Categorical absorption for hereditary orders

Thilo Baumann

22nd May 2025

Abstract

We show that Kuznetsov–Shinder's notion of deformation absorption of singularities leads to a new approach for studying the bounded derived category of a hereditary order on a curve. The starting point is a hereditary order which can be interpreted as a smoothing of the finite-dimensional algebra obtained from the restriction to a ramified point. We construct a triangulated subcategory inside the derived category of this finite-dimensional algebra which provides a deformation absorption of singularities. This allows us to obtain a semiorthogonal decomposition of the bounded derived category of the hereditary order, which is in addition linear over the base.

Contents

1 Introduction

Hereditary orders over a curve form a well-studied class of sheaves of (noncommutative) algebras. Their classification up to étale-local isomorphism [Rei75, Theorem 39.14], and up to Morita equivalence [BDG17, Proposition 7.7] is well-established.

From the geometric point of view, [CI04, Corollary 7.8] provides a dictionary between

a) hereditary orders $\mathscr{A}$ on a smooth curve C, and

b) smooth root stacks $\mathcal{C}$ with coarse moduli space C,

in the sense that there is an equivalence $\mathsf{coh}(C, \mathscr{A}) \simeq \mathsf{coh}(\mathcal{C})$ of abelian categories. The dictionary identifies ramification points of the order $\mathscr{A}$ with the points with non-trivial stabilizer of the root stack $\mathcal{C}$.

The bounded derived category $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathcal{C})$ has been examined on both sides of the dictionary with various objectives in mind. In particular, when trying to decompose these categories, one has the following results if C is projective.

- If $C = \mathbb{P}^1$, the stacky curve $\mathcal{C}$ is a weighted projective line, and $\mathsf{D}^{\mathsf{b}}(\mathcal{C})$ admits a tilting bundle [GL87, Proposition 4.1]. This was recently generalized to a certain class of hereditary orders over non-algebraically closed fields in [Bur24, Theorem 3.12].

- Among other things, [IU15, Theorem 1.2] construct a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(\mathcal{C})$.

We propose a novel approach, using the framework of Kuznetsov–Shinder [KS23] to decompose $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$, by viewing $(C, \mathscr{A})$ as a family of finite-dimensional algebras over the curve C. The idea fits into the general perspective of studying the bounded derived category of families of varieties over a base scheme, see [Kuz23] for a survey. It provides the first example of a deformation absorption of singularities [KS23] in a noncommutative setting.

To state the first main result of the paper, let C be a smooth curve over an algebraically closed field $\mathbf{k}$ of characteristic zero. Consider a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ with ramification locus $\Delta_{\mathscr{A}} = \{o\}$ and ramification index $r \in \mathbb{Z}_{\geq 1}$. Hence, the algebra $\mathscr{A}$ is Azumaya on $C \setminus \{o\}$. The algebra $\mathscr{A}(o) := \mathscr{A} \otimes_C \operatorname{Spec} \mathbf{k}(o)$ is described in Lemma 3.1. The first main result is the existence of a triangulated subcategory in $\mathsf{D}^{\mathsf{b}}(\mathscr{A}(o))$ which absorbs singularities.

Theorem A (Theorem 4.7)**.** *The sequence $(S_1, \dots, S_{r-1})$ of simple $\mathscr{A}(o)$-modules is semiorthogonal in $\mathsf{D}^{\mathsf{b}}(\mathscr{A}(o))$ and absorbs singularities, i.e. the triangulated subcategory*

$$\mathsf{S} = \langle S_1, \dots, S_{r-1} \rangle \subset \mathsf{D}^{\mathsf{b}}(\mathscr{A}(o)) \tag{1}$$

is admissible and both of its complements ${}^{\perp}\mathsf{S}$ and $\mathsf{S}^{\perp}$ are smooth and proper.

More precisely, we show in Lemma 4.8 that the sequence (1) forms a semiorthogonal sequence of $\mathbb{P}^{\infty,2}$-objects. The proof uses the representation theory of finite-dimensional algebras. Appealing to a noncommutative version of [KS23, Theorem 1.8] we use S to provide a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

Theorem B (Theorem 4.11)**.** *Let $\mathsf{i}_o \colon (\operatorname{Spec} \mathbf{k}(o), \mathscr{A}(o)) \to (C, \mathscr{A})$ be the inclusion of $o \in C$. There is a strong C-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathsf{i}_{o,*} S_1, \dots, \mathsf{i}_{o,*} S_{r-1}, \mathsf{D} \rangle, \tag{2}$$

such that

i) the sequence $(\mathsf{i}_{o,*} S_1, \dots, \mathsf{i}_{o,*} S_{r-1})$ is exceptional,

ii) the admissible subcategory D is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$.

In fact, we will show in Lemma 4.14 that D is equivalent to $\mathsf{D}^{\mathsf{b}}(C)$.

The existence of a semiorthogonal decomposition (2) can also be deduced in an indirect way from the stacks–orders dictionary [CI04, Corollary 7.8] and the semiorthogonal decomposition of [BLS16, Theorem 4.7] for root stacks. We explain in Section 4.4 how our results translate to the stacky side in [BLS16, Theorem 4.7].

Categorical absorption and deformation absorption of singularities. We outline the idea of categorical absorption and deformation absorption of singularities from [KS23].

Given a flat family $\mathcal{X} \to (C, o)$ over a smooth pointed curve with a single singular fiber $\mathcal{X}_o = \mathcal{X} \times_C \operatorname{Spec} \mathbf{k}(o)$ and smooth total space $\mathcal{X}$, there is sometimes an interplay of a certain semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathcal{X}_o) = \langle \mathsf{S}, {}^{\perp}\mathsf{S} \rangle$ of the fiber and a C-linear semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(\mathcal{X}) = \langle \mathsf{A}_{\mathsf{S}}, \mathsf{B} \rangle$ of the total space in the following sense.

a) As in Theorem A, the category $\mathsf{S} \subset \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ is admissible and both of its complements ${}^{\perp}\mathsf{S}$ and $\mathsf{S}^{\perp}$ are smooth and proper. In this case [KS23, Definition 1.1] says that S *absorbs singularities of* $\mathcal{X}_o$.

b) The category A_{S} is the (thick) closure of the pushforward of S to $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$. If it is admissible in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$, [KS23, Theorem 1.5] show that its complement B is C-linear, as well as smooth and proper such that

$$\mathsf{B}_p \simeq \begin{cases} {}^{\perp}\mathsf{S} & \text{if } p = o, \\ \mathsf{D}^{\mathsf{b}}(\mathcal{X}_p) & \text{otherwise.} \end{cases} \tag{3}$$

If additionally, (b) is satisfied for S, then Definition 1.4 of *op. cit.* says that S *provides a deformation absorption of singularities of* $\mathcal{X}_o$. Theorem B shows that (b) holds in the noncommutative setting for S defined in Theorem A.

Finding $\mathsf{S} \subsetneq \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ which satisfies (a) and (b) is a non-trivial task. Among other things, Kuznetsov–Shinder thus introduce so-called $\mathbb{P}^{\infty,2}$-objects $S \in \mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$. This is an object S such that its ring of self-extensions is $\mathrm{Ext}^{\bullet}_{\mathcal{X}_o}(S,S) \cong \mathbf{k}[\theta]$ with $\deg\theta = 2$, see Definition 4.1. By [KS23, Theorem 1.8] a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\mathcal{X}_o)$ forms a triangulated subcategory satisfying (a) and (b). Notably, (b) is satisfied independently of the chosen smoothing $\mathcal{X}$ of the singular fiber $\mathcal{X}_o$.

Examples of this phenomenon are odd-dimensional varieties with isolated nodal singularities [KS23, Theorem 6.1]. Generalizing to the notion of compound $\mathbb{P}^{\infty}$-objects, [Var24] observed a similar phenomenon for a projective threefold with non-isolated singularity. An application of categorical absorption in the context of mirror symmetry is given by [LT24]. Interpreting a hereditary order $\mathscr{A}$ as a flat family over its central curve C, we present an example $(C,\mathscr{A}) \to C$ where the role of the singular fiber is played by the restriction $\mathscr{A}(o)$ of $\mathscr{A}$ to o in Section 4.

Noncommutative base change. In Appendix A we will make sense of the fiber B_p from (3) in the derived category of a coherent ringed scheme $(C,\mathscr{A})$ by providing a base change formula which follows from some modifications of the results in [Kuz11].

Let $\mathscr{A}$ be a flat $\mathcal{O}_X$-algebra. Moreover, let $f\colon X \to S$ be a flat morphism of schemes. If $h\colon T \to S$ is a morphism of schemes, we set $X_T = X \times_S T$ and $\mathscr{A}_T = h_T^*\mathscr{A}$, where $h_T\colon X_T \to X$ is the induced morphism. In Lemma 2.4, we will show how $(X_T, \mathscr{A}_T)$ is a fiber product. For its bounded derived category $\mathsf{D}^{\mathsf{b}}(X_T,\mathscr{A}_T)$ we have the following generalization of [Kuz11, Theorem 5.6].

Theorem C (Theorem A.16)**.** *Assume that*

$$\mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m \rangle \tag{4}$$

is an S-linear strong semiorthogonal decomposition such that the projection functors have finite cohomological amplitude. If $\mathscr{A}$ has finite global dimension, then there is a T-linear semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X_T,\mathscr{A}_T) = \langle \mathsf{A}_{1,T}, \dots, \mathsf{A}_{m,T} \rangle \tag{5}$$

compatible with pullback and pushforward.

It should be mentioned that Theorem A and Theorem B are independent from the base change formula in Theorem C.

Organization of the paper. We recall properties of coherent ringed schemes in Section 2. We then apply this point of view to hereditary orders in Section 3. In Section 4, we prove Theorem A and Theorem B. Moreover, we show in Section 4.4 how our results relate to smooth stacky curves using the dictionary [CI04]. In the appendix, we explain how to obtain Theorem C from its commutative version [Kuz11].

Notations and conventions. Throughout the paper $\mathbf{k}$ denotes an algebraically closed field of characteristic zero. Every scheme is assumed to be integral and quasi-projective over $\mathbf{k}$. If X is a scheme, we denote by $\mathbf{k}(X)$ its function field.

Acknowledgements. The author was partially supported by the Luxembourg National Research Fund (PRIDE R-AGR-3528-16-Z). We are grateful to Pieter Belmans for many helpful discussions and for his feedback on drafts of this paper.

2 Coherent ringed schemes

In this section we define coherent ringed schemes $(X,\mathscr{A})$ following the terminology of [YZ06]. Sometimes they are also referred to as 'mild noncommutative schemes', see [DLR25b]. Their derived category $\mathsf{D}(X,\mathscr{A})$

(and bounded versions of it) were for example studied in [K06, Appendix D], [BDG17], [Xie23, Appendix A], [DLR25a; DLR25b].

Definition 2.1. A *coherent ringed scheme* is a pair $(X, \mathscr{A})$ of a scheme X over $\mathbf{k}$ and a coherent sheaf of $\mathcal{O}_X$-algebras $\mathscr{A}$.

A *morphism of coherent ringed schemes* $\mathfrak{f} = (f, f_{\mathrm{alg}})\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ consists of

- a morphism of schemes $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.
- an $\mathcal{O}_X$-algebra morphism $f_{\mathrm{alg}}\colon f^*\mathscr{B} \to \mathscr{A}$.

An $\mathcal{O}_X$-order $\mathscr{A}$, as defined in Section 3, is a coherent ringed scheme $(X, \mathscr{A})$. Lemma 3.1 will show that it is necessary for us to work need in the more general framework of coherent ringed schemes.

Given $\mathfrak{f} = (f, f_{\mathrm{alg}})\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $\mathfrak{g} = (g, g_{\mathrm{alg}})\colon (Y, \mathscr{B}) \to (Z, \mathscr{C})$ we define their *composition* $\mathfrak{h} = (h, h_{\mathrm{alg}}) = \mathfrak{g} \circ \mathfrak{f}\colon (X, \mathscr{A}) \to (Z, \mathscr{C})$ as follows:

- on the underlying schemes $h := g \circ f$, and
- on algebras $h_{\mathrm{alg}} := f_{\mathrm{alg}} \circ f^*(g_{\mathrm{alg}})\colon h^*\mathscr{C} \to f^*\mathscr{B} \to \mathscr{A}$.

This allows us to form the category of coherent ringed schemes $\mathbf{ncSch_k}$. Note that every scheme X is an object in $\mathbf{ncSch_k}$ by setting $\mathscr{A} = \mathcal{O}_X$.

Following [K06, Definition 10.3], we call a morphism $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ *strict* if $f_{\mathrm{alg}} = \mathrm{id}_{f^*\mathscr{B}}$. A morphism $\mathfrak{f}$ is called an *extension* if $X = Y$ and f is the identity.

Remark 2.2. Every coherent ringed scheme $(X, \mathscr{A})$ comes with a *structure morphism* $\mathfrak{f}\colon (X, \mathscr{A}) \to X$. This is an extension, where $f_{\mathrm{alg}}\colon \mathcal{O}_X \to \mathscr{A}$ induces the $\mathcal{O}_X$-algebra structure on $\mathscr{A}$. Since $\mathscr{A}$ is a coherent $\mathcal{O}_X$-algebra, we can view the structure morphism as a *finite noncommutative covering* generalizing the equivalence $\mathsf{coh}(\underline{\mathrm{Spec}}_X(\mathscr{A})) \cong \mathsf{coh}(X, \mathscr{A})$ for commutative $\mathscr{A}$ by allowing noncommutative finite-dimensional $\mathbf{k}$-algebras as fibers, see [L08].

Modules on coherent ringed schemes. Given a coherent ringed scheme $(X, \mathscr{A})$, we denote

- by $\mathsf{QCoh}(X, \mathscr{A})$ the category of right $\mathscr{A}$-modules which are quasicoherent, and
- by $\mathsf{coh}(X, \mathscr{A})$ the subcategory of right $\mathscr{A}$-modules which are coherent as $\mathcal{O}_X$-modules.

Similarly to the commutative case, we have the *sheaf hom-functor* $\mathcal{H}om_{\mathscr{A}}(-,-)$, and the *tensor product of $\mathscr{A}$-modules* $- \otimes_{\mathscr{A}} -$. Note that in the tensor product, the second argument needs to be a left $\mathscr{A}$-module.

Given a morphism $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ of coherent ringed schemes, the *pushforward* $\mathfrak{f}_*M := f_*M$ of $M \in \mathsf{QCoh}(X, \mathscr{A})$ carries an induced $\mathscr{B}$-module structure via

$$f_*M \otimes_Y \mathscr{B} \cong f_*(M \otimes_X f^*\mathscr{B}) \xrightarrow{f_*(f_{\mathrm{alg}})} f_*(M \otimes \mathscr{A}) \xrightarrow{f_*(\mu_M)} f_*M, \tag{6}$$

where $\mu_M\colon M \otimes_X \mathscr{A} \to M$ is the $\mathscr{A}$-module structure of M. The *pullback* of $N \in \mathsf{QCoh}(Y, \mathscr{B})$ along $\mathfrak{f}$ is given by $\mathfrak{f}^*N := f^*N \otimes_{f^*\mathscr{B}} \mathscr{A} \in \mathsf{QCoh}(X, \mathscr{A})$.

