathcal{S}_{\mathsf{can}}, \mathscr{A}_{\mathsf{can}}) \ :F, \tag{4.68}$$

preserve these subcategories of torsion objects. It again suffices to check this complete locally, and the result therefore follows from Lemma 4.3.3. □

*Proof of Theorem 4.3.1.* The first isomorphism (resp. equivalence) in the statement is given by Proposition 4.3.2, the second isomorphism (resp. equivalence) in the statement is given by Proposition 4.3.4. □

### 4.3.2 Properties of the restriction

We will now study the properties of the sheaf of algebras $\mathscr{A}|_C$. We are mostly interested in the case where $\mathscr{A}|_C$ is an order, as the geometry of orders is well-understood. Similar observations to the ones we make have appeared in the literature, albeit under stricter conditions on $\mathcal{A}$, see, e.g., [50, page 59] for the case of a terminal order.

The following is the more precise version of Proposition 4.B(1).

**Proposition 4.3.5.** *Let $S, \mathcal{A}, C$ be as in Theorem 4.3.1, and assume moreover that $S$ is smooth and that $C$ is integral. The sheaf of algebras $\mathscr{A}|_C$ is an order if and only if $C$ is not contained in $\Delta$. Moreover, when $\mathscr{A}|_C$ is an order,*

- *the restriction of $\mathscr{A}|_C$ to the dense open Azumaya locus is split Azumaya,*
- *the discriminant of $\mathscr{A}|_C$ is $C \cap \Delta$.*

*Proof.* The smoothness of $S$ guarantees that the reflexive sheaf $\mathscr{A}$ is locally free.

Let $U := S \setminus \Delta$ be the Azumaya locus of $\mathscr{A}$. If $C$ is not contained in $\Delta$, the restriction $\mathscr{A}|_C$ is Azumaya on the non-empty Zariski-open subset $C \cap U \subseteq C$, so that $\mathscr{A}|_C$ is an order in $\mathscr{A} \otimes_S \mathbf{k}(C)$.

Conversely, if $C$ is contained in $\Delta$, then by the fiberwise characterization of Azumaya algebras [110, Proposition IV.2.1(b)], the fibers of $\mathscr{A}$ at $p \in C$ are not central simple algebras (because otherwise $C \cap U \neq \emptyset$), so $\mathscr{A}|_C$ cannot be Azumaya on some dense open.

For the first point in the moreover part, it suffices to observe that the Brauer group of *any* (integral) curve over an algebraically closed field is trivial, by [56, Proposition 8.5.2]. The second point again follows from the fiberwise characterization [110, Proposition IV.2.1(b)]. □

The following proposition is a more precise version of Proposition 4.B(2).

**Proposition 4.3.6.** *Let $S, \mathcal{A}, C$ be as in Theorem 4.3.1, and assume moreover that $S$ is smooth and that $C$ is integral. Assume that $\mathscr{A}|_C$ is an order. Then the following are equivalent:*

1. *$\mathscr{A}|_C$ is an Azumaya algebra;*
2. *$C \cap \Delta = \emptyset$;*
3. *the morphism $r_C \circ c_C \colon \mathcal{C}_{\mathsf{can}} \to C$ is an isomorphism.*

*In this case, the Azumaya algebra is even split.*

*Proof.* The equivalence (1) ⇔ (2) follows from the characterization of Azumaya algebras in the fibers [110, Proposition IV.2.1(b)], because $\mathcal{A}$ is already locally free, as $S$ is assumed to be smooth.

The equivalence (2) ⇔ (3) follows from Proposition 4.2.13: the maps $r$ and $c$ are an isomorphism if and only if $C \subset S \setminus \Delta$.

The splitness follows as in the proof of Proposition 4.3.5 from the vanishing of $\mathrm{Br}(C)$. □

Finally, the following proposition is a more precise version of Proposition 4.B(3).

**Proposition 4.3.7.** *Let $S, \mathcal{A}, C$ be as in Theorem 4.3.1, and assume moreover that $S$ is smooth and that $C$ is integral. We will use the notation introduced in* (4.47)*.*

*If $C$ is smooth and intersects $\Delta$ transversely in the smooth locus of $\Delta$, then*

1. *The morphism $c_C \colon \mathcal{C}_{\mathsf{can}} \to \mathcal{C}_{\mathsf{root}}$ is an isomorphism, and the stacks are isomorphic to the root stack in the reduced divisor $C \cap \Delta$ with the multiplicities given by the ramification indices of the order $\mathscr{B}$.*
2. *The order $\mathscr{B}$ is hereditary.*

*If $C$ intersects $\Delta$ non-transversely in the smooth locus of $\Delta$, then*

3. *The morphism $c_C \colon \mathcal{C}_{\mathsf{can}} \to \mathcal{C}_{\mathsf{root}}$ is an isomorphism. They are isomorphic to the root stack in the non-reduced divisor $C \cap \Delta$ with the multiplicities given by the ramification indices.*
4. *The order $\mathscr{B}$ is not hereditary.*

*Proof.* In both cases the isomorphism of stacks follows from Proposition 4.2.13. The description of the root stack follows from the functoriality properties of the root stack construction.

Because global dimension is an invariant of the abelian categories, the equivalence (4.48) explains when $\mathscr{B}$ is (not) hereditary. Namely, the root stack $\mathcal{C}$ is smooth if and only if the divisor we consider is reduced, which is determined by the (non-)transversality of the intersection $C \cap \Delta$. □

If $\mathscr{A}$ is terminal (in the sense of [52, Definition 2.5]) and $C$ is smooth, we can upgrade Proposition 4.3.7 to the following. It would be interesting to further extend this to all tame orders and integral curves.

**Proposition 4.3.8.** *Let $S, \mathscr{A}, C$ be as in Theorem 4.3.1 and assume moreover that $C$ is smooth and $\mathscr{A}$ is terminal. Then the following are equivalent.*

1. *The morphism $c_C \colon \mathcal{C}_{\mathsf{can}} \to \mathcal{C}_{\mathsf{root}}$ is an isomorphism, and the stacks are isomorphic to the root stack in the reduced divisor $C \cap \Delta$ with the multiplicities given by the ramification indices of the order $\mathscr{A}|_C$.*
2. *The order $\mathscr{A}|_C$ is hereditary.*
3. *The curve $C$ intersects $\Delta$ transversely in the smooth locus of $\Delta$.*

*Proof.* The equivalence of (1) and (2) follows from Theorem 4.2.6 and Proposition 4.2.13. That (3) implies (1), and that (3) implies (2) follows from Proposition 4.3.7.

For the remaining implication from (2) to (3), we know already from Proposition 4.3.7 that $\mathscr{A}|_C$ is not hereditary if $C$ intersects $\Delta$ non-transversely in the smooth locus.

Hence, assume that $p \in C \cap \Delta$ lies in the singular locus of $\Delta$. We are going to show that the fiber $\mathscr{A}(p) = \mathscr{A}_p \otimes_{\mathcal{O}_{S,p}} \mathcal{O}_{S,p}/\mathfrak{m}_{S,p}$ is not the fiber of a hereditary order over a curve. For this we can use the étale local description of terminal orders (the standard form being given in [52, Definition 2.6]) and describe the fiber as a path algebra with relations isomorphic to the finite-dimensional algebra $\mathscr{A}(p)$. From Lemma 4.3.9 and Lemma 4.3.10 below, it follows that this fiber cannot be the fiber of a hereditary order. Namely, the presence of loops in the quiver of $\mathscr{A}(p)$ implies that any simple $\mathscr{A}(p)$-module has non-trivial first self-extension group, which is not the case for a hereditary order. The description of the fiber of a hereditary order, showing it has no loops, can be obtained using the same methods as for the proof of Lemma 4.3.10, see also [16, Lemma 3.1] or [90, Theorem 3.1]. □

To calculate the fiber of a terminal order at a singular point of the ramification, we work étale locally, and we will recall the standard form from [52, Definition 2.6]. Let $R = \mathbf{k}\{u, v\}$ be the ring of algebraic power series, i.e., the henselization of $\mathbf{k}[x, y]$ at the origin, and $S = R\langle x, y\rangle/(x^e - u, y^e - v, xy - \zeta yx)$, where $e \in \mathbb{N}$ and $\zeta$ is an $e$th root of unity. It follows from the discussion in [52, §2.3] that étale-locally a terminal order $\mathscr{A}$ is isomorphic to

$$\Lambda(n,\zeta) = \begin{pmatrix} S & S & \dots & S & S \\ xS & S & & S & S \\ \vdots & & \ddots & & \vdots \\ xS & xS & \dots & S & S \\ xS & xS & \dots & xS & S \end{pmatrix} \subseteq \mathrm{Mat}_n(S) \tag{4.69}$$

for some $n \in \mathbb{N}$. This allows us to prove the following.

**Lemma 4.3.9.** *The* $\mathbf{k}$*-algebra* $\overline{\Lambda} = \Lambda(n,\zeta) \otimes_R R/(u,v)$ *is a* $n^2e^2$*-dimensional* $\mathbf{k}$*-algebra, with basis spanned by* $\{e_{ij}^{(\alpha,\beta)} \mid 1 \le i,j \le n, 0 \le \alpha,\beta \le e-1\}$ *subject to the multiplication rule*

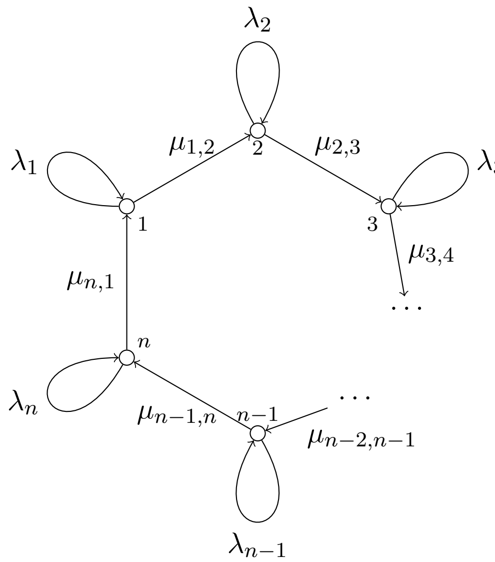
$$e_{ij}^{(\alpha,\beta)} \cdot e_{jk}^{(\gamma,\delta)} = \begin{cases} e_{ik}^{(\alpha+\gamma,\beta+\delta)} & \text{if } i \le j \le k,\ j \le k < i,\ k < i \le j, \\ e_{ik}^{(\alpha+\gamma+1,\beta+\delta)} & \text{otherwise}, \end{cases} \tag{4.70}$$

*where we set* $e_{ik}^{(\alpha,\beta)} = 0$ *if* $\alpha \ge e$ *or* $\beta \ge e$*. Moreover, we have* $e_{ij}^{(\alpha,\beta)} \cdot e_{j'k}^{(\gamma,\delta)} = 0$ *if* $j \ne j'$*.*

*Proof.* The algebra $\Lambda(n,\zeta)$ is freely generated by the elementary matrices multiplied by the element $x^\alpha y^\beta$ for $0 \le \alpha,\beta \le e-1$. These form a $\mathbf{k}$-basis of $\overline{\Lambda}$, where we denote by $e_{ij}^{(\alpha,\beta)}$ the image of the elementary matrix with $x^\alpha y^\beta$ in the $(i,j)$th entry. The multiplication rules follow from the algebra structure of $\Lambda(n,\zeta)$. □

This allows us to obtain the following.

**Lemma 4.3.10.** *With the assumptions from Lemma 4.3.9, the algebra* $\overline{\Lambda}$ *is isomorphic to the quotient of the path algebra* $\mathbf{k}Q/I$*, where* $Q$ *is the quiver*




(4.71)

*and* $I \trianglelefteq \mathbf{k}Q$ *is the ideal generated by the relations*

$$\begin{aligned} (\mu_{i,i+1}\cdots\mu_{i-1,i})^e & \quad \text{for all } i = 1,\dots,n, \\ \lambda_i\mu_{i,i+1} - \mu_{i,i+1}\lambda_{i+1} & \quad \text{for all } i = 1,\dots,n-1, \\ \lambda_n\mu_{n,1} - \zeta\mu_{n,1}\lambda_1. & \end{aligned} \tag{4.72}$$

*Proof.* The $\mathbf{k}$-algebra $\overline{\Lambda}$ is finite-dimensional, basic and connected: a full set of primitive orthogonal idempotents is formed by $e_{11}^{(0,0)},\dots,e_{nn}^{(0,0)}$. It is a straightforward calculation that $\mathsf{rad}(\Lambda)/\mathsf{rad}^2(\Lambda)$ is generated by $e_{ii}^{(0,1)}$, $e_{i,i+1}^{(0,0)}$ and $e_{n,1}^{(0,0)}$. This explains the shape of the quiver.

The relations follow from the multiplication rules in Lemma 4.3.9. □

It would be interesting to upgrade these results. First of all, Proposition 4.3.8 should hold for tame orders, but our proof depends on the local structure of terminal orders for the time being.

Secondly, our results suggest to study what happens in the singular case. There exist other order-theoretic properties one can study, such as being Bass, Gorenstein, tiled, or nodal. These are defined in Section 4.5. We will study these notions in explicit examples, where we have access to additional tools. These examples indicate what the full picture should look like.

## 4.4 Noncommutative plane curves

In what follows we will discuss and extend results from the literature which were obtained through very different methods, but for which the perspective of central curves gives a unified and improved approach. These applications are concerned with algebra quotients of three-dimensional Artin–Schelter regular algebras, thus giving rise to some notion of "noncommutative plane curve" (distinct from the notion of line modules from [9, §6]).

The Artin–Schelter regular algebras from the literature on noncommutative plane curves are (almost) all finite over the center, and this is the setting we will consider. Therefore, we will in Section 4.4.1 recall some properties of the central Proj construction, which turns them into maximal orders on $\mathbb{P}^2$. This makes them amenable to the setup of central curves introduced in Section 4.3. As explained in the introduction, we will mostly focus on the case of noncommutative plane conics and noncommutative plane cubics, in Section 4.5 resp. Section 4.6.

### 4.4.1 Reminder on central Proj construction

To a graded algebra which is finite over its center one can associate an order on a projective variety via the central Proj construction, which we will recall from [104, Section 2].

Let $A$ be a connected graded algebra generated in degree 1, which is moreover finite as a module over a central graded integral domain $R \subset A$. Then there is a projective variety $Y = \operatorname{Proj} R$ and $A$ induces a coherent sheaf of algebras $\mathscr{A} \in \mathsf{coh}\, Y$, defined on each standard open $\mathrm{D}(g) \subset Y$ by

$$\Gamma(\mathrm{D}(g), \mathscr{A}) := A[g^{-1}]_0, \tag{4.73}$$

with the $R[g^{-1}]_0$-linear multiplication induced by the one on $A$. In general, the center $\mathcal{Z} = \mathrm{Z}(\mathscr{A})$ is bigger than $\mathcal{O}_Y$ (see, e.g., [111, Example 5.1]). However, one can consider the affine morphism

$$s\colon \underline{\operatorname{Spec}}_Y(\mathcal{Z}) \longrightarrow Y. \tag{4.74}$$

and there is a sheaf of algebras $\underline{\mathscr{A}} \in \mathsf{coh}\, \underline{\mathrm{Spec}}_Y(\mathcal{Z})$ that is uniquely defined by the property $s_*\underline{\mathscr{A}} = \mathscr{A}$. The pair $(\underline{\mathrm{Spec}}_Y(\mathcal{Z})\,,\underline{\mathscr{A}})$ is called the *central Proj* of $A$. In [104] the sheaf of algebras $\underline{\mathscr{A}}$ was somewhat erroneously denoted $s^*\mathscr{A}$.

It was shown in [104] that $\underline{\mathscr{A}}$ is a well-behaved order under some regularity conditions on $A$. In particular we have the following result [104, Proposition 1].

**Proposition 4.4.1** (Le Bruyn)**.** *If* $\mathrm{gldim}\, A < \infty$ *then* $\underline{\mathscr{A}}$ *is a maximal order over* $\underline{\mathrm{Spec}}_Y(\mathcal{Z})$*. Moreover,* $\underline{\mathrm{Spec}}_Y(\mathcal{Z})$ *is normal and Cohen–Macaulay, and* $\underline{\mathscr{A}}$ *is a sheaf of Cohen–Macaulay modules.*

The noncommutative geometry of the algebra $A$ is captured by both $(Y,\mathscr{A})$ and the central Proj $(\underline{\mathrm{Spec}}_Y(\mathcal{Z})\,,\underline{\mathscr{A}})$ since there are equivalences

$$\mathsf{qgr}\, A \simeq \mathsf{coh}(Y,\mathscr{A}) \simeq \mathsf{coh}(\underline{\mathrm{Spec}}_Y(\mathcal{Z})\,,\underline{\mathscr{A}}). \tag{4.75}$$

There are various algebras $A$ for which the central Proj is simply isomorphic to $Y = \mathrm{Proj}\, \mathrm{Z}(A)$: it was shown by Smith–Tate [127, Lemma 5.1] that this is true if the center $\mathrm{Z}(A)$ is generated by non-zerodivisors of coprime degrees. Moreover, for an algebra $A$ with generators $z_1,\ldots,z_n$ in degree $d_i$ such that $d = \gcd(d_1,\ldots,d_n)$, one can employ the *Veronese construction*

$$A^{(d)} = \bigoplus_{n\in\mathbb{Z}} A_n^{(d)} = \bigoplus_{n\in\mathbb{Z}} A_{nd}. \tag{4.76}$$

As before we define the space $Y^{(d)} = \mathrm{Proj}\, \mathrm{Z}(A^{(d)})$ and a sheaf of algebras $\mathscr{A}^{(d)}$ on $Y^{(d)}$ associated to $A^{(d)}$. Note that the ordinary Veronese subring of $\mathrm{Z}(A)$ is a subring $\mathrm{Z}(A)^{(d)} \subset \mathrm{Z}(A^{(d)})$, and there is therefore a surjective map $Y^{(d)} \to Y$. The following lemma due to Smith–Tate shows this is equal to the central Proj.