Remark 2.3. Since we assume that $\mathscr{A}$ is coherent, it follows that $\mathcal{H}om_{\mathscr{A}}$, $\otimes_{\mathscr{A}}$ and $\mathfrak{f}^*$ map coherent modules to coherent modules. For the pushforward (as in the commutative case), one has to additionally require properness of $f\colon X \to Y$, cf. [Xie23, Lemma A.5].

Fiber products of coherent ringed schemes. We extend fiber products [K06, Lemma 10.37] for Azumaya varieties to fiber products in $\mathbf{ncSch_k}$. Let $\mathfrak{f}_1\colon (X_1, \mathscr{A}_1) \to (X, \mathscr{A})$ and $\mathfrak{f}_2\colon (X_2, \mathscr{A}_2) \to (X, \mathscr{A})$ be morphisms of coherent ringed schemes. Moreover, denote by $p_i\colon X_1 \times_X X_2 \to X_i$ the two canonical projections.

$$\begin{array}{ccc} (X_1 \times_X X_2, p_j^*\mathscr{A}_j) & \xrightarrow{\mathfrak{p}_2} & (X_2, \mathscr{A}_2) \\ \big\downarrow{\scriptstyle \mathfrak{p}_1} & & \big\downarrow{\scriptstyle \mathfrak{f}_2} \\ (X_1, \mathscr{A}_1) & \xrightarrow{\mathfrak{f}_1} & (X, \mathscr{A}) \end{array} \tag{7}$$

Lemma 2.4. *Assume that there exists $i \in \{1,2\}$ such that the morphism $\mathfrak{f}_i$ is strict and let $j \neq i$. The coherent ringed scheme $(X_1 \times_X X_2, p_j^* \mathscr{A}_j)$ is a fibre product of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ in $\mathbf{ncSch_k}$. It is unique up to unique isomorphism.*

Proof. By symmetry, we assume without loss of generality that $\mathfrak{f}_1$ is strict. Since X, X_1, X_2 are noetherian, the $\mathcal{O}_{X_1 \times_X X_2}$-algebra $p_2^* \mathscr{A}_2$ is coherent. The morphism $\mathfrak{p}_2 = (p_2, \mathrm{id}_{p_2^* \mathscr{A}_2})$ is the strict morphism induced from the structure morphism $p_2 \colon X_1 \times_X X_2 \to X_2$ for schemes. The morphism $\mathfrak{p}_1 = (p_1, p_2^*(f_{2,\mathrm{alg}}))$ is given by the structure morphism $p_1 \colon X_1 \times_X X_2 \to X_1$ for schemes and the $\mathcal{O}_{X_1 \times_X X_2}$-algebra homomorphism $p_2^*(f_{2,\mathrm{alg}}) \colon p_1^* \mathscr{A}_1 \to p_2^* \mathscr{A}_2$. Note that $p_1^* \mathscr{A}_1 \cong p_2^* f_2^* \mathscr{A}$ because $\mathfrak{f}_1$ is strict.

It is straightforward to verify the universal property using the universal property for the fiber product of the underlying (commutative schemes). □

Remark 2.5. Note that a morphism $h \colon T \to S$ of schemes is always strict. Hence we can base change a coherent ringed scheme $\mathfrak{f} = (f, f_{\mathrm{alg}}) \colon (X, \mathscr{A}) \to S$ over a commutative base S along $h \colon T \to S$, to obtain a coherent ringed scheme $(X_T, \mathscr{A}_T)$ over T, where $X_T = X \times_S T$ and $\mathscr{A}_T = p_2^* \mathscr{A}$. We will return to this perspective in Section 4.3 for the deformation absorption in Theorem 4.11.

K-injectives and locally projectives. The category $\mathsf{QCoh}(X, \mathscr{A})$ is a Grothendieck abelian category, and therefore every cochain complex $M^\bullet$ of quasicoherent $\mathscr{A}$-modules admits a K-injective resolution, see [Xie23, Lemma A.2]. The notion of locally free modules has to be replaced by locally projective modules.

Definition 2.6. A coherent $\mathscr{A}$-module $P \in \mathsf{coh}(X, \mathscr{A})$ is *locally projective* if there exists a (Zariski-)open covering $X = \bigcup_{i \in I} U_i$ such that $P|_{U_i}$ is a finitely generated projective $\mathscr{A}|_{U_i}$-module for all $i \in I$.

By [Rei75, Corollary 3.23], this definition agrees with [BDG17, Definition 3.6].

Lemma 2.7. *Let $(X, \mathscr{A})$ be a coherent ringed scheme and $P \in \mathsf{coh}(X, \mathscr{A})$ a coherent $\mathscr{A}$-module. Then the following are equivalent:*

i) The $\mathscr{A}$-module P is locally projective.

ii) For every $p \in X$, the $\mathscr{A}_p$-module P_p is projective.

Under the (standing) assumption that X is quasi-projective, [BDG17, Proposition 3.7] show that $\mathsf{coh}(X, \mathscr{A})$ *admits enough locally projectives*, i.e. for every coherent $\mathscr{A}$-module M there exists a locally projective $\mathscr{A}$-module P and an $\mathscr{A}$-module epimorphism $P \twoheadrightarrow M$.

The derived category of a coherent ringed scheme. Let $(X, \mathscr{A})$ be a coherent ringed scheme. We denote by $\mathsf{D}(X, \mathscr{A}) := \mathsf{D}(\mathsf{QCoh}(X, \mathscr{A}))$ the *unbounded derived category of quasicoherent $\mathscr{A}$-modules*. By $\mathsf{D}^*(X, \mathscr{A})$, for $* \in \{+, -, \mathsf{b}\}$, we denote the bounded below, bounded above, resp. bounded derived category of quasicoherent $\mathscr{A}$-modules.

Mostly, we are interested in the *bounded derived category* $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) := \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{QCoh}(X, \mathscr{A}))$. Since $\mathscr{A}$ is coherent, [Xie23, Lemma A.4] provides the useful equivalence

$$\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(X, \mathscr{A})). \tag{8}$$

From time to time, we will have to make the distinction between $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ and the *category of perfect complexes* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$, which is the full triangulated subcategory of $\mathsf{D}(X, \mathscr{A})$ given by cochain complexes that are locally quasi-isomorphic to bounded complexes of locally projective $\mathscr{A}$-modules. For a hereditary order $\mathscr{A}$ on a curve C, we have an equality $\mathsf{D}^{\mathsf{perf}}(C, \mathscr{A}) = \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ by Lemma A.2.

Remark 2.8. It is a straightforward, but important observation, cf. [DLR25a, Example 3.7], that the pushforward $\mathfrak{f}_* \colon \mathsf{D}(X, \mathscr{A}) \to \mathsf{D}(X)$ of the structure morphism $\mathfrak{f} \colon (X, \mathscr{A}) \to X$ is exact as exactness does not depend on the module structure.

Semiorthogonal decompositions linear over the base. Let $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$, and consider a morphism of schemes $\mathfrak{f}\colon (X, \mathscr{A}) \to S$. Recall from [K06, Section 2.6], [Per19, §§2, 3] that a triangulated subcategory $\mathsf{A} \subset \mathsf{T}$ is called *S-linear* if

$$M \otimes_X^{\mathbf{L}} \mathbf{L}f^*\mathcal{F} \in \mathsf{A} \quad \text{for all } M \in \mathsf{A}, \mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S). \tag{9}$$

Remark 2.9. If $\mathsf{T} = \mathsf{D}^-(X, \mathscr{A})$ or $\mathsf{T} = \mathsf{D}(X, \mathscr{A})$, and $\mathsf{A} \subset \mathsf{T}$ is S-linear for $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S)$, then S-linearity automatically holds for $\mathsf{D}^-(S)$, resp. $\mathsf{D}(S)$. This follows from [Kuz11, Lemma 4.5].

Let T be an S-linear triangulated category. If $\mathsf{A} \subset \mathsf{T}$ is a full triangulated subcategory, the *right orthogonal* $\mathsf{A}^\perp$ (resp. *left orthogonal* ${}^\perp\mathsf{A}$) of A is defined as the full subcategory containing all objects $T \in \mathsf{T}$ such that $\mathrm{Hom}_{\mathsf{T}}(A, T[i]) = 0$ (resp. $\mathrm{Hom}_{\mathsf{T}}(T, A[i]) = 0$) for all $A \in \mathsf{A}$ and $i \in \mathbb{Z}$.

The next lemma is a straightforward generalization of [K06, Lemma 2.36].

Lemma 2.10. *Let $\mathfrak{f}\colon (X, \mathscr{A}) \to S$ be a morphism of schemes, and $\mathsf{A} \subset \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ be an S-linear category such that A is S-linear. Then ${}^\perp\mathsf{A}$ and $\mathsf{A}^\perp$ are S-linear as well.*

Recall that an (S-linear) triangulated subcategory A of T is called *right admissible* (resp. *left admissible*) if its embedding functor $\alpha\colon \mathsf{A} \to \mathsf{T}$ admits a right adjoint $\alpha^!$ (resp. a left adjoint α^*). We say that A is *admissible* if it is both, left and right admissible.

A sequence of (S-linear) triangulated subcategories $\mathsf{A}_1, \ldots, \mathsf{A}_m$ is called an *(S-linear) semiorthogonal decomposition* of T if

i) for every $i > j$ one has $\mathsf{A}_j \subset \mathsf{A}_i^\perp$,

ii) the smallest triangulated subcategory containing all A_i is T.

In this case we write $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ and every object $T \in \mathsf{T}$ can be decomposed into distinguished triangles

$$T_\ell \to T_{\ell-1} \to A_\ell \to T_\ell[1] \quad \text{for } 1 \le \ell \le m, \tag{10}$$

such that $T_m = 0$, $T_0 = T$, and $A_\ell \in \mathsf{A}_\ell$.

Admissibility of components of a semiorthogonal decomposition is not automatic. Therefore [Kuz11, Definition 2.6] introduced the notion of strong semiorthogonal decompositions.

Definition 2.11. We call a semiorthogonal decomposition $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ *strong* if the component A_k is admissible in $\langle \mathsf{A}_k, \ldots, \mathsf{A}_m \rangle$.

We finish this section with a relative criterion for S-linear semiorthogonality generalizing [Kuz11, Lemma 2.7] to coherent ringed schemes.

Lemma 2.12. *Let $\mathfrak{f}\colon (X, \mathscr{A}) \to S$ be a morphism and $\mathsf{A}, \mathsf{B} \subseteq \mathsf{D}(X, \mathscr{A})$ be S-linear admissible subcategories. Then $\mathsf{A} \subseteq \mathsf{B}^\perp$ if and only if $\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(\mathsf{B}, \mathsf{A}) = 0$.*

Proof. We start with the assumption that $\mathsf{A} \subset \mathsf{B}^\perp$. By the proof of [Kuz11, Lemma 2.7] it suffices to show that $\mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N, M)) = 0$ for all $P \in \mathsf{D}^{\mathsf{perf}}(S)$, $N \in \mathsf{B}$ and $M \in \mathsf{A}$. Using the adjunction $\mathbf{L}f^* \dashv \mathbf{R}f_*$ and the tensor-hom adjunction, one obtains

$$\mathbf{R}\mathrm{Hom}_S(P, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N, M)) \cong \mathbf{R}\mathrm{Hom}_S(\mathbf{L}f^*P, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(N, M)) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{A}}(\mathbf{L}f^*P \otimes_X^{\mathbf{L}} N, M) = 0.$$

The vanishing in the last step follows from the S-linearity of B. For the converse, note that

$$\mathbf{R}\mathrm{Hom}_S(\mathcal{O}_S, \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{A}}(N, M)) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{O}_X, \mathbf{R}\mathcal{H}om_{\mathscr{A}}(N, M)) \cong \mathbf{R}\mathrm{Hom}_{\mathscr{A}}(N, M) \tag{11}$$

by adjunction and the fact that the pushforward along the structure morphism $(X, \mathscr{A}) \to X$ is exact. $\square$

3 Hereditary orders

Let X be an (integral) scheme. An $\mathcal{O}_X$*-order* is a coherent $\mathcal{O}_X$-algebra $\mathscr{A}$ which is torsion-free as an $\mathcal{O}_X$-module such that $\mathscr{A} \otimes_X \mathbf{k}(X)$ is a central simple $\mathbf{k}(X)$-algebra. Consequently, $(X, \mathscr{A})$ is a coherent ringed scheme.

By definition of an order $\mathscr{A}$ there is a maximal open dense subset $U \subset X$ such that the restriction $\mathscr{A}|_U$ is Azumaya. The complement $\Delta_{\mathscr{A}} = X \setminus U$ is the *ramification locus of* $\mathscr{A}$.

We say that an $\mathcal{O}_X$-order $\mathscr{B}$ is an *overorder of* $\mathscr{A}$ if $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{A} \otimes_X \mathbf{k}(X)$. The order $\mathscr{A}$ is *maximal* if there are no proper overorders of $\mathscr{A}$.

Throughout this section, let C be a smooth quasi-projective curve over $\mathbf{k}$, and $\mathscr{A}$ a hereditary $\mathcal{O}_C$-order with ramification locus $\Delta_{\mathscr{A}}$. Denote by $\mathfrak{f}\colon (C, \mathscr{A}) \to C$ the structure morphism. It is shown in [Rei75, Corollary 3.24, Theorem 40.5] that being a hereditary order is a local property. This means that $\mathscr{A}$ is a hereditary $\mathcal{O}_C$-order if and only if the localization $\mathscr{A}_p$ is a hereditary $\mathcal{O}_{C,p}$-order for every point $p \in C$.

In the following, let $R = \mathcal{O}_{C,p}$ be the local ring at $p \in C$ and $\mathfrak{m} \trianglelefteq R$ be the maximal ideal of R. Since R is the unique maximal R-order in $\mathbf{k}(C)$ by normality of C, it follows from [Rei75, Theorem 39.14] that there is an isomorphism of R-algebras

$$\mathscr{A}_p \cong \begin{pmatrix} R & R & \dots & R \\ \mathfrak{m} & R & \dots & R \\ \vdots & & \ddots & \vdots \\ \mathfrak{m} & \mathfrak{m} & \dots & R \end{pmatrix}^{(n_1,\dots,n_r)} \subseteq \operatorname{Mat}_n(R), \tag{12}$$

where $n^2 = \operatorname{rk} \mathscr{A}$, $n_1 + \ldots + n_r = n$, and the superscript $(n_1, \dots, n_r)$ indicates that the (i, j)-th coordinate of the matrix on the right-hand side is to be read as an $n_i \times n_j$- matrix with entries in $\mathfrak{m}$, or R respectively. We call $(n_1, \dots, n_r) \in \mathbb{N}^r$ the *ramification data* of $\mathscr{A}$ at $p \in C$. If $\mathscr{A}$ is an $\mathcal{O}_C$-order such that the ramification data at $p \in C$ satisfies $r > 1$, then $\mathscr{A}$ is *ramified at* p with *ramification index* r.

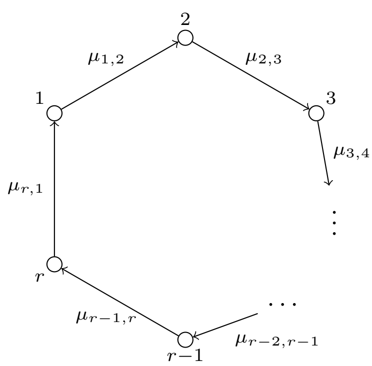
By [Rei75, Theorem 39.23], the indecomposable projective $\mathscr{A}_p$-modules are given (up to isomorphism) by the rows

$$L_p^{(j)} = E_{\alpha_j,\alpha_j} \mathscr{A}_p, \quad j = 1, \dots, r, \tag{13}$$

where $\alpha_j = n_1 + \ldots + n_j$, and $E_{\alpha,\beta} \in \operatorname{Mat}_n(R)$ denote the elementary matrices.

The fibers of hereditary orders. Let $p \in C$. We describe the fiber $\mathscr{A}(p) := \mathscr{A}_p \otimes_R \mathbf{k}(p)$ over p. In the language of Remark 2.5 this is the base change of the structure morphism $\mathfrak{f}\colon (C, \mathscr{A}) \to C$ along the closed immersion $i_p\colon \operatorname{Spec} \mathbf{k}(p) \to C$.

Let Q_r be the cyclic quiver with r vertices $Q_0 = \{1, \dots, r\}$ and r arrows $Q_1 = \{\mu_{i,i+1} \colon i \to i + 1\}$, where here and in the following the numbering has to be understood modulo r. The quiver Q_r can be depicted as follows:



(14)

Denote by $\mu_{[i,j]}\colon i \to i + 1 \to \ldots \to j - 1 \to j$ the shortest path of positive length from i to j and let

$$I = (\operatorname{rad} \mathbf{k}Q_r)^{n-1} \triangleleft \mathbf{k}Q_r \tag{15}$$

be the admissible ideal generated by all cycles $\mu_{[i,i]}$. The quotient

$$\Lambda_r = \mathbf{k}Q_r/I \tag{16}$$

is an r^2-dimensional $\mathbf{k}$-algebra of infinite global dimension.

Lemma 3.1. *Let $(C, \mathscr{A})$ be a hereditary order which is ramified at $p \in C$ with ramification index r. Then the $\mathbf{k}$-algebra $\mathscr{A}(p) = \mathscr{A}_p \otimes_C \mathbf{k}(p)$ is Morita equivalent to Λ_r.*

Proof. By [CI04, Theorem 7.6], we find that $\mathscr{A}_p$ is Morita equivalent to an R-algebra $\Gamma \subset \mathrm{Mat}_r(R)$ as in (12) with $n_1 = \ldots = n_r = 1$. The algebra $\Gamma \otimes_R \mathbf{k}(p)$ is isomorphic to Λ_r by [KSS03, Theorem 3.1], and hence $\mathscr{A}(p)$ is Morita equivalent to Λ_r.

We sketch a different proof of why $\Gamma \otimes_R \mathbf{k}$ is isomorphic to Λ_r using the algorithmic characterization of finite-dimensional basic $\mathbf{k}(p)$-algebras as quotients of path algebras of a quiver by an admissible ideal from [ASS06, Section II.3]. By [Art82, Section 2], there is a $\mathbf{k}$-basis $\{e_{ij}\}_{1 \le i,j \le r}$ of $\Gamma \otimes_R \mathbf{k}(p)$ such that

$$e_{ij} \cdot e_{jk} = \begin{cases} e_{ik} & \text{if } i \le j \le k \text{ or } j \le k < i \text{ or } k < i \le j, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Moreover, one has $e_{ij} \cdot e_{j'k} = 0$ if $j \neq j'$. The algebra $\Gamma \otimes_R \mathbf{k}(p)$ is basic and connected, with $e_{11}, \ldots, e_{rr}$ a complete set of primitive orthogonal idempotents.

Define a $\mathbf{k}$-algebra homomorphism $\varphi \colon \mathbf{k}Q_r \to \Gamma \otimes_R \mathbf{k}(p)$ by sending the lazy path e_i, associated with the vertex $i \in Q_0$, to the idempotent e_{ii}, and the arrow $\mu_{i,i+1} \in Q_1$ to $e_{i,i+1}$. It follows from (17) that this is a surjective $\mathbf{k}$-algebra homomorphism. For dimension reasons and from the relations $e_{i,i+1} \cdot \ldots \cdot e_{i-1,i} = 0$ for each $i = 1, \ldots r$, it follows that φ induces an isomorphism $\Lambda_r \cong \Gamma \otimes_R \mathbf{k}(p)$. □

We can use the representation theory of $\mathbf{k}Q_r/I$ to characterize simple Λ_r-modules. Since there are no non-zero oriented cycles, thanks to the relations, the simple Λ_r-modules up to isomorphism are given by $S_1, \ldots, S_r$ with

$$(S_i)_j = \begin{cases} \mathbf{k} & \text{if } j = i, \\ 0 & \text{if } j \in Q_0 \setminus \{i\}. \end{cases} \tag{18}$$

Remark 3.2. The number of isomorphism classes of simple $\mathscr{A}(p)$-modules does not change under the Morita equivalence from Lemma 3.1. It is straightforward that the simple $\mathscr{A}(p)$-module corresponding to S_i consists of a single n_i-dimensional non-trivial $\mathbf{k}$-vector space at the vertex $i \in Q_0$. In the remainder we will only be interested in properties of the simple Λ_r-modules which are preserved under the Morita equivalence. In particular Theorem 4.7 and Theorem 4.11 do only depend on $\mathscr{A}(p)$ up to Morita equivalence.

Maximal overorders. Denote by $\Delta_{\mathscr{A}} = \{p_1, \ldots, p_m\} \subset C$ the ramification locus of $\mathscr{A}$ and let $r_i > 1$ be the corresponding ramification index of $\mathscr{A}$ at $p_i \in C$.

The inclusion of every maximal overorder $j_{\mathscr{B},\mathrm{alg}} \colon \mathscr{A} \hookrightarrow \mathscr{B}$ gives rise to an extension

$$\mathrm{j}_{\mathscr{B}} = (\mathrm{id}_{\mathscr{B}}, j_{\mathscr{B},\mathrm{alg}}) \colon (C, \mathscr{B}) \to (C, \mathscr{A}). \tag{19}$$

It was shown in [DKR67, Theorem 3.1] that the pushforward $\mathrm{j}_{\mathscr{B},*}\mathscr{B}$ is a locally projective (left and right) $\mathscr{A}$-module. We study the shape of the locally projective $\mathscr{A}$-modules more closely with respect to the indecomposable projective $\mathscr{A}_p$-modules $L_p^{(j)}$ for $j = 1, \ldots, r$ from (13).

Definition 3.3. Let $p \in \Delta_{\mathscr{A}}$ such that $\mathscr{A}$ has ramification index r at p. A nonzero locally projective $\mathscr{A}$-module P is called *purely of type j at p* if

$$P_p \cong L_p^{(j)\oplus k} \quad \text{for some } k \in \mathbb{N}. \tag{20}$$

If the type j is not specified, we say that M is *purely of one type at p*.

As shown in [Rei75, Theorem 40.10], hereditary orders can be characterized by the maximal orders containing them. Using the classification of the indecomposable projective $\mathscr{A}_p$-modules, one can construct all the maximal orders containing $\mathscr{A}$ explicitly.

Proposition 3.4. *Let $(C, \mathscr{A})$ be a hereditary order with ramification data as above.*

i) There are precisely $r_1 \cdot \ldots \cdot r_m$ maximal overorders of $\mathscr{A}$.

ii) Every maximal overorder $\mathscr{B}$ of $\mathscr{A}$ is locally projective as a left and a right $\mathscr{A}$-module.

iii) Every maximal overorder $\mathscr{B} \supseteq \mathscr{A}$ is purely of one type at each ramification point. Moreover, a maximal overorder is uniquely determined by its types at the ramification points.

Proof. The first two parts can be found in [Rei75, Theorem 39.23 and Theorem 40.8]. Let us indicate how to prove these two statements for an algebraically closed field using the classification (13) of the indecomposable projective $\mathscr{A}_p$-modules for $p \in C$.

By Tsen's theorem, $\mathscr{A} \otimes_C \mathbf{k}(C) \cong \mathrm{End}_{\mathbf{k}(C)}(V)$, for some n-dimensional vector space V. Let $\mathscr{B} \supset \mathscr{A}$ be a maximal overorder. Then $\mathscr{B}_p \supset \mathscr{A}_p$ is a maximal overorder. From the classification of maximal orders in discrete valuation rings (see Corollary 17.4 of *op. cit.*), it follows that there exists an $\mathcal{O}_{C,p}$-lattice L_p in V such that $\mathscr{B}_p = \mathrm{End}_{\mathcal{O}_{C,p}}(L_p)$ and L_p has the structure of a right $\mathscr{A}_p$-module from the induced $\mathscr{B}_p$-module structure on L_p. It must be indecomposable as $\mathscr{A}_p$-module, because its endomorphism ring is generically central simple. Therefore L_p is an indecomposable projective $\mathscr{A}_p$-module, e.g. by [Rei75, Theorem 10.6] and the fact that $\mathscr{A}_p$ is hereditary.

Now $\mathscr{B} \subset \mathrm{End}_{\mathbf{k}(C)}(V)$ is coherent and torsion-free, and therefore uniquely determined by the localization at every point $p \in C$, see [BD22, Proposition 6.4]. For a ramified point $p_i \in \Delta_i$ we have r_i choices of $L_{p_i} = L_{p_i}^{(j)}$, $j = 1, \ldots, r_i$, hence r_i different possible specializations of $\mathscr{B}$ at p_i, and we conclude (i).

For the second part, we pass to the local ring at a point $p \in C$ using Lemma 2.7. From the construction of $\mathscr{B}$ above, it is clear that $\mathscr{B}$ is locally projective as a right $\mathscr{A}$-module. The isomorphism as left modules follows similarly using that $\mathrm{Hom}_{\mathcal{O}_{C,p}}(L_{p_i}^{(j)}, \mathcal{O}_{C,p})$ describes indecomposable projective left modules.

The third part can be calculated locally as well. We have at $p_i \in \Delta_{\mathscr{A}}$ that

$$\mathscr{B}_{p_i} \cong \mathrm{End}_{\mathcal{O}_{C,p_i}}(L_{p_i}^{(j)}) \cong \mathrm{Hom}_{\mathcal{O}_{C,p_i}}(\mathcal{O}_{C,p_i}^{\oplus n}, L_{p_i}^{(j)}) \cong {L_{p_i}^{(j)}}^{\oplus n}. \tag{21}$$

The isomorphism as left $\mathscr{A}_{p_i}$-modules is given similarly. Uniqueness follows from the local modification theorem [BD22, Proposition 6.4]. □

The explicit description of $\mathscr{B}$ allows us to draw some powerful conclusions about $\mathrm{j}_{\mathscr{B}}$.

Lemma 3.5. *Let $\mathrm{j}_{\mathscr{B}} \colon (C, \mathscr{B}) \to (C, \mathscr{A})$ be the extension corresponding to a maximal overorder $\mathscr{B} \supset \mathscr{A}$. Then the following hold.*

i) The pullback $\mathrm{j}_{\mathscr{B}}^* \colon \mathsf{QCoh}(C, \mathscr{A}) \to \mathsf{QCoh}(C, \mathscr{B})$ is exact.

ii) The pushforward $\mathrm{j}_{\mathscr{B},*} \colon \mathsf{QCoh}(C, \mathscr{B}) \to \mathsf{QCoh}(C, \mathscr{A})$ is exact and fully faithful.

Proof. The pushforward $\mathrm{j}_{\mathscr{B},*}$ is the restriction of scalars to $\mathscr{A}$, forgetting the $\mathscr{B}$-module structure. Therefore, it does not affect exactness of a sequence, and hence $\mathrm{j}_{\mathscr{B},*}$ is exact.

Since $\mathrm{j}_{\mathscr{B}}$ is an extension, the pullback is given by $\mathrm{j}_{\mathscr{B}}^* = - \otimes_{\mathscr{A}} \mathscr{B}$. By Proposition 3.4, $\mathscr{B}$ is locally projective, hence $\mathrm{j}_{\mathscr{B}}^*$ is exact.

For the second statement it suffices to show that $\mathrm{j}_{\mathscr{B}}^* \mathrm{j}_{\mathscr{B},*} M \cong M$ for all $M \in \mathsf{coh}(C, \mathscr{B})$. As $\mathrm{j}_{\mathscr{B}}^* \mathrm{j}_{\mathscr{B},*} M = M \otimes_{\mathscr{A}} \mathscr{B}$, the $\mathscr{B}$-module structure leads to a homomorphism $M \otimes_{\mathscr{A}} \mathscr{B} \to M$ with inverse (locally) given by $m \mapsto m \otimes 1$. □

4 Deformation absorption applied to hereditary orders

We start by recalling in Section 4.1 the notion of absorption of singularities and deformation absorption of singularities from [KS23] for a flat family of varieties over a curve. A crucial role is played by $\mathbb{P}^{\infty,2}$-objects, which can only exist in the bounded derived category of a singular variety X.

In Section 4.2 and Section 4.3 we apply this theory to hereditary orders, where the role of the singular fiber is played by the restriction of the order to a ramified point. This yields a semiorthogonal decomposition of the bounded derived category of a hereditary order. In Section 4.4 we compare this result to the semiorthogonal decomposition [BLS16, Theorem 4.7] for stacky curves.

4.1 Background

$\mathbb{P}^{\infty}$-objects. Let T be a triangulated category and $q \in \mathbb{Z}_{>0}$. We recall [KS23, Definition 2.6].

Definition 4.1. An object $S \in \mathsf{T}$ is a $\mathbb{P}^{\infty,q}$*-object* if

i) there is a $\mathbf{k}$-algebra isomorphism $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(S,S) \cong \mathbf{k}[\theta]$ with $\deg\theta = q$, and

ii) the induced map $\theta\colon S \to S[q]$ satisfies $\operatorname{hocolim}(S \to S[q] \to S[2q] \to \dots) = 0$ in a cocomplete category $\widehat{\mathsf{T}}$ containing T.

Similarly to [KS23, Remark 2.7] for the bounded derived category of a projective variety the definition simplifies for finite-dimensional $\mathbf{k}$-algebras.