**Lemma 4.4.2** (After [127, Lemma 5.1])**.** *Let $A$ be a graded algebra over* $\mathbf{k}$ *which is finite over its center, and suppose the center is generated by homogeneous elements $z_1,\ldots,z_n$ of degrees $d_1,\ldots,d_n$ which are non-zerodivisors. Writing* $d = \gcd(d_1,\ldots,d_n)$ *there is an isomorphism of ringed spaces*

$$(\underline{\mathrm{Spec}}_Y(\mathcal{Z})\,,\underline{\mathscr{A}}) \cong (Y^{(d)},\mathscr{A}^{(d)}), \tag{4.77}$$

*which commutes with the maps* $s\colon \underline{\mathrm{Spec}}_Y(\mathcal{Z}) \to Y$ *and* $Y^{(d)} \to Y$*.*

*Proof.* In [127, Lemma 5.1] Smith and Tate show that $\mathrm{Z}(A^{(d)}[z_i^{-1}]_0) = \mathrm{Z}(A[z_i^{-1}]_0)$ under the given assumptions, thereby proving $Y^{(d)} \cong \underline{\mathrm{Spec}}_Y(\mathcal{Z})$. It is also immediate from this construction that the isomorphism is compatible with the maps to $Y = \mathrm{Proj}\, \mathrm{Z}(A)^{(d)}$. It remains to show that $\underline{\mathscr{A}}$ corresponds to $\mathscr{A}^{(d)}$, but this is simply the identification

$$\Gamma(\mathrm{D}(z_i),\underline{\mathscr{A}}) = A[z_i^{-1}]_0 = A^{(d)}[z_i^{-1}]_0 = \Gamma(\mathrm{D}(z_i),\mathscr{A}^{(d)}), \tag{4.78}$$

which is also given in the proof of [127, Lemma 5.1]. ∎

Hence we obtained a chain of equivalences

$$\mathsf{qgr}\, A \simeq \mathsf{coh}(Y, \mathscr{A}) \simeq \mathsf{coh}(\underline{\mathrm{Spec}}_Y(\mathcal{Z}), \underline{\mathscr{A}}) \simeq \mathsf{coh}(Y^{(d)}, \mathscr{A}^{(d)}) \simeq \mathsf{qgr}\, A^{(d)}. \tag{4.79}$$

From this point on we will no longer distinguish notation for $\underline{\mathscr{A}}$ and $\mathscr{A}$, rather we will just write $\mathscr{A}$ for the maximal order on $\underline{\mathrm{Spec}}_Y(\mathcal{Z})$, as $Y$ will no longer play a role.

**The central Proj of noncommutative projective planes** In the remainder of this paper we will consider the special case of quadratic 3-dimensional AS-regular algebras, which are coordinate rings of noncommutative projective planes. In this case the central Proj construction gives a particularly pleasant result, which was obtained in full generality by Van Gastel [138, Proposition 4.2], with earlier work by Artin for the Sklyanin case [12, Theorem 5.2], and Mori for the skew case [111, Theorem 4.7].

**Theorem 4.4.3.** *Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra which is finite over its center. Then the central Proj of $A$ is a pair $(\mathbb{P}^2, \mathscr{A})$ with $\mathscr{A}$ a maximal order ramified along a cubic curve which is of one of the following types:*

*(A) elliptic curve*

*(B) nodal cubic*

*(D) union of a conic and a line in general position*

*(E) three lines in general position*

The choice of letters will be explained in Section 4.5.1.

### 4.4.2 Noncommutative plane curves using central curves

If $A$ is any graded algebra which is finite over its center as in Section 4.4.1 and $f \in A$ is a homogeneous central element, then the quotient algebra

$$B := A/(f). \tag{4.80}$$

defines a noncommutative projective variety $\mathsf{qgr}\, B$ that can be viewed as a hypersurface in $\mathsf{qgr}\, A$. We will consider the special case where $\mathsf{qgr}\, A$ is a noncommutative plane.

**Definition 4.4.4.** Let $A$ be a quadratic 3-dimensional Artin–Schelter regular algebra which is finite over its center. Let $f \in \mathrm{Z}(A)_d$ be a central homogeneous element of degree $d$. The noncommutative projective variety $\mathsf{qgr}\, A/(f)$ is a *noncommutative plane curve.*

As explained above, the literature contains results for noncommutative plane curves when $d = 2, 3$, which we will revisit using the machinery of central curves.

In the rest of this section we fix $A$ to be a quadratic 3-dimensional Artin–Schelter regular algebra which is finite over its center $R := \mathrm{Z}(A)$. By Lemma 4.4.2 and Theorem 4.4.3 the central Proj is of the form

$$(\mathbb{P}^2, \mathscr{A}) \cong (\mathrm{Proj}\, \mathrm{Z}(A^{(e)}), \mathscr{A}^{(e)}), \tag{4.81}$$

where $e$ denotes the gcd of the degrees of the homogeneous generators of $R$. We moreover fix a homogeneous central element $f \in R_d$ and consider the noncommutative plane curve $\mathsf{qgr}\, B$ defined by

$$B := A/(f). \tag{4.82}$$

The following reduces the study of the noncommutative plane curve to that of a sheaf of algebras on a curve, not via another application of the central Proj construction (because the graded algebra $B$ lacks some of the nice properties required to do so), but by restricting the output of the central Proj construction for $A$.

**Proposition 4.4.5.** *Let $C = \mathbb{V}(f) \subset \mathbb{P}^2$ be the curve defined by $f \in R_d \subset \mathrm{Z}(A^{(e)})$, and define $\mathscr{B} := \mathscr{A}|_C$. Then there exists an equivalence of categories*

$$\mathsf{qgr}\, B \simeq \mathsf{coh}(C, \mathscr{B}). \tag{4.83}$$

Before we prove this, we first discuss 2 preliminary (and standard) lemmas. For the first we include a proof for completeness' sake, for the second we will prove a stacky generalization in Section A.1.1.

**Lemma 4.4.6.** *With $A, B, f$ as in Proposition 4.4.5, the category $\mathsf{qgr}\, B$ embeds into $\mathsf{qgr}\, A$ as the subcategory*

$$\{M \in \mathsf{qgr}\, A \mid M \text{ is } f\text{-torsion}\} \subset \mathsf{qgr}\, A. \tag{4.84}$$

*Likewise, the category $\mathsf{qgr}\, A^{(e)}/fA^{(e)}$ embeds into $\mathsf{qgr}\, A^{(e)}$ as the subcategory of $f$-torsion modules.*

*Proof.* Observe that the category $\mathrm{gr}\, B$ embeds into $\mathrm{gr}\, A$ as the full subcategory of $f$-torsion modules. Taking the quotient by the Serre subcategory $\mathrm{fd}\, A \subset \mathrm{gr}\, A$, the composed functor

$$\mathrm{gr}\, B \to \mathrm{gr}\, A/\,\mathrm{fd}\, A = \mathsf{qgr}\, A, \tag{4.85}$$

has kernel $\mathrm{fd}\, B$, and therefore yields an embedding of $\mathsf{qgr}\, B = \mathrm{gr}\, B/\,\mathrm{fd}\, B$ into $\mathsf{qgr}\, A$ via the universal property of the Serre quotient.

The proof for the Veronese subalgebra is similar. □

**Lemma 4.4.7.** *With notation as in Proposition 4.4.5, the category $\mathsf{coh}(C, \mathscr{B})$ embeds into $\mathsf{coh}(S, \mathscr{A})$ as the subcategory*

$$\{\mathcal{F} \in \mathsf{coh}(S, \mathscr{A}) \mid \mathcal{I}\mathcal{F} = 0\} \subset \mathsf{coh}(S, \mathscr{A}), \tag{4.86}$$

*where $\mathcal{I} \subset \mathcal{O}_S$ denotes the sheaf of ideals generated by $f \in \mathrm{Z}(A^{(e)})$.*

*Proof.* This is a standard result, alternatively one can consider it a special case of Lemma A.1.2. □

*Proof of Proposition 4.4.5.* The Veronese equivalence $\mathsf{qgr}\, A \simeq \mathsf{qgr}\, A^{(e)}$, which exists because $A$ is generated in degree 1, preserves the $f$-torsion objects, hence it suffices to show that the equivalence $\mathsf{qgr}\, A^{(e)} \simeq \mathsf{coh}(S, \mathscr{A})$ identifies $f$-torsion objects with $\mathcal{I}$-torsion objects, so that we can conclude by Lemma 4.4.7.

Suppose $M \in \mathsf{qgr}\, A^{(e)}$ is $f$-torsion, and let $\mathcal{M} \in \mathsf{coh}(C, \mathscr{B})$ be the corresponding sheaf. Then for any standard open $\mathrm{D}(g) \subset X$, and any local section $\frac{fa}{g^n} \in \mathcal{I}(\mathrm{D}(g)) = (f \cdot \mathrm{Z}(A^{(e)})[g^{-1}])_0$ of $\mathcal{I}$, we have

$$\frac{fa}{g^n} \cdot \mathcal{M}(\mathrm{D}(g)) = \frac{fa}{g^n} \cdot (M[g^{-1}])_0 = \left(\frac{fa}{g^n} \cdot M[g^{-1}]\right)_0 = 0. \tag{4.87}$$

Hence $\mathcal{M}$ is a $\mathcal{I}$-torsion sheaf. Conversely, let $\mathcal{M} \in \mathsf{coh}(C, \mathscr{B})$ be a $\mathcal{I}$-torsion sheaf. Then the corresponding object of $\mathsf{qgr}\, A^{(e)}$ is isomorphic to

$$\Gamma_*(\mathcal{M}) \in \mathsf{qgr}\, \Gamma_*(\mathscr{B}). \tag{4.88}$$

Since $\mathcal{M}$ is $\mathcal{I}$-torsion, it follows that $\Gamma_*(\mathcal{M})$ is torsion for the ideal $\Gamma_*(\mathcal{I}) \subset \Gamma_*(\mathscr{B})$. Viewing $\Gamma_*(\mathcal{M})$ now as an $A^{(e)}$-module via the canonical map $A^{(e)} \to \Gamma_*(\mathscr{B})$, we find that $f$ acts via the global section $f \in \Gamma(\mathcal{I}(n)) \subset \Gamma_*(\mathcal{I})$ with $n = \deg f$. Hence $\Gamma_*(\mathcal{M})$ is an $f$-torsion module, which shows the equivalence. □

## 4.5 Noncommutative conics

The first interesting example of noncommutative plane curves is given by noncommutative conics, i.e., we consider a 3-dimensional Artin–Schelter regular algebra $A$ and a central element $f \in \mathrm{Z}(A)_2$ . These noncommutative conics have been studied and classified in [79, 80] (with a special case already appearing in [135]).

- In [79] the starting point is that of 3-dimensional Calabi–Yau Artin–Schelter regular algebras admitting at least one central element of degree 2, which are subsequently classified in Section 3 of op. cit. All but one of these is finite-over-the-center.
- In [80] on the other hand, the starting point is that of graded Clifford algebras. These naturally admit a net of central elements of degree 2, and are finite-over-their-center, making them suitable for the methods introduced in Section 4.4. In Remark 4.5.6 we explain how these two choices of ambient Artin–Schelter regular algebras compare.

The methods introduced in Section 4.4.2 give a new perspective on noncommutative conics. In Section 4.5.3 we will describe them using sheaves of algebras, and in Section 4.5.4 we will describe them using the geometry of stacks. In Section 4.5.5 we will (re)classify noncommutative conics from our perspective, compare our description to the existing descriptions in the literature, and obtain some corollaries.

### 4.5.1 Graded Clifford algebras and Clifford conics

A natural source for maximal orders on $\mathbb{P}^n$ is given by the central Proj construction applied to graded Clifford algebras, as studied in [104, Section 4]. We will soon specialize to $n = 2$, but for now we will work in the general case.

We start by recalling the definition of a graded Clifford algebra. Let $R = \mathbf{k}[z_0, \ldots, z_n]$ be the polynomial ring in $n+1$ variables with $\deg z_i = 2$. Consider $M \in \mathrm{Mat}_{n+1}(R_2)$ a symmetric matrix with linear entries in the $z_i$. It is useful to write

$$M = M_0 z_0 + \ldots + M_n z_n, \tag{4.89}$$

where $M_i \in \mathrm{Mat}_{n+1}(\mathbf{k})$ are symmetric matrices.

The *graded Clifford algebra* associated to $M$ is defined as $\mathrm{C}\ell(M) = R\langle x_0, \ldots, x_n\rangle / I$ where $I$ is the two-sided ideal generated by

$$x_i x_j + x_j x_i = M_{i,j} = \sum_{k=0}^{n} (M_k)_{i,j} z_k, \quad 0 \leq i, j \leq n. \tag{4.90}$$

By setting $\deg x_i = 1$, this becomes a graded algebra generated in degree 1 and 2 which is finitely generated as a module over the central subring $R$. To study the algebraic properties of $\mathrm{C}\ell(M)$ it is useful to consider its associated geometry.

**A linear system of quadrics associated to a graded Clifford algebra.** Each of the matrices $M_i$ defines via its associated quadratic form a quadric hypersurface $Q_i \subseteq \mathbb{P}^n$. Therefore we can view the data $M_i$ as an $n$-dimensional linear system

$$\mathcal{Q}_M = \mathbf{k}Q_0 + \ldots + \mathbf{k}Q_n \subset \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)). \tag{4.91}$$

The geometry of this linear system tells us when the graded Clifford algebra can be considered as a noncommutative projective space.

**Proposition 4.5.1** ([104, Proposition 7])**.** *The graded Clifford algebra* $\mathrm{C}\ell(M)$ *is an Artin–Schelter regular* $\mathbf{k}$*-algebra if and only if* $\mathcal{Q}_M$ *is basepoint-free.*

Since $R \subseteq \mathrm{C}\ell(M)$ is central, the first step of the central Proj construction from Section 4.4.1 can be carried out over $\mathbb{P}^n = \mathrm{Proj}(R)$. Denote by $(\mathbb{P}^n, \mathscr{A})$ the corresponding noncommutative space and by $s\colon \underline{\mathrm{Spec}}_{\mathbb{P}^n}(\mathcal{Z}) \to \mathbb{P}^n$ the central Proj as defined in (4.74).

If the associated linear system of quadrics $\mathcal{Q}_M$ is basepoint-free, the $\mathcal{O}_{\underline{\mathrm{Spec}}_{\mathbb{P}^n}(\mathcal{Z})}$-order $\mathscr{A}$ is maximal by [104, Proposition 1].

Recall that the *discriminant* of the linear system $\mathcal{Q}_M$ is the closed subscheme $\Delta(\mathcal{Q}_M) \subset \mathbb{P}^n$, where the quadrics are singular. The following lemma explains what the central Proj looks like for $n$ even.

**Lemma 4.5.2.** *Assume that* $n$ *is even and* $\mathcal{Q}_M$ *is basepoint-free. Then the morphism* $s\colon \underline{\mathrm{Spec}}_{\mathbb{P}^n}(\mathcal{Z}) \to \mathbb{P}^n$ *is an isomorphism, and the ramification divisor of the* $\mathcal{O}_{\mathbb{P}^n}$*-order* $\mathscr{A}$ *is given by*

$$S_1 := \{p \in \mathbb{P}^n \mid \mathrm{rk}\, M(p) \leq n\}. \tag{4.92}$$

*Proof.* Since $n$ is even, the center $\mathrm{Z}(\mathrm{C}\ell(M))$ is $R \oplus R\delta = \mathbf{k}[z_0, \ldots, z_n, \delta]$, where $\delta \in \mathrm{C}\ell(M)$ is an element of degree $n+1$ such that $\delta^2 = \det M$. A proof for this result can be found in [125], or more recently in [80, Theorem 3.18].

From Lemma 4.4.2 it follows that the underlying variety of the central Proj construction for the graded Clifford algebra is given by $\mathrm{Proj}(\mathrm{Z}(\mathrm{C}\ell(M))^{(2)}) = \mathrm{Proj}(\mathbf{k}[z_0, \ldots, z_n]) \cong \mathbb{P}^n$, and that $s$ is an isomorphism. By [104, Proposition 9] the ramification divisor of the associated $\mathcal{O}_{\mathbb{P}^n}$-order $\mathscr{A}$ has ramification $\Delta = S_1$. □

**Clifford conics** In the case $n = 2$, we obtain a net of conics as our linear system in the construction of a graded Clifford algebra. By [145, Table 2], for a basepoint-free net of conics $\mathcal{Q}_M$, the discriminant must be one of the following (keeping the notation from op. cit.):

(A) an elliptic curve,

(B) a nodal cubic,

(D) a conic meeting a line in general position, or

(E) three lines in general position.

The classification of nets of conics is sufficient for the description of the ramification divisor of $\mathcal{O}_{\mathbb{P}^2}$-orders obtained from the central Proj construction applied to graded Clifford algebras. More precisely, we have the following, where an order on a smooth surface is said to be *terminal* as in [52, Definition 2.5] if its discriminant $\Delta$ has normal crossings, with the cyclic covers of the components ramified at the nodes, and at each node one cyclic cover is totally ramified of degree $e$ with the other being ramified with index $e$ and degree $ne$ for some $n \geq 2$. This will allow the use of étale local models for terminal orders [52, Section 2.3].

**Lemma 4.5.3.** *The $\mathcal{O}_{\mathbb{P}^2}$-order $\mathscr{A}$ constructed in Lemma 4.5.2 is terminal and its ramification divisor is $\Delta = \Delta(\mathcal{Q}_M)$.*

This explains the labeling of the discriminants in Theorem 4.4.3.