Remark 4.2. Let A be a finite-dimensional $\mathbf{k}$-algebra. An object $S \in \mathsf{D}^{\mathsf{b}}(A)$ is a $\mathbb{P}^{\infty,q}$-object if and only if $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(S,S) \cong \mathbf{k}[\theta]$.

Indeed, since $\mathsf{D}(A) \supset \mathsf{D}^{\mathsf{b}}(A)$ is cocomplete with a compact generator A which satisfies for every $M \in \mathsf{D}^{\mathsf{b}}(A)$ that $\mathrm{Ext}^{\bullet}_{A}(A,M) \cong \mathrm{H}^{\bullet}(M)$ is bounded above, we can apply [KS23, Lemma 2.3].

Every $\mathbb{P}^{\infty,q}$-object comes with the distinguished triangle

$$M \to S \xrightarrow{\theta} S[q] \to M[1] \tag{22}$$

induced by the non-trivial morphism $\theta \in \mathrm{Hom}_{\mathsf{T}}(S,S[q]) = \mathrm{Ext}^{q}_{\mathsf{T}}(S,S)$. Following [KS23, Definition 2.8], we call this triangle the *canonical self-extension* of S.

Remark 4.3. The object M can be used to detect whether the triangulated subcategory $\langle S\rangle \subset T$ generated by S is admissible by [KS23, Lemma 2.10]. For right (resp. left) admissibility one needs M to be *homologically left* (resp. *right*) *finite-dimensional*, that is, $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(M,N)$, (resp. $\mathrm{Ext}^{\bullet}_{\mathsf{T}}(N,M)$) is finite-dimensional for all $N \in \mathsf{T}$, see [KS25, §4.1] for a definition.

Overview and definitions. A (sequence of) $\mathbb{P}^{\infty,q}$-object(s) detects the difference between smooth and singular varieties resp. categories of finite and infinite global dimension. In special cases, they generate a proper subcategory $\mathsf{S} \subset \mathsf{T}$ which absorbs the singularities in the sense of [KS23, Definition 1.1] for projective varieties. We present an analogous definition for $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(\Lambda)$, where Λ is a finite-dimensional $\mathbf{k}$-algebra.

Definition 4.4. A triangulated subcategory $\mathsf{S} \subset \mathsf{T}$ absorbs singularities of T if S is admissible and its complements $\mathsf{S}^{\perp}$ as well as ${}^{\perp}\mathsf{S}$ are smooth and proper.

Remark 4.5. Note that by [ELS20, Theorem A] $\mathsf{T} = \mathsf{D}^{\mathsf{b}}(\Lambda)$ is always smooth, but not proper unless Λ has finite global dimension. This is the same phenomenon as for $\mathsf{D}^{\mathsf{b}}(X)$, where X is a projective variety.

We propose the following definition for a noncommutative smoothing.

Definition 4.6. Assume that $\mathfrak{f}\colon (\mathcal{X},\mathscr{B}) \to C$ is a morphism of coherent ringed schemes such that C is a smooth pointed curve with fixed (closed) point $o \in C$. If the fiber over $o \in C$ is

$$(\mathcal{X} \times_C \operatorname{Spec}\mathbf{k}(o), \mathscr{B}_{\operatorname{Spec}\mathbf{k}(o)}) = (X,\mathscr{A}) \tag{23}$$

we say that $\mathfrak{f}\colon (\mathcal{X},\mathscr{B}) \to C$ is a *smoothing for* $(X,\mathscr{A})$ if the following conditions are satisfied:

- for each $p \in C \setminus \{o\}$ the fiber $(\mathcal{X}|_{\operatorname{Spec}\mathbf{k}(p)}, \mathscr{B}_{\operatorname{Spec}\mathbf{k}(p)})$ is a smooth scheme with an Azumaya algebra $\mathscr{B}_{\operatorname{Spec}\mathbf{k}(p)}$,
- the total space $\mathcal{X}$ is smooth and f is flat, and
- the $\mathcal{O}_X$-algebra $\mathscr{B}$ flat as an $\mathcal{O}_X$-module, and $\operatorname{gldim}\mathscr{B} = \dim\mathcal{X}$.

Kuznetsov–Shinder [KS23] provided powerful results for the derived category of such a smoothing, when $\mathscr{A} = \mathcal{O}_X$ and $\mathscr{B} = \mathcal{O}_{\mathcal{X}}$. We recall some of their results, stressing the link to $\mathbb{P}^{\infty,2}$-objects.

If a singular projective variety X admits a semiorthogonal collection $S_1, \ldots, S_r$ of $\mathbb{P}^{\infty,2}$-objects such that the triangulated subcategory $\mathsf{S} \subset \mathsf{D}^{\mathsf{b}}(X)$ generated by these objects absorbs singularities of X, then by [KS23, Theorem 1.8]:

- the pushforward of $S_1, \ldots, S_r$ to any smoothing $\mathcal{X} \to B$ defines a collection of exceptional objects in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$, and
- the triangulated subcategory in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$ *provides a deformation absorption of singularities of X with respect to any smoothing $\mathcal{X} \to B$*, i.e. it is admissible in $\mathsf{D}^{\mathsf{b}}(\mathcal{X})$.

In particular the second point implies by Theorem 1.5 of *op. cit.* that there is a B-linear semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\mathcal{X}) = \langle \mathsf{A}_S, \mathsf{B}\rangle \tag{24}$$

where A_{S} is the triangulated subcategory generated by the pushforward of S, and B is smooth and proper such that the base change satisfies

$$\mathsf{B}_p \simeq \begin{cases} {}^{\perp}\mathsf{S} & \text{if } p = o, \\ \mathsf{D}^{\mathsf{b}}(\mathcal{X}_p) & \text{else.} \end{cases} \tag{25}$$

In Theorem 4.11 we provide the last two results in the special case where a hereditary order is viewed as the smoothing of the finite-dimensional algebra Λ_r defined in (16).

4.2 Absorption of singularities for the fiber over a ramified point

Let $\mathscr{A}$ be a hereditary order over a curve C. Let $o \in C$ be a ramified point with ramification index r. From Lemma 3.1 we know that the fiber $\mathscr{A}(o)$ is Morita equivalent to the algebra Λ_r defined in (16). For each $i \in Q_0 = \{1, \ldots, r\}$, denote by S_i the simple Λ_r-module as defined in (18), and by P_i the unique indecomposable projective Λ_r-module such that $P_i/\operatorname{rad}P_i \cong S_i$.

Theorem 4.7. *With the notation as above let $i \in Q_0$.*

i) There is a semiorthogonal collection $(S_{i+1}, \ldots, S_{i-1})$ of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$.

ii) Let

$$\mathsf{S}_i = \langle S_{i+1}, \ldots, S_{i-1}\rangle \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r) \tag{26}$$

be the triangulated subcategory generated by $S_{i+1}, \ldots, S_{i-1}$. Then there is a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle \mathsf{S}_i, P_i\rangle. \tag{27}$$

iii) The triangulated subcategory S_i absorbs singularities of Λ_r.

Note that every S_i has infinite projective dimension and admits a 2-periodic projective resolution of the form

$$C_i^{\bullet} := (\ldots \xrightarrow{\mu_{[i+1,i]}} P_{i+1} \xrightarrow{\mu_{i,i+1}} P_i \xrightarrow{\mu_{[i+1,i]}} P_{i+1} \xrightarrow{\mu_{i,i+1}} P_i), \tag{28}$$

where the maps in the cochain complex are given by left multiplication with the elements written above the arrows. See (14) for their definition.

Lemma 4.8. *Let $i \in Q_0$.*

i) The simple Λ_r-module S_i is a $\mathbb{P}^{\infty,2}$-object.

ii) The triangulated subcategory $\langle S_i\rangle \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ generated by S_i is admissible.

Proof. Let $i \in Q_0$. By Remark 4.2, it suffices to show that $\operatorname{Ext}^\bullet_{\Lambda_r}(S_i, S_i) \cong \mathbf{k}[\theta]$ with $\deg\theta = 2$. Since $\operatorname{Hom}_{\Lambda_r}(P_{i+1}, S_i) = 0$, it follows from (28) that

$$\operatorname{Ext}^k_{\Lambda_r}(S_i, S_i) \cong \begin{cases} \mathbf{k}\theta^k & \text{if } k \in 2\mathbb{Z}_{\geq 0}, \\ 0 & \text{else.} \end{cases} \tag{29}$$

The map $\theta\colon S_i \to S_i[2]$, can be explicitly described, using the projective resolution $C_i^\bullet$, as the morphism of cochain complexes which is the identity map in each degree below -1.

For the admissibility of $\langle S_i \rangle$, we use that the cochain complex $M_i = (P_{i+1} \to P_i) \in \mathsf{D}^{\mathsf{perf}}(\Lambda_r)$ of projective Λ_r-modules, which is concentrated in degrees $\{-1, 0\}$, represents the third object in the canonical self-extension (22) of S_i. The algebra Λ_r is Gorenstein in the sense of [Jin20, Assumption 0.1], because every injective Λ_r-module is projective as well. Hence by [KS25, Proposition 6.9] the cochain complex M_i is homologically left and right finite-dimensional. It follows from [KS23, Lemma 2.10] that the triangulated category $\langle S_i \rangle$ generated by S_i is admissible. □

Lemma 4.9. *For all $1 \leq i, j \leq r$, one has*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\Lambda_r)}(S_j, S_i[\ell]) = 0 \quad \textit{for all } \ell \in \mathbb{Z}, \tag{30}$$

if and only if $j \notin \{i, i+1\}$.

Proof. From (28) and $\operatorname{Hom}_{\Lambda_r}(P_j, S_k) = 0$ unless $j = k$, one obtains that $\operatorname{Ext}^\bullet_{\Lambda_r}(S_j, S_k) = 0$ unless $k \in \{j, j+1\}$. Therefore, the collection is semiorthogonal. □

Lemma 4.10. *With the notation from Theorem 4.7, let P_i be the indecomposable projective and I_i the indecomposable injective Λ_r-module associated with the vertex $i \in Q_0$. Then*

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle \mathsf{S}_i, P_i \rangle, \tag{31}$$

$$\mathsf{D}^{\mathsf{b}}(\Lambda_r) = \langle I_i, \mathsf{S}_i \rangle. \tag{32}$$

are semiorthogonal decompositions of $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$.

Proof. By rotational symmetry, it suffices to consider $i = r$. Both collections are semiorthogonal by Lemma 4.9, and because

$$\operatorname{Hom}_{\Lambda_r}(P_r, S_j) = \operatorname{Hom}_{\Lambda_r}(S_j, I_r) = 0 \quad \text{for all } 1 \leq j < r. \tag{33}$$

It remains to show that $S_1, \ldots, S_{r-1}, P_r$ generate $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$. Since $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$ is generated by Λ_r, it suffices to show that all indecomposable projective Λ_r-modules $P_1, \ldots, P_r$ are in the smallest triangulated subcategory T generated by $S_1, \ldots, S_{r-1}, P_r$.

From the canonical self-extension for S_i, we obtain that $M_1 = (P_2 \to P_1), \ldots, M_{r-1} = (P_r \to P_{r-1})$ belong to T. For every $i \in 1, \ldots, r-1$, one obtains a distinguished triangle

$$P_i \to M_i \to P_{i+1} \to P_i[1]. \tag{34}$$

Since T is closed under taking cones and $P_r \in \mathsf{T}$, the claim follows. □

We are now ready to prove the main assertion of this section.

Proof of Theorem 4.7. Part (i) is a consequence of Lemma 4.8 and Lemma 4.9. Statement (ii) is part of Lemma 4.10.

Lemma 4.10 implies in particular that S_i is admissible, and both ${}^\perp\mathsf{S}_i$ as well as $\mathsf{S}_i^\perp$ are generated by an exceptional object, hence smooth and proper. Therefore, S_i absorbs singularities. □

4.3 Deformation absorption of singularities

Throughout this section let $(C, \mathscr{A})$ be a pair of a curve C, and a hereditary $\mathcal{O}_C$-order $\mathscr{A}$ with ramification locus $\Delta_{\mathscr{A}} = \{o\}$ a single closed point of ramification index $r \geq 1$. Since orders are generically Azumaya, this can always be achieved for every hereditary $\mathcal{O}_C$-order $\mathscr{A}$ by shrinking C to a Zariski open neighborhood around a chosen ramification point.

Given a closed point $p \in C$, we denote by $i_p \colon \operatorname{Spec} \mathbf{k}(p) \to C$ the inclusion. Building on the perspective of Remark 2.5, we consider the base change diagram

$$\begin{array}{ccc} (\operatorname{Spec} \mathbf{k}(p), \mathscr{A}(p)) & \xrightarrow{\mathfrak{i}_p = (i_p, \mathrm{id}_{\mathscr{A}(p)})} & (C, \mathscr{A}) \\ \downarrow \mathfrak{f}_p & & \downarrow \mathfrak{f} \\ \operatorname{Spec} \mathbf{k}(p) & \xhookrightarrow{\quad i_p \quad} & C \end{array} \tag{35}$$

of the structure morphism $\mathfrak{f} \colon (C, \mathscr{A}) \to C$ along i_p. If $p \neq o$, the $\mathbf{k}(p)$-algebra $\mathscr{A}(p)$ is a matrix algebra and therefore Morita equivalent to $\mathbf{k}(p)$. From Lemma 3.1, one knows that $\mathscr{A}(o)$ is Morita equivalent to the algebra Λ_r from (16).

Since $\mathfrak{f}$ is an extension and every $\mathcal{O}_C$-order is flat, the morphism i_p is faithful for $\mathfrak{f}$ by Lemma A.10. Moreover, using Lemmas A.2 and A.4 for hereditary $\mathcal{O}_C$-orders, we may appeal to Theorem A.16 for the base change of strong semiorthogonal decompositions of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ along $\mathfrak{i}_p$. Our main result of this section is the following.

Theorem 4.11. *Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order ramified over $\Delta_{\mathscr{A}} = \{o\}$ with ramification index $r \in \mathbb{Z}_{\geq 1}$. For each $i \in \{1, \ldots, r\}$ there is a strong C-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathfrak{i}_{o,*} S_{i+1}, \ldots, \mathfrak{i}_{o,*} S_{i-1}, \mathsf{D} \rangle \tag{36}$$

such that

i) the sequence $\mathfrak{i}_{o,*} S_{i+1}, \ldots, \mathfrak{i}_{o,*} S_{i-1}$ is exceptional,

ii) the admissible subcategory D is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$,

iii) the fibers of D over $p \in C$ are equivalent to $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} \mathbf{k}(p))$.

In light of [KS23, Theorem 1.5] we say that $\mathsf{S}_i = \langle S_{i+1}, \ldots, S_{i-1} \rangle$ (from Theorem 4.7) *provides a deformation absorption of singularities of Λ_r with respect to $(C, \mathscr{A})$*, i.e. it is admissible in $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$.