**Remark 4.5.4.** In fact, as pointed out by the referee, by using the Artin–Mumford sequence, one can show that any maximal order on $\mathbb{P}^2$ ramified along a cubic divisor is terminal, and thus this holds for all orders obtained from the central Proj construction for a quadratic 3-dimensional Artin–Schelter-regular algebra, as in Theorem 4.4.3. This follows from [54, Proposition 24(1)].

*Proof.* The ramification of the central Proj of a graded Clifford algebra was determined in Lemma 4.5.2. It remains to show that the $\mathcal{O}_{\mathbb{P}^2}$-order is terminal. Since $\mathcal{Q}_M$ is basepoint-free by assumption, the ramification divisor is a normal crossing divisor.

Given a point $p \in \mathbb{P}^2$ of codimension 1, the $\mathbf{k}(p)$-algebra $\mathscr{A}_p / \operatorname{rad} \mathscr{A}_p$ is simple artinian and $\operatorname{rad} \mathscr{A}_p$ is principal. Therefore, the quotient $\mathscr{A}_p / \operatorname{rad} \mathscr{A}_p$ is either isomorphic to $\mathrm{Mat}_2(\mathbf{k}(p))$, since $\mathbf{k}$ is algebraically closed and thus Tsen's theorem applies to $\mathbf{k}(p)$, or to a field extension $L/\mathbf{k}(p)$ of degree 2. To see this, one can use the classification of quaternion $\mathbf{k}$-algebras of [53, Theorem 4.5].

In the first case the $\mathcal{O}_{\mathbb{P}^2,p}$-order is Azumaya and hence unramified. In the second case, the ramification index is $e_p = 2$ and we obtain a two-fold covering over each irreducible component of the ramification divisor which is ramified over the points of intersection of the irreducible components. Because we only have double covers, the $\mathcal{O}_{\mathbb{P}^2}$-order is terminal. □

For more on the geometry of graded Clifford algebras from the perspective of sheaves of orders, one is referred to [28].

Because 3-dimensional graded Clifford algebras have a net of central elements of degree 2, we can define noncommutative conics. We introduce the following terminology.

**Definition 4.5.5.** Let $A$ be the 3-dimensional graded Clifford algebra associated to the basepoint-free net of conics $\mathcal{Q}_M$. Let $f \in \mathrm{Z}(A)_2 = \mathbf{k}z_0 + \mathbf{k}z_1 + \mathbf{k}z_2$ be a central element of degree 2, and set $B := A/(f)$. We call the abelian category $\mathsf{qgr}\, B$ a *Clifford conic.*

This choice of terminology sets them apart from other sources of noncommutative conics studied in the literature. The first class of noncommutative conics in the literature are "skew conics". These are defined with respect to a skew polynomial algebra $\mathbf{k}\langle x_0, x_1, x_2\rangle/(x_i x_j - q_{i,j} x_j x_i)$ where $q_{i,j} = \pm 1$ [135]. The second source are noncommutative conics inside Calabi–Yau quantum projective planes [79]. We will compare these classes of noncommutative conics to Clifford conics in Section 4.5.5.

**Remark 4.5.6.** By [80, Theorem 1.2] a 3-dimensional graded Clifford algebra is Calabi–Yau. Conversely, the 4 relevant isomorphism types of graded Calabi–Yau algebras with a central net of conics from [79, Theorem 3.6] are graded Clifford algebras, by using the symmetric matrices

$$\begin{pmatrix} 2z_0 & 0 & 0 \\ 0 & 2z_1 & 0 \\ 0 & 0 & 2z_2 \end{pmatrix}, \begin{pmatrix} 2z_0 & 0 & 0 \\ 0 & 2z_1 & -z_0 \\ 0 & -z_0 & 2z_2 \end{pmatrix}, \begin{pmatrix} 2z_0 & 0 & -z_1 \\ 0 & 2z_1 & -z_0 \\ -z_1 & -z_0 & 2z_2 \end{pmatrix}, \begin{pmatrix} 2z_0 & -\lambda z_2 & -\lambda z_1 \\ -\lambda z_2 & 2z_1 & -\lambda z_0 \\ -\lambda z_1 & -\lambda z_0 & 2z_2 \end{pmatrix}, \tag{4.93}$$

ordered as in [79].

### 4.5.2 Properties of orders on curves

In order to understand non-smooth noncommutative curves, we recall the notion of Gorenstein and Bass orders. In what follows let $C$ denote a smooth curve equipped with an $\mathcal{O}_C$-order $\mathscr{A}$, and denote by

$$\omega_{\mathscr{A}} = \mathcal{H}om_{\mathcal{O}_C}(\mathscr{A}, \mathcal{O}_C) \tag{4.94}$$

the dualizing bimodule. Moreover, let $\mathscr{A}_\eta$ be the localization of $\mathscr{A}$ at the generic point $\eta$ of $C$. Note that $\mathscr{A}_\eta$ is always isomorphic to a matrix algebra over the function field of $C$

since $\mathbf{k}$ is algebraically closed. Following [55], we give the following definition of Gorenstein and Bass orders.

**Definition 4.5.7.** Let $\mathscr{A}$ be an $\mathcal{O}_C$-order.

1. We say that $\mathscr{A}$ is *Gorenstein* if the dualizing bimodule $\omega_{\mathscr{A}}$ is locally free of rank one as a left and a right $\mathscr{A}$-module.
2. If every order $\mathscr{A}' \supset \mathscr{A}$ in $\mathscr{A}_\eta$ containing $\mathscr{A}$ is Gorenstein, we say that $\mathscr{A}$ is *Bass*.

**Remark 4.5.8.** Sometimes, it is required that $\omega_{\mathscr{A}}$ is locally free of rank one as an $\mathscr{A}$-bimodule. For the discussion of Clifford conics this is equivalent to the above definition by [143, Main Theorem 16.7.7]. In [112, Theorem 2.1] the reader can find a detailed discussion on the difference of these two notions. We stick to the weaker definition as above because for example the definition of Bass orders agrees with the one in [59].

Nodal orders have been studied in a series of papers [40, 41, 43]. To define a *nodal order* globally we first need to recall the local definition from [43, Definition 3.1].

Let $R$ be a discrete valuation ring and $\Lambda \subset \mathrm{Mat}_n(\mathrm{Frac}(R))$ be an $R$-order. We say that $\Lambda$ is *nodal* if it is contained in a hereditary $R$-order $\Lambda'$ in $\mathrm{Mat}_n(\mathrm{Frac}(R))$ such that $\mathsf{rad}(\Lambda') = \mathsf{rad}(\Lambda)$ and for every simple left $\Lambda$-module $S$ the length of $\Lambda' \otimes_\Lambda S$ as a $\Lambda$-module is bounded by 2.

With this setup we are able to define nodal orders on a curve, see [43, Definition 4.1].

**Definition 4.5.9.** An $\mathcal{O}_C$-order $\mathscr{A}$ is *nodal* if for every point $p \in C$ the $\mathcal{O}_{C,p}$-order $\mathscr{A}_p$ is nodal.

For an order $\mathscr{A}$ on a curve $C$ we thus have the following implications

$$\begin{array}{ccccccc} \mathscr{A} \text{ maximal} & \Longrightarrow & \mathscr{A} \text{ hereditary} & \Longrightarrow & \mathscr{A} \text{ Bass} & \Longrightarrow & \mathscr{A} \text{ Gorenstein.} \\ & & \Downarrow & & & & \\ & & \mathscr{A} \text{ nodal} & & & & \end{array} \tag{4.95}$$

**Remark 4.5.10.** It will become clear from Lemma 4.5.12 that every nodal order of degree 2 is Bass. However, this is not true in higher degree. Let $R$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$. Consider the non-Gorenstein $R$-order

$$\Lambda = \begin{pmatrix} R & \mathfrak{m} & R \\ \mathfrak{m} & R & R \\ \mathfrak{m} & \mathfrak{m} & R \end{pmatrix} \subset \begin{pmatrix} R & R & R \\ R & R & R \\ \mathfrak{m} & \mathfrak{m} & R \end{pmatrix} = \Lambda' \tag{4.96}$$

from [112, Example 5]. By the classification of nodal orders in [41, Theorem 3.13], the $R$-order is nodal and corresponds to the set $\Omega = \{1 \approx 1, 2\}$. Note that the hereditary $R$-order $\Lambda'$ satisfies $\mathsf{rad}\,\Lambda' = \mathsf{rad}\,\Lambda$.

We note that these four properties of orders can be checked locally.

**Lemma 4.5.11.** *Let $\mathscr{A}$ be an order on a smooth curve $C$. The following are equivalent:*

1. *The $\mathcal{O}_C$-order $\mathscr{A}$ is maximal, resp. hereditary, Bass, or Gorenstein.*
2. *For each $p \in C$ the $\mathcal{O}_{C,p}$-order $\mathscr{A}_p$ is maximal, resp. hereditary, Bass, or Gorenstein.*
3. *For each $p \in C$ the $\widehat{\mathcal{O}}_{C,p}$-order $\widehat{\mathscr{A}_p}$ is maximal, resp. hereditary, Bass, or Gorenstein.*

*Proof.* The equivalence (1) ⇔ (2) can be found in the literature, see, e.g. [143, Lemma 10.4.3 and §21.4.4, 24.2.2, and 24.5.2]. The equivalence (1) ⇔ (3) is given in [119, Corollary (11.6)] for maximal orders and in [119, Theorem (40.5)] for hereditary orders.

The equivalence (2) ⇔ (3) for Gorenstein and Bass orders follows from the inclusion-preserving one-to-one correspondence [119, Theorem (5.2)] between lattices over a discrete valuation ring and its completion. □

**Properties of quaternion orders** In the following we analyze how the above properties can be recognized in the case of quaternion orders, which will be the setting of Section 4.5.3. Since it suffices to consider the complete-local structure, we fix a point $p \in C$ and consider quaternion orders over the complete local ring $R = \mathbf{k}[\![t]\!] = \widehat{\mathcal{O}}_{C,p}$. We will repeatedly write

$$(t^{e_1}, t^{e_2})_2^R := R\langle i, j\rangle / (i^2 - t^{e_1}, j^2 - t^{e_2}, ij + ji) \tag{4.97}$$

for the quaternion $R$-order in $\mathrm{Mat}_2(\mathbf{k}((t)))$ for $e_1, e_2 \in \mathbb{Z}_{\geq 0}$, where $e_1 \leq e_2$. In particular the product of the generators satisfies $(ij)^2 = -t^{e_1+e_2}$. Note that

$$(t^{e_1}, t^{e_2})_2^R \cong (t^{f_1}, t^{f_2})_2^R \tag{4.98}$$

if and only if $(e_1, e_2) = (f_1, f_2)$. The following describes the local properties of these orders.

An interesting (complete) local property of an order in a matrix algebra is whether it is tiled. Recall from [132, §2] that an $R$-order $\Lambda$ in $\mathrm{Mat}_n(\mathbf{k}((t)))$ is *tiled* if it has a complete set of $n$ primitive orthogonal idempotents.

**Lemma 4.5.12.** *Let $e_1, e_2 \geq 0$ and denote by $\Lambda := (t^{e_1}, t^{e_2})_2^R$ the cyclic quaternion $R$-order.*

*i) The $R$-order $\Lambda$ is maximal if and only if $e_1 = e_2 = 0$.*

*ii) The $R$-order $\Lambda$ is hereditary if and only if $e_1 = 0$ and $e_2 \leq 1$.*

*iii) The $R$-order $\Lambda$ is Bass if and only if $e_1 \leq 1$.*

*iv) The $R$-order $\Lambda$ is Gorenstein for all $e_1, e_2 \in \mathbb{Z}_{\geq 0}$, and the dualizing bimodule satisfies $\omega_\Lambda^2 = (t^{-e_1-e_2})\Lambda$.*

*v) The $R$-order $\Lambda$ is nodal if and only if $(e_1, e_2) \in \{(0,0), (0,1), (0,2), (1,1)\}$.*

*vi) The $R$-order $\Lambda$ is tiled if and only if $e_1 = 0$.*

*vii) At the closed point we have*

$$\Lambda/\operatorname{rad}\Lambda \cong \begin{cases} \mathrm{Mat}_2(\mathbf{k}) & \text{if } e_1 = e_2 = 0, \\ \mathbf{k}\times\mathbf{k} & \text{if } 0 = e_1 < e_2, \\ \mathbf{k} & \text{if } 0 < e_1 \le e_2. \end{cases} \tag{4.99}$$

**Remark 4.5.13.** The $R$-order satisfying $\Lambda/\operatorname{rad}\Lambda \cong \mathbf{k}\times\mathbf{k}$ are also called *residually split*, whereas the $R$-orders satisfying $\Lambda/\operatorname{rad}\Lambda \cong \mathbf{k}$ are called *residually ramified*, see [143, Definition 24.3.2]. We list these local properties in Table 4.1.

*Proof.* The first two statements follow from the classification of maximal (resp. hereditary) orders over complete discrete valuation rings, see [119, Theorem (17.3), Theorem (39.14)].

For the next two statements, one can use the functorial one-to-one correspondence [142, Corollary 3.20] between nondegenerate quadratic forms and ternary quadratic modules. The associated bilinear form to $\Lambda$ is given by

$$Q\colon R^{\oplus 3} \to R, \quad (\alpha_1,\alpha_2,\alpha_3) \mapsto -t^{e_2}\alpha_1^2 - t^{e_1}\alpha_2^2 + \alpha_3^2. \tag{4.100}$$

We see that the quadratic form is always primitive, hence $\Lambda$ is Gorenstein, by [143, Theorem 24.2.10]. Moreover, combining (vii) and [143, Exercise 24.13(a)], statement (iii) follows.

Using [143, Proposition 15.6.7], one can present the dualizing bimodule as the $\Lambda$-$\Lambda$-submodule

$$\omega_\Lambda \cong R \oplus (t^{-e_1})\cdot i \oplus (t^{-e_2})\cdot j \oplus (t^{-e_1-e_2})\cdot ji. \tag{4.101}$$

inside $\operatorname{Frac} R\cdot\Lambda$. With this presentation it is a straightforward calculation that $\omega_\Lambda^2 \subseteq (t^{-e_1-e_2})\Lambda$ because $t^{-e_1-e_2} = -(t^{-e_1-e_2}\cdot ji)^2$. This settles (iv).

Statement (v) follows from the classification [41, Theorem 3.19] of nodal orders over $R$ up to conjugation. Following [41, §3.2] we identify four possible triples $(\Omega,\approx,\sigma)$, where $\approx$ is a symmetric binary relation on $\Omega$, and $\sigma \in S_{|\Omega|}$ is a cycle of length $|\Omega|$. The cycle $\sigma$ has to be of maximal length because the nodal order $\Lambda$ is an order in $\mathrm{Mat}_2(\mathbf{k}((t)))$. The cardinality is $|\Omega| \le 2$ by rank reasons. Finally, we see that in the case where $|\Omega| = 2$ reflexivity is not allowed, again by rank reasons. Therefore we get the following correspondence:

- $\Omega = \{1\}$ corresponds to $(e_1,e_2) = (0,0)$,
- $\Omega = \{1\}$ with $1 \approx 1$ corresponds to the $(e_1,e_2) = (0,2)$,
- $\Omega = \{1,2\}$ corresponds to $(e_1,e_2) = (0,1)$, and
- $\Omega = \{1,2\}$ with $1 \approx 2$ corresponds to $(e_1,e_2) = (1,1)$.

The sixth statement follows from the fact that tiled quaternion $R$-orders are of the form

$$\begin{pmatrix} R & R \\ (a) & R \end{pmatrix} \tag{4.102}$$

for $a \in R$ non-zero.

The last statement follows from a straightforward calculation of the radical. □

### 4.5.3 Clifford conics as orders

We will now describe Clifford conics as defined in Definition 4.5.5 from the perspective of orders by applying Proposition 4.4.5. We thus consider a graded Clifford algebra associated to a basepoint-free net of conics in $\mathbb{P}^2$, and we will let $(\mathbb{P}^2, \mathscr{A})$ denote the central Proj of the graded Clifford algebra $\mathrm{C}\ell(M)$ over $R = \mathbf{k}[z_0, z_1, z_2]$, where $M$ defines the basepoint-free net $\mathcal{Q}_M$ of conics.

**Definition 4.5.14.** Let $(\mathbb{P}^2, \mathscr{A})$ be the central Proj of a graded Clifford algebra. Let $L \cong \mathbb{P}^1$ be a line in $\mathbb{P}^2$ and denote by $\iota\colon L \hookrightarrow \mathbb{P}^2$ the closed immersion. We call the noncommutative space $(L, \mathscr{B})$ a *Clifford conic order*, where $\mathscr{B} := \mathscr{A}|_L$.

We denote by $\Delta_L = \iota^{-1}(\Delta)$ the pullback of the ramification divisor of $\mathscr{A}$. If $L$ is not contained in $\Delta$, it follows from Proposition 4.3.5 that $\Delta_L$ is the ramification divisor of the $\mathcal{O}_L$-order $\mathscr{B}$.

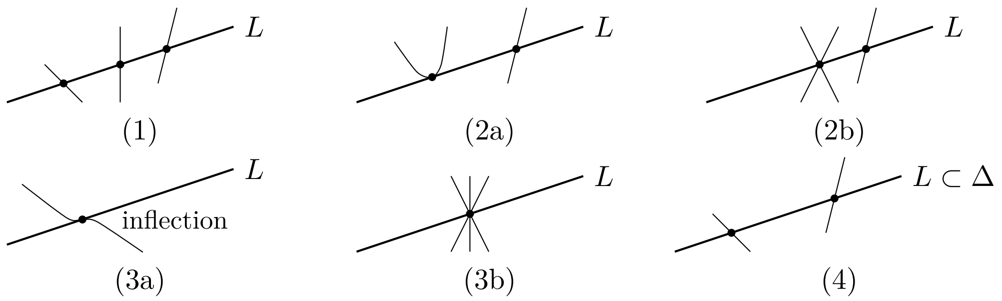



Figure 4.1: The 6 possible intersections $L \cap \Delta$

Clifford conics can be classified by inspecting the geometry of the ramification divisor $\Delta$ of the $\mathcal{O}_{\mathbb{P}^2}$-order and using the complete-local model of terminal orders. The first classification theorem describes Clifford conics in the language of sheaves of orders. The 6 possible types will be referred to as (1), (2a), (2b), (3a), (3b), and (4), and a pictorial representation is given in Figure 4.1.

For a point $p \in L$, we denote by $\mathfrak{m}_p \trianglelefteq \mathcal{O}_{L,p}$ the maximal ideal and by $t$ a uniformizer of $\mathfrak{m}_p$.

**Theorem 4.5.15.** *Let $A$ be a graded Clifford algebra, let $f$ define a Clifford conic. The Clifford conic order $(L, \mathscr{B})$ obtained from restricting $(\mathbb{P}^2, \mathscr{A})$ belongs to one of the following Morita equivalence classes, and has the following complete-local structure:*

*(1) If $\#\Delta_L = 3$, then $\mathscr{B}$ is a hereditary $\mathcal{O}_L$-order ramified over three points such that for every $p \in \Delta_L$*

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (1,t)_2^{\widehat{\mathcal{O}}_{L,p}}. \tag{4.103}$$

*(2) If $\#\Delta_L = 2$, then $\mathscr{B}$ is a Bass $\mathcal{O}_L$-order ramified over two points. We distinguish the following two cases:*

*(2a)* Tangent case. *Assume that $L \cap \Delta^{\mathrm{sing}} = \emptyset$, then over a ramified point $p \in \Delta_L$ one has*

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong \begin{cases} (1, t^2)_2^{\widehat{\mathcal{O}}_{L,p}} & \text{if } \mathrm{mult}_p(\Delta, L) = 2, \\ (1, t)_2^{\widehat{\mathcal{O}}_{L,p}} & \text{if } \mathrm{mult}_p(\Delta, L) = 1. \end{cases} \tag{4.104}$$

*(2b)* Singular case. *Assume that $L \cap \Delta^{\mathrm{sing}}$ is non-empty, then over a ramified point $p \in \Delta_L$ one has*

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong \begin{cases} (t, t)_2^{\widehat{\mathcal{O}}_{L,p}} & \text{if } p \in \Delta^{\mathrm{sing}}, \\ (1, t)_2^{\widehat{\mathcal{O}}_{L,p}} & \text{if } p \notin \Delta^{\mathrm{sing}}. \end{cases} \tag{4.105}$$

*(3) If $\#\Delta_L = 1$, then $\mathscr{B}$ is a Bass $\mathcal{O}_L$-order ramified over a single point $p \in L$. We distinguish the following two cases:*

*(3a)* Tangent case. *Assume that $L \cap \Delta^{\mathrm{sing}} = \emptyset$, then $\mathcal{O}_L$-order $\mathscr{B}$ restricts over $p$ to*

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (1, t^3)_2^{\widehat{\mathcal{O}}_{L,p}}. \tag{4.106}$$

*(3b)* Singular case. *Assume that $L \cap \Delta^{\mathrm{sing}} = \Delta_L$, then $\mathcal{O}_L$-order $\mathscr{B}$ restricts over $p$ to*

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (t, t^2)_2^{\widehat{\mathcal{O}}_{L,p}}. \tag{4.107}$$

*(4) If $L \subset \Delta$, then $\mathscr{B}$ is not an order. There exists an isomorphism $\mathscr{B} \cong \mathrm{C}\ell^0(\mathcal{E}, q, \mathcal{O}_{\mathbb{P}^1}(1))$ with the even Clifford algebra of the vector bundle $\mathcal{E} \cong \mathcal{O}_L^{\oplus 3}$ and a quadratic form which is of rank 2 everywhere except at two points, where it is of rank 1. This corresponds to the pencil of conics with Segre symbol[2] $[1,1;\ ;1]$.*

In Table 4.1 we summarize the results of this theorem and some subsequent propositions.

The point scheme of the graded Clifford algebra (or equivalently, the discriminant of the net of conics) prescribes which isomorphism classes are feasible for a given algebra: not all types are possible for any given graded Clifford algebra. This can again be read off from Table 4.1.

We can in fact bootstrap from this result, and classify Morita equivalence classes of Clifford conics for all graded Clifford algebras simultaneously. This will be explained in Section 4.5.5.

[2]See Section 4.5.5 for more on Segre symbols. In the symbol $[1,1;\ ;1]$, the last entry encodes that the pencil is a cone over a pencil of length-2 subschemes, degenerating to a double point in two points of the base.

*Proof.* By definition of a Clifford conic and Lemma 4.5.3, the $\mathcal{O}_L$-order $\mathscr{B}$ is the restriction of a terminal $\mathcal{O}_{\mathbb{P}^2}$-order $\mathscr{A}$. We start with the complete-local description of the algebra structure of $\mathscr{B}$. To do so, denote for $p \in \mathbb{P}^2$ by $\mathbf{k}[\![x,y]\!]$ the completion of the local ring $\mathcal{O}_{\mathbb{P}^2,p}$ at $p$.

Assume first that $L$ does not lie in the ramification divisor $\Delta$. From [52, Section 2.3], it follows that complete-locally terminal orders of rank 4 can only take the following two forms

- If $p \in \Delta^{\mathrm{sm}}$, then

$$\widehat{\mathscr{A}_p} = \mathscr{A} \otimes_{\mathcal{O}_{\mathbb{P}^2,p}} \widehat{\mathcal{O}}_{\mathbb{P}^2,p} \cong \begin{pmatrix} \mathbf{k}[\![x,y]\!] & \mathbf{k}[\![x,y]\!] \\ (x) & \mathbf{k}[\![x,y]\!] \end{pmatrix}. \tag{4.108}$$

- If $p \in \Delta^{\mathrm{sing}}$, then

$$\widehat{\mathscr{A}_p} = \mathscr{A} \otimes_{\mathcal{O}_{\mathbb{P}^2,p}} \widehat{\mathcal{O}}_{\mathbb{P}^2,p} \cong \frac{\mathbf{k}[\![x,y]\!]\langle i,j\rangle}{(i^2-x, j^2-y, ij+ji)}. \tag{4.109}$$

Assume that $p \in L \cap \Delta$, where we identify $L$ with its image $\iota(L)$. If $\mathcal{I}_L \trianglelefteq \mathcal{O}_{\mathbb{P}^2,p}$ is the ideal sheaf of $L$, then

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong \widehat{\mathscr{A}_p}/\widehat{\mathcal{I}}_L\widehat{\mathscr{A}_p}. \tag{4.110}$$

Thus the description of $\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p}$ is a matter of distinguishing the possible intersections of a line with the ramification divisor $\Delta$, which is of the form (A), (B), (D) or (E) as in Theorem 4.4.3. There are precisely six cases how $\Delta$ and $L$ can meet at $p$, depicted in Figure 4.1.

Since $\Delta$ is a curve of degree 3, there are three cases when $p \in L \cap \Delta^{\mathrm{sm}}$: the intersection is either transversal, tangential or an inflection point. After completion the ramification divisor is mapped to a line $\Delta = \mathbb{V}(x) \subset \operatorname{Spec} \mathbf{k}[\![x,y]\!]$ in the model (4.108). This in turn implies that $L$ is a curve of degree 1, 2 or 3 going through the origin transversely, tangentially or as an inflection point.

If the intersection is transversal, the ideal sheaf of the curve, which we can assume to be $(y)$ as we work complete-locally, and $(x)$ generate the maximal ideal at $\mathbf{k}[\![x,y]\!]$, and therefore we obtain

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (1,t)_2^{\widehat{\mathcal{O}}_{L,p}}, \tag{4.111}$$

which corresponds to case (1). This follows also from Proposition 4.3.5.

If the intersection is tangential (of degree 2 or 3), then after completion we can assume that $L$ is mapped to a conic (resp. cubic) of the form $\mathbb{V}(x-y^i)$. We obtain the orders

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong \begin{pmatrix} \mathbf{k}[\![y]\!] & \mathbf{k}[\![y]\!] \\ (y^i) & \mathbf{k}[\![y]\!] \end{pmatrix} \cong (1,t^i)_2^{\widehat{\mathcal{O}}_{L,p}}. \tag{4.112}$$

This describes the cases (2a) and (3a).

In the situation $p \in L \cap \Delta^{\mathrm{sing}}$, there are two possibilities. The first we consider is when $L$ intersects each of the branches of the ramification divisor transversely at $p$. After completion we can assume that $L = \mathbb{V}(x-y)$. This leads to the case (2b)

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (t,t)_2^{\widehat{\mathcal{O}}_{L,p}}. \tag{4.113}$$

The last case is when $L$ is tangent to a branch of the ramification divisor (and consequently passes transversely through the other branch). The complete-local picture is given by $L = \mathbb{V}(x-y^2)$ and hence

$$\mathscr{B} \otimes_{\mathcal{O}_L} \widehat{\mathcal{O}}_{L,p} \cong (t,t^2)_2^{\widehat{\mathcal{O}}_{L,p}}, \tag{4.114}$$

which yields the case (3b). We show that the orders are Bass in Corollary 4.5.16.

Finally, it is possible that $L \subset \Delta$. In that case $L$ intersects the other irreducible component(s) of $\Delta$ in 2 points. The description in the statement of the theorem follows readily from the description of the pencil of conics and functoriality of the even Clifford algebra construction.

Finally, we explain how this complete-local description gives a classification into Morita equivalence classes. For this we appeal to [42, Theorem 7.8] which in particular shows that orders $\mathcal{B}$ and $\mathcal{B}'$ on smooth curves $C$ and $C'$ are Morita equivalent if[3] there exists an isomorphism $f\colon C \to C'$ such that

- there exists an isomorphism $\mathcal{B}_\eta \cong \mathcal{B}'_{f(\eta)}$ of central simple algebras over $\mathbf{k}(C) \cong \mathbf{k}(C')$, where $\eta$ is the generic point of $C$;
- their ramification divisors are identified by $f$;
- for every point $p$ in the ramification divisor of $\mathcal{B}$ there exists an isomorphism between $\mathcal{B} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}_{C,p}$ and $\mathcal{B}' \otimes_{\mathcal{O}_{C'}} \widehat{\mathcal{O}}_{C',f(p)}$.

If $L$ and $L'$ are of the same type (excluding case (4)), which is determined by how they intersect $\Delta$, then by the 3-transitivity of the automorphism group of $\mathbb{P}^1$ we can find an isomorphism between $L$ and $L'$ that identifies the ramification divisors and the complete-local descriptions.

Because case (4) is not an order, one has to use a different tool for the classification of Morita equivalence classes. However, the classification is possibly non-trivial only when the discriminant is a triangle of lines. In this case, the graded Clifford algebra admits an $\mathrm{S}_3$-symmetry exchanging the lines, and thus showing the 3 different $\mathcal{B}$ are even isomorphic. □

The local information about Clifford conics can be used to upgrade the commutative result that hypersurfaces in smooth varieties are Gorenstein to a version for Clifford conics.

**Corollary 4.5.16.** *With the notation from Theorem 4.5.15, assume that $(L, \mathscr{B})$ is a Clifford conic order such that $\Delta_L$ is finite. Then $\mathscr{B}$ is a Bass $\mathcal{O}_L$-order. In particular $\mathscr{B}$, is Gorenstein.*

[3] In fact, the sufficient conditions in loc. cit. are weaker than what we state here. See Remark 4.5.17.

*Proof.* By Lemma 4.5.11 one can check (complete) locally whether an order is Bass, resp. Gorenstein. Therefore the result follows from the local description of the orders in Theorem 4.5.15 and Lemma 4.5.12 (iii). □

Note in particular, that by Lemma 4.5.12 (ii) none of the orders in the cases (2a), (2b), (3a) and (3b) are hereditary.

We collect the properties of the orders in the first three points in Table 4.1.

**Remark 4.5.17.** The appeal to [42, Theorem 7.8] in the proof of Theorem 4.5.15 for the Morita equivalences is made using generic and complete-local isomorphisms. However, for a Morita equivalence it would be sufficient by loc. cit. to have suitable generic and complete-local Morita equivalences. It remains desirable to understand whether one can upgrade the classification of Theorem 4.5.15 (and the subsequent classification in Theorem 4.5.24) to one of *isomorphism classes* of sheaves of algebras. One would like to apply [42, Proposition 6.4] for this purpose, however, this result requires not just an isomorphism generically and complete-locally, but an *equality*, which requires a more explicit study.

### 4.5.4 Clifford conics as stacks

To the noncommutative space $(\mathbb{P}^2, \mathscr{A})$ obtained from a graded Clifford algebra we can apply the bottom-up construction from [62] as recalled in Section 4.2.3 and obtain

$$\mathcal{S}_{\mathsf{can}} \xrightarrow{c} \mathcal{S}_{\mathsf{root}} \xrightarrow{r} \mathbb{P}^2. \tag{4.115}$$

We will consider the structure of $\mathcal{S}_{\mathsf{can}}$ relative to a complete-local neighborhood $\operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^2,p})$ of a point $p \in \mathbb{P}^2$. We write $\widehat{\mathcal{S}}_{\mathsf{can},p} := \mathcal{S}_{\mathsf{can}} \times_{\mathbb{P}^2} \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^2,p})$ for the restriction to the neighborhood.

**Proposition 4.5.18.** *For $p \in \mathbb{P}^2$ the restriction $\widehat{\mathcal{S}}_{\mathsf{can},p} \to \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^2,p})$ is equivalent to one of the following:*

1. *if $p \in \mathbb{P}^2 \setminus \Delta$ then it is equivalent to $\operatorname{Spec} \mathbf{k}[\![x,y]\!] \xrightarrow{\sim} \operatorname{Spec} \mathbf{k}[\![x,y]\!]$,*
2. *if $p \in \Delta^{\mathrm{sm}}$ then it is equivalent to a root stack*

$$\sqrt{\operatorname{Spec} \mathbf{k}[\![x,y]\!]; \mathbb{V}(x)} \to \operatorname{Spec} \mathbf{k}[\![x,y]\!], \tag{4.116}$$

   *where $\mathbb{V}(x)$ is identified with the ramification divisor at $p$,*

3. *if $p \in \Delta^{\mathrm{sing}}$ then it is equivalent to the stack*

$$\left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![x,y,v,w]\!]}{(v^2 - x, w^2 - y)}\right) / \mu_2 \times \mu_2\right] \to \operatorname{Spec} \mathbf{k}[\![x,y]\!], \tag{4.117}$$

   *where the $\mu_2 \times \mu_2$ acts with weights $(1,1)$ and $(1,0)$ on $v$ and $w$ respectively and $\mathbb{V}(xy)$ describes the ramification divisor at $p$.*

*Proof.* The case (1) follows directly from Proposition 4.2.13 since the map $r \circ c \colon \mathcal{S}_{\mathsf{can}} \to \mathbb{P}^2$ restricts to an isomorphism over any point outside $\Delta$, and the complete-local neighborhood is of the form $\operatorname{Spec}(\mathbf{k}[\![x,y]\!])$.

Since $\mathrm{C}\ell(M)$ defines a terminal order on $\mathbb{P}^2$, as in the proof of Theorem 4.5.15, there are two different possibilities for the complete-local structure corresponding to the other cases.

In case (2) we can choose an isomorphism $\widehat{\mathcal{O}}_{\mathbb{P}^2,p} \cong \mathbf{k}[\![x,y]\!]$ such that $\mathrm{C}\ell(M)$ restricts to the $\mathbf{k}[\![x,y]\!]$-order (4.108), which is ramified in $\mathbb{V}(x)$ with index 2. By Proposition 4.2.13 the restriction $\mathcal{S}_{\mathsf{can},p}$ is isomorphic to the restriction of the root stack, which is a root stack of degree 2 in $\mathbb{V}(x)$ by Corollary 4.2.4.

In case (3) we can choose an isomorphism $\widehat{\mathcal{O}}_{\mathbb{P}^2,p} \cong \mathbf{k}[\![x,y]\!]$ such that $\mathrm{C}\ell(M)$ restricts to the $\mathbf{k}[\![x,y]\!]$-order (4.109), which is ramified in $\mathbb{V}(xy)$ with index 2 in both components. It follows by Corollary 4.2.4 that

$$\mathcal{S}_{\mathsf{root}} \times_{\mathbb{P}^2} \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathbb{P}^2,p}) \cong \left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![x,y,u]\!]}{(u^2-xy)}\right)/\mu_2\right]. \tag{4.118}$$

We can present $\mathbf{k}[\![x,y,u]\!]/(u^2-xy)$ as the invariant ring $\mathbf{k}[\![v,w]\!]^H$ for $H=\mu_2$ acting by $v\mapsto -v$ and $w\mapsto -w$; this identifies $x\mapsto v^2$, $y\mapsto w^2$, and $u\mapsto vw$. This presentation can be made equivariant for the original $\mu_2$-action on $\mathbf{k}[\![x,y,u]\!]/(u^2-xy)$ via the action[4] $v\mapsto -v$ and $w\mapsto w$. It then follows from Section A.1.3 that the canonical stack is obtained as the map

$$\widehat{\mathcal{S}}_{\mathsf{can},p} \cong [\operatorname{Spec}(\mathbf{k}[\![v,w]\!])/\mu_2\times\mu_2] \to \left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![x,y,u]\!]}{(u^2-xy)}\right)/\mu_2\right]. \tag{4.119}$$

Composing with the map down to $\operatorname{Spec}\mathbf{k}[\![x,y]\!]$ yields the result. ☐

As before, we can consider the restriction to a line, which gives the following stacky incarnation Definition 4.5.14.