We split the proof of Theorem 4.11 into three steps. In the first step, we show that, as in the commutative case of [KS23, Theorem 1.8], a semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects on the singular fiber pushes forward to an exceptional collection along the noncommutative smoothing. The idea of the proof is similar, but requires the extension to $\mathbf{ncSch}_{\mathbf{k}}$. Let us spell out the details.

Lemma 4.12. *The semiorthogonal collection of $\mathbb{P}^{\infty,2}$-objects $(S_{i+1}, \ldots, S_{i-1})$ of $\mathsf{D}^{\mathsf{b}}(\Lambda_r)$ pushes forward to an exceptional collection*

$$(\mathfrak{i}_{o,*} S_{i+1}, \ldots, \mathfrak{i}_{o,*} S_{i-1}) \quad \text{in } \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}). \tag{37}$$

Proof. By the rotational symmetry of the question, it suffices to consider the case $i = r$.

Since $\mathfrak{i}$ is the base change of $(C, \mathscr{A}) \to C$ along $i_o \colon \operatorname{Spec} \mathbf{k}(o) \to C$, the object $\mathbf{L}\mathfrak{i}^* \mathfrak{i}_{o,*} S_k$ fits into the distinguished triangle

$$S_k[1] \to \mathbf{L}\mathfrak{i}_o^* \mathfrak{i}_{o,*} S_k \to S_k \xrightarrow{\theta} S_k[2] \tag{38}$$

for Cartier divisors with trivial normal bundle [KS23, Section 4.2]. The map $S_k \to S_k[2]$ is non-zero, because the pullback $\mathbf{L}\mathfrak{i}^*$ preserves perfect complexes and $S_k \notin \mathsf{D}^{\mathsf{perf}}(\Lambda_r)$. It follows from the comparison of this triangle to the canonical self-extension (22) that

$$\operatorname{Ext}^{\bullet}_{\mathscr{A}}(\mathfrak{i}_{o,*} S_k, \mathfrak{i}_{o,*} S_k) \cong \operatorname{Ext}^{\bullet}_{\Lambda_r}(\mathbf{L}\mathfrak{i}^* \mathfrak{i}_{o,*} S_k, S_k) \cong \operatorname{Ext}^{\bullet}_{\Lambda_r}(M_k, S_k) \cong \mathbf{k}[0]. \tag{39}$$

Therefore, each object is exceptional.

Similarly, the distinguished triangle (38) leads to the vanishing of $\operatorname{Ext}^\bullet_{(C,\mathscr{A})}(\mathsf{i}_{o,*}S_k, \mathsf{i}_{o,*}S_j)$ for $1 \leq j < k \leq r-1$, because $M_k = (P_{k+1} \to P_k)$ maps only trivially to S_j. ☐

Together with Theorem 4.7 this lemma already shows that $\mathsf{S}_i \subset \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ provides a deformation absorption of singularities with respect to $(C, \mathscr{A})$. In the next step we prove that (36) is a C-linear semiorthogonal decomposition.

Lemma 4.13. *There is a C-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathsf{i}_{o,*}S_{i+1}, \dots, \mathsf{i}_{o,*}S_{i-1}, \mathsf{D}\rangle. \tag{40}$$

Proof. A sequence of exceptional objects generates an admissible subcategory by [Bon89, Theorem 3.2]. Hence we obtain a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ with $\mathsf{D} = {}^{\perp}\langle \mathsf{i}_{o,*}S_{i+1}, \dots, \mathsf{i}_{o,*}S_{i-1}\rangle$.

As before we consider only the case $i = r$. For the C-linearity, we use that each of the admissible subcategories $\langle \mathsf{i}_{o,*}S_k\rangle$ is C-linear. More precisely, for each $S_k \in \mathsf{D}^{\mathsf{b}}(\Lambda_r)$ and every $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(C)$, the projection formula implies

$$(\mathsf{i}_{o,*}S_k) \otimes^{\mathbf{L}}_C f^*\mathcal{F} \cong \mathsf{i}_{o,*}\left(S_k \otimes_{\mathbf{k}} \mathbf{L}(f_o \circ i_o)^*\mathcal{F}\right). \tag{41}$$

Since $\mathbf{L}(f_o \circ i_o)^*\mathcal{F}$ can be represented by a bounded cochain complex of $\mathbf{k}$-vector spaces, one obtains $S_k \otimes_{\mathbf{k}} \mathbf{L}(f_o \circ i_o)^*\mathcal{F} \in \langle S_k\rangle$. Since $\mathsf{i}_{o,*}$ is exact and commutes with direct sums, the C-linearity of $\langle \mathsf{i}_{o,*}S_k\rangle$ follows. The C-linearity of the complement D follows from Lemma 2.10. ☐

For the last lemma recall the notion of locally projective $\mathscr{A}$-modules purely of one type from Definition 3.3 and the characterization of maximal overorders of $\mathscr{A}$ from Proposition 3.4. For $i \in Q_0 = \{1, \dots, r\}$ denote by $\mathscr{B}_i$ the (unique) maximal overorder of $\mathscr{A}$ which is purely of type i at o.

Lemma 4.14. *The component D in the semiorthogonal decomposition from Lemma 4.13 is equivalent to $\mathsf{D}^{\mathsf{b}}(C)$ given by the thick closure of the embedding*

$$\mathsf{j}_{\mathscr{B}_i,*} \colon \mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_i) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}), \tag{42}$$

where $\mathscr{B}_i$ is purely of type i at o. In particular, D is an admissible subcategory.

Proof. Each of the maximal orders $\mathscr{B}_i$ is in fact Azumaya. We consider the case $i = r$. Since $\mathbf{k}$ is algebraically closed, we have $\mathsf{coh}(C, \mathscr{B}_r) \simeq \mathsf{coh}(C)$.

By Lemma 3.5, the pushforward and the pullback are both exact, and the pushforward is fully faithful. This implies that

$$\mathsf{j}_{\mathscr{B}_r,*} \colon \mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_r) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \tag{43}$$

is fully faithful on the bounded derived category as well.

We need to show that $\mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_r) = \mathsf{D}$. Using Lemma 2.12, one has to show that $M \in \mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_r)$ if and only if $\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathsf{i}_{o,*}S_k) = 0$ for every $k = 1, \dots, r-1$. Note that $\mathfrak{f}$ is an extension. Therefore $\mathbf{R}f_*$ is the identity.

Since $M \in \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$, we can calculate $\mathbf{R}\mathcal{H}om_{\mathscr{A}}(M, \mathsf{i}_{o,*}S_k)$ by replacing M by a bounded cochain complex of projective $\mathscr{A}$-modules. Moreover $\mathsf{i}_{o,*}S_k \in \mathsf{coh}(C, \mathscr{A})$ and it is supported at $o \in C$. Thus, it suffices to show

$$\mathbf{R}\mathrm{Hom}_{\mathscr{A}_o}(M_o, \mathsf{i}_{o,*}S_k) = 0. \tag{44}$$

Assume that $M \in \mathsf{j}_{\mathscr{B}_r,*}\mathsf{D}^b(C, \mathscr{B}_r)$. As $\mathscr{B}_r$ is purely of type r at o, it follows from Proposition 3.4 and its proof that $\mathscr{B}_r \cong \mathrm{End}_{\mathcal{O}_{C,o}}(L_o^{(r)})$, where $L_o^{(r)}$ is the indecomposable projective $\mathscr{A}_{C,o}$-module defined in (13). Hence M_o can be expressed as an iterated cone of direct sums of $L_o^{(r)}$. Since $k \neq r$, it follows that $M \in \mathsf{D}$ as

$$\mathbf{R}\mathrm{Hom}_{\mathscr{A}_o}(L_o^{(r)}, \mathsf{i}_{o,*}S_k) = \mathrm{Hom}_{\mathscr{A}_o}(L_o^{(r)}, \mathsf{i}_{o,*}S_k) = 0. \tag{45}$$

Vice versa, assume that $M \in \mathsf{D}$. Restricting to $(\mathrm{Spec}(\mathcal{O}_{C,o}), \mathscr{A}_o)$, we can represent M_o by a bounded cochain complex $Q^\bullet$ of projective $\mathscr{A}_o$-modules. Since we have by assumption that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathscr{A}_o)}(Q^\bullet, \mathfrak{i}_{o,*}S_k) = 0$ for every $i = 1, \dots, r-1$, the cochain complex $Q^\bullet$ belongs to the subcategory generated by $L_o^{(r)}$. It follows that M must belong to the C-linear subcategory of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ generated by $\mathscr{B}_r$, i.e. $M \in \mathfrak{j}_{r,*}\mathsf{D}^{\mathsf{b}}(C, \mathscr{B}_r)$.

The right admissibility of D is automatic and left admissibility follows from the adjunction $\mathfrak{j}^*_{\mathscr{B}_r} \dashv \mathfrak{j}_{\mathscr{B}_r,*}$. ☐

Proof of Theorem 4.11. The fact that the semiorthogonal decomposition (36) is strong follows from Lemma 4.14. We have shown in Lemma 4.12 that $(\mathfrak{i}_{o,*}S_{i+1}, \dots, \mathfrak{i}_{o,*}S_{i-1})$ is an exceptional collection in $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$. From the equivalence $\mathsf{D} \simeq \mathsf{D}^{\mathsf{b}}(C)$ of Lemma 4.14, it follows that D is smooth and proper over $\mathsf{D}^{\mathsf{b}}(C)$. For the third point, note that over each point $p \neq o$, the restriction $\mathscr{A}(p)$ is Azumaya, and hence Morita equivalent to a point. Moreover, D_o is the admissible subcategory generated by the r-th indecomposable projective Λ_r-module, which is exceptional. ☐

Remark 4.15. There is a two-dimensional analogue given by *tame* orders of global dimension 2. By [RV89, Theorem 1.1] a tame order on a surface is uniquely determined by its overorders. Moreover Theorem 1.14 of *op. cit.* hints to a similar decomposition of the derived category of a tame order as in Lemma 4.14 using a maximal overorder. The precise shape of such a decomposition must take into account that maximal orders on surfaces are not necessarily Azumaya. In light of the recently developed stacks–orders dictionary [Fab+] in dimension two, such a decomposition would be interesting.

Periodicity of the semiorthogonal decomposition. As an application we show that the semiorthogonal decomposition (36) of $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ is $2r$-periodic. Let

$$\mathsf{T} = \langle \mathsf{A}, \mathsf{B} \rangle \tag{46}$$

be a semiorthogonal decomposition. We denote the *right dual semiorthogonal decomposition* by $\mathsf{T} = \langle \mathsf{B}, \mathbb{R}_{\mathsf{B}}\mathsf{A} \rangle$, where $\mathbb{R}_{\mathsf{B}}$ is the right mutation functor as defined in [Bon89, §2]. By [BD24, Definition 4.2] a semiorthogonal decomposition $\mathsf{T} = \langle \mathsf{A}, \mathsf{B} \rangle$ is *N-periodic* if the Nth right dual is again the original decomposition.

In [BD24, Section 4] the periodicity of a semiorthogonal decomposition for the derived category of a root stack is studied. We provide the same result for the semiorthogonal decomposition (36) thereby explaining the connection between the r different versions of (36).

Theorem 4.16. *The semiorthogonal decomposition* (36) *is $2r$-periodic.*

Proof. Start with the semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) = \langle \mathfrak{i}_{o,*}S_{i+1}, \dots, \mathfrak{i}_{o,*}S_{i-1}, \mathfrak{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \rangle. \tag{47}$$

By the proof of Lemma 4.14, the category $\mathfrak{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ is generated by $\{P_i \otimes_C \mathcal{L}_\alpha\}$, where $\{\mathcal{L}_\alpha\}_\alpha$ is a generating set of $\mathsf{D}^{\mathsf{b}}(C)$, and P_i is the locally projective $\mathscr{A}$-module purely of type i at o such that $P_{i,o} \cong L_o^{(i)}$ is indecomposable. See (13) for a definition.

By [VV84, Theorem 1] $\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ possesses a Serre functor $\mathbb{S}_{\mathscr{A}} \colon \mathsf{D}^{\mathsf{b}}(C, \mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$ given by $\mathbb{S}_{\mathscr{A}}(M) = M \otimes_{\mathscr{A}} \omega_{\mathscr{A}}[1]$ with dualizing bimodule $\omega_{\mathscr{A}} = \mathcal{H}om_X(\mathscr{A}, \omega_X)$. The Serre functor satisfies $\mathbb{S}_{\mathscr{A}}(P_i) = P_{i+1}[1]$, and $\mathbb{S}_{\mathscr{A}}(\mathfrak{i}_{o,*}S_i) = \mathfrak{i}_{o,*}S_{i+1}$.

If we denote by $\mathsf{A} = \langle \mathfrak{i}_{o,*}S_{i+1}, \dots, \mathfrak{i}_{o,*}S_{i-1} \rangle$, and by $\mathsf{B} = \mathfrak{j}_{\mathscr{B}_{i,*}}\mathsf{D}^{\mathsf{b}}(C, \mathscr{A})$, it follows from [BK89, Proposition 3.6] that $\mathbb{R}_{\mathsf{B}}(\mathsf{A}) = \mathbb{S}(\mathsf{A})$. Hence, after $2r$ times taking the right dual of B, the category B is replaced by $\mathbb{S}^r(\mathsf{B})$. Since $\mathbb{S}^r(\mathsf{B})$ is generated by $\{\mathbb{S}^r(P_i) \otimes_C \mathcal{L}_\alpha\}_\alpha$, and B is triangulated, it follows that $\mathbb{S}^r(\mathsf{B}) = \mathsf{B}$. Similarly, we have that $\mathbb{S}^r(\mathsf{A}) = \mathsf{A}$. ☐

4.4 The dictionary between hereditary orders and smooth root stacks

There is a stacks–orders dictionary in dimension one and two [CI04; Fab+; BBG24]. Let C be a quasi-projective curve. By [CI04, Corollary 7.8], resp. [BBG24, §2.2] in the language of root stacks, the dictionary relates the following two objects.

i) Let $\mathscr{A}$ be a hereditary $\mathcal{O}_C$-order with ramification divisor $\Delta = \{p_1, \ldots, p_m\}$ and ramification indices $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, and

ii) denote by $\pi\colon \sqrt[\mathbf{r}]{C;\Delta} \to C$ the iterated root stack over C by doing the r_i-th root construction at $p_i \in \Delta_{\mathscr{A}}$.

Theorem 4.17 (Chan–Ingalls)**.** *With the notation as above, there is an equivalence of categories*

$$\mathsf{coh}(C, \mathscr{A}) \simeq \mathsf{coh}(\sqrt[\mathbf{r}]{C;\Delta_{\mathscr{A}}}). \tag{48}$$

Therefore, there is a stacky version of Theorem 4.11. Assume that $\Delta = \{o\}$ and do the r-th root construction at o. Then the singular fiber is described by

$$\operatorname{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta} \cong \left[\operatorname{Spec}\left(\frac{\mathbf{k}[t]}{(t^r)}\right)/\mu_r\right], \tag{49}$$

where μ_r is the group scheme of r-th roots of unity, and the r-th primitive root acts by multiplication on t. Coherent sheaves on the singular fiber are given by

$$\mathsf{coh}([\operatorname{Spec}(\mathbf{k}[t]/(t^r))/\mu_r]) \simeq \mathsf{coh}^{\mu_r}(\mathbf{k}[t]/(t^r)) \simeq \mathsf{coh}(\mathbf{k}[t]/(t^r) * \mu_r), \tag{50}$$

where $\mathbf{k}[t]/(t^r) * \mu_r$ is the skew group algebra.