**Definition 4.5.19.** Let $(\mathbb{P}^2,\mathscr{A})$ be the central Proj of a graded Clifford algebra, with associated stacky surface $\mathcal{S}_{\mathsf{can}}$. Let $L\cong\mathbb{P}^1$ be a line in $\mathbb{P}^2$. We call the Deligne–Mumford stack $\mathcal{L} := L\times_{\mathbb{P}^2}\mathcal{S}_{\mathsf{can}}$ a *stacky Clifford conic.*

**Remark 4.5.20.** We will omit the restriction of the (non-trivial) Azumaya algebra $\mathscr{A}_{\mathsf{can}}$ on $\mathcal{S}_{\mathsf{can}}$ for now, and drop it from the notation. The question whether the Azumaya algebra $\mathscr{B}_{\mathsf{can}}$ on $\mathcal{L}$ is split or not is however relevant. We will for now assume that $\mathscr{B}_{\mathsf{can}}$ is split, and come back to this question in Remark 4.5.23.

[4] There is another action where $\mu_2$ fixes $v$ and acts on $w$ by multiplication by $-1$, and it appears as if we need to make a non-canonical choice. However, the two possible $\mu_2\times\mu_2$-actions obtained are related by an automorphism of $\mu_2\times\mu_2$ and the quotient stacks defined by these two actions are isomorphic relative to the base.

The points with non-trivial stabilizer coincide with $\Delta_L$. If $\Delta_L$ is finite, it follows by the results in Section A.1.2 that the isomorphism class of $\mathcal{L} \to L$ is determined uniquely by the complete-local structure

$$\widehat{\mathcal{L}}_p := \mathcal{L} \times_L \operatorname{Spec}(\widehat{\mathcal{O}}_{L,p}) \tag{4.120}$$

over each of the points $p \in \Delta_L$. The complete-local structure is described by the following analogue of Theorem 4.5.15, where we continue to use the labeling described in Figure 4.1.

**Theorem 4.5.21.** *Let $A$ be a graded Clifford algebra, let $f$ define a Clifford conic. The stacky Clifford conic $\mathcal{L}$ is one of the following isomorphism classes of Deligne–Mumford stacks:*

*(1) If $\#\Delta_L = 3$, then $\mathcal{L} \to L$ is the smooth root stack $\mathcal{L} \cong \sqrt[2]{L; \Delta_L}$.*

*(2) If $\#\Delta_L = 2$, there are the following two cases:*

*(2a)* Tangent case. *If $L \cap \Delta^{\mathrm{sing}} = \emptyset$, so that $\Delta$ meets $L$ in two points $p, q$ with* $\operatorname{mult}_q(\Delta, L) = 2$ *and* $\operatorname{mult}_p(\Delta, L) = 1$, *then $\mathcal{L} \to L$ is the singular root stack*

$$\mathcal{L} \cong \sqrt[2]{L; 2[q] + [p]}. \tag{4.121}$$

*(2b)* Singular case. *If $L \cap \Delta^{\mathrm{sing}}$ is non-empty, then $\mathcal{L} \to L$ is the singular stacky curve with stacky structure over* $\operatorname{Spec}(\widehat{\mathcal{O}}_{L,p}) = \operatorname{Spec}(\mathbf{k}[\![t]\!])$ *for $p \in \Delta_L$ given by*

$$\widehat{\mathcal{L}}_p \cong \begin{cases} \left[\operatorname{Spec}\left(\dfrac{\mathbf{k}[\![t,u]\!]}{(u^2-t)}\right)/\mu_2\right] & \text{if } p \in \Delta^{\mathrm{sm}}, \\ \left[\operatorname{Spec}\left(\dfrac{\mathbf{k}[\![t,v,w]\!]}{(w^2-v^2, t-w^2)}\right)/\mu_2 \times \mu_2\right] & \text{if } p \in \Delta^{\mathrm{sing}}, \end{cases} \tag{4.122}$$

*where in the first case $\mu_2$ acts with weight $1$ on $u$, and in the second case $\mu_2 \times \mu_2$ acts with weights $(1,1)$ and $(1,0)$ on $v, w$.*

*(3) If $\#\Delta_L = 1$, there are two cases:*

*(3a)* Tangent case. *If $L$ meets $\Delta$ in an inflection point $p \in \Delta^{\mathrm{sm}}$, then $\mathcal{L} \to L$ is the singular root stack*

$$\mathcal{L} \cong \sqrt[2]{\mathbb{P}^1; 3[p]}. \tag{4.123}$$

*(3b)* Singular case. *If $L \cap \Delta^{\mathrm{sing}} = \Delta_L$, then $\mathcal{L}$ is the singular stacky curve over $L$ with stacky structure over* $\operatorname{Spec}(\widehat{\mathcal{O}}_{L,p}) = \operatorname{Spec}(\mathbf{k}[\![t]\!])$ *for the unique point $p \in \Delta_L$ given by*

$$\widehat{\mathcal{L}}_p \cong \left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![t,v,w]\!]}{(w^4-v^2, t-w^2)}\right)/\mu_2 \times \mu_2\right], \tag{4.124}$$

*where $\mu_2 \times \mu_2$ acts with weights $(1,1)$ and $(1,0)$ on $v, w$.*

*(4) If $L \subset \Delta$, then $\mathcal{L} \to L$ is non-reduced and has generic stabilizer $\mu_2$. The stacky structure over* $\operatorname{Spec}(\widehat{\mathcal{O}}_{L,p}) = \operatorname{Spec}(\mathbf{k}[\![t]\!])$ *for $p \in L$ is given by*

$$\widehat{\mathcal{L}}_p \cong \begin{cases} \left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![t,u]\!]}{(u^2)}\right)/\mu_2\right] & \text{if } p \in \Delta^{\mathrm{sm}}, \\ \left[\operatorname{Spec}\left(\frac{\mathbf{k}[\![t,v,w]\!]}{(v^2, t-w^2)}\right)/\mu_2 \times \mu_2\right] & \text{if } p \in \Delta^{\mathrm{sing}}. \end{cases} \tag{4.125}$$

*where in the first case $\mu_2$ acts with weight* $1$ *on $u$, and in the second case $\mu_2 \times \mu_2$ acts with weights* $(1,1)$ *and* $(1,0)$ *on $v, w$.*

The proof follows by restricting the local structure in Proposition 4.5.18 along the equation of a line $L$. This is a case-by-case analysis, analogous to the proof of Theorem 4.5.15, first describing the complete-local structure, and subsequently bootstrapping to global isomorphisms.

*Proof.* For the cases (1), (2a), and (3a) the line $L$ hits $\Delta$ in points $p \in \Delta^{\mathrm{sm}}$ with multiplicity at most degree 3. The local structure of $\mathcal{S}_{\mathsf{can},p}$ is therefore described by Proposition 4.5.18(2) and is a root stack over $\operatorname{Spec}(\widehat{\mathcal{O}}) = \operatorname{Spec}(\mathbf{k}[\![x,y]\!])$ in the divisor $\mathbb{V}(x)$. Up to a change of coordinates the line $L$ is described near $p$ by one of the equations $x = y$, $x = y^2$, or $x = y^3$ depending on the multiplicity at $p$. Choosing $t = y$ as the local coordinate of $L$ at $p$, one obtains a root stack in $\mathbb{V}(t)$, $\mathbb{V}(t^2)$, or $\mathbb{V}(t^3)$ respectively. This also describes the structure of $\mathcal{L}$ over the point $p \in \Delta^{\mathrm{sm}}$ in the case (2b).

For the points $p \in L \cap \Delta^{\mathrm{sing}}$ the local structure of $\mathcal{S}_{\mathsf{can},p}$ over $\operatorname{Spec}(\mathbf{k}[\![x,y]\!])$ is given by Proposition 4.5.18(3). The line $L$ maps to a curve of at most degree 2 in $\operatorname{Spec}(\mathbf{k}[\![x,y]\!])$, and either meets both components $\mathbb{V}(x)$ and $\mathbb{V}(y)$ of $\Delta$ transversely (for (2b)), or is tangent to one of them (for (3b)). Up to isomorphism of $\operatorname{Spec}(\mathbf{k}[\![x,y]\!])$ the line $L$ is given by $x = y$ in the former case, and is given by $x = y^2$ in the latter case. Choosing the local coordinate $t = y$ for $L$ at $p$ then yields the given stacks after base change.

Finally, if $L \subset \Delta$ then for a generic point $p \in \Delta^{\mathrm{sm}} \cap L$ the complete-local structure is given by restricting the stack in Proposition 4.5.18(2) along the equation $x = 0$. For the two points $p \in \Delta^{\mathrm{sing}} \cap L$ the complete-local structure is given by restricting the stack in Proposition 4.5.18(3) along one of the components, say $x = 0$. Since $x \mapsto v^2$ under the structure morphism $\mathcal{S}_{\mathsf{can},p} \to \operatorname{Spec}(\mathbf{k}[\![x,y]\!])$ we obtain the given stack after base change.

By Proposition A.1.3, we have that stacky curves with the same coarse moduli space and complete-local isomorphisms relative to the coarse moduli space are in fact isomorphic. For the stacks of type (1), (2a), (2b), (3a) and (3b), we can rearrange the stacky loci by the 3-transitivity of the automorphism group of $\mathbb{P}^1$, and thus conclude that our complete-local description induces a classification up to isomorphism. The same reasoning works in case (4), except that we rearrange the points on $\mathcal{L}$ corresponding to the singular points of the discriminant. $\square$

We have also summarized these results in Table 4.2.

**Remark 4.5.22.** From Lemma 4.5.12 (v) it follows that the cases (2a) and (2b) of Theorem 4.5.15 are nodal orders which are not hereditary, but are contained in a unique hereditary order with the same radical. As explained in [41, Section 4], the inclusion of the nodal non-hereditary order into the hereditary order can be globalized. On the stacky side, this is explained as follows.

In the case (2a) the inclusion maps into the hereditary $\mathcal{O}_L$-order ramified in one point. This translates to a map of root stacks $\sqrt[2]{\mathbb{P}^1; 2q+p} \to \sqrt[2]{\mathbb{P}^1; p}$. The vanishing of the stackiness at $q$ follows from the fact that, writing $R = \mathscr{A} \otimes_{\mathcal{O}_{\mathbb{P}^1,q}} \widehat{\mathcal{O}}_{\mathbb{P}^1,q} \cong \mathbf{k}[[t]]$, the radical of $(1,t^2)_2^R$ is, up to conjugation by an element of $\mathrm{Mat}_2(\mathbf{k}((t)))$, the radical of the maximal order $\mathrm{Mat}_2(R)$, see e.g. [143, §23.4.15].

In the case (2b) the nodal order is mapped into a hereditary order ramified over the same two points. This inclusion corresponds to the map $\mathcal{L} \to \sqrt[2]{\mathbb{P}^1; p+q}$ from the canonical stack to the root stack in Theorem 4.5.21.

**Remark 4.5.23.** By [1, Proposition 7.10] the Brauer group of the stacks appearing in cases (1), (2a) and (3a) is trivial. Indeed, there is no generic stabilizer, and the stack is locally Brauerless in the sense of [1, Definition 4.5] by [1, Example 4.9]. However, the same reasoning does not work for cases (2b) and (3b): by [1, Remark 4.10] we cannot use that the stacks are locally Brauerless in the sense of op. cit. We have not found a direct argument to show the Azumaya algebra on the singular stacky curve is split.

### 4.5.5 Classification and comments

We end our discussion of noncommutative conics, by explaining how our results recover those in [79, 80], and how they give more information about the derived categories of noncommutative conics.

**The global classification of Clifford conics** The classifications in Theorems 4.5.15 and 4.5.21 take place relative to a fixed graded Clifford algebra $A$. However, [79, Theorem 5.11] gives a classification of noncommutative conics into 6 isomorphism types (where isomorphism refers to equivalence of the qgr), taking all possible ambient algebras $A$ into account. We will now explain how the algebra-per-algebra classification in Theorems 4.5.15 and 4.5.21 actually gives a global classification.

Observe that, if $f$ is a Clifford conic inside the graded Clifford algebra $\mathrm{C}\ell(M)$, where $M$ is a net of conics, the restriction to $L \subset \mathbb{P}^2$ gives a pencil of conics. Segre symbols are a means of classifying pencils of quadric hypersurfaces, and thus in particular conics. We refer to [78, Chapter XIII] for their definition, or [21] for a survey by the second author.

**Theorem 4.5.24.** *Let $A, A'$ be two graded Clifford algebras, and let $f, f'$ define two Clifford conics. The following are equivalent:*

1. $\mathsf{qgr}\, A/(f)$ *and* $\mathsf{qgr}\, A'/(f')$ *are isomorphic as noncommutative projective varieties, i.e., they are equivalent categories;*

| type | hereditary | Gorenstein | tiled | nodal | residually | appearance |
|---|---|---|---|---|---|---|
| (1) | yes | yes | yes | yes | split | A, B, D, E |
| (2a) | no | yes | | yes | | A, B, D |
| $\mathrm{mult}_p(\Delta, L) = 2$ | no | | yes | yes | split | |
| $\mathrm{mult}_p(\Delta, L) = 1$ | yes | | yes | yes | split | |
| (2b) | no | yes | | yes | | B, D, E |
| $p \in \Delta^{\mathrm{sing}}$ | no | | no | yes | ramified | |
| $p \notin \Delta^{\mathrm{sing}}$ | yes | | yes | yes | split | |
| (3a) | no | yes | yes | no | split | A, B |
| (3b) | no | yes | no | no | ramified | B, D |
| (4) | no | | *not an order* | | | D, E |

Table 4.1: Summary of the (local) properties of the Clifford conics at $p \in \Delta_L$

| type | smooth | generically scheme | root stack | appearance |
|---|---|---|---|---|
| (1) | yes | yes | yes | A, B, D, E |
| (2a) | no | yes | yes | A, B, D |
| $\mathrm{mult}_p(\Delta, L) = 2$ | no | | | |
| $\mathrm{mult}_p(\Delta, L) = 1$ | yes | | | |
| (2b) | no | yes | no | B, D, E |
| $p \in \Delta^{\mathrm{sing}}$ | no | | no | |
| $p \notin \Delta^{\mathrm{sing}}$ | yes | | yes | |
| (3a) | no | yes | yes | A, B |
| (3b) | no | yes | no | B, D |
| (4) | no | no | no | D, E |

Table 4.2: Summary of the properties of the stacky Clifford conics $\mathcal{L}$

| type | $\#\mathbf{E_B}$ | $\mathrm{C}(B)$ | Segre symbol |
|---|---|---|---|
| (1) | 6 | $\mathbf{k}^4$ | $[1,1,1]$ |
| (2a) | 4 | $\mathbf{k}[t]/(t^2)\times\mathbf{k}^2$ | $[2,1]$ |
| (2b) | 3 | $(\mathbf{k}[t]/(t^2))^2$ | $[(1,1),1]$ |
| (3a) | 2 | $\mathbf{k}[t]/(t^3)\times\mathbf{k}$ | $[3]$ |
| (3b) | 1 | $\mathbf{k}[t]/(t^4)$ | $[(2,1)]$ |
| (4) | $\infty$ | $\mathbf{k}[s,t]/(s^2,t^2)$ | $[1,1;;1]$ |

Table 4.3: Dictionary between our types and the classification in [79, Table 4] and [80, Table 3]

2. *they have the same type from the set* $\{(1),(2a),(2b),(3a),(3b),4\}$,
3. *the Segre symbols of the pencils of conics attached to* $f$ *and* $f'$ *are equal.*

*Proof.* For the equivalence of (1) and (2), recall that by Proposition 4.4.5 we have that $\mathsf{qgr}\,A/(f)\cong\mathsf{coh}(L,\mathcal{B})$, where $\mathcal{B}$ is the sheaf of algebras classified in Theorem 4.5.15. It now suffices to observe that the proof of the classification into Morita equivalence classes uses a complete-local description which is seen to be independent of $f$, only the type from the set $\{(1),(2a),(2b),(3a),(3b),4\}$ plays a role.

The equivalence of (1) and (3) follows from [80, Theorem 3.28 and Remark 3.30][5]. □

In Table 4.3 we compare our point-of-view with the classification in [79, 80]. The point scheme $E_B$ refers to [79, Table 4], where it is denoted $E_A$, and it is determined in Example 4.6 and Theorem 4.13 of op. cit.

**Derived categories of noncommutative conics** As a part of their classification [79, Theorem 5.10], Hu–Matsuno–Mori obtain a description of the derived category of a noncommutative conic in terms of Orlov's semiorthogonal decomposition [115, Theorem 16]

$$\mathsf{D}^{\mathsf{b}}(\mathsf{qgr}\,B)=\langle\mathscr{B},\mathsf{D}^{\mathsf{b}}(\mathrm{C}(B))\rangle, \tag{4.126}$$

using the result by Smith–Van den Bergh [128, Theorem 5.2] that the singularity category is actually given by the bounded derived category of modules over the finite-dimensional $\mathbf{k}$-algebra $\mathrm{C}(B)$. Here $B:=A/(f)$ is the homogeneous coordinate ring of a noncommutative conic as before. The algebras $\mathrm{C}(B)$ for the six cases of noncommutative conics can be found in [79, Table 4], as recalled in Table 4.3.

[5] Note that it must be pointed out that not all possible Segre symbols arise: by assumption the net of conics (and thus the pencil of conics) is basepoint-free, which excludes two types.

However, from our perspective we have another natural semiorthogonal decomposition, which (assuming the vanishing of the Brauer class) in 3 out of 5 cases gives a more explicit description of the derived category because the gluing data can be explicitly read off.

Namely, let $\pi\colon \mathcal{L} \to L$ be the coarse moduli space for the stacky curve attached to the noncommutative conic. Then $\mathbf{R}\pi_*\mathcal{O}_{\mathcal{L}} \cong \mathcal{O}_L$, so that by the projection formula we can obtain the semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathcal{L}) = \langle \pi^*\mathsf{D}^{\mathsf{b}}(\mathbb{P}^1), \mathcal{T}\rangle, \tag{4.127}$$

where $\mathcal{T}$ is defined as the semiorthogonal complement. Because $\mathcal{L}$ is a root stack for types (1), (2a) and (3a), the semiorthogonal decomposition for root stacks from [29, Theorem 4.7] gives us

$$\mathcal{T} \simeq \begin{cases} \mathsf{D}^{\mathsf{b}}(k^3) & \text{type (1)} \\ \mathsf{D}^{\mathsf{b}}(k \times k[t]/(t^2)) & \text{type (2a)} \\ \mathsf{D}^{\mathsf{b}}(k[t]/(t^3)) & \text{type (3a)} \end{cases} \tag{4.128}$$

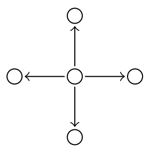
The sheaf of algebras $\mathscr{A}|_L$ corresponds to an Azumaya algebra on the stack $\mathcal{L}$. For type (1) we have that it is split, as in Corollary 4.2.10. We expect that some version of the vanishing of the Brauer group of a curve as in [56, Proposition 7.3.2] holds for the stacky curves appearing in Theorem 4.3.1, at least when there is no generic stabilizer. This would give rise to an equivalence

$$\mathsf{D}^{\mathsf{b}}(\mathsf{qgr}\,B) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}\,\mathcal{L}), \tag{4.129}$$

and thus (4.127) really is an alternative to (4.126).

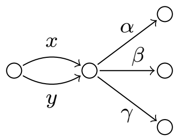
**Remark 4.5.25.** Even in the nicest possible case of type (1), the semiorthogonal decompositions (4.126) and (4.127) are different. The former gives the path algebra of the extended Dynkin quiver $\widetilde{\mathrm{D}}_4$,

$\widetilde{\mathrm{D}}_4$ :



(4.130)

whereas the latter is the quotient of the path algebra for the squid quiver




(4.131)

by the ideal of relations $(\alpha x, \beta y, \gamma x - \gamma y)$. It is a classical fact that these are derived equivalent.

## 4.6 Noncommutative skew cubics

Going beyond the case of noncommutative conics, we consider noncommutative plane cubics. The noncommutative projective planes we will consider are given by skew polynomial algebras $A = \mathbf{k}_q[x,y,z]$ with coefficient matrix

$$q = \begin{pmatrix} 1 & q_{1,2} & q_{1,3} \\ q_{1,2}^{-1} & 1 & q_{2,3} \\ q_{1,3}^{-1} & q_{2,3}^{-1} & 1 \end{pmatrix} \tag{4.132}$$

where $q_{i,j} \in \mathbf{k}^\times$ are cube roots of unity. This guarantees the presence of a natural net of central cubics. The algebra $A$ is always finite over its center, which is described explicitly in [111], and the central Proj is always given by a maximal order $\mathscr{A}$ on $\mathbb{P}^2$. See Section 4.4.2 for more details.

There are two distinct cases for $(\mathbb{P}^2, \mathscr{A})$ which depend on whether the condition

$$q_{1,2}^{-1}q_{1,3}^{-1} = q_{1,2}q_{2,3}^{-1} = q_{1,3}q_{2,3} \tag{4.133}$$

is satisfied or not. As in the conic setting of Section 4.5, we can therefore study noncommutative curves by considering quotients by an element in the net

$$f \in \mathrm{Z}(A)_3 = \langle x^3, y^3, z^3 \rangle. \tag{4.134}$$

In [84], the quotient by the central element $f = x^3 + y^3 + z^3 \in \mathrm{Z}(A)_3$ is considered[6] to describe a noncommutative "Fermat" cubic $\mathsf{qgr}\, A/(f)$. It is shown in op. cit. that this quotient is Calabi–Yau if and only if the condition (4.133) holds. We will generalize these results to arbitrary noncommutative cubics, and independent of whether condition (4.133) holds.

**Proposition 4.6.1.** *Let $A = \mathbf{k}_q[x,y,z]$ be a skew polynomial algebra where the $q_{i,j}$ are cube roots of unity satisfying* (4.133)*. For every $f = ax^3 + by^3 + cz^3$ defined by some $[a:b:c] \in \mathbb{P}^2$, and writing $B := A/(f)$, we have*

$$\mathsf{qgr}\, B \simeq \mathsf{coh}\, E \tag{4.135}$$

*where $E \subset \mathbb{P}^2_{[x:y:z]}$ is the cubic curve with equation $f$.*

Proposition 4.6.1 is an upgrade of [84, Theorem 1.1]: by the classification of 1-Calabi–Yau categories in [123] one knows that $\mathsf{qgr}\, B$ for $f = x^3 + y^3 + z^3$ must be $\mathsf{coh}\, E$ for *some* elliptic curve $E$. In Section 4.6.1 we prove that it is the Fermat elliptic curve, and moreover extend the description to *all* elements in the net $\langle x^3, y^3, z^3 \rangle$ of central cubics.

Note that this net includes singular fibres: the curve $E$ is singular when $abc = 0$. If only one of the $a, b, c$ vanishes, then $E$ are three concurrent lines, if two of the $a, b, c$ vanish, then $E$ is the non-reduced triple line.

We also consider the case where the condition of [84] does not hold, which is new. In Section 4.6.2 we prove the following result, giving a complete description of the elements in the net of noncommutative cubics.

[6] Op. cit. also considers higher-dimensional versions of this setup.

**Proposition 4.6.2.** *Let $A = \mathbf{k}_q[x, y, z]$ be a skew polynomial algebra where the $q_{i,j}$ are cube roots of unity such that* (4.133) *does not hold. For every $f = ax^3 + by^3 + cz^3$ defined by some $[a : b : c] \in \mathbb{P}^2$, and writing $B := A/(f)$, we have*

$$\mathsf{qgr}\, B \simeq \mathsf{coh}(\mathcal{C}, \mathscr{B}), \tag{4.136}$$

*where $\mathcal{C}$ is a separated Deligne–Mumford stack of dimension 1 with coarse moduli space $\mathbb{P}^1$, and $\mathscr{B}$ is an Azumaya algebra on $\mathcal{C}$, with the following description:*

1. *if $abc \neq 0$, then $\mathcal{C}$ is isomorphic to the (smooth) root stack $\mathcal{C} \cong \sqrt[3]{\mathbb{P}^1; 0 + 1 + \infty}$ and $\mathscr{B}$ is split;*
2. *if $abc = 0$ and exactly one of $a, b, c$ is 0, then $\mathcal{C}$ has trivial generic stabilizer and two stacky points: it is the union in the Zariski topology of the (smooth) root stack $\sqrt[3]{\mathbb{A}^1; 0}$ and a singular quotient stack;*
3. *if exactly two of $a, b, c$ are 0, then $\mathcal{C}$ has generic stabilizer $\mu_3$ and two points with non-generic behavior.*

Hence, we now obtain a family of *stacks* over $\mathbb{P}^1$, some of which are singular, and isotrivial on the complement of $abc = 0$. We give a full description of the singular fibres in the net in Section 4.6.2. We expect that the Azumaya algebra $\mathscr{B}$ is split also for singular fibres.

### 4.6.1 Kanazawa's skew Fermat cubic is the Fermat cubic

When the condition (4.133) holds, [111, §4] can be used to show the following easy lemma.

**Lemma 4.6.3.** *If $A$ is the skew polynomial algebra with coefficient matrix as in* (4.132) *satisfying* (4.133) *then $A$ is a twisted polynomial ring, i.e., there exists an isomorphism of algebras*

$$\phi\colon A \to \mathbf{k}[X, Y, Z]^\theta, \quad x \mapsto X,\ y \mapsto Y,\ z \mapsto Z, \tag{4.137}$$

*where $\mathbf{k}[X, Y, Z]^\theta = (\mathbf{k}[X, Y, Z], *)$ has the twisted product defined by $f * g = fg^{\theta^m}$ on $f \in \mathbf{k}[X, Y, Z]_m$ homogeneous and extended linearly, for the ring automorphism $\theta\colon \mathbf{k}[X, Y, Z] \to \mathbf{k}[X, Y, Z]$ with $X^\theta = X$, $Y^\theta = q_{1,2}Y$, and $Z^\theta = q_{1,3}Z$.*

With this lemma in hand, the proof of Proposition 4.6.1 is easy.

*Proof of Proposition 4.6.1.* The twisted polynomial ring has center generated by elements of degree 3, thus by Lemma 4.4.2 the central Proj is constructed from the 3-Veronese $A^{(3)}$. It follows by Lemma 4.6.3 that the 3-Veronese is given by

$$A^{(3)} \cong (\mathbf{k}[X, Y, Z]^\theta)^{(3)} = \mathbf{k}[X, Y, Z]^{(3)}, \tag{4.138}$$

where the final equality is [111, Corollary 4.5]. In particular the central Proj is simply the commutative plane

$$(\operatorname{Proj} \mathrm{Z}(A^{(3)}), \mathscr{A}^{(3)}) \cong (\operatorname{Proj} \mathbf{k}[X, Y, Z]^{(3)}, \mathcal{O}_{\operatorname{Proj} \mathbf{k}[X,Y,Z]^{(3)}}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}). \tag{4.139}$$

Under the algebra isomorphism from Lemma 4.6.3 the central element $f = ax^3 + by^3 + cz^3$ is mapped to

$$\phi(f) = aX * X * X + bY * Y * Y + cZ * Z * Z = aX^3 + bq_{1,2}^2 Y^3 + cq_{1,3}^2 Z^3. \qquad (4.140)$$

Multiplying $Y$ and $Z$ by an appropriate 9th root of unity we recover the equation $aX^3 + bY^3 + cZ^3$, which defines a cubic curve $E \subset \mathbb{P}^2$. We conclude that

$$\mathsf{qgr}\, A/(f) \simeq \mathsf{qgr}\left(A^{(3)}/(f)\right) \simeq \mathsf{qgr}\left(\mathbf{k}[X,Y,Z]/(aX^3 + bY^3 + cZ^3)\right) \simeq \mathsf{coh}(E), \qquad (4.141)$$

as claimed. □

**Remark 4.6.4.** Proposition 4.6.1 explains the need for a caveat to the statement of [84, Proposition 3.7], in that we have to exclude skew Fermat cubic curves from loc. cit., as we do not obtain a genuinely noncommutative Calabi–Yau curve. In fact, no such thing exists, by the classification in [123].

### 4.6.2 The other skew cubics as (degenerations of) tubular weighted projective lines

Now suppose that $A = \mathbf{k}_q[x, y, z]$ is a skew polynomial algebra for which the condition (4.133) does not hold. The central Proj of $A$ can be described using [111, §3] and is as follows.

**Lemma 4.6.5.** *If $A = \mathbf{k}_q[x, y, z]$ with coefficient matrix as in* (4.132) *which does not satisfy* (4.133)*, then $\mathrm{Z}(A)$ has generators of coprime degree and the central Proj is given by*

$$(\mathbb{P}^2, \mathscr{A}) = (\operatorname{Proj} \mathrm{Z}(A)^{(3)}, \mathscr{A}^{(3)}) = (\operatorname{Proj} \mathbf{k}[x^3, y^3, z^3]^{(3)}, \mathscr{A}^{(3)}). \qquad (4.142)$$

*Proof.* By inspection the elements $x^3, y^3, z^3$ are generators of $\mathrm{Z}(A)$ of degree 3, which implies that the greatest common denominator of the degrees of the generators is either 1 or 3. As in [111, §3] we take a primitive 3rd root of unity $\zeta \in \mathbf{k}^*$ and present

$$q_{1,2} = \zeta^{-c}, \quad q_{1,3} = \zeta^{b}, \quad q_{2,3} = \zeta^{-a}, \qquad (4.143)$$

for some $1 \le a, b, c \le 3$. One checks that the condition (4.133) translates to $a + b + c \equiv 0 \bmod 3$. Because this condition fails by assumption, it follows that $a+b+c$ is not divisible by 3, and it therefore follows by [111, Corollary 3.4] that the gcd of the generators of $\mathrm{Z}(A)$ is also not divisible by 3. Hence $\mathrm{Z}(A)$ is generated in coprime degrees, and it follows from Lemma 4.4.2 that the central Proj is given by

$$(\operatorname{Proj} \mathrm{Z}(A)^{(3)}, \mathscr{A}^{(3)}) = (\operatorname{Proj} \mathbf{k}[x^3, y^3, z^3]^{(3)}, \mathscr{A}^{(3)}), \qquad (4.144)$$

where the order $\widetilde{A}^{(3)}$ corresponds to $A^{(3)} \in \mathsf{qgr}\, \mathrm{Z}(A)^{(3)}$. □

To simplify the notation and avoid confusion, we write $\mathrm{Z}(A)^{(3)} = \mathbf{k}[X, Y, Z]$ where $X$, $Y$, $Z$ are the degree-1 elements corresponding to $x^3$, $y^3$, $z^3$.

**Lemma 4.6.6.** *The order $\mathscr{A}^{(3)}$ has ramification index 3 on the three lines $\mathbb{V}(XYZ) \subset \mathbb{P}^2$.*

*Proof.* It is straightforward to see that $A^{(3)}$ is generated as a module over $\mathbf{k}[X,Y,Z]$ by monomials $x^i y^j z^k$ with $i,j,k < 3$ and $i+j+k$ divisible by 3, which are the following 9 elements:

$$1, \quad x^2y, \quad xy^2, \quad x^2z, \quad xz^2, \quad y^2z, \quad yz^2, \quad xyz, \quad x^2y^2z^2. \tag{4.145}$$

Therefore, if we restrict the order $\mathscr{A}$ to the complement $U = \mathbb{P}^2 \setminus \mathbb{V}(XYZ)$ we obtain the algebra $\Lambda = A^{(3)}_{(XYZ)}$ generated over $R = \mathbf{k}[(\frac{Y}{X})^{\pm}, (\frac{Z}{X})^{\pm}]$ by the elements

$$1, \quad \frac{x^2y}{X}, \quad \frac{xy^2}{X}, \quad \frac{x^2z}{X}, \quad \frac{xz^2}{X}, \quad \frac{y^2z}{X}, \quad \frac{yz^2}{X}, \quad \frac{xyz}{X}, \quad \frac{x^2y^2z^2}{X^2}. \tag{4.146}$$

Each of these generators is invertible: for example $\frac{x^2y}{X} \cdot \frac{xy^2}{X} = q_{1,2}^{-1}\frac{Y}{X}$ and $\frac{xyz}{X} \cdot \frac{x^2y^2z^2}{X^2} = q_{1,2}q_{2,3}q_{1,3}\frac{YZ}{X}$ are units in $R$, and the other cases can be checked similarly. Hence if $\mathfrak{p} \subset R$ is a prime of codimension 1 then the localization $\Lambda_{\mathfrak{p}}$ has radical $\operatorname{rad}\Lambda_{\mathfrak{p}} = \mathfrak{p}\Lambda_{\mathfrak{p}}$ and $\mathscr{A}$ is therefore unramified on $U$.

Now we check that the order has ramification index 3 on each line. By symmetry it suffices to consider the line $Z = 0$ on the open chart $U = \mathbb{P}^2 \setminus \mathbb{V}(XY)$.

The restriction of $\mathscr{A}$ to $\{XY \neq 0\}$ is the algebra $\Lambda = A^{(3)}_{(XY)}$, generated as a module over $R = \mathbf{k}[(\frac{Y}{X})^{\pm}, \frac{Z}{X}]$ by the same elements (4.146). We claim that the localization at $\mathfrak{p} = (\frac{Z}{X})$ has radical given by the two-sided ideal

$$\operatorname{rad}\Lambda_{\mathfrak{p}} = \mathfrak{p}\Lambda_{\mathfrak{p}} + R_{\mathfrak{p}}\left\{\frac{x^2z}{X}, \frac{xz^2}{X}, \frac{y^2z}{X}, \frac{yz^2}{X}, \frac{xyz}{X}, \frac{x^2y^2z^2}{X^2}\right\}. \tag{4.147}$$

Indeed, the generators $1, \frac{x^2y}{X}, \frac{xy^2}{X}$ are already invertible in $\Lambda$, and if $g$ is any other generator in (4.146), then $g^3$ is divisible by $Z = z^3$ and therefore lies in $\operatorname{rad}\Lambda_{\mathfrak{p}}$. Moreover, any product $g_1g_2g_3$ of three such generators is divisible by $Z = z^3$, which shows that

$$(\operatorname{rad}\Lambda_{\mathfrak{p}})^3 \subset \mathfrak{p}\Lambda_{\mathfrak{p}}. \tag{4.148}$$

In fact this is an equality, as, e.g., $(\frac{xyz}{X})^3 = \frac{YZ}{X^2}$ is a generator for $\mathfrak{p}\Lambda_{\mathfrak{p}}$. It follows that $\Lambda$ has ramification index 3 on the line $Z = 0$, which shows the result. $\square$

*Proof of Proposition 4.6.2.* Let $B = A/(f)$ where $f = ax^3 + by^3 + cz^3$ is a noncommutative cubic. Then it follows from Proposition 4.4.5 that there is an equivalence of categories

$$\operatorname{qgr} B \simeq \operatorname{coh}(C, \mathscr{B}), \tag{4.149}$$

where $C = \mathbb{V}(aX + bY + cZ) \subset \mathbb{P}^2$ and $\mathscr{B} = \mathscr{A}|_C$. Because $\mathscr{A}$ is a tame order, it follows by Theorem 4.3.1 that

$$\operatorname{coh}(C, \mathscr{B}) \simeq \operatorname{coh}(\mathcal{C}_{\mathsf{root}}, \mathscr{B}_{\mathsf{root}}) \simeq \operatorname{coh}(\mathcal{C}_{\mathsf{can}}, \mathscr{B}_{\mathsf{can}}), \tag{4.150}$$

where $\mathcal{C}_{\mathsf{root}}$, and $\mathcal{C}_{\mathsf{can}}$ are the restriction of the root stack and canonical stack of $\mathscr{A}$ to $C$. The first part of the statement therefore follows taking $\mathcal{C} = \mathcal{C}_{\mathsf{can}}$.

The precise form of this stack depends on the intersection of $C$ with the ramification locus $\Delta$, which is of the form $\Delta = \mathbb{V}(XYZ)$ by Lemma 4.6.6. We make do a case-by-case analysis.

Case (1). If $abc \neq 0$ then the line $C = \mathbb{V}(aX + bY + cZ)$ meets $\Delta = \mathbb{V}(XYZ)$ transversely in three points, and we can choose an isomorphism $C \cong \mathbb{P}^1$ so that $C \cap \Delta$ maps to $\{0, 1, \infty\}$. By Proposition 4.3.7(1) we therefore obtain the root stack

$$\mathcal{C} = \mathcal{C}_{\mathsf{can}} \cong \mathcal{C}_{\mathsf{root}} \cong \sqrt[3]{\mathbb{P}^1; 0 + 1 + \infty}. \tag{4.151}$$

Case (2). If $abc = 0$ but only one of $a, b, c$ is 0 then $C = \mathbb{V}(aX + bY + cZ)$ does not lie in $\Delta$, and $\mathcal{C}_{\mathsf{can}} \to C$ is an isomorphism outside the two intersection points $C \cap \Delta = \{p_1, p_2\}$, where $p_1 \notin C \cap \Delta_{\mathrm{sing}}$ and $p_2 \in C \cap \Delta_{\mathrm{sing}}$. By symmetry we may assume that $c = 0$.

By Proposition 4.2.13 the map $\mathcal{C}_{\mathsf{can}} \to \mathcal{C}_{\mathsf{root}}$ is an isomorphism over the locus $C \setminus \{p_2\} \cong \mathbb{A}^1$, and is therefore (locally) given by the root stack

$$\mathcal{C}_{\mathsf{root}} \times_C (C \setminus \{p_2\}) \cong \sqrt[3]{C \setminus \{p_2\}; p_1} \cong \sqrt[3]{\mathbb{A}^1; 0}. \tag{4.152}$$

and that $p_2$ is the unique singular point of $\Delta = \mathbb{V}(XYZ)$ on the chart $\mathbb{P}^2|_{Z \neq 0}$. Let $u = \frac{X}{Z}$ and $v = \frac{Y}{Z}$ denote the coordinates on this chart, so that the ramification locus is $\mathbb{V}(uv) \subset \mathbb{A}^2_{u,v}$. It follows by Lemma 4.2.3 and Corollary 4.2.4 that $\mathcal{C}_{\mathsf{root}}$ is given on this chart by

$$\mathcal{S}_{\mathsf{root}} \times_{\mathbb{P}^2} \mathbb{P}^2|_{Z \neq 0} \cong \left[\operatorname{Spec} \frac{\mathbf{k}[u, v, t]}{(t^3 - uv)} / \mu_3\right]. \tag{4.153}$$

The stack on the right-hand side has an étale cover by $\operatorname{Spec} \mathbf{k}[r, s]^{\mu_3}$ where $\mu_3$ acts via $r \mapsto \zeta_3 \cdot r$ and $s \mapsto \zeta_3^2 \cdot s$ for $\zeta_3$ a primitive 3rd root of unity: the map is induced by the ring isomorphism

$$\frac{\mathbf{k}[u, v, t]}{(t^3 - uv)} \xrightarrow{\sim} \mathbf{k}[r, s]^{\mu_3}, \quad u \mapsto r^3, \quad v \mapsto s^3, \quad t \mapsto rs. \tag{4.154}$$

The canonical stack can then be described over this same locus via the étale cover

$$[\operatorname{Spec} \mathbf{k}[r, s] / \mu_3] \to \mathcal{S}_{\mathsf{root}} \times_{\mathbb{P}^2} \mathbb{P}^2|_{Z \neq 0}. \tag{4.155}$$

In the coordinates $u, v$ the restriction of $C$ to the chart is given by $\mathbb{V}(au + bv)$, so after restricting the canonical stack we obtain an étale cover of $\mathcal{C}_{\mathsf{can}} \times_C (C \setminus \{p_1\})$ of the form

$$\left[\operatorname{Spec} \frac{\mathbf{k}[r, s]}{(ar^3 + bs^3)} / \mu_3\right] \longrightarrow \mathcal{C}_{\mathsf{can}} \times_C (C \setminus \{p_1\}). \tag{4.156}$$

Thus $\mathcal{C}_{\mathsf{can}}$ is singular around the point $p_2$, being the quotient of three concurrent lines.

Case (3) If two of $a, b, c$ are 0, then by symmetry we may assume that $b = c = 0$, so that $C = \mathbb{V}(X)$ is a component of the ramification divisor $\Delta$. Writing $U = \mathbb{P}^2|_{Z \neq 0} \cong \operatorname{Spec} \mathbf{k}[u, v]$

with coordinates as above, the canonical stack $\mathcal{S}_{\mathsf{can}}$ is again described by the étale cover in (4.155). The curve $C$ is cut out by the equation $u = 0$ on the chart $U$, which maps $u \mapsto r^3$ under the isomorphism (4.154). The stack $\mathcal{C}_{\mathsf{can}}$ can therefore be described by the étale cover

$$\left[\operatorname{Spec} \frac{\mathbf{k}[r,s]}{(r^3)}/\mu_3\right] \longrightarrow \mathcal{C}_{\mathsf{can}} \times_{\mathbb{P}^2} U, \tag{4.157}$$

Zariski-locally exhibiting $\mathcal{C}_{\mathsf{can}}$ as a non-reduced quotient stack, with generic stabilizer $\mu_3$, and special behavior at $r = 0$. The description of the other chart $Y \neq 0$ is identical. □

The stacky curve $\mathcal{C}$ is a weighted projective line of type $(3,3,3)$, which is called *tubular* [70, §5.4.2]. It is one of the (rigid) classes of weighted projective lines with Euler characteristic 0, thus it shares some properties with cubic curves, but it is only fractionally Calabi–Yau of dimension 3/3.

It is clear from the examples in Sections 4.5 and 4.6 that one can study many other examples of central curves on noncommutative surfaces using these methods. We illustrate this by giving 2 more examples.

**Example 4.6.7.** Consider a graded Clifford algebra $A$ as in Section 4.5, where we will freely use the results obtained in that section. Let $f \in \mathrm{Z}(A)_4$ be a central element of degree 4. The space of such elements is spanned by $x^4, y^4, z^4, x^2y^2, x^2z^2, y^2z^2$, and $f$ defines a conic $C$ in the projective plane underlying the central Proj. Assume that $C$ is smooth, and that $C \cap \Delta$ is a transverse intersection in 6 distinct points. Then by Proposition 4.3.7 we have an equivalence

$$\mathsf{qgr}\, A/(f) \simeq \mathsf{coh}\, \sqrt[2]{\mathbb{P}^1; C \cap \Delta}, \tag{4.158}$$

where on the right-hand side we have a weighted projective line of type $(2,2,2,2,2,2)$. By varying $f$ we can vary the location of the points and obtain a non-isotrivial family of weighted projective lines, because we can only choose coordinates to fix 3 of the points. This contrasts with the isotrivial family of tubular weighted projective line earlier in this subsection.

For a non-isotrivial family of tubular weighted projective lines arising from our construction, we turn our attention towards an exotic del Pezzo order on $\mathbb{P}^2$. Unlike the previous example, it does not arise from a central Proj construction applied to a quadratic Artin–Schelter regular algebra.

**Example 4.6.8.** Let $\mathscr{A}$ be a quaternion order on $\mathbb{P}^2$ ramified along a smooth quartic $\Delta$. See, e.g., [48, §6] for an explicit construction of such an order using the noncommutative cyclic covering trick. Let $L \subset \mathbb{P}^2$ be a line such that $L \cap \Delta$ is a transverse intersection in 4 distinct points. Then by Proposition 4.3.7 we have an equivalence

$$\mathsf{qgr}\, A/(f) \simeq \mathsf{coh}\, \sqrt[2]{\mathbb{P}^1; C \cap \Delta}, \tag{4.159}$$

where on the right-hand side we have a weighted projective line of type $(2,2,2,2)$. By varying $L$ we can vary the location of the points as in the previous example, and we obtain a non-isotrivial family of *tubular* weighted projective lines. They are fractionally Calabi–Yau of dimension 2/2.

## A.1 Some auxiliary results involving stacks

### A.1.1 Closed embeddings of noncommutative Deligne–Mumford stacks

We consider a stacky version of the noncommutative schemes considered in [44, 146].

**Definition A.1.1.** A *noncommutative Deligne–Mumford stack* is a pair $(X, \mathscr{R}_X)$ of a Deligne–Mumford stack $X$ equipped with a quasi-coherent sheaf $\mathscr{R}_X \in \mathsf{QCoh}(X)$ endowed with an algebra structure $m_X \colon \mathscr{R}_X \otimes \mathscr{R}_X \to \mathscr{R}_X$. A morphism between noncommutative stacks $(X, \mathscr{R}_X)$ and $(Y, \mathscr{R}_Y)$ is a pair $\mathbf{f} = (f, f^\sharp)$ of morphisms

$$f \colon X \to Y, \quad f^\sharp \colon f^*\mathscr{R}_Y \to \mathscr{R}_X, \tag{160}$$

with $f$ a morphism of stacks, and $f^\sharp$ a morphism of algebra objects in $\mathsf{QCoh}(X)$.

In what follows we will always assume that all the sheaves of algebras are coherent. With this assumption, any morphism $\mathbf{f} = (f, f^\sharp) \colon (X, \mathscr{R}_X) \to (Y, \mathscr{R}_Y)$ with $f$ proper defines pushforward and pullback morphisms

$$\mathbf{f}_* \colon \mathsf{coh}(X, \mathscr{R}_X) \to \mathsf{coh}(Y, \mathscr{R}_Y), \quad \mathbf{f}^* \colon \mathsf{coh}(Y, \mathscr{R}_Y) \to \mathsf{coh}(X, \mathscr{R}_X), \tag{161}$$

where for $\mathcal{M} \in \mathsf{coh}(X, \mathscr{R}_X)$ the module $\mathbf{f}_*\mathcal{M}$ is given by $f_*\mathcal{M}$ endowed with a $\mathscr{R}_Y$-module structure via

$$f_*\mathcal{M} \otimes_Y \mathscr{R}_Y \to f_*(\mathcal{M} \otimes_Y f^*\mathscr{R}_Y) \xrightarrow{f_*(\mathrm{id}_{\mathcal{M}} \otimes_Y f^\sharp)} f_*(\mathcal{M} \otimes_Y \mathscr{R}_X) \xrightarrow{f_* m_X} f_*\mathcal{M}, \tag{162}$$

and the pullback of $\mathcal{N} \in \mathsf{coh}(Y, \mathscr{R}_Y)$ is given by the right $\mathscr{R}_X$-module $\mathbf{f}^*\mathcal{N} := f^*\mathcal{N} \otimes_{f^*\mathscr{R}_Y} \mathscr{R}_X$. These functors form an adjoint pair as in [146, Lemma A.5].

Given a noncommutative stack $(X, \mathscr{R}_X)$ and a morphism $f \colon X \to Y$, there is an induced noncommutative stack structure on $Y$ given by $f_*\mathscr{R}_X$ and a natural map $\mathbf{f} \colon (X, \mathscr{R}_X) \to (Y, f_*\mathscr{R}_X)$ defined by the counit

$$f^\sharp \colon f^* f_* \mathscr{R}_X \to \mathscr{R}_X, \tag{163}$$

of the adjunction between $f_*$ and $f^*$. Likewise, if $(Y, \mathscr{R}_Y)$ is a noncommutative stack and $f \colon X \to Y$ a morphism, there is an induced noncommutative stack structure on $X$ defined by $f^*\mathscr{R}_Y$, and

$$\mathbf{f} = (f, \mathrm{id}_{f^*\mathscr{R}_Y}) \colon (X, f^*\mathscr{R}_Y) \to (Y, \mathscr{R}_Y) \tag{164}$$

is a well-defined map of noncommutative stacks.

**Lemma A.1.2.** *Let $(X, \mathscr{R})$ be a noncommutative stack, and $i\colon V \to X$ a closed immersion. Then*

$$\mathbf{i}_*\colon\ \mathsf{coh}(V, i^*\mathscr{R}) \hookrightarrow \mathsf{coh}(X, \mathscr{R}) \tag{165}$$

*embeds $\mathsf{coh}(V, i^*\mathscr{R})$ as the full subcategory of modules whose underlying sheaf lies in $i_* \mathsf{coh}(V)$.*

*Proof.* Since $i\colon V \to X$ is a closed embedding of stacks, we claim that the counit $i^*i_* \to \mathrm{id}$ of the adjunction is a natural isomorphism. To see this, we can pull back $i$ to a morphism $i'\colon V \times_X U \to U$, where $U \to X$ is an atlas. By [15, Lemma 04XC] the map $i'$ is a closed immersion of algebraic spaces, and $(i')^*(i')_* \to \mathrm{id}$ is a natural isomorphism by standard results [15, Lemma 04CJ]. Then for any sheaf $\mathcal{F}$ on $V$ the morphism $i^*i_*\mathcal{F} \to \mathcal{F}$ pulls back to the isomorphism $(i')^*(i')_*\mathcal{F} \to \mathcal{F}$, hence $i^*i_*\mathcal{F} \to \mathcal{F}$ is also an isomorphism by [15, Lemma 0GQF].

For the morphism $\mathbf{i}$, we note that the left adjoint to $\mathbf{i}_*$ is simply given by

$$\mathbf{i}^*(-) = i^*(-) \otimes_{i^*\mathscr{R}} i^*\mathscr{R} \cong i^*(-), \tag{166}$$

which agrees with the left adjoint of $i_*\colon \mathsf{coh}(S) \to \mathsf{coh}(T)$ on the underlying sheaves. In particular, for any $\mathcal{M} \in \mathsf{coh}(V, i^*\mathscr{R})$ the natural map

$$\mathbf{i}^*\mathbf{i}_*\mathcal{M} = i^*i_*\mathcal{M} \xrightarrow{\sim} \mathcal{M} \tag{167}$$

is an isomorphism. Hence, the transformation $\mathbf{i}^*\mathbf{i}_* \to \mathrm{id}$ is a natural isomorphism and hence $\mathbf{i}_*$ is fully faithful.

Finally, it is clear by construction that the objects in the image of $\mathbf{i}_*$ have underlying sheaf in the image of $i_*$. Conversely, if $\mathcal{M} \in \mathsf{coh}(V, i^*\mathscr{R})$ is a module with underlying sheaf $i_*\mathcal{F}$ then the natural map $\mathcal{M} \to \mathbf{i}_*\mathbf{i}^*\mathcal{M} = i_*i^*i_*\mathcal{F} \cong \mathcal{M}$ is an isomorphism, so $\mathcal{M}$ is the image of $\mathbf{i}^*\mathcal{M}$. □

### A.1.2 Stacky curves are determined complete locally

It is well-known that a *smooth* Deligne–Mumford stack with trivial generic stabilizer is determined by its coarse moduli space and the complete local structure over the stacky points; this follows for example by the bottom-up characterization of [71]. Although it is probably known to experts that the following result holds in the singular setting, we could not find a reference and have chosen to include a proof below.

In what follows we fix a curve $C$ and consider stacky curves $\mathcal{C} \to C$ with coarse moduli scheme $C$ in the sense of Definition 4.2.5. Given a point $c \in C$ we let $\widehat{\mathcal{C}}_c := \mathcal{C} \times_C \operatorname{Spec} \widehat{\mathcal{O}}_{C,c}$ denote the completion over $c$.

**Proposition A.1.3.** *Let $\mathcal{X} \to C$ and $\mathcal{Y} \to C$ be stacky curves over $C$. Suppose that for every closed point $c \in C$ there is an isomorphism $\widehat{\mathcal{X}}_c \cong \widehat{\mathcal{Y}}_c$ relative to $C$. Then $\mathcal{X} \cong \mathcal{Y}$ as stacks over $C$.*

**Corollary A.1.4.** *Suppose $\mathcal{C} \to C$ is a stacky curve which is complete locally isomorphic to a root stack $\sqrt[e_1,\ldots,e_n]{C; D_1, \ldots, D_n}$ with $D_i = m_i[p_i]$ (possibly non-reduced) effective divisors supported on $p_i \in C$. Then $\mathcal{C}$ is isomorphic to this root stack as a stack over $C$.*

To prove Proposition A.1.3 we use an Artin approximation argument adapted from [2, Theorem 4.19] to the Hom-stack defined in [3] to pass from the complete-local to the étale local setting. Given two algebraic stacks $\mathcal{X} \to C$ and $\mathcal{Y} \to C$ the Hom-stack is defined by the 2-functor

$$\underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})\colon \mathrm{Sch}/C \to \mathrm{Grpd}, \quad T \mapsto \mathrm{Hom}_T(\mathcal{X}\times_C T, \mathcal{Y}\times_C T) \cong \mathrm{Hom}_C(\mathcal{X}\times_C T, \mathcal{Y}). \tag{168}$$

By [3, §3.2] this defines a stack which is limit-preserving, in the sense that there is a natural equivalence $\operatorname{colim} \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(T_i) \simeq \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(\lim T_i)$ for any inverse system of affine schemes $T_i \in \mathrm{Sch}/C$. This allows us to establish the following helpful lemma.

**Lemma A.1.5.** *Let $\mathcal{X} \to C$ and $\mathcal{Y} \to C$ be algebraic stacks over $C$ as above and let $c \in C$ be a point for which there exists an isomorphism $\widehat{\xi}\colon \widehat{\mathcal{X}}_c \xrightarrow{\sim} \widehat{\mathcal{Y}}_c$ of stacks over $C$. Then there exists an affine étale neighborhood $U \to C$ of $c$ and an étale map*

$$\xi\colon \mathcal{X} \times_C U \to \mathcal{Y} \tag{169}$$

*of stacks over $C$ which induces an isomorphism on stabilizer groups.*

*Proof.* By taking connected components of groupoids, the Hom-stack defines a set-valued functor

$$F(-) = \pi_0 \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})\colon \mathrm{Sch}/C \to \mathrm{Sets}, \tag{170}$$

such that for any inverse system of affine schemes $T_i \in \mathrm{Sch}/C$ the natural map $\operatorname{colim} F(T_i) \to F(\lim T_i)$ is an isomorphism. Hence $F$ is a functor of finite presentation in the sense of [4]. The isomorphism $\widehat{\xi}$ relative to $C$ defines a class $[\widehat{\xi}] \in F(\operatorname{Spec} \widehat{\mathcal{O}}_{C,c})$ and the Artin approximation theorem [4, Theorem 1.12] therefore shows that there exists an étale neighborhood $U \to C$ of $c$ and a class $[\xi] \in F(U)$ of a map

$$\xi \in \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(U) = \mathrm{Hom}_C(\mathcal{X} \times_C U, \mathcal{Y}) \tag{171}$$

which is equivalent to $\widehat{\xi}$ when restricted to the second-order neighborhood $\operatorname{Spec} \mathcal{O}_{C,c}/\mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal of the local ring at $c$. The proof of [2, Theorem 4.19] then shows that (possibly after shrinking $U$) the map $\xi$ is étale and induces an isomorphism on stabilizers. □

The proof now follows by applying the above lemma to each of the stacky points on the curve and gluing the resulting étale maps to an isomorphism of stacks in the étale topology.

*Proof of Proposition A.1.3.* Denote the coarse moduli maps by $p\colon \mathcal{X} \to C$ and $q\colon \mathcal{Y} \to C$. Since $\mathcal{X}$ and $\mathcal{Y}$ have trivial generic stabilizer, there is an open $U_0 \subset C$ such that the maps

$p|_{U_0}\colon \mathcal{X} \times_C U_0 \to U_0$ and $q|_{U_0}\colon \mathcal{Y} \times_C U_0 \to U_0$ are isomorphisms. In particular, there is an isomorphism

$$\xi_0 = q|_{U_0}^{-1} \circ p|_{U_0}\colon \mathcal{X} \times_C U_0 \xrightarrow{\sim} \mathcal{Y} \times_C U_0. \tag{172}$$

Because $C$ is a curve, the complement $C \setminus U_0$ consists of a finite number of points $c_1, \ldots, c_n$ and by Lemma A.1.5 there are étale neighborhoods $U_i \to C$ with maps $\xi_i\colon \mathcal{X} \times_C\colon U_i \to \mathcal{Y}$ inducing isomorphisms on stabilizers at the points $c_i$. This yields a tuple of elements

$$(\xi_0, \ldots, \xi_n) \in \prod_{i=0,\ldots,n} \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(U_i). \tag{173}$$

The maps $\{U_i \to C\}$ are jointly surjective, hence form a cover in the étale topology on $\mathrm{Sch}/C$. Because $\underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})$ is a stack it is a sheaf over $\mathrm{Sch}/C$ and such a tuple glues to a global element in $\underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(C)$ if on the overlaps $U_{ij} = U_i \times_C U_j$ there is an equality[7]

$$\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}. \tag{174}$$

Shrinking the neighborhoods if necessary, we can and will assume that each $U_{ij} \to C$ factors through $U_0 \to C$ along a map $f_{ij}\colon U_{ij} \to U_0$. It then follows that the restrictions (174) factor as the top row in a diagram of the form

$$\begin{array}{ccccc} \mathcal{X} \times_C U_{ij} & \xrightarrow{\zeta} & \mathcal{Y} \times_C U_0 & \longrightarrow & \mathcal{Y} \\ \downarrow{\scriptstyle p|_{U_{ij}}} & & \downarrow{\scriptstyle q|_{U_0}} & & \downarrow{\scriptstyle q} \\ U_{ij} & \xrightarrow{f_{ij}} & U_0 & \longrightarrow & C \end{array} \tag{175}$$

where $\zeta$ is the map induced by either $\xi_i$ or $\xi_j$ respectively via the universal property of the fibre product $\mathcal{Y} \times_C U_0$. Although $\zeta$ might a priori be different for $\xi_i$ and $\xi_j$, the fact that $q|_{U_0}$ is an isomorphism shows that for either choice of $\zeta$

$$\zeta = q|_{U_0}^{-1} \circ f_{ij} \circ p|_{U_{ij}} \tag{176}$$

Therefore the top row is the same for $\xi_i$ and $\xi_j$ and the two elements agree on the nose on $U_{ij}$. As a result, the tuple $(\xi_0, \ldots, \xi_n)$ glues to a global element $\xi \in \underline{\mathrm{Hom}}_C(\mathcal{X}, \mathcal{Y})(C) = \mathrm{Hom}_C(\mathcal{X}, \mathcal{Y})$ which is étale map $\mathcal{X} \to \mathcal{Y}$ inducing an isomorphism of stabilizers at every point. Because both stacks have the same coarse moduli scheme $\xi$ also induces an equivalence on closed points, and it follows by [15, Tag 0DUD] that $\xi$ is an isomorphism. □

### A.1.3 A description of the canonical stack

The goal of this section is to find an explicit complete-local description of the canonical stack associated to a root stack. We start with a review of the canonical stack.

[7] The actual stack condition asks for an natural isomorphism satisfying the cocycle condition instead of equality, but equality certainly suffices.

**The canonical stack** Let $X$ be a variety over a field of characteristic 0 with quotient singularities, then there is a canonical stack $\mathcal{X}^{\mathsf{can}}$ constructed by [141]. This is a smooth separated Deligne–Mumford stack with coarse moduli space $X$ which is characterized in [63, Theorem 4.6] by a universal property: every map $\mathcal{Z} \to X$ from a smooth separated Deligne–Mumford stacks of finite type, which is dominant and codimension-preserving at the level of coarse moduli spaces, factors 2-uniquely:

$$\mathcal{Z} \overset{\exists !}{\dashrightarrow} \mathcal{X}^{\mathsf{can}} \longrightarrow X \tag{177}$$

In particular, this shows that the canonical stack is unique and can be constructed étale (or smooth) locally.

In [62] the canonical stack of a stack $\mathcal{X}$ is defined by applying the construction of [141] to an étale cover: if $\mathcal{X}$ is a finite type Deligne–Mumford stack with an étale cover $\{X_i \to \mathcal{X}\}$ such that each $X_i$ has only quotient singularities, then $\mathcal{X}_i^{\mathsf{can}} \to X_i$ exists for each $i$. The universal property guarantees that the canonical stacks glue to a global stack $\mathcal{X}^{\mathsf{can}}$ with $\mathcal{X}^{\mathsf{can}} \times_{\mathcal{X}} X_i \cong \mathcal{X}_i^{\mathsf{can}}$. We claim that the canonical stack again satisfies a universal property.

**Lemma A.1.6.** *Let $\mathcal{X}$ be a Deligne–Mumford stack admitting a canonical stack $\mathcal{X}^{\mathsf{can}}$. If $\mathcal{Z}$ is a smooth separated Deligne–Mumford stack of finite type with a map $\mathcal{Z} \to \mathcal{X}$ which induces a dominant codimension-preserving map on coarse moduli spaces, then there is a unique factorization*

$$\mathcal{Z} \overset{\exists !}{\dashrightarrow} \mathcal{X}^{\mathsf{can}} \longrightarrow \mathcal{X} \tag{178}$$

*Proof.* Let $\{X_i \to \mathcal{X}\}$ be an étale cover admitting canonical stacks $\mathcal{X}_i^{\mathsf{can}} \to X_i$. Then the pullbacks $\mathcal{Z} \times_{\mathcal{X}} X_i \to X_i$ are again dominant codimension-preserving, and therefore factor uniquely over the canonical stack $\mathcal{X}_i^{\mathsf{can}} \to X_i$, yielding $\mathcal{Z} \times_{\mathcal{X}} X_i \to \mathcal{X}_i^{\mathsf{can}}$ in the commutative diagram:

$$\begin{array}{ccccc} \mathcal{Z} \times_{\mathcal{X}} X_i & \longrightarrow & \mathcal{X}_i^{\mathsf{can}} & \longrightarrow & X_i \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z} & \dashrightarrow & \mathcal{X}^{\mathsf{can}} & \longrightarrow & \mathcal{X}. \end{array} \tag{179}$$

The universal property guarantees that these compositions $\mathcal{Z} \times_{\mathcal{X}} X_i \to \mathcal{X}^{\mathsf{can}}$ glue uniquely to the dashed arrow factoring the map $\mathcal{Z} \to \mathcal{X}$. □

The universal property then characterizes the canonical stack as follows:

**Corollary A.1.7.** *Let $\mathcal{X}$ be a Deligne–Mumford stack admitting a coarse moduli stack glued from canonical stacks $\mathcal{X}_i^{\mathsf{can}} \to X_i$ along an étale cover $\{X_i \to X\}$. If $\mathcal{Z} \to \mathcal{X}$ is a morphism inducing an isomorphism on coarse moduli spaces which fits into cartesian* *diagrams*

$$\begin{array}{ccc} \mathcal{X}_i^{\mathsf{can}} & \longrightarrow & X_i \\ \big\downarrow & & \big\downarrow \\ \mathcal{Z} & \longrightarrow & \mathcal{X}, \end{array} \tag{180}$$

*then this morphism factors over an isomorphism* $\mathcal{Z} \cong \mathcal{X}^{\mathsf{can}}$*.*

In other words: the canonical stack is just the one which pulls back to the canonical stack along an étale cover of $\mathcal{X}$. The same remains true after taking a completion relative to a map $\mathcal{X} \to X$ and a point $x \in X$.

**Our case** We want to find an explicit expression for the canonical stack associated to the complete-local chart of the root stack $[\operatorname{Spec} R_e/\mu_e] \to \operatorname{Spec} R$. This is constructed by taking $S = \mathbf{k}[\![x, y]\!]$ and $H < \mathrm{SL}_2(\mathbf{k})$ a finite group such that such that $R_e = S^H$. Then the completion of the canonical stack fits into the cartesian diagram

$$\begin{array}{ccc} [\operatorname{Spec} S/H] & \xrightarrow{\alpha} & \operatorname{Spec} R_e \\ \big\downarrow & & \big\downarrow \beta \\ \mathcal{S}_{\mathsf{can}} \times_S \operatorname{Spec} R & \longrightarrow & [\operatorname{Spec} R_e/\mu_e] \end{array} \tag{181}$$

where $\alpha$ is the coarse moduli map and $\beta$ is the quotient map. This diagram determines the canonical stack uniquely among the smooth Deligne–Mumford stacks with the same coarse moduli space as $[\operatorname{Spec} R_e/\mu_e]$. We claim that the stack $[\operatorname{Spec} S/(H \times \mu_e)]$ satisfies these properties and therefore has to be isomorphic to the canonical stack. This follows from the following general result.

**Proposition A.1.8.** *Let $X$ be an affine scheme with an action of a finite group scheme $G \times H$, and let $Y = X /\!\!/ H$ be the affine GIT quotient. Then there is a cartesian square*

$$\begin{array}{ccc} [X/H] & \xrightarrow{\alpha} & Y \\ \big\downarrow \beta & & \big\downarrow \beta \\ [X/(G \times H)] & \xrightarrow{\alpha} & [Y/G] \end{array} \tag{182}$$

Before we give the proof, recall that a quotient stack $[X/H]$ can be defined as the category fibered in groupoids over $\mathsf{Sch}/\mathbf{k}$ with objects over a scheme $T$ given by pairs

$$(p\colon P \to T,\ f\colon P \to X), \tag{183}$$

where $p$ is an $H$-torsor and $f$ is an $H$-equivariant map. The stacks $[Y/G]$ and $[X/(G \times H)]$ have an analogous description, but the objects of the latter can be described more conveniently as products.

**Lemma A.1.9.** *Let $q\colon Q \to T$ be a $G \times H$-torsor. Then there is a $G \times H$-equivariant $T$-scheme isomorphism $Q \cong Q_1 \times_T Q_2$ where $Q_1 \to T$ is a $G$-torsor and $Q_2 \to T$ is a $H$-torsor.*

*Proof.* The quotients $Q_1 = Q/H$ and $Q_2 = Q/G$ are torsors because the actions of $G$ and $H$ commute. The quotient maps yield a $G \times H$-equivariant morphism $Q \to Q/H \times_T Q/G$ of $T$-schemes which is then automatically an isomorphism. □

In view of the above, we will write all $G \times H$-torsors as fibre products $Q_1 \times_T Q_2$.

The top map denoted $\alpha$ in (182) is the coarse moduli map, described by the functor mapping a pair (183) to the morphism

$$T \xrightarrow{\overline{p}^{-1}} P/H \xrightarrow{\overline{f}} X/\!\!/H = Y, \tag{184}$$

where $\overline{p}$ and $\overline{f}$ are the maps induced by the GIT quotient of $P$ and $X$ by the action of $H$. The functor $\alpha\colon [X/(G \times H)] \to [Y/G]$ on the bottom is defined similarly as

$$(q\colon Q_1 \times_T Q_2 \to T,\ g\colon Q_1 \times_T Q_2 \to X) \quad \mapsto \quad (\overline{q}\colon Q_1 \to T,\ \overline{g}\colon Q_1 \to Y), \tag{185}$$

where the maps are induced by the quotient $(Q_1 \times_T Q_2)/H \cong Q_1$.

The vertical maps given by the inclusion of the trivial $G$-torsors: the functor $\beta\colon Y \to [Y/G]$ sends a morphism $h\colon T \to Y$ to the trivial torsor $T \times G \to T$ equipped with the map

$$a \circ (h \times \mathrm{id})\colon T \times G \xrightarrow{h\times\mathrm{id}} Y \times G \xrightarrow{a} Y, \tag{186}$$

where $a$ denotes the action map. The functor $\beta\colon [X/H] \to [X/(G \times H)]$ similarly maps a pair (183) to the product torsor $Q_1 \times_T Q_2 = (T \times G) \times_T P$ equipped with the map

$$(T \times G) \times_T P \xrightarrow{\sim} P \times G \xrightarrow{f\times\mathrm{id}} X \times G \xrightarrow{a} X, \tag{187}$$

where $a$ again denotes the action map. Note that this morphism is $G \times H$-equivariant because the actions of $G$ and $H$ commute.

*Proof of Proposition A.1.8.* We want to show that $[X/H]$ is isomorphic to the fibre product of $Y$ and $[X/(G \times H)]$ over $[Y/G]$. The maps above yield a functor $[X/H] \to [X/(G \times H)] \times_{[Y/G]} Y$ via the universal property, so it suffices to construct an inverse functor. The 2-categorical fibre product has objects over a scheme $T$ given by triples

$$\Big((q\colon Q_1 \times_T Q_2 \to T,\ g\colon Q_1 \times_T Q_2 \to X),\ h\colon T \to Y,\ u\colon T \times G \xrightarrow{\sim} Q_1\Big). \tag{188}$$

of objects in $[X/(G \times H)]$ and $Y$, together with an isomorphism $u$ in $[Y/G]$ between their images. Such a triple defines an object of $[X/H]$ over $T$ given by the $H$-torsor $\overline{q}\colon Q_2 \to T$ equipped with the map

$$Q_2 \cong T \times_T Q_2 \xrightarrow{(s,\mathrm{id})} (T \times G) \times_T Q_2 \xrightarrow{(u,\mathrm{id})} Q_1 \times_T Q_2 \xrightarrow{g} X, \tag{189}$$

where $s\colon T \to T \times G$ is the unit section $t \mapsto (t, 1_G)$. It is clear that this is $H$-equivariant, because $g$ is $G \times H$-equivariant for the $H$ action on the factor $Q_2$.

To see that this is an inverse construction, note that a pair (183) is mapped to the triple as in (188) of the product torsor $(T \times G) \times_T P \to T$ with the map $f \circ \mathrm{pr}_P\colon (T \times G) \times_T P \to X$, the morphism $\overline{f} \circ \overline{p}^{-1}\colon T \to Y$, and the isomorphism $u = \mathrm{id}_{T\times G}$. This is mapped to the pair of the $H$-torsor $Q_2 = P \to T$ equipped with the map

$$P \cong T \times_T P \xrightarrow{(s,\mathrm{id})} (T \times G) \times_T P \xrightarrow{f \circ \mathrm{pr}_P} X, \tag{190}$$

which is equal to the map $f\colon P \to X$. Hence the composition

$$[X/H] \to [X/(G \times H)] \times_{[Y/G]} Y \to [X/H] \tag{191}$$

is equal to the identity on the nose. To see that $[X/H] \to [X/(G \times H)] \times_{[Y/G]} Y$ is an equivalence it therefore suffices to show that it is essentially surjective. But this is immediate, because every triple as in (188) is isomorphic to the triple

$$((q\circ(u\times \mathrm{id})\colon (T\times G)\times_T Q_2 \to T, g\circ(u\times \mathrm{id})\colon (T\times G)\times_T Q_2 \to T, h\colon T \to Y,\ \mathrm{id}_{T\times G}), \tag{192}$$

with the isomorphism induced by the map $u \times \mathrm{id}\colon (T \times G) \times_T Q_2 \to Q_1 \times Q_2$. □

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