On the other hand each closed point in $\sqrt[r]{C;\Delta}$ (i.e. a morphism $\operatorname{Spec}\mathbf{k} \to \sqrt[r]{C;\Delta}$) factors through its residual gerbe $\mathrm{BAut}(p)$, where $p \in C$ is obtained by postcomposing with the map to the coarse moduli space C. The residual gerbe is only nontrivial at $o \in C$, where we have $\mathrm{B}\mu_r = [(\operatorname{Spec}\mathbf{k}(o))/\mu_r]$, and

$$\mathsf{coh}(\mathrm{B}\mu_r) \simeq \mathsf{mod}(\mathbf{k}[\mu_r]). \tag{51}$$

The group algebra $\mathbf{k}[\mu_r]$ is semisimple with r simple modules. Following the terminology of [LP21, Section 2.2], we obtain therefore r generalized points $(o, \zeta_1), \ldots, (o, \zeta_r)$, each corresponding to one character of $\mathbf{k}[\mu_r]$. Denote by $\mathcal{O}_{o,\zeta_i}$ the irreducible μ_r-representation corresponding to (o, ζ_i).

By [Cad07, Example 2.4.3], the residual gerbe over o embeds as a closed substack into the fiber $\operatorname{Spec}\mathbf{k}(o) \times_C \sqrt[r]{C;\Delta}$ so that we obtain a commutative diagram

$$\begin{array}{ccccc} [\operatorname{Spec}\mathbf{k}(o)/\mu_r] & \xrightarrow{\;j\;} & [\operatorname{Spec}(\mathbf{k}[t]/(t^r))/\mu_r] & \xrightarrow{\;\iota_o\;} & \sqrt[r]{C;\Delta} \\ & \searrow & \downarrow & & \downarrow{\scriptstyle \pi} \\ & & \operatorname{Spec}\mathbf{k}(o) & \xrightarrow{\;i_o\;} & C \end{array} \tag{52}$$

On the level of algebras, j corresponds to the $\mathbf{k}$-algebra homomorphism $\mathbf{k}[t]/(t^r) * \mu_r \to \mathbf{k}[\mu_r]$, which identifies the irreducible module $\mathcal{O}_{o,\zeta_i}$ with a simple $\mathbf{k}[t]/(t^r) * \mu_r$-module, denoted by $\widetilde{\mathcal{O}}_{o,\zeta_i}$. We can use the modules $\widetilde{\mathcal{O}}_{o,\zeta_i}$ for the version of Theorem 4.7 for stacky curves.

Corollary 4.18. *Let $i \in \{1, \ldots, r\}$. The collection $\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \ldots, \widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ (counted modulo r) is a semi-orthogonal collection of $\mathbb{P}^{\infty,2}$-objects in $\mathsf{D}^{\mathsf{b}}(\operatorname{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta})$.*

Moreover, the smallest triangulated category $\mathsf{O}_i \subset \mathsf{D}^{\mathsf{b}}(\operatorname{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta})$ containing $\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \ldots, \widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ absorbs singularities of $\operatorname{Spec}(\mathbf{k}(o)) \times_C \sqrt[r]{C;\Delta}$.

Proof. This follows from Theorem 4.7 and the $\mathbf{k}$-algebra isomorphism $\mathbf{k}[t]/(t^r) * \mu_r \cong \Lambda_r$, which identifies the simple $\mathbf{k}[t]/(t^r) * \mu_r$-module $\widetilde{\mathcal{O}}_{o,\zeta_i}$ with the simple Λ_r-module S_i. □

Using [BLS16, Theorem 4.7] we obtain the deformation absorption result for $\sqrt[r]{C;\Delta}$ as well.

Theorem 4.19. *Let $\sqrt[r]{C;o} \to C$ be smooth stacky curve with nontrivial stabilizer μ_r over the closed point $o \in C$. For each $i \in \{1, \ldots, r\}$ there is a strong C-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{b}}(\sqrt[r]{C;o}) = \langle \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \ldots, \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i-1}}, \mathsf{D} \rangle \tag{53}$$

such that

i) the sequence $\iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i+1}}, \dots, \iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_{i-1}}$ *is exceptional,*

ii) the admissible subcategory D *is smooth and proper over* $\mathsf{D}^{\mathsf{b}}(C)$*,*

iii) the fibers of D *over* $p \in C$ *are equivalent to* $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\mathbf{k}(p))$*.*

In other words O_i *provides a deformation absorption of singularities of* $[\mathrm{Spec}(\mathbf{k}[t]/(t^r))/\mu_r]$ *with respect to the smoothing* $\sqrt[r]{C;o} \to C$*.*

Proof. We have that $\iota_{o,*}\widetilde{\mathcal{O}}_{o,\zeta_k} = \mathbf{R}(\iota_{o,*} \circ j)\mathcal{O}_{o,\zeta_k}$. Therefore, [BLS16, Theorem 4.7] provides the semiorthogonal decomposition (53) for $i = 0$, where $\mathsf{D} = \mathbf{L}\pi^*\mathsf{D}^{\mathsf{b}}(C)$. For $i > 0$ the semiorthogonal decomposition follows by [BD24, Theorem 4.3]. □

A A noncommutative base change formula

The goal of the appendix is to give a version of Kuznetsov's base change formula [Kuz11, Theorem 5.6] for the bounded derived category $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ of a coherent ringed scheme $(X, \mathscr{A})$. After restricting our attention to coherent $\mathcal{O}_X$-algebras $\mathscr{A}$ of finite global dimension, we only have to make small modifications of the proofs of *op. cit.* to arrive at Theorem A.16, the base change formula for noncommutative schemes. We denote

- by $\mathsf{D}(X, \mathscr{A}) = \mathsf{D}(\mathsf{QCoh}(X, \mathscr{A}))$ the *unbounded derived category of quasicoherent* $\mathscr{A}$*-modules*, and by $\mathsf{D}^*(X, \mathscr{A})$, for $* \in \{+, -, \mathsf{b}\}$, its bounded below, bounded above, resp. bounded derived category;
- by $\mathsf{D}_{\mathsf{coh}}(X, \mathscr{A}) = \mathsf{D}_{\mathsf{coh}}(\mathsf{QCoh}(X, \mathscr{A}))$ the *derived category of quasicoherent* $\mathscr{A}$*-modules with coherent cohomology*, and by $\mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) := \mathsf{D}^{b}_{\mathsf{coh}}(\mathsf{QCoh}(X, \mathscr{A}))$ the *bounded derived category* of $(X, \mathscr{A})$;
- by $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ the *category of perfect complexes* consisting of objects which are represented by complexes that are locally quasi-isomorphic to bounded complexes of locally projective $\mathscr{A}$-modules;
- by $\mathsf{D}^{[a,b]}(X, \mathscr{A}) \subset \mathsf{D}(X, \mathscr{A})$, for $a \leq b \in \mathbb{Z}$, all $M \in \mathsf{D}(X, \mathscr{A})$ such that the cohomology sheaf $\mathcal{H}^i(M)$ vanishes for $i \notin [a, b]$. If $a = -\infty$, we write $\mathsf{D}^{\leq b}(X, \mathscr{A})$, and if $b = \infty$, we write $\mathsf{D}^{\geq a}(X, \mathscr{A})$.

Perfect complexes

Recall that an object $P \in \mathsf{D}(X, \mathscr{A})$ is *compact* if $\mathrm{Hom}_{\mathsf{D}(X,\mathscr{A})}(P, -)$ commutes with filtered colimits.

Remark A.1. For $(X, \mathscr{A})$ a coherent ringed scheme, [BDG17, Proposition 3.14] shows that $\mathsf{D}(X, \mathscr{A})$ is compactly generated, and the compact objects are $\mathsf{D}(X, \mathscr{A})^{\mathsf{c}} = \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$.

Using the identification (8), it is straightforward to extend [TT90, Proposition 2.3.1] to coherent ringed schemes $(X, \mathscr{A})$, where $\mathscr{A}$ has finite global dimension. For $\mathscr{A} = \mathcal{O}_X$, it says that a perfect complex on a quasi-projective scheme X is quasi-isomorphic to a bounded cochain complex of locally free sheaves.

Lemma A.2. *Let* $(X, \mathscr{A})$ *be a coherent ringed scheme.*

i) If $\mathscr{A}$ *has finite global dimension or* $X = \mathrm{Spec}\,\mathbf{k}$ *is a point, each perfect complex is globally quasi-isomorphic to a bounded cochain complex of locally projective modules.*

ii) If $\mathscr{A}$ *has finite global dimension, then* $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$*.*

Proof. Let $M \in \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$. Since X is quasi-compact and the cohomology sheaf $\mathcal{H}^\ell(M)$ does only depend on the $\mathcal{O}_X$-module structure, it follows that $\mathcal{H}^\ell(M)$ is coherent for all $\ell \in \mathbb{Z}$, and non-zero for only finitely many ℓ. From (8) one sees that $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A}) \simeq \mathsf{D}^{\mathsf{b}}(\mathsf{coh}(X, \mathscr{A}))$ since $\mathscr{A}$ is coherent. Hence every perfect complex is quasi-isomorphic to a bounded cochain complex of coherent $\mathscr{A}$-modules.

If $X = \mathrm{Spec}\,\mathbf{k}$ the statements follow from the fact that every module $\mathscr{A}$-module admits a projective resolution and by definition of perfect complexes. Hence, we pass to $\mathscr{A}$ of finite global dimension. Let $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$. By [BDG17, Proposition 3.7], there is a locally projective resolution $P^\bullet \xrightarrow{\sim} M$, with $P^\bullet \in \mathsf{D}^-(\mathsf{coh}(X, \mathscr{A}))$ bounded above. Since M is bounded on both sides, there exists $n \ll 0$ such that $\mathcal{H}^k(P^\bullet) = 0$ for all $k \leq n$. Then the canonical truncation $N^\bullet = \tau_{\geq n}P^\bullet$ (cf. [SP, Section 0118]) is quasi-isomorphic to $P^\bullet$. The canonical truncation consists of locally projective $\mathscr{A}$-modules except at the n-th position, we have $N^n = \operatorname{Coker}(d^{n-1}\colon P^{n-1} \to P^n) \cong \operatorname{Im}(d^n)$. Since N^n is a coherent $\mathscr{A}$-module, it admits a locally projective resolution $\varepsilon\colon Q^\bullet \to N^n$, which is of finite length if $\mathscr{A}$ is of finite global dimension. Replacing N^n by its locally projective resolution, we obtain a bounded cochain complex

$$L^\bullet = \left(\ldots \to Q^{-1} \to Q^0 \xrightarrow{\iota\circ\varepsilon} P^{n+1} \to \ldots\right) \tag{54}$$

of locally projective $\mathscr{A}$-modules and an induced morphism of cochain complexes $L^\bullet \to N^\bullet$. Here $\iota\colon N^n \to P^{n+1}$ is the inclusion of the image of d^n. This is a quasi-isomorphism. All in all, we obtain that M quasi-isomorphic to $L^\bullet \in \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$. □

Recall from [Orl06, Definition 1.6] that an object $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ is *homologically finite* if for every $N \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ the $\mathbf{k}$-vector space $\bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}^i_{\mathscr{A}}(M, N)$ is finite-dimensional. The following generalizes [Orl06, Proposition 1.11] to the noncommutative setting.

Lemma A.3. *Let $(X, \mathscr{A})$ be a coherent ringed scheme such that $\mathscr{A}$ is of finite global dimension or $X = \operatorname{Spec}\mathbf{k}$ is a point. An object $M \in \mathsf{D}^{\mathsf{b}}(X, \mathscr{A})$ is perfect if and only if it is homologically finite.*

Proof. We explain how to modify the proof in [Orl06, Proposition 1.11] so that it works in our setting. Since every homologically finite object is bounded, by [Kuz08, Proposition 2.9] it is perfect. Vice versa, assume that M is perfect. By Lemma A.2 it is represented by a bounded cochain complex $P^\bullet$ of locally projective $\mathscr{A}$-modules. The spectral sequence

$$\mathrm{H}^p(X, \mathcal{H}^q(\mathcal{H}om_{\mathscr{A}}(P^\bullet, N))) \quad\Rightarrow\quad \operatorname{Ext}^{p+q}_{\mathscr{A}}(M, N) \tag{55}$$

is concentrated in $p \in [0, \dim X]$. It is zero in the q-direction outside the bounds of $P^\bullet$. Moreover, each $\mathbf{k}$-vector spaces on the E_2-page is finite-dimensional, because $\mathscr{A}$ is coherent. Therefore, $\operatorname{Ext}^i_{\mathscr{A}}(M, N)$ is finite-dimensional for all $i \in \mathbb{Z}$ and non-zero for only finitely many $i \in \mathbb{Z}$. □

Next, we come to a generalization of [Kuz08, Proposition 2.5] to coherent ringed schemes. Let T be a triangulated category with a t-structure $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$, and $\mathsf{C} \subset \mathsf{D}(X, \mathscr{A})$ be a triangulated subcategory. A functor $\Phi\colon \mathsf{C} \to \mathsf{T}$ has *finite cohomological amplitude* if there are a, b such that

$$\Phi(\mathsf{C} \cap \mathsf{D}^{\geq 0}(X, \mathscr{A})) \subset \mathsf{T}^{\geq a}, \quad \text{and} \quad \Phi(\mathsf{C} \cap \mathsf{D}^{\leq 0}(X, \mathscr{A})) \subset \mathsf{T}^{\leq b}. \tag{56}$$

We refer to Section 2.3 of *op. cit.* for a short introduction to (bounded) t-structures.

Lemma A.4. *Let $(X, \mathscr{A})$ be a coherent ringed scheme, where X is a quasi-projective variety. Let T be a triangulated category which admits a bounded t-structure. Then every functor $\Phi\colon \mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) \to \mathsf{T}$ has finite cohomological amplitude.*

Proof. The proof of [Kuz08, Proposition 2.5] needs only a small modification. Let $\mathcal{O}_X(1)$ be an ample line bundle on X, and denote by $\mathscr{A}(n) = \mathscr{A} \otimes_X \mathcal{O}_X(n)$. Since T has a bounded t-structure, and X is quasi-projective, one finds $a, b \in \mathbb{Z}$ such that $\Phi(\mathscr{A}(n)) \in \mathsf{T}^{[a,b]}$ for all $n \in \mathbb{Z}$. The remainder works as in [Kuz08, Proposition 2.5]. □

Induced semiorthogonal decompositions

Now, we may look at the question when semiorthogonal decompositions on $\mathsf{D}(X, \mathscr{A})$ and its bounded versions are induced from each other. The exposition is close to [Kuz11, Chapter 4]. In order to compare semiorthogonal decompositions we need the notion fo compatibility from [Kuz11, Section 3] for exact functors.

Definition A.5. Let $\Phi\colon \mathsf{T} \to \mathsf{T}'$ be an exact functor between triangulated categories with semiorthogonal decompositions $\mathsf{T} = \langle \mathsf{A}_1, \ldots, \mathsf{A}_m \rangle$ and $\mathsf{T}' = \langle \mathsf{A}'_1, \ldots, \mathsf{A}'_m \rangle$. We say that Φ *is compatible with the semiorthogonal decompositions* if $\Phi(\mathsf{A}_k) \subset \mathsf{A}'_k$.

Throughout the section we let $\mathfrak{f}\colon (X,\mathscr{A}) \to S$ be a morphism of coherent ringed schemes. The first lemma follows as in [Kuz11, Proposition 4.1] using Lemma A.3.

Lemma A.6. *Let $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m\rangle$ be a strong S-linear semiorthogonal decomposition. If $\mathscr{A}$ has finite global dimension or $X = \operatorname{Spec} \mathbf{k}$, then there is a unique S-linear decomposition*

$$\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}}\rangle, \tag{57}$$

which is compatible with the inclusion $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) \subset \mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$. The components are given by $\mathsf{A}_i^{\mathsf{perf}} = \mathsf{A}_i \cap \mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$.

The next step was already provided for ∞-enhanced categories by [Per19, Lemma 3.12]. We restrict ourselves to the derived category of coherent ringed schemes. Recall from (10) that a semiorthogonal decomposition comes with distinguished triangles. This gives rise to *projection functors* $\mathrm{pr}_\ell\colon \mathsf{T} \to \mathsf{T}$, $T \mapsto A_\ell$. If the decomposition is S-linear, it follows from [Kuz11, Lemma 3.1] that $\mathrm{pr}_\ell(M \otimes_X \mathbf{L}f^*\mathcal{F}) = \mathrm{pr}_\ell(M) \otimes_X \mathbf{L}f^*\mathcal{F}$ for $M \in \mathsf{T}$, $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(S)$. In other words, the projection functors are S-linear.

Lemma A.7. *Assume that $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) = \langle \mathsf{A}_1, \dots, \mathsf{A}_m\rangle$ is an S-linear semiorthogonal decomposition.*

i) There is a unique S-linear semiorthogonal decomposition $\mathsf{D}(S,\mathscr{A}) = \langle \widehat{\mathsf{A}}_1, \dots, \widehat{\mathsf{A}}_m\rangle$ compatible with the inclusion $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) \subset \mathsf{D}(S,\mathscr{A})$.

ii) If the semiorthogonal decomposition was induced by one of $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$, where the projection functors $\mathrm{pr}_i\colon \mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) \to \mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$ have finite cohomological amplitude, then the semiorthogonal decomposition of $\mathsf{D}(X,\mathscr{A})$ is compatible with the inclusion $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) \subset \mathsf{D}(X,\mathscr{A})$ as well.

Proof. Part (i) carries over analogously from [Kuz11, Proposition 4.2] using Remark A.1 that $\mathsf{D}(X,\mathscr{A})^{\mathsf{c}} = \mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$ and $\mathsf{D}(X,\mathscr{A})$ is compactly generated. Note that $\widehat{\mathsf{A}}_i$ is obtained from $\mathsf{A}_i^{\mathsf{perf}} \subset \mathsf{D}(S,\mathscr{A})$ as the smallest triangulated category closed under (arbitrary) direct sums and cones containing $\mathsf{A}_i^{\mathsf{perf}}$. Part (ii) also follows as in [Kuz11, Proposition 4.2], because every object in $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$ admits a resolution by locally projective $\mathscr{A}$-modules, see [BDG17, Proposition 3.7]. □

The projection functors $\mathrm{pr}_i\colon \mathsf{A}_i^{\mathsf{perf}} \to \mathsf{A}_i^{\mathsf{perf}}$ have finite cohomological amplitude by Lemma A.4. Therefore, as in [Kuz11, Proposition 4.3], we obtain an induced semiorthogonal decomposition on $\mathsf{D}^-(X,\mathscr{A})$.

Lemma A.8. *Assume that $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}}\rangle$ is an S-linear semiorthogonal decomposition. There is a unique S-linear semiorthogonal decomposition*

$$\mathsf{D}^-(S,\mathscr{A}) = \langle \mathsf{A}_1^-, \dots, \mathsf{A}_m^-\rangle, \tag{58}$$

with components $\mathsf{A}_i^- = \widehat{\mathsf{A}}_i \cap \mathsf{D}^-(S,\mathscr{A})$, compatible with the embeddings $\mathsf{D}^{\mathsf{perf}}(S,\mathscr{A}) \subset \mathsf{D}^-(S,\mathscr{A}) \subset \mathsf{D}(S,\mathscr{A})$.

Base change of semiorthogonal decompositions

We generalize the base change formulas [Kuz11, §5] for semiorthogonal decompositions to the noncommutative setting. For $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$ and $\mathsf{D}(X,\mathscr{A})$ this can be seen as a special case of [Per19, Lemma 3.15]. There have been several generalizations of Kuznetsov's base change formula, notably [Bay+21, Theorem 3.17], and [BOR24, Theorem 3.5] weakening the assumptions on the schemes. Besides the base change formula [K06, Theorem 2.46] for certain Azumaya varieties, we are not aware of results for the bounded derived category $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A})$. However, most of the proofs can be adapted from the commutative case.

We specialize to the situation mentioned in Remark 2.5 Assume that $h\colon T \to S$ is a morphism of schemes and $\mathfrak{f}\colon (X,\mathscr{A}) \to S$ is a morphism of coherent ringed schemes. Consider the base change

$$\begin{array}{ccc} (X_T,\mathscr{A}_T) & \xrightarrow{\mathfrak{h}_T} & (X,\mathscr{A}) \\ \downarrow{\scriptstyle \mathfrak{f}_T} & & \downarrow{\scriptstyle \mathfrak{f}} \\ T & \xrightarrow{h} & S \end{array} \tag{59}$$

of $\mathfrak{f}$ along $h\colon T \to S$. Lemma 2.4 implies that $\mathscr{A}_T = h_T^*\mathscr{A}$. Then, there is a natural transformation of functors

$$\mathbf{L}h^* \circ \mathbf{R}\mathfrak{f}_* \Rightarrow \mathbf{R}\mathfrak{f}_{T,*} \circ \mathbf{L}\mathfrak{h}_T^*. \tag{60}$$

from $\mathsf{D}(X, \mathscr{A})$ to $\mathsf{D}(T)$.

This follows from the push-pull adjunction [Xie23, Lemma A.6]. Then, the construction of the natural transformation translates verbatim from [SP, Section 02N6 and Remark 08HY].

Definition A.9. A morphism $h\colon T \to S$ is *faithful* for $\mathfrak{f}$ if $\mathbf{L}h^* \circ \mathbf{R}\mathfrak{f}_* \Rightarrow \mathbf{R}\mathfrak{f}_{T,*} \circ \mathbf{L}\mathfrak{h}_T^*$ from $\mathsf{D}(X, \mathscr{A})$ to $\mathsf{D}(T)$ is an equivalence.

If $\mathscr{A}$ is flat over X, a K-flat resolution $F^\bullet$ of an $\mathscr{A}$-module $M \in \mathsf{D}(X, \mathscr{A})$ is also flat over X. Hence, on has $\mathbf{L}\mathfrak{h}_T^* = \mathbf{L}h_T^*$, and we can appeal to commutative base change, to obtain the following.

Lemma A.10. *If $\mathscr{A}$ is flat over X and $f\colon X \to S$ is flat, then each morphism $h\colon T \to S$ is faithful for $\mathfrak{f}$.*

In the following we will always assume that $h\colon T \to S$ is faithful for $\mathfrak{f}\colon (X, \mathscr{A}) \to S$.

Perfect complexes. For base changing perfect complexes, one uses that $\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ is generated by 'box tensors' of the components $\mathsf{D}^{\mathsf{perf}}(T)$ and $\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A})$ inside the unbounded derived category $\mathsf{D}(X_T, \mathscr{A}_T)$. This holds in the full generality of ∞-categories by [Per19, Lemma 2.7]. We provide the small modifications to apply the proof of [Kuz11, Lemma 5.2].

Lemma A.11. *The category $\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ is the minimal triangulated subcategory of $\mathsf{D}(X, \mathscr{A}_T)$ closed under taking direct summands which is generated by the objects*

$$\mathbf{L}\mathfrak{h}_T^* M \otimes_T^{\mathbf{L}} \mathbf{L}f_T^*\mathcal{F}, \quad \textit{where } M \in \mathsf{D}^{\mathsf{perf}}(S, \mathscr{A}) \textit{ and } \mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T). \tag{61}$$

Proof. Since $h\colon T \to S$ is a quasi-projective morphism, every coherent $\mathcal{O}_{X_T}$-module $\mathcal{F}$ admits a surjection $f_T^*\mathcal{O}_h(n)^{\oplus k} \twoheadrightarrow \mathcal{F}$ for some $n, k \in \mathbb{Z}$, where $\mathcal{O}_h(1)$ is an h-ample line bundle on T. Note that by [SP, Lemma 0893], the line bundle $f_T^*\mathcal{O}_h(1)$ is h_T-ample.

Therefore, every coherent $\mathscr{A}_T$-module M admits a surjection $f_T^*\mathcal{O}_h(n)^{\oplus k} \otimes_T \mathscr{A}_T \twoheadrightarrow M$ of $\mathscr{A}_T$-modules, where we use the right $\mathscr{A}_T$-module structure on M. Since $\mathfrak{h}_T$ is strict, $\mathscr{A}_T$ is the pullback of (the locally projective $\mathscr{A}$-module) $\mathscr{A}$. For this reason we find for every $M \in \mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T)$ a bounded above locally projective $\mathscr{A}_T$-resolution $P^\bullet \to M$ such that for each $P^i \cong \mathfrak{h}_T^* Q^i \otimes_T f_T^*\mathcal{E}^i$, where Q^i is locally projective, and $\mathcal{E}^i$ is a locally free $\mathcal{O}_T$-module. This allows us to proceed as in [Kuz11, Lemma 5.2]. □

With this lemma, we are ready to define the base change of admissible subcategories in the category of perfect complexes. Given an S-linear semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{perf}}(X, \mathscr{A}) = \langle \mathsf{A}_1^{\mathsf{perf}}, \dots, \mathsf{A}_m^{\mathsf{perf}} \rangle, \tag{62}$$

we define $\mathsf{A}_{iT}^{\mathsf{perf}}$ to be the smallest triangulated subcategory, closed under direct summands containing all objects of the form $\mathbf{L}\mathfrak{h}_T^* M \otimes_{X_T}^{\mathbf{L}} \mathbf{L}f_T^*\mathcal{F}$for $M \in \mathsf{A}_i^{\mathsf{perf}}$ and $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T)$.

Proposition A.12. *From the S-linear semiorthogonal decomposition* (62) *one obtains a T-linear semiorthogonal decomposition*

$$\mathsf{D}^{\mathsf{perf}}(X_T, \mathscr{A}_T) = \langle \mathsf{A}_{1T}^{\mathsf{perf}}, \dots, \mathsf{A}_{mT}^{\mathsf{perf}} \rangle \tag{63}$$

compatible with $\mathbf{L}\mathfrak{h}_T^*$.

Proof. Using Lemma 2.12, it follows from a standard manipulations that $\mathsf{A}_{jT}^{\mathsf{perf}} \subset {\mathsf{A}_{iT}^{\mathsf{perf}}}^\perp$. The construction of $\mathsf{A}_{iT}^{\mathsf{perf}}$ implies T-linearity and $\mathbf{L}\mathfrak{h}_T^*(\mathsf{A}_i^{\mathsf{perf}}) \subseteq \mathsf{A}_{iT}^{\mathsf{perf}}$. Generation follows Lemma A.11. □

The unbounded derived category. We extend [Kuz11, Proposition 5.3] to our setting.

Proposition A.13. *Let $\mathsf{D}(X,\mathscr{A}) = \langle \widehat{\mathsf{A}}_1,\dots,\widehat{\mathsf{A}}_m\rangle$ be an S-linear semiorthogonal decomposition. Then*

$$\mathsf{D}(X_T,\mathscr{A}_T) = \langle \widehat{\mathsf{A}}_{1,T},\dots,\widehat{\mathsf{A}}_{m,T}\rangle \tag{64}$$

is a T-linear semiorthogonal decomposition compatible with $\mathbf{L}\mathfrak{h}_{T,*}$ and $\mathbf{L}\mathfrak{h}_T^*$. Moreover, if the decomposition is induced from a semiorthogonal decomposition (62) on $\mathsf{D}^{\mathsf{perf}}(X,\mathscr{A})$, the semiorthogonal decomposition (64) is compatible with the one constructed in Proposition A.12.

Proof. Recall from the proof of Lemma A.7 that by construction $\widehat{A}_{iT}$ is obtained as the closure of $\mathsf{A}_{iT}^{\mathsf{perf}}$ in $\mathsf{D}(X_T,\mathscr{A}_T)$ under direct sums and iterated cones. In particular, it contains all homotopy colimits of perfect complexes in $\mathsf{A}_{iT}^{\mathsf{perf}}$. The compatibility with the pullback follows now directly from $\mathbf{L}\mathfrak{h}_T^*\mathsf{A}_i^{\mathsf{perf}} \subset \mathsf{A}_{iT}^{\mathsf{perf}} \subset \widehat{\mathsf{A}}_{iT}$, where the first inclusion follows from the compatible base change in Proposition A.12.

Next consider the pushforward. By Lemma A.11, we can consider $M \in \mathsf{A}_i^{\mathsf{perf}}$ and $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T)$. It is to show that $\mathbf{R}\mathfrak{h}_{T,*}(\mathbf{L}\mathfrak{h}_T^* M \otimes^{\mathbf{L}}_{X_T} \mathbf{L}f_T^*\mathcal{F}) \in \widehat{\mathsf{A}}_i$. Because $\mathbf{L}\mathfrak{h}_T^* M \otimes^{\mathbf{L}}_{X_T} \mathbf{L}f_T^*\mathcal{F} \cong \mathbf{L}\mathfrak{h}_T^* M \otimes^{\mathbf{L}}_{\mathscr{A}_T} \mathbf{L}\mathfrak{f}_T^*\mathcal{F}$ in $\mathsf{D}^{\mathsf{b}}(X_T,\mathscr{A}_T)$, we can use the noncommutative projection formula [Xie23, Proposition A.6] to obtain

$$\mathbf{R}\mathfrak{h}_{T,*}(\mathbf{L}\mathfrak{h}_T^* M \otimes^{\mathbf{L}}_{\mathscr{A}_T} \mathbf{L}\mathfrak{f}_T^*\mathcal{F}) = M \otimes^{\mathbf{L}}_{\mathscr{A}} (\mathbf{R}\mathfrak{h}_{T,*}\mathbf{L}f_T^*\mathcal{F}). \tag{65}$$

The right hand side lies in $\widehat{A}_i$ as $\mathbf{R}h_{T,*}\mathbf{L}f_T^*\mathcal{F} \cong \mathbf{L}f^*\mathbf{R}h_*\mathcal{F} \in \mathsf{D}(X)$, because h is faithful for $\mathfrak{f}$, and $\widehat{\mathsf{A}}_i$ is S-linear. Note that by Remark 2.9, each component is S-linear for pullbacks of all quasi-coherent sheaves $\mathcal{F} \in \mathsf{D}(S)$. □

Similarly to [Kuz11, Proposition 5.3], we can give a more precise description of the components $\widehat{\mathsf{A}}_{iT}$. For $M \in \mathsf{D}(X_T,\mathscr{A}_T)$ we find that $M \in \widehat{\mathsf{A}}_{iT}$ if and only if $\mathbf{R}\mathfrak{h}_{T,*}(M \otimes^{\mathbf{L}}_T \mathbf{L}f_T^*\mathcal{F}) \in \widehat{\mathsf{A}}_i$ for all $\mathcal{F} \in \mathsf{D}^{\mathsf{perf}}(T)$. With statement (ii) of the next lemma this can be shown as in [Kuz11, Proposition 5.3].

Lemma A.14. *Let $\mathfrak{f}\colon (X,\mathscr{A}) \to (Y,\mathscr{B})$ be a morphism of coherent ringed schemes, such that f is quasi-projective $M \in \mathsf{D}(X,\mathscr{A})$ and $\mathcal{O}_f(1)$ an f-ample line bundle on X.*

- *i) Then $M \in \mathsf{D}^{[p,q]}(X,\mathscr{A})$ if and only if there is a sequence $\mathcal{O}_f(k_1) \to \mathcal{O}_f(k_2) \to \dots$ with $k_i \to \infty$ such that* hocolim $\mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y,\mathscr{B})$.
- *ii) Moreover, $M = 0$ if and only if there is a sequence $\mathcal{O}_f(k_1) \to \mathcal{O}_f(k_2) \to \dots$ with $k_i \to \infty$ such that* hocolim $\mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) = 0$.

Proof. Given a quasi-coherent $\mathscr{A}$-module M, we have $M \in \mathsf{D}^{[p,q]}(X,\mathscr{A})$ if and only if $M \in \mathsf{D}^{[p,q]}(X)$, because the cohomology sheaf of M does not depend on its $\mathscr{A}$-module structure. By [Kuz11, Lemma 5.4], this holds if and only if hocolim $\mathbf{R}f_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y)$. Since $\mathbf{R}\mathfrak{f}_* = \mathbf{R}f_*$, we find in particular that hocolim $\mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k_i)) \in \mathsf{D}^{[p,q]}(Y,\mathscr{B})$. Similarly, Lemma 5.4 of *op. cit.* implies (ii). □

Remark A.15. With the assumption that the projection functors in the semiorthogonal decompositions of Proposition A.12 have finite cohomological amplitude one obtains a semiorthogonal decomposition of the bounded above derived category $\mathsf{D}^-(T,\mathscr{A}_T) = \langle \mathsf{A}_{1T}^-,\dots,\mathsf{A}_{mT}^-\rangle$ compatible with derived pushforward and pullback. The T-linear components are $\mathsf{A}_{iT}^- = \widehat{\mathsf{A}}_{iT} \cap \mathsf{D}^-(T,\mathscr{A}_T)$.

The bounded derived category. We are now ready to formulate the main statement of this section, the generalization of [Kuz11, Theorem 5.6] to coherent ringed schemes.

Theorem A.16. *Let $\mathfrak{f}\colon (X,\mathscr{A}) \to S$ and $h\colon T \to S$ be morphisms of coherent ringed schemes. Assume that $\mathsf{D}^{\mathsf{b}}(X,\mathscr{A}) = \langle \mathsf{A}_1,\dots,\mathsf{A}_m\rangle$ is an S-linear strong semiorthogonal decomposition such that the projection functors have finite cohomological amplitude and h is faithful for $\mathfrak{f}$. If $\mathscr{A}$ has finite global dimension, then*

$$\mathsf{D}^{\mathsf{b}}(X_T,\mathscr{A}_T) = \langle \mathsf{A}_{1,T},\dots,\mathsf{A}_{m,T}\rangle \tag{66}$$

is a T-linear semiorthogonal decomposition compatible with

- *the induced semiorthogonal decomposition on* $\mathsf{D}(X_T, \mathscr{A}_T)$ *and* $\mathsf{D}^-(X_T, \mathscr{A}_T)$,
- *the pullback* $\mathbf{L}\mathfrak{h}_T^*\colon \mathsf{D}^{\mathsf{b}}(S, \mathscr{A}) \to \mathsf{D}^-(X_T, \mathscr{A}_T)$ *and the pushforward* $\mathbf{R}\mathfrak{h}_{T,*}\colon \mathsf{D}^{\mathsf{b}}(X_T, \mathscr{A}_T) \to \mathsf{D}(S, \mathscr{A})$.

The proof follows as in [Kuz11, Theorem 5.6] with the following lemma for the approximation of the pushforward of bounded quasi-coherent complexes $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$.

Lemma A.17. *Let* $\mathfrak{f}\colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ *be a morphism of coherent ringed schemes, such that* f *is quasi-projective* $M \in \mathsf{D}(X, \mathscr{A})$ *and* $\mathcal{O}_f(1)$ *an* f*-ample line bundle on* X. *If* $M \in \mathsf{D}^{[p,q]}(X, \mathscr{A})$ *and* $k \gg 0$, *then there is a direct system* $\{N_m\}$ *in* $\mathsf{D}^{[p,q]}(Y, \mathscr{B})$ *such that* $\mathbf{R}\mathfrak{f}_*(M \otimes_X \mathcal{O}_f(k)) \cong \operatorname{hocolim} N_m$.

Since every quasi-coherent $\mathscr{A}$-module is a colimit of its coherent submodules, one can use the same argument as in [Kuz11, Lemma 2.20] to prove this lemma.

References

[Art82] M. Artin. 'Left ideals in maximal orders'. In: *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*. Vol. 917. Lecture Notes in Math. Springer, Berlin-New York, 1982, pp. 182–193. MR: 657429.

[ASS06] Ibrahim Assem, Daniel Simson and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*. Vol. 65. London Mathematical Society Student Texts. Techniques of representation theory. Cambridge University Press, Cambridge, 2006, pp. x+458. MR: 2197389.

[Bay+21] Arend Bayer, Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry and Paolo Stellari. 'Stability conditions in families'. In: *Publ. Math. Inst. Hautes Études Sci.* 133 (2021), pp. 157–325. MR: 4292740.

[BBG24] Thilo Baumann, Pieter Belmans and Okke van Garderen. *Central curves on noncommutative surfaces.* To appear in Transactions of the American Mathematical Society. 2024. arXiv: 2410.07620.

[BD22] Igor Burban and Yuriy Drozd. 'Morita theory for non-commutative noetherian schemes'. In: *Adv. Math.* 399 (2022), Paper No. 108273, 42. MR: 4385131.

[BD24] A. Bodzenta and W. Donovan. 'Root stacks and periodic decompositions'. In: *Manuscripta Math.* 175.1-2 (2024), pp. 53–73. MR: 4790551.

[BDG17] Igor Burban, Yuriy Drozd and Volodymyr Gavran. 'Minors and resolutions of non-commutative schemes'. In: *Eur. J. Math.* 3.2 (2017), pp. 311–341. MR: 3695056.

[BK89] A. I. Bondal and M. M. Kapranov. 'Representable functors, Serre functors, and reconstructions'. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 53.6 (1989), pp. 1183–1205, 1337. MR: 1039961.

[BLS16] Daniel Bergh, Valery A. Lunts and Olaf M. Schnürer. 'Geometricity for derived categories of algebraic stacks'. In: *Selecta Math. (N.S.)* 22.4 (2016), pp. 2535–2568. MR: 3573964.

[Bon89] A. I. Bondal. 'Representations of associative algebras and coherent sheaves'. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 53.1 (1989), pp. 25–44. MR: 992977.

[BOR24] Pieter Belmans, Shinnosuke Okawa and Andrea T. Ricolfi. *Moduli spaces of semiorthogonal decompositions in families.* 2024. arXiv: 2002.03303.

[Bur24] Igor Burban. *Exceptional hereditary curves and real curve orbifolds.* 2024. arXiv: 2411.06222.

[Cad07] Charles Cadman. 'Using stacks to impose tangency conditions on curves'. In: *Amer. J. Math.* 129.2 (2007), pp. 405–427. MR: 2306040.

[CI04] Daniel Chan and Colin Ingalls. 'Non-commutative coordinate rings and stacks'. In: *Proc. London Math. Soc. (3)* 88.1 (2004), pp. 63–88. MR: 2018958.

[DKR67] Ju. A. Drozd, V. V. Kiričenko and A. V. Roĭter. 'Hereditary and Bass orders'. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 31 (1967), pp. 1415–1436. MR: 219527.

[DLR25a] Timothy De Deyn, Pat Lank and Kabeer Manali Rahul. *Approximability and Rouquier dimension for noncommutative algebras over schemes.* 2025. arXiv: 2408.04561.

[DLR25b] Timothy De Deyn, Pat Lank and Kabeer Manali Rahul. *Descent and generation for noncommutative coherent algebras over schemes.* 2025. arXiv: 2410.01785.

[ELS20] Alexey Elagin, Valery A. Lunts and Olaf M. Schnürer. 'Smoothness of derived categories of algebras'. In: *Mosc. Math. J.* 20.2 (2020), pp. 277–309. MR: 4088795.

[Fab+] Eleonore Faber, Colin Ingalls, Shinnosuke Okawa and Matthew Satriano. *On stacky surfaces and noncommutative surfaces.* In: Trans. Amer. Math. Soc., electronically published on April 4, 2025, (to appear in print).

[GL87] Werner Geigle and Helmut Lenzing. 'A class of weighted projective curves arising in representation theory of finite-dimensional algebras'. In: *Singularities, representation of algebras, and vector bundles (Lambrecht, 1985)*. Vol. 1273. Lecture Notes in Math. Springer, Berlin, 1987, pp. 265–297. MR: 915180.

[IU15] Akira Ishii and Kazushi Ueda. 'The special McKay correspondence and exceptional collections'. In: *Tohoku Math. J. (2)* 67.4 (2015), pp. 585–609. MR: 3436544.

[Jin20] Haibo Jin. 'Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations'. In: *Adv. Math.* 374 (2020), pp. 107338, 59. MR: 4133519.

[K06] A. G. Kuznetsov. 'Hyperplane sections and derived categories'. In: *Izv. Ross. Akad. Nauk Ser. Mat.* 70.3 (2006), pp. 23–128. MR: 2238172.

[KS23] Alexander Kuznetsov and Evgeny Shinder. 'Categorical absorptions of singularities and degenerations'. In: *Épijournal Géom. Algébrique* (2023), Art. 12, 42. MR: 4698898.

[KS25] Alexander Kuznetsov and Evgeny Shinder. 'Homologically finite-dimensional objects in triangulated categories'. In: *Selecta Math. (N.S.)* 31.2 (2025), Paper No. 27, 45. MR: 4868123.

[KSS03] Steffen Koenig, Katsunori Sanada and Nicole Snashall. 'On Hochschild cohomology of orders'. In: *Arch. Math. (Basel)* 81.6 (2003), pp. 627–635. MR: 2029238.

[Kuz08] Alexander Kuznetsov. 'Lefschetz decompositions and categorical resolutions of singularities'. In: *Selecta Math. (N.S.)* 13.4 (2008), pp. 661–696. MR: 2403307.

[Kuz11] Alexander Kuznetsov. 'Base change for semiorthogonal decompositions'. In: *Compos. Math.* 147.3 (2011), pp. 852–876. MR: 2801403.

[Kuz23] Alexander Kuznetsov. 'Semiorthogonal decompositions in families'. In: *ICM—International Congress of Mathematicians. Vol. 2. Plenary lectures.* EMS Press, Berlin, 2023, pp. 1154–1200. MR: 4680279.

[L08] Lieven Le Bruyn. *Noncommutative geometry and Cayley-smooth orders.* Vol. 290. Pure and Applied Mathematics (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2008, pp. lxiv+524. MR: 2356702.

[LP21] Bronson Lim and Alexander Polishchuk. 'Bondal-Orlov fully faithfulness criterion for Deligne-Mumford stacks'. In: *Manuscripta Math.* 165.3-4 (2021), pp. 469–481. MR: 4280492.

[LT24] Yanki Lekili and Jenia Tevelev. *Deformations of Kalck–Karmazyn algebras via Mirror Symmetry.* 2024. arXiv: 2412.09724.

[Orl06] D. O. Orlov. 'Triangulated categories of singularities, and equivalences between Landau-Ginzburg models'. In: *Mat. Sb.* 197.12 (2006), pp. 117–132. MR: 2437083.

[Per19] Alexander Perry. 'Noncommutative homological projective duality'. In: *Adv. Math.* 350 (2019), pp. 877–972. MR: 3948688.

[Rei75] I. Reiner. *Maximal orders.* Vol. No. 5. London Mathematical Society Monographs. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1975, pp. xii+395. MR: 393100.

[RV89] Idun Reiten and Michel Van den Bergh. 'Two-dimensional tame and maximal orders of finite representation type'. In: *Mem. Amer. Math. Soc.* 80.408 (1989), pp. viii+72. MR: 978602.

[SP] The Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu. 2018.

[TT90] R. W. Thomason and Thomas Trobaugh. 'Higher algebraic K-theory of schemes and of derived categories'. In: *The Grothendieck Festschrift, Vol. III.* Vol. 88. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR: 1106918.

[Var24] Aporva Varshney. *Categorical absorption of a non-isolated singularity.* 2024. arXiv: 2402.18513.

[VV84] M. Van den Bergh and J. Van Geel. 'A duality theorem for orders in central simple algebras over function fields'. In: *J. Pure Appl. Algebra* 31.1-3 (1984), pp. 227–239. MR: 738217.

[Xie23] Fei Xie. 'Residual categories of quadric surface bundles'. In: *J. Reine Angew. Math.* 796 (2023), pp. 161–199. MR: 4554471.

[YZ06] Amnon Yekutieli and James J. Zhang. 'Dualizing complexes and perverse sheaves on noncommutative ringed schemes'. In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 137–177. MR: 2244264.

Thilo Baumann, thilo.baumann@uni.lu
Department of Mathematics, Université de Luxembourg, 6, avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg