PhD-FSTM-2025-047
Faculty of Science, Technology and Medicine

DISSERTATION

Defence held on 20 May 2025 in Esch-sur-Alzette

to obtain the degree of

DOCTEUR DE L'UNIVERSITÉ DU LUXEMBOURG

EN MATHÉMATIQUES

by

Vincent WOLFF
Born on 23 December 1996 in Esch-sur-Alzette (Luxembourg)

Polydifferential properads, graph complexes and the Grothendieck-Teichmüller group

Dissertation defence committee

Prof. Dr. Sarah Scherotzke, Chairman
Université du Luxembourg

Prof. Dr. Sergey Shadrin, Vice Chairman
Universiteit van Amsterdam

Prof. Dr. Vladimir Dotsenko
Université de Strasbourg

Prof. Dr. Sergei Merkulov, Supervisor
Université du Luxembourg

Prof. Dr. Bruno Teheux,
Université du Luxembourg

Abstract

The first major theme of this thesis is the deformation theory of morphisms of certain, important in applications, properads and its relationship to the theory of graph complexes. The second major theme of this thesis is the study of interrelations between different graph complexes in order to prove a long standing conjecture about them.

A large part of the thesis is devoted to the computation of the cohomology groups of two chain complexes. One complex controls deformations of the standard morphism from the operad of Lie algebras to its polydifferential closure and another (more complicated) one controls deformations of a morphism of the properad of Lie bialgebras to its polydifferential closure. We prove that in both cases these complexes are quasi-isomorphic to the famous Kontsevich graph complex. In particular we conclude that the Grothendieck-Teichmüller group acts transitively on the homotopy classes of both morphisms.

We also study interrelations between the Kontsevich graph complex $\mathbf{GC}_d$ and its important oriented version $\mathbf{OGC}_{d+1}$ which were proven to be isomorphic at the cohomology level by Thomas Willwacher in 2015. This result is extended to the level of dg Lie algebras. More precisely, we prove that $\mathbf{GC}_d$ and $\mathbf{OGC}_{d+1}$ are $\mathbf{Lie}_\infty$ quasi-isomorphic.

Acknowledgments

First and foremost I want to thank my supervisor Sergei Merkulov for his support over the many years I had the honor of working with him. I will be forever grateful for his patience while I struggled my way through the theory of graph complexes.

Next I want to thank each of the jury members for agreeing to be part of the dissertation committee.

I need to thank Sergey Shadrin and Vladimir Dotsenko for many discussions during their many visits. Their suggestions greatly improved this thesis.

I am highly indebted to Thomas Willwacher for shaping my understanding of graph cohomology and for the opportunity to work with him.

A very special thanks goes to all the people at the university who supported me during the last four years. There are too many names I need to mention, so here are a select few: Oskar for our joint struggle with operads and graph complexes; Thilo for many coffee breaks; Antigona for listening to me when I needed it; Charles whose office was always open. I apologize to anyone I forgot, but be sure that I will not forget these last four years we spent together.

One person I am not allowed to forget is Fabio. I will never be able to pay you back for the countless hours we spend talking and his support during the more stressful parts of the last four years.

Last but not least I need to thank my parents for their support over these many years. Without them I would have never undertaken this adventure.

Contents

Chapter 1

Introduction

1.1 Motivation and historical background

There are two famous deformation quantization theories. First the deformation quantization of Poisson structures defined in [BFF$^+$78] and whose existence was proved in [Kon03] and second the deformation quantization of Lie bialgebras [Dri92, EK96, MW18b]. These theories can be reformulated in terms of morphisms from suitable operads/PROPs to polydifferential closures of some other operads/PROPs (see [MW20] and [AM22]). The polydifferential closures are constructed with the help of the polydifferential endofunctor $\mathcal{D}$ in the category of PROPs [MW20]

$$\mathcal{D} : \mathbf{Props} \longrightarrow \mathbf{Props}$$

and its reduced version

$$\mathcal{O} : \mathbf{Props} \longrightarrow \mathbf{Operads}$$

which was introduced in an earlier paper [MW15]. The key property of the functor $\mathcal{D}$ is the following one: for any representation

$$\mathcal{P} \to \mathbf{End}_V$$

in a dg vector space V, there is an associated representation of its polydifferential closure

$$\mathcal{D}(\mathcal{P}) \to \mathbf{End}_{\odot V}$$

in the symmetric tensor algebra $\odot V$ of V, given in terms of polydifferential (with respect to the standard product in $\odot V$) operators.

Let $\mathbf{Lie}_d$ be the operad governing graded Lie algebras with Lie bracket of degree $1-d$ (so that $d=1$ corresponds to usual Lie algebras and is often denoted by $\mathbf{Lie} := \mathbf{Lie}_1$). By applying the functor $\mathcal{O}$ to the (PROP closure of the) operad $\mathbf{Lie}_d$ we obtain an operad $\mathcal{O}(\mathbf{Lie}_d)$ which occurs naturally in the study of the well-known universal enveloping functor

$\mathfrak{U}$ from the category of Lie algebras to the category of associative algebras. This functor can be understood as a morphism of operads

$$\mathcal{U} : \mathbf{Ass} \to \mathcal{O}(\mathbf{Lie}), \tag{1.1}$$

satisfying some natural non-triviality condition (see § 4.3 below).

Our starting observation is that the polydifferential operad $\mathcal{O}(\mathbf{Lie}_d)$ comes equipped with a natural morphism of operads

$$i : \mathbf{Lie}_d \to \mathcal{O}(\mathbf{Lie}_d). \tag{1.2}$$

The first main result of this thesis is the computation of the cohomology (of the connected part) of the deformation complex $\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}(\mathbf{Lie}_d))$ in terms of graph complexes. More precisely we prove that

$$H^\bullet(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))) \cong H^\bullet(\mathbf{fGC}_d)$$

where $\mathbf{fGC}_d$ is the famous M. Kontsevich graph complex. In particular, this result implies that for $d = 2$ the mysterious Grothendieck-Teichmüller group acts (up to homotopy) non-trivially and almost faithfully as a symmetry group of the map i.

This result gives us one of the simplest incarnations of the Grothendieck-Teichmüller group, and explains, perhaps, why it occurs in two seemingly different deformation quantization problems, the universal quantization of Poisson structures (solved by M. Kontsevich in [Kon03]) and the universal quantization of Lie bialgebras (formulated by V. Drinfeld and solved by Etingov-Kazhdan in [EK96]) as both these theories involve $\mathcal{O}(\mathbf{Lie})$ as a sub-structure.

The second incarnation of the operad $\mathcal{O}(\mathbf{Lie})$ is via the universal enveloping construction associating to a Lie algebra V the corresponding associative algebra $\mathfrak{U}(V)$ which is isomorphic by the Poincaré-Birkhoff-Witt theorem to the space of symmetric tensors $\odot V$. Hence it is not that surprising that this construction can be described in terms of the operad $\mathcal{O}(\mathbf{Lie})$. Indeed, following M. Kontsevich [Kon03], S. Gutt [Gut11] and V. Kathotia [Kat98], one first interprets the universal enveloping algebra construction $\mathfrak{U}(V)$ as a star product construction on $\odot V$ given in terms of polydifferential operators. Second, one notices that this interpretation can be encoded as a morphism of operads

$$\mathbf{Ass} \to \mathbf{End}_{\odot V}$$

satisfying some non-triviality condition. The point is that this map factors through some morphism

$$\mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$$

and the canonical map

$$\mathcal{O}(\mathbf{Lie}) \to \mathbf{End}_{\odot V}$$

induced by the functor $\mathcal{O}$ applied to the given representation of **Lie** in V. Hence all subtleties of the universal enveloping functor $\mathfrak{U}$ get encoded into the morphism of operads

$$\mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$$

satisfying some non-triviality condition (see Section 4.3 below).

We study in this thesis the deformation complex $\mathbf{Def}(\mathbf{Ass} \stackrel{\mathcal{U}}{\to} \mathcal{O}(\mathbf{Lie}))$ and prove that its cohomology is one-dimensional, the unique cohomology class corresponding to the rescaling freedom of the Lie bracket. The conclusion is that the morphism $\mathcal{U}$ is unique (up to gauge equivalence). This result is not surprising, of course. One can infer it from the classification theorem of Kontsevich formality maps given in terms of the graph complex $\mathbf{GC}_2$ [Dol21]. So we just give a new very short proof of this uniqueness. This proof has a small advantage in that it carries over to $\mathbf{Lie}_\infty$ straightforwardly.

There are several constructions generalizing the universal enveloping functor from Lie algebras (controlled by the operad **Lie**) to strongly homotopy Lie algebras (which are controlled by $\mathbf{Lie}_\infty$, the minimal resolution of **Lie**). All constructions involve the notion of strongly homotopy associative algebras, which was introduced by J. Stasheff in [Sta63] and which are controlled by the dg operad $\mathbf{Ass}_\infty$, the minimal resolution of **Ass**. One such generalization of the functor $\mathfrak{U}$ is offered by M. Kontsevich's formality map applied to linear polyvector fields. That generalization uses, in general, graphs with wheels. However, as has been proven by B. Shoikhet [Sho01], the graphs with wheels can be removed so that one gets a strongly homotopy extension of the functor $\mathfrak{U}$ which works well for infinite dimensional $\mathbf{Lie}_\infty$ algebras. Another construction was given by V. Baranovsky [Bar08] as the cobar construction of the Cartan-Chevalley-Eilenberg coalgebra associated to an $\mathbf{Lie}_\infty$ algebra, by J. M. Moreno-Fernández [MF22] as an $\mathbf{Ass}_\infty$ algebra isomorphic as graded vector spaces to the free symmetric algebra associated to an $\mathbf{Lie}_\infty$ algebra and by T. Lada and M. Markl in [LM95].

These constructions can be understood as a morphism of operads

$$\mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie}_\infty).$$

We study the deformation complex of any of these maps and prove that it is quasi-isomorphic to the deformation complex of the map considered in (1.1), implying that all constructions are gauge equivalent. Moreover, any other attempt to construct such a generalization satisfying some natural conditions must be gauge equivalent to these ones as the cohomology of the complex $\mathbf{Def}(\mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie}_\infty))$ is one-dimensional (see Corollary 4.3.8). This cohomology result is in full agreement with the derived Poincaré-Birkhoff-Witt theorem established by A. Khoroshkin and P. Tamaroff in [KT23].

The second part of this thesis is devoted to the theory of (degree shifted) Lie bialgebras. Lie bialgebras were introduced by V. Drinfeld in the context of quantum groups [Dri87] and their deformation quantization theory was constructed by P. Etingov and D. Kazhdan in

[EK96]. Lie bialgebras have found applications in many areas of mathematics, for example in string topology [CS04], integrable systems [BD82] and the study of Poisson-Lie groups [KS97].

Our interest lies in the properad $\mathbf{Lieb}_{c,d}$ controlling a degree shifted version of Lie bialgebras which were defined in [MW15]. By applying the polydifferential functor $\mathcal{D}$ to the (PROP closure of) the properad $\mathbf{Lieb}_{c,d}$ we obtain a PROP $\mathcal{D}(\mathbf{Lieb}_{c,d})$. This PROP is very useful in the study of the deformation theory of Lie bialgebras [Dri92, EK96], as one can interpret any universal deformation quantization of Lie bialgebras as a morphism of PROPs [MW20]

$$\mathbf{Assb} \longrightarrow \mathcal{D}(\mathbf{Lieb}_{1,1})$$

satisfying certain non-triviality conditions; here **Assb** stands for the PROP of associative bialgebras. This interpretation of the deformation quantization theory was used in [MW20] to classify all homotopy non-trivial universal quantizations in terms of the Kontsevich graph complex.

The PROP $\mathcal{D}(\mathbf{Lieb}_{c,d})$ contains naturally a properad $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$ spanned by connected decorated graphs, called the properad of polydifferential Lie bialgebras. Similarly as in the Lie case, there is a natural morphism of properads

$$i : \mathbf{Lieb}_{c,d} \longrightarrow \mathcal{D}_c(\mathbf{Lieb}_{c,d}) \tag{1.3}$$

(see Lemma 5.2.2 for the explicit formulae).The main result of this part is the following:

Theorem 1.1.1. *There is an isomorphism, up to one class, of cohomology groups*

$$H^\bullet(\mathbf{GC}_{c+d}) \longrightarrow H^{\bullet+1}(\mathbf{Def}(\mathbf{Lieb}_{c,d} \to \mathcal{D}_c(\mathbf{Lieb}_{c,d}))).$$

The connection between the two parts above goes via the different types of graph complexes. We recall that the original graph complex $\mathbf{GC}_d$ was defined by M. Kontsevich in [Kon97] while studying the deformation quantization of Poisson structures and the formality theorem: the cohomology group $H^0(\mathbf{GC}_2)$ measures the choices while $H^1(\mathbf{GC}_2)$ measures the possible obstructions to formality maps. Little is known about their cohomology in general, the biggest leap forward was done in the seminal paper [Wil15a] where T. Willwacher computed the zeroth cohomology group and proved an isomorphism of Lie algebras

$$H^0(\mathbf{GC}_2) \cong \mathfrak{grt}_1$$

where $\mathfrak{grt}_1$ denotes the Grothendieck-Teichmüller Lie algebra introduced by V. Drinfeld in [Dri90].

The graph complexes above and their cohomology have seen many recent application in homological algebra, algebraic geometry [CGP21, AWZ20], algebraic topology [FW20], and the Lie theory [AT12].

There is a directed version $\mathbf{dfGC}_{d+1}$ spanned by graphs with fixed directions on edges. In [Wil15a] the author proved that the directed and undirected version are quasi-isomorphic as dg Lie algebras and thus we gain nothing new on the level of cohomology. However there are interesting subcomplexes, the most important one for this thesis being the oriented graph complex $\mathbf{OGC}_{d+1}$ controlling deformations of the properad of Lie bialgebras [MW18a]. An isomorphism on the level of Lie algebras between the cohomology of $\mathbf{OGC}_{d+1}$ and the cohomology of $\mathbf{GC}_d$ was first constructed in [Wil15b]. Later explicit quasi-isomorphisms of complexes, but not of dg Lie algebras, have been constructed by M. Živković [Ž20] (on the dual graph complexes) and S. Merkulov [Mer25]. The main contribution of this thesis is to upgrade this isomorphism on cohomology to a zigzag of explicit quasi-isomorphisms of dg Lie algebras.

1.2 Structure of the thesis

In Chapter 2 we recall the basic theory of properads and the construction of the different operads and properads we encounter throughout this thesis.

Chapter 3 consists of an introduction to the deformation theory of morphisms of properads. We also present the main examples: the deformation complex of the operad of Lie algebras, the deformation complex of the properad of Lie bialgebras and the full graph complex. Afterwards we introduce the different versions of said complex which are used in this thesis.

There are no new results in the two introductory chapters above.

In Chapter 4 we study the natural morphism $\mathbf{Lie}_d \to \mathcal{O}(\mathbf{Lie}_d)$ and compute the cohomology of the associated deformation complex. We also define the notion of (homotopy) S. Gutt quantization and compute the cohomology of their deformation complexes.

In Chapter 5 we prove Theorem 1.1.1 by introducing a new graph complex of entangled graphs and computing its cohomology.

In Chapter 6 we construct the $\mathbf{Lie}_\infty$ quasi-isomorphism between $\mathbf{GC}_d$ and $\mathbf{OGC}_{d+1}$.

In Chapter 7 we introduce two Lie brackets on so-called reduced graph complexes $\overline{\mathbf{GC}}_{d+1}$ and show that the induced Lie brackets on cohomology coincide.

1.3 Notation

We work over a field $\mathbb{K}$ of characteristic 0. We denote by $\mathbf{dgVect}_{\mathbb{K}}$ the category of differential graded (or dg for short) vector spaces, which for this thesis means cohomologically graded chain complexes. All properads encountered will live in the category $\mathbf{dgVect}_{\mathbb{K}}$. For any graded vector space V, the vector space $V[k]$ is defined by $V[k]^i = V^{i+k}$. For any element $v \in V^i$ we write $|v| = i$.

For a positive integer n we abbreviate $[n] = \{1, \cdots, n\}$ and denote by $\mathbb{S}_n$ its automorphism groups. We write 1_n respectively sgn_n for the one-dimensional trivial respectively sign representation. The cardinality of a finite set A is denoted by $\#A$ while its linear span over a field $\mathbb{K}$ by $\mathrm{span}\langle A\rangle$. The top degree skew-symmetric tensor power of $\mathrm{span}\langle A\rangle$ is denoted by $\det A$; it is assumed that $\det A$ is a 1-dimensional Euclidean space associated with the unique Euclidean structure on span $\langle A \rangle$ in which the elements of A serve as an orthonormal basis; in particular, $\det A$ contains precisely two vectors of unit length.

For any group G and G-module V, we write V_G for the space of G-coinvariants and V^G for the space for G-invariants.

Chapter 2

On the theory of operads and properads

2.1 Properads as $\mathcal{G}$-algebras

Following [BM08, Mer10] we introduce the notion of $\mathcal{G}$-algebras where $\mathcal{G}$ is some chosen set of graphs.

For starters we give the definition of graphs we use throughout this thesis.

Definition 2.1.1. A graph with hairs is a triple $\Gamma = (H(\Gamma), \sqcup, \tau)$ where

1) $H(\Gamma)$ is a finite set of *half-edges*,

2) $\sqcup$ is a partition of $H(\Gamma)$

$$H(\Gamma) = \bigsqcup_{v \in V(\Gamma)} H(v),$$

parametrized by a set $V(\Gamma)$ called the *set of vertices* of Γ. For a vertex v the set $H(v)$ is called the *set of half-edges attached to* v. The valency of a vertex v is defined to be the cardinality of $H(v)$.

3) $\tau : H(\Gamma) \to H(\Gamma)$ is an involution. The orbits of cardinality two are called the *(internal) edges* and the set of edges is denoted by $E(\Gamma)$. The orbits of cardinality one are called *hairs* (or *legs*) and the set of hairs is denoted by $L(\Gamma)$

If $L(\Gamma) = \varnothing$, then Γ is simply called a *graph*. A graph Γ is called *directed* if each edge $e = (h, \tau(h))$ comes with a choice of an ordering of half-edges. A directed graph is called *oriented* if there are no directed cycles.

A *subgraph* γ of a graph Γ is a choice of a subset $V(\gamma) \subset V(\Gamma)$ and a choice for each $v \in V(\gamma)$ of a subset $H_\gamma(v) \subset H(v)$ such that

$$H(\gamma) := \bigsqcup_{v \in V(\gamma)} H_\gamma(v)$$

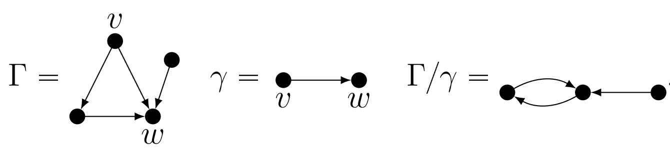
is stable under the involution τ.

For any graph Γ with a subgraph γ, denote by Γ/γ the graph given as follows: $H(\Gamma/\gamma)$ is obtained by deleting all half-edges of $H(\Gamma)$ corresponding to edges in γ. We then collect all the remaining half-edges attached to the vertices of γ into a single vertex v and define the set of vertices of Γ/γ by

$$V(\Gamma/\gamma) = V(\Gamma)\backslash V(\gamma) \sqcup \{v\}.$$

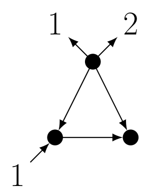
We say that the graph Γ/γ is obtained by contracting γ into a single vertex.

Example 2.1.2.



We call a directed graph Γ an (m, n)-*graph* if the set of hairs is partitioned $L(\Gamma) = L_{in} \sqcup L_{out}$ of cardinality m and n respectively together with bijective maps $f_{in} : [m] \to L_{in}$ and $f_{out} : [n] \to L_{out}$. These maps are called *labelling maps*. Denote by $\mathcal{G}^{\circlearrowright}(m, n)$ the set of all directed (m, n)-graphs.

Example 2.1.3.



$$\in \mathcal{G}^{\circlearrowright}(1, 2).$$

Consider an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m, n)\}$ and let $\Gamma \in \mathcal{G}^{\circlearrowright}$. One can associate to $\mathcal{P}$ and Γ a vector space as follows. For a vertex v we define

$$\mathcal{P}(\mathrm{Out}_v, \mathrm{In}_v) := \langle \mathrm{Out}_v \rangle \otimes_{\mathbb{S}_{\#\mathrm{Out}_v}} \mathcal{P}(\#\mathrm{Out}_v, \#\mathrm{In}_v) \otimes_{\mathbb{S}_{\#\mathrm{In}_v}} \langle \mathrm{In}_v \rangle$$

where Out_v and In_v stand for the set of outgoing respectively incoming half-edges of v, $\langle \mathrm{Out}_v \rangle$ and $\langle \mathrm{In}_v \rangle$ stand for the vector space spanned by the bijections $[\#\mathrm{Out}_v] \to \mathrm{Out}_v$ respectively spanned by the bijections $[\#\mathrm{In}_v] \to \mathrm{In}_v$.

This space carries an action of the automorphism groups of Out_v and In_v and thus the following unordered tensor product

$$\bigotimes_{v \in V(\Gamma)} \mathcal{P}(\mathrm{Out}_v, \mathrm{In}_v) := \Big(\bigoplus_{\sigma : [k] \to V(\Gamma)} \mathcal{P}(\mathrm{Out}_{\sigma(1)}, \mathrm{In}_{\sigma(1)}) \otimes \cdots \otimes \mathcal{P}(\mathrm{Out}_{\sigma(k)}, \mathrm{In}_{\sigma(k)})\Big)_{\mathbb{S}_k},$$

where $k = \#V(\Gamma)$, is a representation space of $\mathrm{Aut}(\Gamma)$. Here $\mathrm{Aut}(\Gamma)$ is the subgroup of the symmetry group of Γ fixing the hairs and is called the automorphism group of Γ. This now allows us to associate to $\mathcal{P}$ and Γ the vector space

$$\Gamma\langle \mathcal{P} \rangle := \Big(\bigotimes_{v \in V(\Gamma)} \mathcal{P}(\mathrm{Out}_v, \mathrm{In}_v)\Big)_{\mathrm{Aut}(\Gamma)}$$

called space of *decorated (by $\mathcal{P}$) graphs.*

If $\mathcal{P}$ comes with a differential δ then for any graph $\Gamma \in \mathcal{G}^{\circlearrowright}(m,n)$ there is an induced $\mathbb{S}_m \times \mathbb{S}_n$-equivariant differential δ_Γ on the vector space $\Gamma\langle\mathcal{P}\rangle$ such that the collection $\{\bigoplus_{\Gamma\in\mathcal{G}^{\circlearrowright}(m,n)}\Gamma\langle\mathcal{P}\rangle\}$ is also a dg $\mathbb{S}$-bimodule.

Consider $\mathcal{G} \subset \mathcal{G}^{\circlearrowright}$ some subset. Let $\Gamma \in \mathcal{G}$ and call a subgraph γ of Γ *admissible* if both $\gamma \in \mathcal{G}$ and $\Gamma/\gamma \in \mathcal{G}$.

A $\mathbb{S}$-bimodule $\mathcal{P}$ is called a $\mathcal{G}$-algebra if there are $\mathbb{S}_m \times \mathbb{S}_n$-equivariant linear maps

$$\{\phi_\Gamma : \Gamma\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{\Gamma\in\mathcal{G}(m,n)}$$

satisfying for any graph Γ and admissible subgraph γ the associativity condition

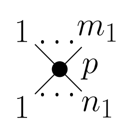
$$\phi_\Gamma = \phi_{\Gamma/\gamma} \circ \phi'_\gamma.$$

We are mainly interested in *properads*, which is the term for $\mathcal{G}$-algebras, where $\mathcal{G}$ is the subset of connected oriented graphs. If we restrict ourselves to graphs of genus zero with exactly one output hair, we get the notion of an *operad.*

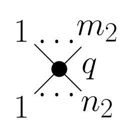
Remark 2.1.4.
1) If $\mathcal{G}$ is spanned by graphs with no directed cycles, but we drop the connectivity condition, we obtain the notion of a PROP.

2) If we allow directed cycles, we obtain the notion of *wheeled* PROPs or properads.

Let $\mathcal{P}$ be a properad and take elements $p \in \mathcal{P}(m_1,n_1)$ and $q \in \mathcal{P}(m_2,n_2)$. We can represent p and q pictorially by

$p \equiv$ 

and

$q \equiv$ 

.

Let $1 \leqslant k \leqslant \min(n_1,m_2)$, let $i_1,\cdots,i_k$ be some collection of input hairs of p and let $j_1,\cdots,j_k$ be some collection of output hairs of q. Define a (m_1+m_2-k, n_1+n_2-k)-graph graph Γ obtained by gluing the hairs labelled i_l and j_l for any $1 \leqslant l \leqslant k$. We can define now the properadic composition $p\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ q$ by

$$p\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ q = \phi_\Gamma(p \otimes q).$$

Remark 2.1.5. If $\mathcal{P}$ is an operad, then $m_2 = k = 1$ and we write $\circ_i$ instead of ${}_i\circ_1$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be properads. A *morphism of properads* is a collection $f = \{f(m,n) : \mathcal{P}(m,n) \to \mathcal{Q}(m,n)\}$ of homogeneous morphisms of $\mathbb{S}_m \times \mathbb{S}_n$-bimodules which respect properadic compositions, i.e.

$$f(p\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ q) = f(p)_{i_1,\cdots,i_k} \circ_{j_1,\cdots,j_k} f(q)$$

for any $p, q \in \mathcal{P}$.

The category of properads will be written as **Properads** and the subcategory of operads will be written as **Operads**.

2.2 Properadic derivations

Consider a collection $D = \{D(m,n)\}$ of homogeneous morphisms of $\mathbb{S}_m \times \mathbb{S}_n$ bimodules

$$D(m,n) : \mathcal{P}(m,n) \longrightarrow \mathcal{P}(m,n).$$

We extend D as a derivation on the tensor algebra generated by the $\mathcal{P}(m,n)$. For any graph Γ, D can be extended to a linear map $\Gamma\langle\mathcal{P}\rangle \to \Gamma\langle\mathcal{P}\rangle$ defined by

$$D(\Gamma\langle\mathcal{P}\rangle) = (D \otimes_{v \in V(\Gamma)} \mathcal{P}(\mathrm{Out}_v, \mathrm{In}_v))_{\mathrm{Aut}(\Gamma)}.$$

D is called a derivation of degree k if each map $D(m,n)$ is homogeneous of degree k and

$$D \circ \phi_\Gamma = \phi_\Gamma \circ D.$$

Derivations of the properad $\mathcal{P}$ form a Lie algebra $\mathbf{Der}(P)$ with Lie bracket given for homogeneous derivations D_1 and D_2 by

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1.$$

Here $|D_1|$ stands for the degree of D_1.

A differential δ is a derivation of degree 1 such that $\delta^2 = 0$. The fact that δ is a derivation can be expressed using the composition morphisms leading to the following definition.

Definition 2.2.1. A dg properad $\mathcal{P}$ is a properad such that for any $m, n \geqslant 1$ the vector space $\mathcal{P}(m,n)$ is equipped with a differential δ satisfying

$$\delta(p\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ q) = \delta(p)\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ q + (-1)^{|p|} p\ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k}\ \delta(q).$$

We define the cohomology properad $H(\mathcal{P})$ by the family

$$\{H(\mathcal{P}(m,n))\}_{m,n \geqslant 1}$$

and the compositions induced on the cohomology due to the above relation.

The differential on a free properad $(\mathrm{Free}\langle E\rangle, \delta)$ is uniquely defined by its value on generators and thus by sums of the form

$$\delta(\overset{1 \cdots m}{\underset{1 \cdots n}{\times}}) = \sum_{k\geqslant 1}\sum_{\Gamma\in G_k} \lambda_\Gamma \Gamma,$$

where G_k is a family of graphs with exactly k vertices and $\lambda_\Gamma \in \mathbb{K}$. We call the differential δ *minimal* if $G_1 = \varnothing$ and *quadratic* if $G_k = \varnothing$ for any $k \geqslant 3$.

2.3 Derivation complex of a morphism of properads

Let $f : (\mathcal{P}, \delta_{\mathcal{P}}) \to (\mathcal{Q}, \delta_{\mathcal{Q}})$ be a morphism of dg properads. A collection $D = \{D(m,n)\}$ of homogeneous morphisms of $\mathbb{S}_m \times \mathbb{S}_n$ bimodules is called a *derivation for the morphism f* if for any composition $p \;{}_{i_1,\cdots,i_k}\!\circ_{j_1,\cdots,j_k}\; q$ of elements in $\mathcal{P}$ the following equation is satisfied:

$$D(p \;{}_{i_1,\cdots,i_k}\!\circ_{j_1,\cdots,j_k}\; q) = D(p) \;{}_{i_1,\cdots,i_k}\!\circ_{j_1,\cdots,j_k}\; f(q) + (-1)^{|D||p|} f(p) \;{}_{i_1,\cdots,i_k}\!\circ_{j_1,\cdots,j_k}\; D(q).$$

Denote by $\mathbf{Der}(\mathcal{P} \xrightarrow{f} \mathcal{Q})$ the set of all derivations. It comes equipped with a differential δ defined by

$$\delta(D) = \delta_{\mathcal{Q}} \circ D - (-1)^{|D|} D \circ \delta_{\mathcal{P}}. \tag{2.1}$$

The space $(\mathbf{Der}(\mathcal{P} \xrightarrow{f} \mathcal{Q}), \delta)$ is called the *derivation complex*.

Remark 2.3.1. If we consider the identity map $id : \mathcal{P} \to \mathcal{P}$ we recover the derivations defined in Section 2.2. In particular we have $\mathbf{Der}(\mathcal{P} \xrightarrow{id} \mathcal{P}) = \mathbf{Der}(\mathcal{P})$.

2.4 Free properad and minimal resolutions

We consider a $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n\geqslant 1}$ and define the *free properad* $\mathrm{Free}\langle E\rangle$ as the vector space spanned by decorated graphs, i.e. elements in the space

$$\Gamma\langle E\rangle := \Big(\bigotimes_{v\in V(\Gamma)} E(\mathrm{Out}_v, \mathrm{In}_v)\Big)_{\mathrm{Aut}(G)}$$

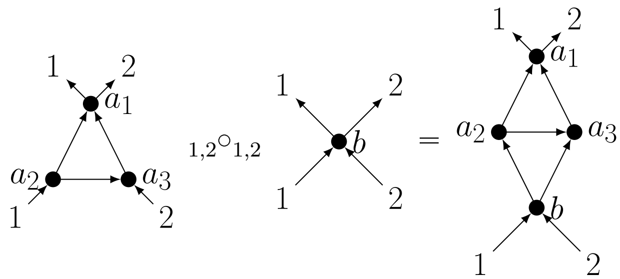
where Γ is a connected oriented (m,n)-graph. The properadic compositions are given by gluing and taking tensor products.

More precisely for any decorated (m_1,n_1)-graph Γ_1, for any decorated (m_2,n_2)-graph Γ_2, for any $1 \leqslant k \leqslant \min(n_1,m_2)$, for any collection of input hairs labeled $i_1,\cdots,i_k$ of Γ_1 and for any collection of output hairs $j_1,\cdots,j_k$ the properadic composition

$$\Gamma_1 \;{}_{i_1,\cdots,i_k}\!\circ_{j_1,\cdots,j_k}\; \Gamma_2$$

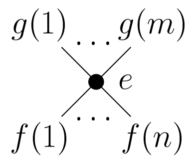
is given by the decorated graph Γ_3 obtained by gluing for any $1 \leqslant l \leqslant k$ the pairs (i_l, j_l) of input and output hairs (and keeping the decorations on vertices).

Example 2.4.1.



where $a_1, b \in E(2,2)$, $a_2 \in E(2,1)$ and $a_3 \in E(1,2)$.

The free properad is generated by graphs of the form



for $e \in E(m,n)$ and maps $f \in \mathbb{S}_n, g \in \mathbb{S}_m$. These are called *generating corollas.*

Remark 2.4.2. If $E(m,n) = 0$ for any $m \geqslant 2$, we call $\mathrm{Free}\langle E\rangle$ the *free operad.*

Any collection of equivariant maps $E \to \mathrm{Free}\langle E\rangle$ extends to a derivation of $\mathrm{Free}\langle E\rangle$ and vice-versa any derivation of $\mathrm{Free}\langle E\rangle$ is of this form.

Definition 2.4.3. Let $\mathcal{P}$ be a dg properad. A *quasi-free resolution* of $\mathcal{P}$ is a dg free properad $(\mathrm{Free}\langle E\rangle, \delta)$ equipped with an epimorphism

$$\pi : \mathrm{Free}\langle E\rangle \longrightarrow \mathcal{P}$$

which is a quasi-isomorphism, i.e. the induced morphism of cohomology properads

$$H(\mathrm{Free}\langle E\rangle) \longrightarrow H(\mathcal{P})$$

is an isomorphism.
We call $(\mathrm{Free}\langle E\rangle, \delta)$ *minimal* if the differential is minimal.

2.5 Endomorphism properad and degree shifted properads

Let V be any vector space. The *endomorphism operad* $\mathbf{End}_V$ is defined as the $\mathbb{S}$-module

$$\mathbf{End}_V(m,n) = \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V^{\otimes n}, V^{\otimes m}).$$

The properadic compositions are defined by composition of maps. More precisely for any maps $f : V^{\otimes n_1} \to V^{\otimes m_1}$ and $g : V^{\otimes n_2} \to V^{\otimes m_2}$, for any $1 \leqslant k \leqslant \min(n_1, m_2)$ and for any collection of inputs $1 \leqslant i_1, \cdots, i_k \leqslant n_1$ and outputs $1 \leqslant j_1, \cdots, \leqslant m_2$ we define the properadic composition

$$f \ {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k} g$$

by inserting for any $1 \leqslant l \leqslant k$ the function $g_{j_l} : V^{\otimes n_2} \to V^{\otimes m_2}$ into the i_l coordinate of f. The functions g_{j_l} are defined by

$$g(y_1, \cdots, y_{n_2}) = g_1(y_1, \cdots, y_{n_2}) \otimes \cdots \otimes g_{m_2}(y_1, \cdots, y_{n_2}).$$

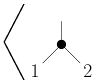
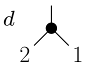
Given a properad $\mathcal{P}$ and a vector space V, we define a *representation of* $\mathcal{P}$ *in* V as a morphism of properads

$$\mathcal{P} \longrightarrow \mathbf{End}_V.$$

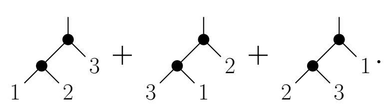
For a properad $\mathcal{P}$, the *degree d shifted* properad $\mathcal{P}\{d\}$ is defined by the property that any representation of $\mathcal{P}\{d\}$ in some graded vector space V corresponds to a representation of $\mathcal{P}$ in $V[d]$.

2.6 Operad of Lie algebras

Let $E = \{E(n)\}$ be the $\mathbb{S}$-module defined by $E(n) = 0$ except that

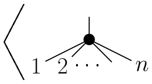
$E(2) = \mathrm{sgn}_2^{\otimes |d|}[d-1] = \mathrm{span}\Big\langle$  $= (-1)^d$  $\Big\rangle$.

The operad of degree d shifted Lie algebras is defined to be the quotient $\mathbf{Lie}_d := \mathrm{Free}\langle E\rangle / I$, where I is generated by the element



When studying linear combinations of graphs built from this generating corolla, it is useful, not to make sign mistakes, to view each such a corolla as a degree d vertex with two degree $-d$ incoming half-edges and one degree 1 outgoing half-edge. Hence for d odd, such a graph has vertices of odd degree (and thus an ordering of this set, up to a permutation σ and multiplication by $\mathrm{sgn}(\sigma)$, has to be chosen), while internal edges have degree $1 - d$. In the case d even, the vertices are even, but the internal edges are odd so that an ordering of this set is chosen (up to permutation).

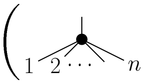
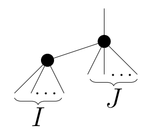
The minimal resolution of $\mathbf{Lie}_d$ is given by the dg quasi-free operad $\mathbf{HoLie}_d = (\mathbf{Free}\langle E\rangle, \delta)$ generated by the $\mathbb{S}$-module

$$E = \{E(n) = (\mathrm{sgn}_n)^{\otimes |d|}[nd-d-1] = \mathrm{span}\Big\langle$$  $$\Big\rangle\}_{n\geqslant 2},$$

for $n \geqslant 2$, where

$$(\mathrm{sgn}_n)^{\otimes |d|} = \begin{cases} \mathrm{sgn}_n & \text{if } d \text{ is odd} \\ 1_n & \text{if } d \text{ is even.} \end{cases}$$

The differential is given on generators by [MŽ22]

$$\delta\Big($$  $$\Big) = \sum_{\substack{I \sqcup J = [n] \\ |I|\geqslant 2, |J|\geqslant 1}} (-1)^{d(\#I + \mathrm{sgn}(I,J))}$$  $$, \qquad (2.2)$$

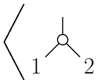
where $\mathrm{sgn}(I,J)$ denotes the sign of the shuffle I, J.

In the case $d = 1$ we will write $\mathbf{Lie} := \mathbf{Lie}_1$ and $\mathbf{Lie}_\infty := \mathbf{HoLie}_1$.

If we relax the restriction $n \geqslant 2$ on the generators above to $n \geqslant 1$, and the restrictions in the definition of the differential from $\#I \geqslant 2$ to $\#I \geqslant 1$ and $\#J \geqslant 1$ to $\#J \geqslant 0$, then [Mer23] one gets a dg free operad $\mathbf{HoLie}^+_{d+1}$ which is acyclic but is often very helpful in building important deformation complexes. The newly added generator  controls deformations of the differential in representations of $\mathbf{HoLie}^+_{d+1}$ in dg vector spaces.

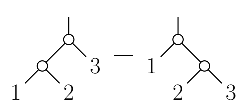
2.7 Operad of associative algebras

Let $E = \{E(n)\}$ be the $\mathbb{S}$-module defined by $E(n) = 0$ except that

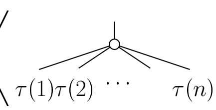
$$E(2) = \mathrm{id}_2 = \mathrm{span}\Big\langle$$  $$\Big\rangle,$$

where id_2 is the regular representation.

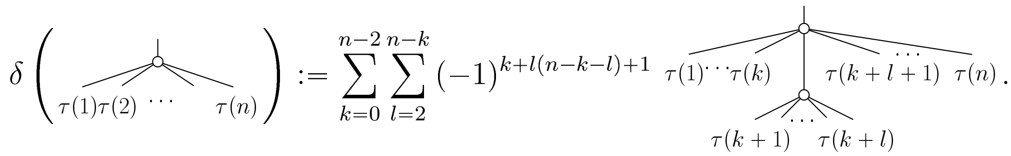
The operad of associative algebras is defined by the quotient $\mathbf{Ass} := \mathrm{Free}\langle E\rangle / I$ where I is generated by



Its minimal resolution is given by the dg quasi-free operad $\mathbf{Ass}_\infty := (\mathrm{Free}\langle E\rangle, \delta)$, where E is the $\mathbb{S}$-module generated by

$$E = \{E(n) = \mathbb{K}[\mathbb{S}_n][n-2] = \mathrm{span}\Big\langle$$  $$\Big\rangle_{\tau\in\mathbb{S}_n}\}_{n\geqslant 2}$$

and the differential is given on generators by

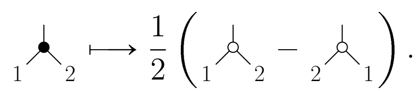


Representations of $\mathbf{Ass}_\infty$ in a dg vector space V are precisely the strongly homotopy associative algebras introduced by J. Stasheff in [Sta63].

The operads **Ass** and **Lie** are related via the following morphism of operads

$$\mathbf{Lie} \longrightarrow \mathbf{Ass}$$

defined on the generating corolla by



This map will play a role in Chapter 4.

2.8 Operad of graphs

The operad of graphs was first constructed in [Wil15a]. Denote by $G_{v,e}$ the set of connected directed graphs with no hairs, with v vertices labelled by $\{1, \cdots, v\}$, and e edges labelled by $\{1, \cdots, e\}$. We define an $\mathbb{S}_v$ module

$$\mathbf{Gra}_d(v) := \begin{cases} \bigoplus_{e \geqslant 0} \operatorname{span}\langle G_{v,e} \rangle \otimes_{\mathbb{S}_e \ltimes (\mathbb{S}_2)^e} \operatorname{sgn}_e[e(d-1)] & \text{if } d \text{ is even,} \\ \bigoplus_{e \geqslant 0} \operatorname{span}\langle G_{v,e} \rangle \otimes_{\mathbb{S}_e \ltimes (\mathbb{S}_2)^e} \operatorname{sgn}_2^{\otimes e}[e(d-1)] & \text{if } d \text{ is odd} \end{cases}$$

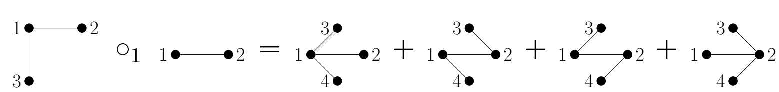
where d is an integer, and thus an operad $\mathbf{Gra}_d = \{\mathbf{Gra}_d(v)\}_{v \geqslant 1}$ called the operad of graphs.

If d is even, elements of $\mathbf{Gra}_d$ can be seen as undirected graphs with edges having degree $1-d$ together with an ordering of the edges up to an even permutation, while an odd permutation acts by multiplication by -1.

If d is odd, $\mathbf{Gra}_d$ consists of directed graphs where changing the direction of an edge yields a multiplication by -1.

The operadic composition $\Gamma_1 \circ_v \Gamma_2$ works by substituting the graph Γ_2 in the vertex v of Γ_1 and by summing over all possibilities of attaching the half-edges of v to the vertices of Γ_2.

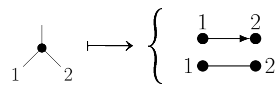
Example 2.8.1.



There is a morphism of operads

$$\mathbf{Lie}_d \longrightarrow \mathbf{Gra}_d \tag{2.3}$$

by mapping



if d is odd,
if d is even. (2.4)

This map will be used to define the Kontsevich graph complex in Chapter 3.

By forgetting the action on edges we obtain an operad $\mathbf{dGra}_d$ whose elements are graphs with edges having fixed directions. There is a morphism of operads

$$\mathbf{Gra}_d \longrightarrow \mathbf{dGra}_d$$

defined by (skew)-symmetrizing edges

$$\longrightarrow \;\longmapsto\; \longrightarrow + (-1)^d \longleftarrow .$$

2.9 Properad of Lie bialgebras

A Lie bialgebra structure on a vector space V is given by a Lie bracket $[-,-] : V \otimes V \to V$ and a Lie cobracket $\Delta : V \to V \otimes V$ satisfying a compatibility condition

$$\Delta([v,w]) = \sum v_1 \otimes [v_2, w] + [v, w_1] \otimes w_2 - (-1)^{|v||w|}([w, v_1] \otimes v_2 + w_1 \otimes [w_2, v])$$

where $v, w \in V$ and $\Delta(v) = \sum v_1 \otimes v_2$, $\Delta(w) = \sum w_1 \otimes w_2$.

Lie bialgebras have been introduced by V. Drinfeld in the context of quantum groups, see for example [Dri87].

They have been generalized [MW15] to Lie n-bialgebras (the case $n = 1$ has been studied previously in [Mer06, Mer09]). They are given on some graded vector space V by a Lie bracket $[-,-] : V \otimes V \to V$ of degree 0 and a Lie cobracket $\Delta : V[n] \to V[n] \otimes V[n]$ of degree $-n$ satisfying a modified compatibility condition

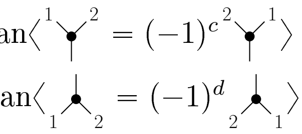
$$\Delta([v,w]) = \sum v_1 \otimes [v_2, w] + [v, w_1] \otimes w_2 - (-1)^{(|v|+n)(|w|+n)}([w, v_1] \otimes v_2 + w_1 \otimes [w_2, v]).$$

Write $\mathbf{Lieb}_n$ for the properad governing Lie n-bialgebras. Our main interest lies in the degree shifted properad of Lie bialgebras $\mathbf{Lieb}_{c,d}$ defined by

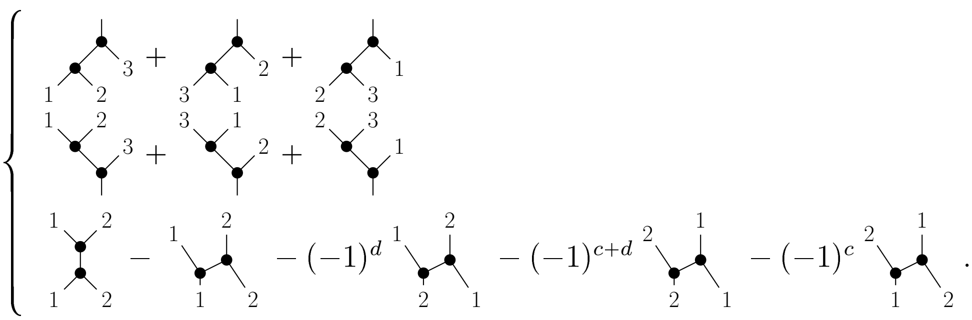
$$\mathbf{Lieb}_{c,d} := \mathbf{Lieb}_{c+d-2}\{1-c\}.$$

The properad $\mathbf{Lieb}_{c,d}$ can be described more explicitly as follows: consider the free properad E spanned by the $\mathbb{S}$ bimodule $E = \{E(m,n)\}$ given by

$$E(m,n) := \begin{cases} \mathrm{sgn}_2^{\otimes |c|} \otimes 1_2[c-1] = \mathrm{span}\langle \cdots \rangle & \text{if } m=2, n=1, \\ 1_2 \otimes \mathrm{sgn}_2^{\otimes |d|}[d-1] = \mathrm{span}\langle \cdots \rangle & \text{if } m=1, n=2, \\ 0 & \text{otherwise.} \end{cases}$$

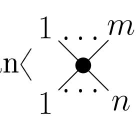


The properad of Lie bialgebras $\mathbf{Lieb}_{c,d}$ is given by the quotient of the free properad $\mathrm{Free}\langle E \rangle / I$ by the ideal I generated by the elements below:

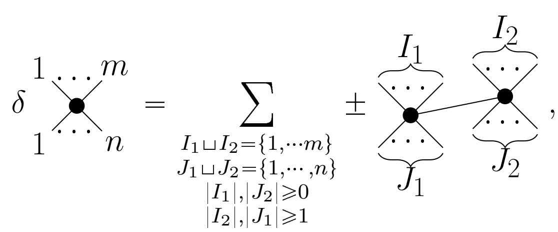


Its minimal resolution $\mathbf{HoLieb}_{c,d}$ is generated by the $\mathbb{S}$-bimodule $E = \{E(m,n)\}_{m,n \geqslant 1, m+n \geqslant 3}$ given by

$$E(m,n) = \mathrm{sgn}_m^{\otimes |c|} \otimes \mathrm{sgn}_n^{\otimes |d|}[-(1 + c(1-m) + d(1-n))] = \mathrm{span}\langle \cdots \rangle.$$



Its differential acts on the generators by



$$\sum_{\substack{I_1 \sqcup I_2 = \{1, \cdots m\} \\ J_1 \sqcup J_2 = \{1, \cdots, n\} \\ |I_1|, |J_2| \geqslant 0 \\ |I_2|, |J_1| \geqslant 1}} \pm \qquad (2.5)$$

where the explicit signs can be found in [Mer06].

2.10 Coloured operads

Let $C = \{c_1, \cdots, c_N\}$ be a collection of colours and let $\mathcal{G} \subset \mathcal{G}^{\circlearrowright}$ be the collection of directed (m,n)-graphs of genus 0 and with exactly one output hair. For any $\Gamma \in \mathcal{G}$ we assume that there is a partition of the set of half-edges

$$H(\Gamma) = H_{c_1}(\Gamma) \sqcup \cdots \sqcup H_{c_N}(\Gamma),$$

i.e. we decorate half-edges with elements of C. We require in addition that any two half-edges forming an edge in Γ are coloured by the same element in C. We denote the set of such graphs $\mathcal{G}_C$ and define a *C-coloured operad* $\mathcal{P}$ as a $\mathcal{G}_C$ algebra.

More intuitively, a C-coloured operad $\mathcal{P}$ can be seen as an operad whose inputs and output are decorated by elements of C and such that for any $p, q \in \mathcal{P}$ the composition $p \circ_i q$ is zero unless the ith input of p and the output of q have the same colour.

For example, let V and W be two vector spaces. Then the the 2-coloured endomorphism operad $\mathbf{End}_{V,W}$ is defined by the $\mathbb{S}$-bimodule

$$\mathbf{End}_{V,W}(n) = \bigoplus_{k=0}^{n} (\mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V^{\otimes k} \otimes W^{\otimes n-k}, V) \oplus \mathrm{Hom}_{\mathbf{Vect}_{\mathbb{K}}}(V^{\otimes k} \otimes W^{\otimes n-k}, W)).$$

Coloured operads play a crucial role in Chapter 6 where we define a 2-coloured version of the operad of graphs.

We refer the reader to [Mar04, Yau16] for additional details.

2.11 Polydifferential functors

In the paper [MW20] the authors constructed an endofunctor in the category of (augmented) PROPs

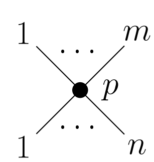
$$\mathcal{D} : \mathbf{Props} \longrightarrow \mathbf{Props}$$

with the main property that for any PROP $\mathcal{P}$ and any representation $\mathcal{P} \to \mathbf{End}_V$ in some dg vector space V, there is an induced representation of $\mathcal{D}(\mathcal{P}) \to \mathbf{End}_{\odot V}$ in the symmetric tensor algebra $\odot V$, given in terms of polydifferential (with respect to the standard product in $\odot V$) operators.

By passing to the PROP closure of a properad $\mathcal{P}$ one obtains an endofunctor

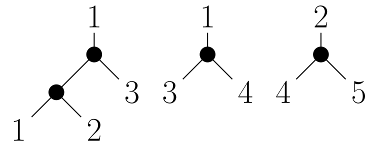
$$\mathcal{D} : \mathbf{Properads} \longrightarrow \mathbf{Properads}$$

and the properad $\mathcal{D}(\mathcal{P})$ can be described as follows: any element $p \in \mathcal{P}$ can be pictorially represented by a decorated graph

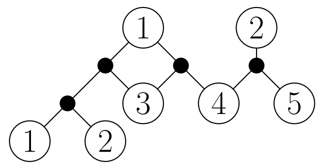


.

Generators of $\mathcal{D}(\mathcal{P})$ are given by such decorated graphs together with partitions of the input and output edges, for example

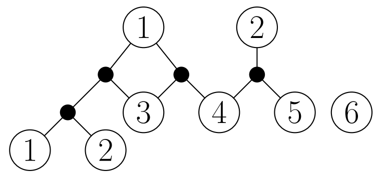


By joining all the input or output edges of each partition class we obtain the following graph



$\in \mathcal{D}(\mathcal{P})(2,5)$.

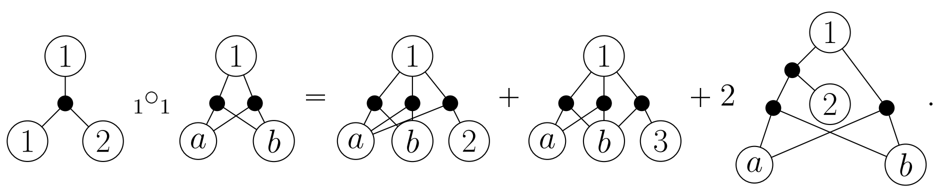
Note that we allow input or output vertices which are isolated, i.e.



$\in \mathcal{D}(\mathcal{P})(2,6)$.

The properadic composition $p\ {}_j{\circ}_i\, q$ of two elements $p, q \in \mathcal{D}(\mathcal{P})$ is done in multiple steps : we delete the output vertex j of q and the input vertex i of p. We then sum over all attachments of the output vertices labelled j to the input edges labelled i or to the output vertices in the connected component of the vertex i in p. Last we sum over all possible attachments of the remaining input edges labelled i to the input vertices of the connected component of the vertex j in q.

For example



Remark 2.11.1. The general properadic composition $p\ {}_{i_1,\cdots,i_k}{\circ}_{j_1,\cdots,j_k}\ q$ is defined as above except that we follow the above procedure for each pair (i_l, j_l) simultaneously.

The properad $\mathcal{D}(\mathcal{P})$ contains a suboperad $\mathcal{O}(\mathcal{P})$ given for $n \geqslant 1$ by

$$\mathcal{O}(\mathcal{P})(n) := \mathcal{D}(\mathcal{P})(1, n).$$

This operad was studied in an earlier paper [MW15] as a functor

$$\mathcal{O} : \mathbf{Props} \longrightarrow \mathbf{Operads}$$

having a similar property to the functor $\mathcal{D}$: for any representation of a PROP $\mathcal{P} \to \mathbf{End}_V$ of some PROP $\mathcal{P}$ in a dg vector space V, there is an induced representation $\mathcal{O}(\mathcal{P}) \to \mathbf{End}_{\odot V}$ of the polydifferential operad $\mathcal{O}(\mathcal{P})$ in the symmetric tensor algebra $\odot V$.

We will use the following fact in the proof of Corollary 4.3.8:

Fact 2.11.2. [MW20] The functor $\mathcal{D}$ is exact.

By applying the functor $\mathcal{O}$ to the (PROP closure of the) operad $\mathbf{Lie}_d$ we obtain the operad $\mathcal{O}(\mathbf{Lie}_d)$ of *polydifferential Lie algebras* whose study is the main focus of Chapter 4.

The second important example is the properad $\mathcal{D}(\mathbf{Lieb}_{c,d})$ of *polydifferential Lie bialgebras*, studied in Chapter 5.

Chapter 3

On graph complexes and deformation theory of properads

3.1 Deformation complexes

We follow [MV09] to introduce deformation complexes of morphisms of properads, which are of central importance for the remainder of this thesis.
We restrict ourselves now to the case $\mathcal{P} = \mathrm{Free}\langle E\rangle$ for some $\mathbb{S}$-bimodule E. In this case the space of derivations can be identified as a graded vector space with

$$\prod_{m,n\geqslant 1} \mathrm{Hom}(E(m,n), \mathcal{Q}(m,n))$$

where $\mathrm{Hom}(E(m,n), \mathcal{Q}(m,n))$ is the set of all morphisms of $\mathbb{S}_m \times \mathbb{S}_n$-bimodules from $E(m,n)$ to $\mathcal{Q}(m,n)$. The *deformation complex* $\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q})$ is defined as a graded vector space by

$$\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q}) = \prod_{m,n\geqslant 1} \mathrm{Hom}(E(m,n), \mathcal{Q}(m,n))[-1].$$

Theorem 3.1.1. *[MV09]*
The deformation complex $\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q})$ *is equipped with the structure of a filtered* $\mathbf{Lie}_\infty$ *algebra whose differential is given by (2.1). If* $\mathcal{P}$ *is quadratic then the deformation complex is equipped with a dg Lie algebra structure.*

Remark 3.1.2. If $\mathcal{P}$ is not free, we will pass to a minimal resolution $\pi : \tilde{\mathcal{P}} \to \mathcal{P}$ and define the deformation complex of f as the deformation complex of the composition $f \circ \pi$, i.e.

$$\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q}) := \mathbf{Def}(\tilde{\mathcal{P}} \xrightarrow{f\circ\pi} \mathcal{Q}).$$

The following result will be used in the proof of Corollary 4.3.8:

Fact 3.1.3. [MV09]
Let $f : \mathcal{P} \to \mathcal{Q}_1$, $s : \mathcal{Q}_1 \to \mathcal{Q}_2$ be morphism of dg operads such that s is a quasi-isomorphism. Then there is a quasi-isomorphism

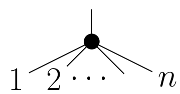
$$\mathbf{Def}(\mathcal{P} \xrightarrow{f} \mathcal{Q}_1) \cong \mathbf{Def}(\mathcal{P} \xrightarrow{s\circ f} \mathcal{Q}_2).$$

3.2 Deformation complex of the operad of Lie algebras

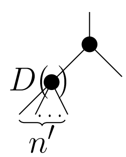
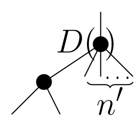
We consider the deformation complex of the identity morphism $id : \mathbf{Lie}_d \to \mathbf{Lie}_d$. As a graded vector space it is given by

$$\begin{aligned} \mathbf{Def}(\mathbf{Lie}_d \xrightarrow{id} \mathbf{Lie}_d) &= \mathbf{Def}(\mathbf{HoLie}_d \xrightarrow{id\circ\pi} \mathbf{Lie}_d) &= \prod_{n\geqslant 1} [E(n)^* \otimes \mathbf{Lie}_d(n)]^{\mathbb{S}_n}[-1] \\ & &= \prod_{n\geqslant 1} [\mathrm{sgn}_n^{\otimes |d|} \otimes \mathbf{Lie}_d(n)]^{\mathbb{S}_n}[d - dn]. \end{aligned}$$

The map $id \circ \pi : \mathbf{HoLie}_d \to \mathbf{Lie}_d \to \mathbf{Lie}_d$ induces a differential as follows: an element $F \in [\mathrm{sgn}_n^{\otimes |d|} \otimes \mathbf{Lie}_d(n)]^{\mathbb{S}_n}$ can be interpreted as the image of

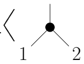


under a derivation $D : \mathbf{HoLie}_d \to \mathbf{Lie}_d$. The differential is given by

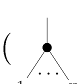
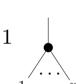
$$\delta D = \sum_{n=n'+1}$$  $$- (-1)^{|D|} \sum_{n=n'+2}$$  .

The cohomology of this deformation complex is quite simple:

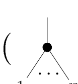
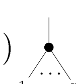
Fact 3.2.1. [Wil15a]

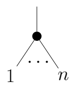
$$H(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{id} \mathbf{Lie}_d)) \cong \mathrm{span}\langle$$  $$\rangle.$$

By using Fact 3.1.3 we see that the cohomology of $\mathbf{Def}(\mathbf{HoLie}_d \xrightarrow{id} \mathbf{HoLie}_d)$ is one-dimensional. Now consider for $\lambda \in \mathbb{K}^*$ the map ϕ_λ, defined by

$$\phi_\lambda($$  $$) := \lambda^{n-1}$$  .

This is an automorphism of $\mathbf{HoLie}_d$. We compute that,

$$\frac{d\phi_\lambda}{d\lambda}\Big|_{\lambda=1}($$  $$) = (n-1)$$  ,

which implies that $\Lambda := \sum_{n\geqslant 2}(n-1)$  $\in \mathbf{Def}(\mathbf{HoLie}_d \to \mathbf{HoLie}_d)$ is a cohomology class and that

$$H^0(\mathbf{Def}(\mathbf{HoLie}_d \to \mathbf{HoLie}_d)) \cong \mathrm{span}\langle\Lambda\rangle.$$

3.3 Deformation complex of the properad of Lie bialgebras

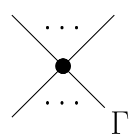
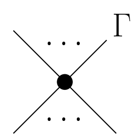
We are interested in the deformation complexes $\mathbf{Def}(\mathbf{Lieb}_{c,d} \overset{id}{\to} \mathbf{Lieb}_{c,d})$ which will be encountered in Chapter 5. By Fact 3.1.3 we know that there is a quasi-isomorphism

$$\mathbf{Def}(\mathbf{HoLieb}_{c,d} \overset{id}{\to} \mathbf{HoLieb}_{c,d}) \longrightarrow \mathbf{Def}(\mathbf{Lieb}_{c,d} \overset{id}{\to} \mathbf{Lieb}_{c,d}).$$

The complex $\mathbf{Def}(\mathbf{HoLieb}_{c,d} \overset{id}{\to} \mathbf{HoLieb}_{c,d})$ is given as a graded vector space by

$$\mathbf{Def}(\mathbf{HoLieb}_{c,d} \overset{id}{\to} \mathbf{HoLieb}_{c,d}) = \prod_{m,n\geqslant 1}(\mathbf{HoLieb}_{c,d}(m,n)\otimes \mathrm{sgn}_m^{\otimes|c|}\otimes \mathrm{sgn}_n^{\otimes|d|})^{\mathbb{S}_m\times\mathbb{S}_n}[c(1-m)+d(1-n)]$$

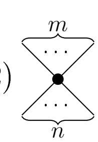
Thus an element of $\mathbf{Def}(\mathbf{HoLieb}_{c,d} \overset{id}{\to} \mathbf{HoLieb}_{c,d})$ can be understood as a formal sum of elements of $\mathbf{HoLieb}_{c,d}$ whose outputs (respectively inputs) are (skew)symmetrized according to the parity of c (respectively d). The differential δ acts on an element Γ by

$$\delta(\Gamma) = \delta_1(\Gamma) \pm \sum$$  $$\pm \sum$$ 

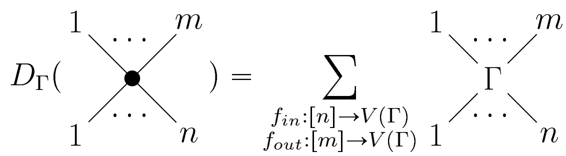
where δ_1 acts as in (2.5).
The authors of [MW18a] constructed an explicit morphism of dg Lie algebras

$$\mathbf{OGC}_{c+d+1} \longrightarrow \mathbf{Der}(\mathbf{HoLieb}_{c,d}) \simeq \mathbf{Def}(\mathbf{HoLieb}_{c,d} \overset{id}{\to} \mathbf{HoLieb}_{c,d})[1], \qquad (3.1)$$

which is a quasi-isomorphism up to one class represented by

$$\sum_{m,n}(m+n-2)$$  .

Here $\mathbf{OGC}_{c+d+1}$ is the oriented graph complex. It can be explicitly described as follows: given a graph $\Gamma \in \mathbf{OGC}_{c+d+1}$ the associated derivation D_Γ is defined by its image on the generators of $\mathbf{Lieb}_\infty$



where the last sum is taken over all attachments of labelled input and output hairs to the vertices of Γ under the condition that all vertices have valency at least 3 and at least one input or output hair attached.

3.4 The different flavours of graph complexes

3.4.1 Undirected graph complexes

Recall that there is a morphism of operads $\mathbf{Lie}_d \to \mathbf{Gra}_d$ given by (2.4). We define the *full graph complex* as the deformation complex

$$\mathbf{fGC}_d := \mathbf{Def}(\mathbf{Lie}_d \longrightarrow \mathbf{Gra}_d).$$

Elements of $\mathbf{fGC}_d$ can be interpreted as connected graphs together with an ordering of the edges, up to an even permutation when d is even, while for d odd, we assume an ordering of the vertices, up to even permutations, together with a choice on the orientation of each edge, up to a flip yielding a multiplication by -1.

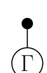
The cohomological degree of a graph Γ is given by

$$|\Gamma| = d\#V(\Gamma) + (1-d)\#E(\Gamma) - d$$

and its loop number is given by

$$g(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1.$$

Its differential can be represented by

$$\delta(\Gamma) = -2\sum_{v\in V(\Gamma)}$$  $$+ \sum_{v\in V(\Gamma)} \Gamma \circ_v$$,

where the first term is given by summing over all attachments of a univalent vertex to a vertex of Γ. There is a pre-Lie algebra structure $\circ$ on $\mathbf{fGC}_d$ defined for graphs Γ_1 and Γ_2 by

$$\Gamma_1 \circ \Gamma_2 = \sum_{v\in V(\Gamma_1)} \Gamma_1 \circ_v \Gamma_2 \tag{3.2}$$

where $\Gamma_1 \circ_v \Gamma_2$ is defined by inserting Γ_2 into a vertex of Γ_1. In particular there is a dg Lie algebra structure on $\mathbf{fGC}_d$ given by

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \circ \Gamma_2 - (-1)^{|\Gamma_1||\Gamma_2|}\Gamma_2 \circ \Gamma_1.$$

We consider the subcomplex $\mathbf{GC}_d \subset \mathbf{fGC}_d$ spanned by connected graphs with vertices having valency at least 3. Let $\mathbf{GC}_d^2 := (\bigoplus_{\substack{p \geqslant 1 \\ p \equiv 2d+1 \mod 4}} \mathbb{K}[d-p], 0)$. It has been shown [Wil15a] that the natural projection $\mathbf{fGC}_d \to \mathbf{GC}_d^{\geqslant 2} := \mathbf{GC}_d \oplus \mathbf{GC}_d^2$ is a quasi-isomorphism, i.e. $H(\mathbf{GC}_d^{\geqslant 2}) = H(\mathbf{fGC}_d)$.

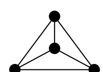
One of the main results known about graph cohomology was obtained by Thomas Willwacher in [Wil15a] and states the following:

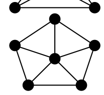
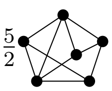
Theorem 3.4.1. *For $d = 2$ one has*

$$H^0(\mathbf{GC}_2) \cong \mathfrak{grt}_1,$$

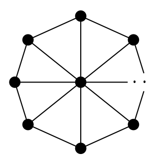
where $\mathfrak{grt}_1$ denotes the Grothendieck-Teichmüller Lie algebra (see 3.5 for the definition).

It has been shown [RW14] that $H(\mathbf{GC}_2)$ contains the so-called *wheel classes* $\mathfrak{w}_{2n+1}$ given by

$\mathfrak{w}_3 =$ 

$\mathfrak{w}_5 =$  $+ \frac{5}{2}$ 

$\cdots = \cdots$

$\mathfrak{w}_{2n+1} =$  $+ \sum_{p=4}^{2n-1} \lambda_p \Gamma^p,$

where Γ^p is a linear combination of graphs with exactly one vertex of valency p and all other vertices of valency strictly less than p and λ_p is some coefficient.

3.4.2 Directed graph complexes

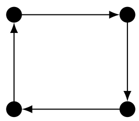
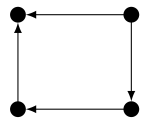
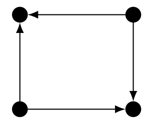
There is a directed version $\mathbf{dGC}_{d+1}$ of $\mathbf{GC}_{d+1}^{\geqslant 2}$ where directions on edges are fixed. As there is a quasi-isomorphism of dg Lie algebras [Wil15a] which sends every edge to a linear combination

$$\longrightarrow \ \mapsto \ \longrightarrow + (-1)^{d+1} \longleftarrow$$

we do not gain anything on the cohomological level. There are however multiple subcomplexes which are of interest for this thesis, the most notable one being the *oriented* graph complex $\mathbf{fOGC}_{d+1}$ spanned by graphs which are connected and oriented, and $\mathbf{OGC}_{d+1} \subset \mathbf{fOGC}_{d+1}$ where we require that each vertex has to be of valency at least 2 and passing vertices are forbidden, i.e. vertices of the form

$$\longrightarrow\bullet\longrightarrow .$$

Example 3.4.2.

 $\notin \mathbf{fOGC}_{d+1}$,  $\notin \mathbf{OGC}_{d+1}$,  $\in \mathbf{OGC}_{d+1}$.

The cohomology of $\mathbf{OGC}_{d+1}$ has been computed first in [Wil15b] where it was shown that

$$H(\mathbf{GC}_d^{\geqslant 2}) \cong H(\mathbf{OGC}_{d+1})$$

as graded Lie algebras. A second proof of this isomorphism has been found in [Ž20] by constructing an explicit morphism

$$(\mathbf{GC}_d)^* \longrightarrow (\overline{\mathbf{OGC}}_{d+1})^*$$

between the dual graph complexes. Note that this map does not respect the Lie co-algebra structures and thus the isomorphism on cohomology is only of graded vector spaces. Here $\overline{\mathbf{OGC}}_{d+1}$ denotes a *reduced* version introduced in op. cit. A third proof of this isomorphism of graded vector spaces was given in [Mer25].

3.4.3 Reduced graph complexes

Let $\overline{\mathbf{GC}}_{d+1}$ be a complex spanned by graphs with all vertices of valency at least 3 and two types of edges, solid edges $\longrightarrow$ with fixed direction and of degree $-d$ and dotted edges $\cdots\!\!\blacktriangleright = (-1)^d \blacktriangleleft\!\!\cdots$ of degree $1-d$.

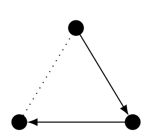
The cohomological degree of a graph Γ is given by the formula

$$|\Gamma| = (d+1)\#V(\Gamma) - d\#E_{sol}(\Gamma) - (1-d)\#E_{dot}(\Gamma) - (d+1).$$

The differential $\delta' + \delta$ consists of a part δ' which changes types of edges

while δ acts by vertex splitting

We consider an oriented subcomplex $\overline{\mathbf{OGC}}_{d+1}$ which is spanned by graphs which have no closed cycled of directed edges, i.e. dotted edges are considered impassable. For example the "cycle"



is allowed.

There is an explicit quasi-isomorphism [Ž20]

$$\overline{\mathbf{OGC}}_{d+1} \longrightarrow \mathbf{OGC}_{d+1} \tag{3.3}$$

sending each dotted edge to the linear combination

$$\bullet \cdots\!\!\rightarrow \bullet \longmapsto \frac{1}{2}(\bullet \!\longrightarrow\! \bullet \!\longleftarrow\! \bullet \; - \; \bullet \!\longleftarrow\! \bullet \!\longrightarrow\! \bullet).$$

It was shown in [Mer25] that the same map defines a quasi-isomorphism

$$\overline{\mathbf{GC}}_{d+1} \longrightarrow \mathbf{dGC}_{d+1}.$$

In Chapter 7 we will construct a Lie bracket on $\overline{\mathbf{GC}}_{d+1}$ (and $\overline{\mathbf{OGC}}_{d+1}$) such that (3.3) is a quasi-isomorphism of dg Lie algebras.

We refer the reader to [Mer25] for a more detailed treatment of these complexes.

3.5 The Grothendieck-Teichmüller group and its Lie algebra

The Grothendieck-Teichmüller group $\mathbf{GRT}_1$ and its Lie algebra $\mathfrak{grt}_1$ were introduced by V. Drinfeld in [Dri90] in the study of quasi-Hopf algebras and number theory. The group $\mathbf{GRT}_1$ found applications in deformation quantization of Poisson structures [Kon03] and of Lie bialgebras [Dri92, EK96, MW18b], in the Lie theory [AT12] and many other areas of mathematics.

The Lie algebra $\mathfrak{grt}_1$ found applications in the theory of moduli spaces of curves [CGP21] and is of special importance for this thesis. The Deligne–Drinfeld–Ihara conjecture states that $\mathfrak{grt}_1$ is the completion of a free Lie algebra generated by formal variables of degree $2n+1$ for $n \geqslant 1$. In view of Theorem 3.4.1, the wheel classes $\mathfrak{w}_{2n+1}$ studied in [RW14] are the conjectural generators. The inclusion of such a free Lie algebra has been proved by F. Brown in [Bro12].

We recall for completeness the definition of the group $\mathbf{GRT}_1$ following [Wil15a].

Consider $\mathbb{F}_2 = \mathbb{K}\langle\langle X, Y\rangle\rangle$ the completed free associative algebra generated by X, Y. Define a coproduct Δ by setting $\Delta X = X \otimes 1 + 1 \otimes X$ and $\Delta Y = Y \otimes 1 + 1 \otimes Y$. An element $\alpha \in \mathbb{F}_2$ is called *group-like* if $\Delta\alpha = \alpha \otimes \alpha$.

We also need the Drinfeld-Kohno Lie algebra t_n defined by generators $t_{ij} = t_{ji}$ for $1 \leqslant i, j \leqslant n$,$i \neq j$, which satisfy $[t_{ij}, t_{ik} + t_{kj}] = 0$ and $[t_{ij}, t_{kl}] = 0$ for any distinct i, j, k, l.

The Grothendieck-Teichmüller group $\mathbf{GRT}_1$ is defined to be set of group-like elements $\alpha \in \mathbb{F}_2$ which satisfy the following relations

$$\begin{aligned} \alpha(t_{12}, t_{23} + t_{24})\alpha(t_{13} + t_{23}, t_{34}) &= \alpha(t_{23}, t_{34})\alpha(t_{13} + t_{13}, t_{24} + t_{34})\alpha(t_{12}, t_{23}) \\ 1 &= \alpha(t_{13}, t_{12})\alpha(t_{13}, t_{23})^{-1}\alpha(t_{12}, t_{23}) \\ \alpha(x, y) &= \alpha(y, x)^{-1}. \end{aligned}$$

The group structure is given by

$$\alpha_1(X,Y)\cdot\alpha_2(Y,X)=\alpha_1(X,Y)\alpha_2(X,\alpha_1(X,Y)^{-1}Y\alpha_1(X,Y)).$$

The Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$ is given by the elements $\alpha\in\hat{\mathbb{F}}_2(X,Y)$, the completed free Lie algebra generated by X,Y, satisfying

$$\begin{aligned}\alpha(t_{12},t_{23}+t_{24})+\alpha(t_{13}+t_{23},t_{34}) &= \alpha(t_{23},t_{24})+\alpha(t_{12}+t_{13},t_{24}+t_{34})+\alpha(t_{12},t_{23})\\ \alpha(X,Y)+\alpha(Y,-X-Y)+\alpha(-X-Y,X) &= 0\\ \alpha(X,Y) &= -\alpha(Y,X).\end{aligned}$$

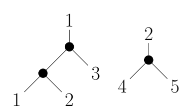
Chapter 4

Polydifferential Lie algebras and graph cohomology

This section is taken from the paper [Wol23]:

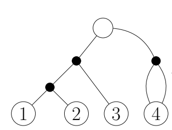
4.1 Operad of polydifferential Lie algebras

When $\mathcal{P}$ is an operad we consider its PROP closure, which we denote $\bar{\mathcal{P}}$. The elements of the PROP closure are given by disjoint unions of elements of $\mathcal{P}$ with inputs and outputs labelled differently. For example



$\in \mathbf{Lie}_d(2,5)$.

We are mainly interested in the operad $\mathcal{O}(\mathbf{Lie}_d)$ obtained by applying the functor $\mathcal{O}$ to the PROP closure of the operad of degree d shifted Lie algebras. Elements of $\mathcal{O}(\mathbf{Lie}_d)$ are obtained joining the outputs of an element of the PROP closure of $\mathbf{Lie}_d$ to a new white vertex and also by partitioning all of the input legs into groups and joining each group them to a new labelled white vertex. For example

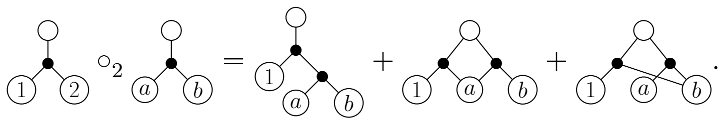


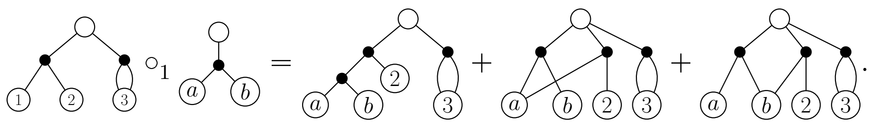
$. \in \mathcal{O}(\mathbf{Lie}_d)(4)$.

The black vertices are called *internal vertices* and edges between internal vertices are called *internal edges*. By erasing all white vertices we obtain a collection of disjoint trees which will be called *internal irreducible components* (i.i.c.).

Note that for d even we implicitly assume an ordering of the internal edges as well as an ordering of the in-edges of the white out-vertex (up to a sign). For d odd, we assume implicitly an ordering of the internal vertices as well as an ordering (up to a sign) of the out-edges attached to each white in-vertex.

The compositions $\Gamma_1 \circ_i \Gamma_2$ in $\mathcal{O}(\mathbf{Lie}_d)$ work as follows: first we erase the white out-vertex of Γ_2 and the ith white in-vertex of Γ_1. This step creates many "hanging" out-edges in Γ_2 and in-edges in Γ_1. Second we sum over all possible attachments of the hanging out-edges of Γ_2 to the hanging in-edges of Γ_1 and the white out-vertex of Γ_1. Finally we sum over all attachments of remaining in-edges of Γ_1 to the white in-vertices of Γ_2.

Example 4.1.1. 1) 

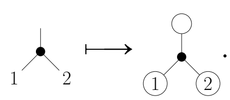
2) 

We define $\mathcal{O}_c(\mathbf{Lie}_d)$ to be the suboperad spanned by connected graphs, i.e. we assume that the graphs remain connected when we erase the white output vertex.

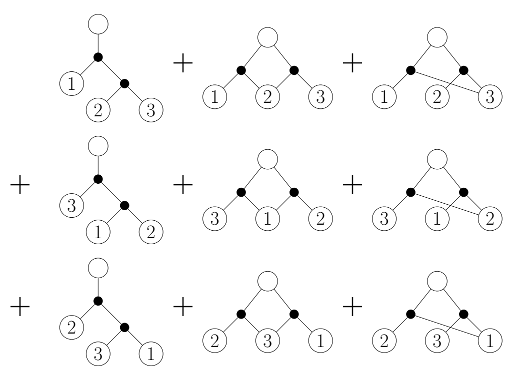
Lemma 4.1.2. *There is a morphism of operads*

$$i : \mathbf{Lie}_d \longrightarrow \mathcal{O}_c(\mathbf{Lie}_d) \tag{4.1}$$

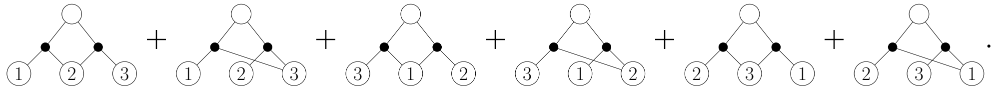
given by



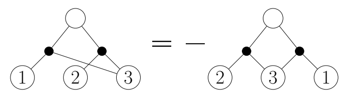
Proof. It suffices to check that the Jacobi identity is mapped to 0. Indeed we need that



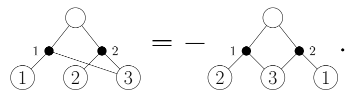
vanishes. This expression reduces to



If d is odd we observe that



and thus for any permutation of $\{1, 2, 3\}$. For d even we see that

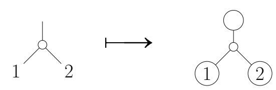


In any case we see that the Jacobi identity is mapped to zero.

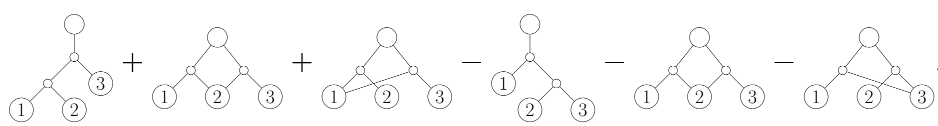
□

Remark 4.1.3. The map

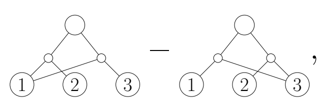
$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Ass})$$



is not a morphism of operads as the associativity condition is not mapped to zero. Indeed the associativity condition would require that the following relation vanishes:



This relation simplifies to



and thus it does not vanish.

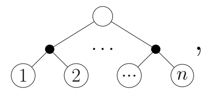
4.2 Kontsevich graph complex from the operad of Lie algebras

There is a relation between the operad of graphs and the operad $\mathcal{O}(\mathbf{Lie}_d)$.

Lemma 4.2.1. *Let I be the ideal in the operad $\mathcal{O}_c(\mathbf{Lie}_d)$ linearly spanned by graphs having at least one internal edge. Then the quotient operad can be identified with the operad $\mathbf{Gra}_d$, i.e.*

$$\mathcal{O}_c(\mathbf{Lie}_d)/I \cong \mathbf{Gra}_d.$$

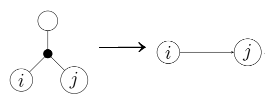
Proof. The operad $\mathcal{O}(\mathbf{Lie}_d)/I$ is generated by elements Γ of the form



i.e. graphs where each irreducible component has a unique black vertex.
Consider the linear map

$$\begin{array}{rccc} \alpha: & \mathcal{O}_c(\mathbf{Lie}_d)/I & \longrightarrow & \mathbf{Gra}_d \\ & \Gamma & \longmapsto & \alpha(\Gamma), \end{array}$$

where $\alpha(\Gamma)$ is a graph in $\mathbf{Gra}_d$ which by definition has labelled white vertices identical to the white vertices of Γ, while the edges between vertices in $\alpha(\Gamma)$ correspond to irreducible components in Γ. If d is even the ordering of edges attached to the white output vertex of Γ induces an ordering of the edges in $\alpha(\Gamma)$. For d odd, an orientation of the edges in $\alpha(\Gamma)$ is given by the ordering of the output edges attached to the white input vertices,



Conversely given a generator $g \in \mathbf{Gra}_d$ we can construct a generator $\Gamma \in \mathcal{O}_c(\mathbf{Lie}_d)$ with white input vertices corresponding the vertices of g and each edge corresponds to one irreducible component between the two output vertices. It follows that α defines a bijection on generators and it remains to show that it respects operadic compositions. By its very definition as qoutient operad, for generators $\Gamma_1, \Gamma_2 \in \mathcal{O}_c(\mathbf{Lie}_d)$, the operadic composition $\Gamma_1 \circ_i \Gamma_2$ is given by erasing input vertex i of Γ_1 and summing over all attachments of the hanging edges to the input vertices of Γ_2. This corresponds by definition of $\mathbf{Gra}_d$ to the operadic composition $\alpha(\Gamma_1) \circ_i \alpha(\Gamma_2)$ and the result follows. □

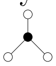
This map α induces a map $\mathcal{O}_c(\mathbf{Lie}_d) \to \mathcal{O}_c(\mathbf{Lie}_d)/I \to \mathbf{Gra}_d$ and thus a map of the deformation complexes

$$F : \mathbf{Def}(\mathbf{Lie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)) \to \mathbf{fGC}_d.$$

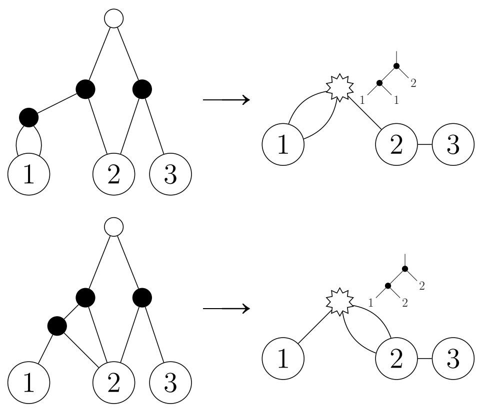
We can now state the main theorem:

Theorem 4.2.2. *The map F is a quasi-isomorphism.*

Proof. Recall that the natural projection $\mathbf{fGC}_d \to \mathbf{GC}_d^{\geqslant 2} = \mathbf{GC}_d^2 \oplus \mathbf{GC}_d$ is a quasi-isomorphism and thus it is enough to show that the map $C := \mathbf{Def}(\mathbf{Lie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)) \to \mathbf{GC}_d^{\geqslant 2}$ is a quasi-isomorphism.
We notice that the generators of $\mathcal{O}_c(\mathbf{Lie}_d)$ can be suitably represented as connected graphs having two types of vertices, the white vertices corresponding to the input vertices and the star vertices corresponding to the irreducible connected components. This graph is obtained by first erasing the output vertex and all attached edges. Then we contract the internal

vertices of each irreducible component to a single vertex, denoted , which is decorated by the internal irreducible component it originates. Last, the star vertices corresponding to  is replaced by an edge between the associated white vertices.

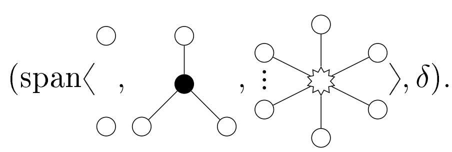
Example 4.2.3.

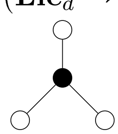
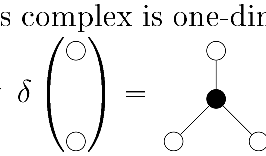


An edge between a white vertex and a star vertex is called *star edge*. Note that by construction, there are no edges between star vertices.

The complex C splits as a direct sum $C = (C^{\leqslant 1}, \delta) \oplus (C^{\geqslant 2}, \delta)$ where $C^{\leqslant 1}$ is generated by graphs with all white vertices having valency less or equal to 1 and $C^{\geqslant 2}$ is generated by graphs with at least one white vertex of valency at least 2.

The complex $C^{\leqslant 1}$ is acyclic. Indeed this complex is equal to



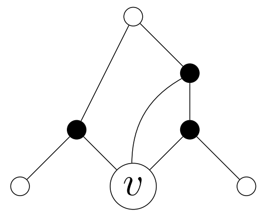
As the differential δ cannot create univalent white vertices we see that the complex where we omit the first graph is isomorphic to $\mathbf{Def}(\mathbf{Lie}_d \to \mathbf{Lie}_d)$. As this complex is one-dimensional with unique cohomology class given by , the equality  and the above isomorphism imply that $C^{\leqslant 1}$ is acyclic.

Hence it is enough to prove that the restriction $f : (C^{\geqslant 2}, \delta) \to (\mathbf{GC}_d^2, 0) \oplus (\mathbf{GC}_d, \delta)$ is a quasi-isomorphism. On the right-hand side we consider a filtration on the number of vertices. Then on the first page the induced differential vanishes and the second page is equal to $(\mathbf{GC}_d^2, 0) \oplus (\mathbf{GC}_d, \delta)$.

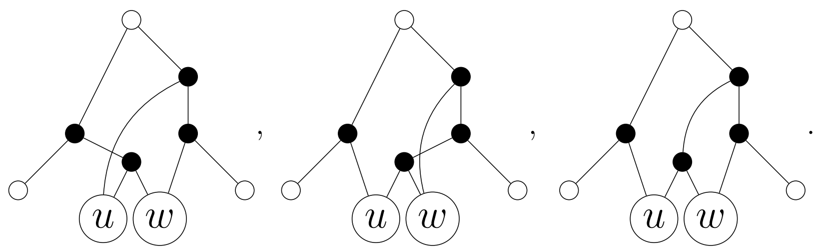
On the left-hand side we consider a filtration defined by

$$F_{-p} = \text{span of graphs with } \#\{\text{white vertices of valency at least 3}\} + 3\#\{\text{internal edges}\} \geqslant p.$$

There is exactly one situation where the number of vertices of valency at least 3 is decreasing. Let v be a vertex of valency exactly 3, e.g. consider



Then the differential can split v in to two bivalent vertices u and w. In the above example the differential creates (up to changing u and w) three such terms



Hence the number of white vertices of valency at least 3 can decrease by 1 but note that an internal edge needs to be created. In this case the number F_{-p} increases by 1 and hence is respected by the differential.

As the differential does not create new univalent vertices we see that the differential δ_0 on the first page can only create new white bivalent vertices. As the number of star vertices is preserved, we have a direct sum decomposition $\mathrm{gr} C^{\geqslant 2} = \bigoplus_{N \geqslant 0} C_N$ where C_N is the subcomplex spanned by graphs with N star vertices. It has been proven in [Wil15a] that $H(C_0, \delta_0) = \mathbf{GC}_d^2 \oplus \mathbf{GC}_d$. It remains to show that $(\bigoplus_{N \geqslant 1} C_N, \delta_0)$ is acyclic.

The differential δ_0 can be represented by

$$\delta_0(\Gamma) = -2 \sum_{v \in V_{\text{in}}(\Gamma)} \overset{\circ}{\underset{\textcircled{\scriptsize\Gamma}}{|}} + \sum_{v \in V_{\text{in}}(\Gamma)} \Gamma \circ_v \circ\!\!-\!\!\circ$$

where $V_{\text{in}}(\Gamma)$ denotes the set of white vertices of Γ. The first part is given by summing over all attachments of a new univalent white vertex to a white vertex of Γ. The term $\Gamma\, o'_v \circ\!\!-\!\!\circ$ is given by splitting the white vertex v and summing over all attachments of the half-edges pf v to the two newly created vertices. As the graphs Γ are connected this differential is equal to

$$\delta_0(\Gamma) = \sum_{\substack{v \in V_{\text{in}}(\Gamma) \\ \text{valency of } v \geqslant 2}} \Gamma \circ_v \circ\!\!-\!\!\circ,$$

where we omit the summands creating a univalent vertex.

We show that the complex spanned by graphs having at least one star vertex and at least one white vertex with valency at least 2 is acyclic. It is enough to show that the complex

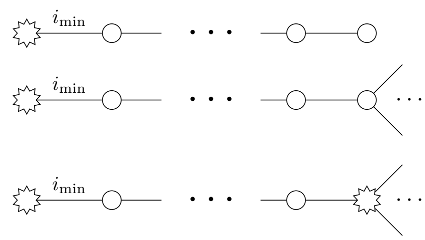
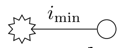
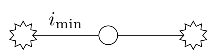


Figure 4.1: The different types of antennas.

spanned by such graphs with a labelling of star edges with integers is acyclic. In particular the set of star edges is totally ordered. Consider the star edge with minimum label which is attached to a white vertex of valency at least 2. We call *antenna* the sequence of white vertices starting from 

and ending with a white vertex of valency different from 2 or a star vertex. The two-valent white vertices of this sequence are called *antenna vertices*. Consider a filtration by the number of non-antenna vertices. To finish the argument it is sufficient to observe that the graphs with no antenna vertices and graphs with an antenna given by



are not cocycles and we conclude that $(\bigoplus_{N\geqslant 1} C_N, \delta_0)$ is acyclic.
On the second page of the spectral sequence, there is a map

$$(H^*(C_0,\delta_0),\delta_1) \longrightarrow (\mathbf{GC}_d^2, 0) \oplus (\mathbf{GC}_d, \delta),$$

where $H^*(C_0,\delta_0) \cong \mathbf{GC}_d^2 \oplus \mathbf{GC}_d$ as vector spaces and δ_1 is the differential which increases the value of the parameter of the filtration by 1. Hence δ_1 cannot create new star vertices (as this gives an increase by at least 2) and thus δ_1 can only split white vertices as the usual differential in $\mathbf{GC}_d$. Finally, this shows that we have an isomorphism of complexes on the second page and the result follows.

□

As a corollary we obtain

Corollary 4.2.4.

$$H^\bullet(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))) \cong \mathbf{GC}_d^2 \oplus H^\bullet(\mathbf{GC}_d)$$

In particular

$$H^0(\mathbf{Def}(\mathbf{Lie}_2 \to \mathcal{O}(\mathbf{Lie}_2))) \cong \mathfrak{grt}_1.$$

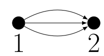
Another application is the following: we know that $\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))$ controls the infinitesimal homotopy non-trivial deformations of i, which are given by $H^1(\mathbf{Def}(\mathbf{Lie}_d \xrightarrow{i} \mathcal{O}_c(\mathbf{Lie}_d))) \cong H^1(\mathbf{GC}_d^2) \oplus H^1(\mathbf{GC}_d)$, while the obstructions to extending an infinitesimal deformation Δ of i to a genuine morphism of (completed) operads

$$i^{\Delta} : \mathbf{HoLie}_d \to \mathcal{O}_c(\mathbf{Lie}_d)$$

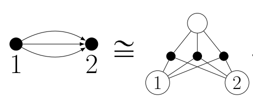
lie in $H^2(\mathbf{GC}_d^2) \oplus H^2(\mathbf{GC}_d)$.

Remark 4.2.5. We assume from now on that our operads are completed in the sense that we allow formal infinite sums.

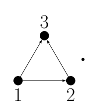
We now consider the case $d = 1$ corresponding to the operad of usual Lie algebras. It is noticed in [Wil15b] that $H^1(\mathbf{GC}_1^2) = 0$ while $H^1(\mathbf{GC}_1)$ is one-dimensional and is generated by the theta graph

$\Delta :=$  .

Therefore the standard morphism $i : \mathbf{Lie} \to \mathcal{O}_c(\mathbf{Lie})$ admits precisely one homotopy non-trivial infinitesimal deformation corresponding to the above mentioned theta graph, which in our approach is incarnated as the following element in $\mathcal{O}_c(\mathbf{Lie})$:



Moreover the second cohomology group is generated by



This obstruction cohomology class cannot be hit when we try to extend the infinitesimal deformation Δ to a genuine deformation (cf. [Wil15b] Section 5). Hence we conclude that the standard morphism $i : \mathbf{Lie} \to \mathcal{O}_c(\mathbf{Lie})$ admits precisely one (up to homotopy equivalence) non-trivial deformation. Remarkably, this unique non-trivial deformation is directly related to the universal enveloping construction and the PBW theorem as we show in Section 4.4.

4.3 Gutt quantizations

Let V be a Lie algebra with Lie bracket $[-,-]$. This datum is equivalent to a morphism of operads

$$\rho : \mathbf{Lie} \longrightarrow \mathbf{End}_V.$$

Using the functor $\mathcal{O}$, we see that there is an associated morphism of operads

$$\hat{\rho} : \mathcal{O}(\mathbf{Lie}) \longrightarrow \mathbf{End}_{\odot V}$$

given by polydifferential operators.

The space $\odot V$ can be canonically given a structure of an associative algebra, with product denoted by $*$, as follows: denote by $\mathcal{U}V$ the universal enveloping algebra of V, i.e.

$$\mathcal{U}V := \mathcal{T}V/\langle v_1 \otimes v_2 - v_2 \otimes v_1 - [v_1, v_2] | v_1, v_2 \in V\rangle,$$

where $\mathcal{T}V := \bigoplus_{n\geqslant 0} \bigotimes^n V$ is the tensor algebra. Then $\mathcal{U}V$ is an associative algebra with product $\odot$ given by $\otimes$ mod I.

The spaces $\odot V$ and $\mathcal{U}V$ are related as vector spaces by the Poincaré-Birkhoff-Witt (PBW) theorem:

Theorem 4.3.1. *As a vector space $\odot V \cong \mathcal{U}V$.*

More precisely consider the linear map

$$\begin{array}{cccc} \sigma : & \odot V & \longrightarrow & \mathcal{U}V \\ & v_1 \odot \cdots \odot v_n & \longmapsto & \sum_{\tau \in \mathbb{S}_n} \frac{1}{n!} v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)}. \end{array}$$

By PBW this is an isomorphism of vector spaces and this allows us to define a product of $p, q \in \odot V$ by

$$p * q := \sigma^{-1}(\sigma(p) \circ \sigma(q)).$$

It has been shown [Kat98, Gut11] that

$$p * q = p \odot q + \sum_{m,n \geqslant 1} B_{m,n}(p, q), \tag{4.2}$$

where $B_{m,n}$ are bi-differential operators of order m on p and order n on q. The product $*$ is called $\mathcal{U}$-star product.

If p and q are polynomials, then the right-hand side is a finite sum. If we try to extend (4.2) to general smooth functions on V^*, then the sum can be infinite and thus we run into convergence issues. This can be solved by introducing a parameter $\hbar$ and by working with

$$\mathcal{U}_\hbar V := \mathcal{T}V[[\hbar]]\langle v_1 \otimes v_2 - v_2 \otimes v_1 - \hbar[v_1, v_2] | v_1, v_2 \in V\rangle.$$

Example 4.3.2. 1) If $v_1, v_2 \in V$, then

$$v_1 * v_2 = v_1 \odot v_2 + \frac{\hbar}{2}[v_1, v_2]. \tag{4.3}$$

2) Let $e^{v_1}, e^{v_2} \in C^\infty(V^*)$ for $v_1, v_2 \in V$. Then

$$e^{v_1} * e^{v_2} = e^{\mathrm{CBH}(v_1,v_2)},$$

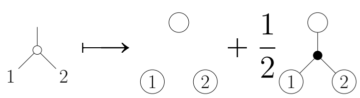
where $\mathrm{CBH}(v_1, v_2)$ denotes the Campbell-Baker-Hausdorff series

$$\mathrm{CBH}(v_1, v_2) = v_1 + v_2 + \frac{\hbar}{2}[v_1, v_2] + \frac{\hbar^2}{12}([v_1, [v_1, v_2]] + [[v_1, v_2], v_2]) + \cdots .$$

We conclude that the $\mathcal{U}$-star product gives us a map $\mathcal{U}_V : \mathbf{Ass} \to \mathbf{End}_{\odot V}$. As this map does not depend on the choice of V and is given by polydifferential operators, we infer that it factors through $\mathcal{O}(\mathbf{Lie})$, i.e.

$$\begin{array}{ccc} \mathbf{Ass} & \xrightarrow{\ \mathcal{U}\ } & \mathcal{O}(\mathbf{Lie}) \\ & {\scriptstyle \mathcal{U}_V} \searrow & \downarrow {\scriptstyle \hat\rho} \\ & & \mathbf{End}_{\odot V} \end{array},$$

for some morphism $\mathcal{U}$ of operads satisfying the non-triviality condition



+ terms with at least two internal vertices. (4.4)

This motivates the following definition:

Definition 4.3.3. Any morphism of operads $\mathcal{U} : \mathbf{Ass} \to \mathcal{O}(\mathbf{Lie})$ satisfying (4.4) is called an *S. Gutt quantization.*

That the standard universal enveloping construction gives us an example of such a quantization was first noticed by Simone Gutt in [Gut11]. We classify below all such quantizations up to homotopy equivalence, and prove that they are all gauge equivalent to the universal enveloping one.

Theorem 4.3.4. *Let $\mathcal{U}$ be an S. Gutt quantization. Then the cohomology of* $\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie}))$ *is one-dimensional.*

Proof. Recall that by its very definition the deformation complex $\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie}))$ is given by

$$\prod_{n\geqslant 1} \mathbf{Ass}(n)^* \otimes (\mathcal{O}(\mathbf{Lie})(n))^{\mathbb{S}_n}[-1] = \prod_{n\geqslant 1} (\mathcal{O}(\mathbf{Lie})(n))^{\mathbb{S}_n}[1-n]$$

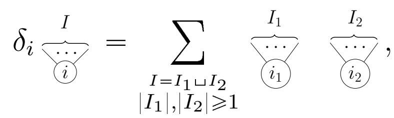
equipped with the differential d associated to $\mathcal{U}$.

Consider a filtration by the number of internal vertices. As the differential d acts (up to signs) by the composition of an element $\Gamma \in \mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}\mathbf{Lie})$ with the right-hand side of

(4.4) we see that d respects the filtration and we consider the associated spectral sequence $(\mathcal{E}_n, d_n)_{n\geqslant 0}$. The differential on the first page is only induced by the first term of $\mathcal{U}$. We observe that if Γ is connected, then the surviving terms are also connected and the differential is given by

$$d_0\Gamma = \sum_i \delta_i(\Gamma),$$

where i ranges over the number of white vertices and δ_i acts by



This follows from the fact that the two terms with a disconnected white vertex only differ swapping i_1 and i_2 and thus by the skew-symmetrization of labels of white vertices these terms are opposites. In particular $\delta_i(\Gamma) = 0$ and $d_0(\Gamma) = 0$ if all white vertices are univalent.

In [Mer11] it has been shown that the cohomology on the first page $H(\mathcal{E}_0, d_0)$ is linearly spanned by graphs with all white vertices univalent and skew-symmetrized.

This complex can be identified with the one in which white vertices are removed and where the outputs are symmetrized and the inputs are skew-symmetrized. This space can be identified with the deformation complex of the PROP extension of **Lie**, $\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{PROP}}$ which in turn is isomorphic, as proven in [MV09], to $\odot(\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{operad}}[-1])[1]$. Since we are working over a field we see that the cohomology of this space is one-dimensional as the cohomology of $\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie})_{\mathrm{operad}}$ is concentrated in degree 1.

□

Corollary 4.3.5. *Let U be an S. Gutt quantization. Then the map*

$$\mathbf{Def}(\mathbf{Lie} \to \mathbf{Lie}) \longrightarrow \mathbf{Def}(\mathbf{Ass} \xrightarrow{U} \mathcal{O}(\mathbf{Lie}))$$

is a quasi-isomorphism.

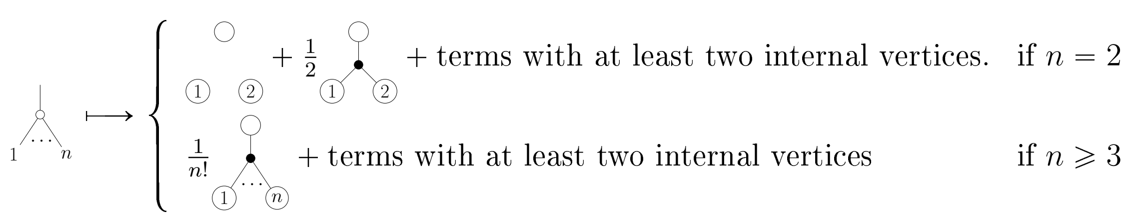
We will now generalize the above results to $\mathbf{Lie}_\infty$ and $\mathbf{Ass}_\infty$ structures.

Let V be a $\mathbb{Z}$-graded vector space.

Definition 4.3.6. A homotopy S. Gutt quantization is a morphism of dg operads

$$\varphi : \mathbf{Ass}_\infty \longrightarrow \mathcal{O}(\mathbf{Lie}_\infty)$$

satisfying



Example 4.3.7. 1) The constructions given by Baranovsky [Bar08], Moreno-Fernández [MF22] and Lada and Markl [LM95] can be interpreted as such maps.

2) The Kontsevich formality map [Kon03] implies this formula, but as a map to a wheeled version $\mathcal{O}(\mathbf{Lie}_\infty^\circlearrowright)$ which is ill-defined for infinite-dimensional Lie algebras. This was resolved by Shoikhet in [Sho01] by showing that wheels can be set to zero and thus we get a map of the desired form.

Corollary 4.3.8. *Let φ be a homotopy S. Gutt quantization. Then the complex* $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{Lie}_\infty))$ *is one-dimensional. In particular the map*

$$\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{Lie}_\infty)) \cong \mathbf{Def}(\mathbf{Lie}_\infty \to \mathbf{Lie}_\infty).$$

is a quasi-isomorphism.

Proof. Since the canonical map $\pi : \mathbf{HoLie}_1 \to \mathbf{Lie}$ is a quasi-isomorphism and the functor $\mathcal{O}$ is exact, we see that the map $\mathcal{O}(\pi)$ is a quasi-isomorphism. Hence the deformation complexes $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\varphi} \mathcal{O}(\mathbf{HoLie}_1))$ and $\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\mathcal{O}(\pi)\varphi} \mathcal{O}(\mathbf{Lie}))$ are quasi-isomorphic. In addition the map $\mathcal{O}(\pi)\varphi : \mathbf{Ass}_\infty \to \mathcal{O}(\mathbf{Lie})$ factors through $\mathbf{Ass}$, i.e.

$$\begin{array}{ccc} \mathbf{Ass}_\infty & \xrightarrow{\pi'} & \mathbf{Ass} \\ & \searrow^{\mathcal{O}(\pi)\varphi} & \downarrow \mathcal{U} \\ & & \mathcal{O}(\mathbf{Lie}) \end{array} ,$$

where $\mathcal{U}$ is a S. Gutt quantization. Thus

$$H(\mathbf{Def}(\mathbf{Ass}_\infty \xrightarrow{\mathcal{O}(\pi)\varphi} \mathcal{O}(\mathbf{Lie}))) \cong H(\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie}))).$$

By Theorem 4.3.4 the cohomology $H(\mathbf{Def}(\mathbf{Ass} \xrightarrow{\mathcal{U}} \mathcal{O}(\mathbf{Lie})))$ is one-dimensional and the result follows.

□

This result fully agrees with the PBW theorem obtained in [KT23] using the bar-cobar duality in the homotopy theory of operads.

4.4 On the unique non-trivial deformation of the map i

Let $\mathfrak{g}$ be a Lie algebra over some field $\mathbb{K}$ with a countable basis $\{t_i\}_{i\in I}$. Then $\odot\mathfrak{g}$ can be identified with the polynomial ring $\mathbb{K}[t_I]$. The Gutt quantization formula or the PBW quantization formula applied to polynomials $P(t)$ and $Q(t)$ can be given in terms of differential operators as follows (see Theorem 5 in [Bek05]):

$$P(t) * Q(t) = \exp(\sum_i t_i m^i(\frac{\partial}{\partial u}\frac{\partial}{\partial v}))\ P(u)Q(v)|_{u=v=t}\,,$$

where m^i is defined as follows: Let $X = \sum_i x^i t_i$, $Y = \sum_i y^i t_i$ be arbitrary elements of $\mathfrak{g}$ (we understand the numerical coefficients x^i and y^i as formal parameters). Then define

$$m(X,Y) := \log(e^X e^Y) - X - Y = \sum t_i m^i_{j_1,\cdots,j_p,k_1,\cdots,k_q} x^{j_1} \cdots x^{j_p} y^{k_1} \cdots y^{k_q}$$

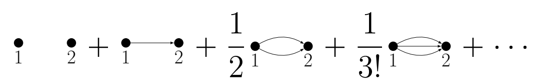
and set $m(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ to be the differential operator obtained from the above power series by replacing each $x^{j_1} \cdots x^{j_p}$ by $\frac{\partial}{\partial u_{j_1}} \cdots \frac{\partial}{\partial u_{j_p}}$ and similarly for $y^{k_1} \cdots y^{k_q}$. It is hard in general to rewrite this formula in terms of our graphs as an explicit morphism of operads

$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie})$$

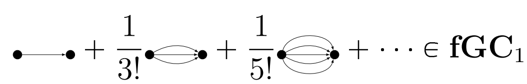
but for our purposes it is enough to see its quotient modulo the graphs $I \subset \mathcal{O}(\mathbf{Lie})$ containing at least one internal edge

$$\mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie})/I \cong \mathbf{Gra}_1$$

whose value on the generator of $\mathbf{Ass}$ is given by the following element in $\mathbf{Gra}_1$



which was found earlier in Appendix A of [KWŽ17].
After skew-symmetrization of the indices 1 and 2 this series simplifies as follows



As we see the first non-trivial term is precisely our theta class in $H^1(\mathbf{GC}_1)$, we complete the proof that the composition

$$\mathbf{Lie} \longrightarrow \mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie}),$$

is precisely the unique homotopy non-trivial deformation of the naive map i. Thus we proved the following theorem:

Theorem 4.4.1. *In the case $d = 1$ the natural map $i : \mathbf{Lie} \to \mathcal{O}(\mathbf{Lie})$ has a unique homotopy non-trivial deformation $J : \mathbf{Lie} \to \mathcal{O}(\mathbf{Lie})$ corresponding to the composition*

$$\mathbf{Lie} \longrightarrow \mathbf{Ass} \longrightarrow \mathcal{O}(\mathbf{Lie}),$$

where the first arrow is the canonical map of operads, and the second arrow is the universal enveloping map.

Chapter 5

Polydifferential Lie bialgebras and deformation theory of Lie bialgebras

This section is taken from the paper [Wol25]:

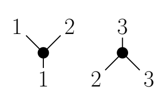
5.1 Polydifferential functor on PROPs

In [MW20] there is an endofunctor in the category of (augmented) PROPs

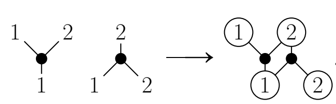
$$\begin{array}{rccc} \mathcal{D}: & \mathbf{Props} & \longrightarrow & \mathbf{Props} \\ & P & \longmapsto & \mathcal{D}(P), \end{array}$$

with the property that for any representation $P \to \mathbf{End}_V$ of a PROP P in some dg vector space V there is an associated representation $\mathcal{D}(P) \to \mathbf{End}_{\odot V}$ of the polydifferential PROP $\mathcal{D}(P)$ in the symmetric tensor algebra $\odot V$, such that elements of P acts as polydifferential operators.

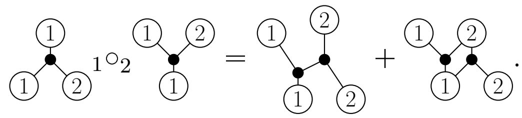
Our interest lies in the properad $\mathcal{D}(\mathbf{Lieb}_{c,d})$ (technically in a sub-properad derived from said properad, see below) obtained by applying the above functor to the PROP closure $\overline{\mathbf{Lieb}}_{c,d}$ of the properad of Lie bialgebras. Generators of the PROP closure are given by disjoint unions of elements in $\mathbf{Lieb}_{c,d}$, e.g.

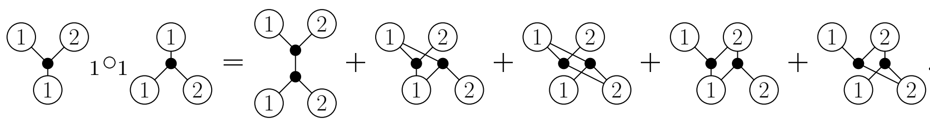


Given $p \in \overline{\mathbf{Lieb}}_{c,d}$ and a partition of the input and output vertices respectively, we obtain an element of $\mathcal{D}(\mathbf{Lieb}_{c,d})$ by combining all input respectively output vertices in each partition class into a single vertex. For example



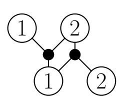
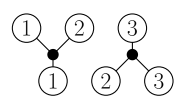
For elements $\Gamma_1, \Gamma_2 \in \mathcal{D}(\mathbf{Lieb}_{c,d})$ the properadic compositions $\Gamma_1 \, {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k} \, \Gamma_2$ are defined as follows: We first erase the input vertices $i_1, \cdots i_k$ of Γ_1 and output vertices $j_1, \cdots, j_k$ of Γ_2. Then if the input vertex i is paired with the output j , we first sum over all possible attachments of the input vertices labelled i and and output edges labelled j. Secondly we sum over all possible attachments of the remaining input edges labelled i to the input vertices of Γ_2 and similarly for the remaining output edges labelled j.

Example 5.1.1. a) 

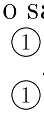
b) 

5.2 Properad of polydifferential Lie bialgebras

Given the polydifferential properad $\mathcal{D}(\mathbf{Lieb}_{c,d})$ associated to the properad of Lie bialgebras, we consider a sub-properad $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$, called properad of polydifferential Lie bialgebras, whose elements are required to be connected as graphs and remain connected if we delete either the input or the output vertices. For example

 $\in \mathcal{D}_c(\mathbf{Lieb}_{c,d})$,  $\notin \mathcal{D}_c(\mathbf{Lieb}_{c,d})$.

An element of $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$ consists of three types of vertices, the output and input vertices, which are at the top respectively at the bottom. We call the black vertices the *internal vertices* and any edge between two internal vertices will be called an *internal edge*. The graph induced by deleting the input and output vertices is in general not connected and each connected component will be called an *irreducible connected component*.

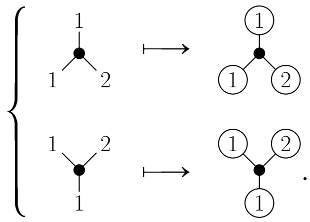
Remark 5.2.1. The condition that the elements of $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$ are connected is in this case equivalent to say that there is at least one irreducible connected component, i.e. we exclude the element .

We observe that the compositions in $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$ does not decrease the number of internal edges and thus we can consider the quotient properad $\mathcal{D}_c(\mathbf{Lieb}_{c,d})/I$ where I is the ideal generated by all elements with at least one internal edge. This quotient can be fully described using hairy graphs (see §5.3).

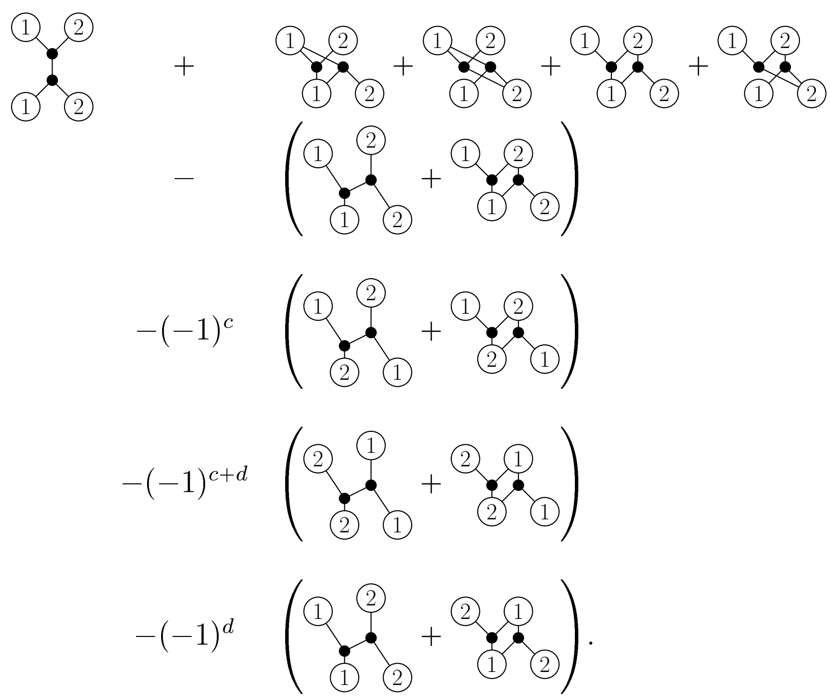
Lemma 5.2.2. *There is a morphism of properads*

$$i : \mathbf{Lieb}_{c,d} \to \mathcal{D}_c(\mathbf{Lieb}_{c,d})$$

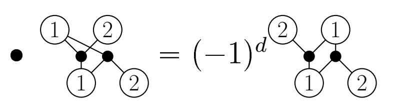
given by

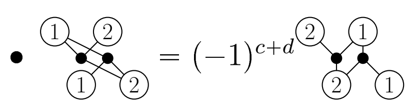


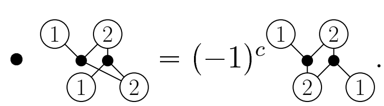
Proof. The proof of the Jacobi and co-Jacobi identity follows from Lemma 4.1.2. It remains to show that the compatibility condition is satisfied in $\mathcal{D}_c(\mathbf{Lieb}_{c,d})$:



This now follows from the observations







□

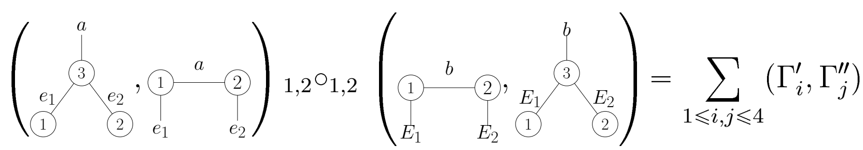
Remark 5.2.3. The above morphism is not well-defined for any properad generated by two operations. Indeed if we consider the properad **Assb** of associative bialgebras, then it was shown in Remark 4.1.3 that already the associativity condition fails.

The aim of this chapter is to study the deformation complex of the above morphism.

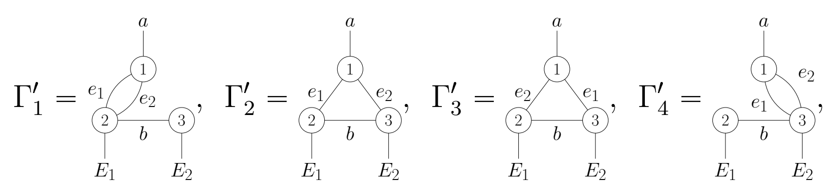
5.3 Complex of entangled graphs

Definition 5.3.1. We define the *properad of entangled graphs* $\mathbf{Gra}_{c,d}$ as the properad whose elements are pairs of graphs (Γ_1, Γ_2) where Γ_1 has an orientation induced for even c by an ordering of the undirected edges $e_1 \wedge \cdots \wedge e_{\#E(\Gamma_1)}$ up to an even permutation, while an odd permutation gives a multiplication by -1. For c odd we assume that edges are directed up to a flip and multiplication by -1. Similarly for Γ_2. In addition we assume that Γ_1 has hairs labelled by the edges of Γ_2 and vice-versa. Given two pairs (Γ_1, Γ_2) and (γ_1, γ_2) and given vertices $i_1, \cdots, i_k \in V(\Gamma_1)$ and $j_1, \cdots, j_k \in V(\gamma_2)$ we define the properadic composition $(\Gamma_1, \Gamma_2)\, {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k} (\gamma_1, \gamma_2)$ is defined to be the pair (G_1, G_2) where G_1 is the graph obtained by erasing the vertices $i_1, \cdots, i_k$ of Γ_1 and summing over all possible attachments of the half-edges to the vertices of γ_1. Similarly G_2 is obtained by erasing the vertices $j_1, \cdots, j_k$ of γ_2 and summing over all possible attachments of the half-edges to the vertices of Γ_2.

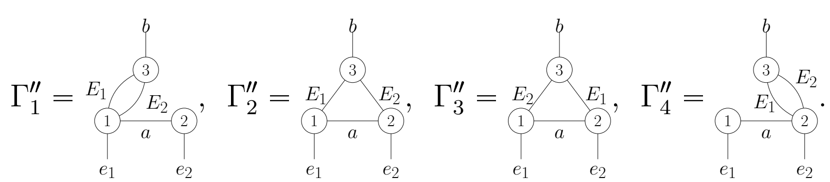
Example 5.3.2. The properadic composition



with



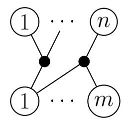
and



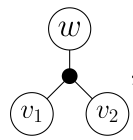
Lemma 5.3.3. *Let $I \subset \mathcal{D}_c(\mathbf{Lieb}_{c,d})$ be the ideal linearly spanned by graphs with at least one internal edge. There is an explicit isomorphism of properads*

$$\mathcal{D}_c(\mathbf{Lieb}_{c,d})/I \xrightarrow{F} \mathbf{Gra}_{c,d}.$$

Proof. For a generator G



we associate a pair of graphs (Γ_1, Γ_2) as follows: the vertices of Γ_1 respectively the vertices of Γ_2 correspond to the input respectively output vertices of G. There is an edge between two vertices v_1 and v_2 of Γ_1 if the corresponding input vertices are connected in G by a tree of the form

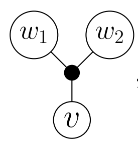


,

for some output vertex w.

In addition each such tree points to exactly one output vertex of Γ_2 to whom we add an hair labelled by that edge.

Similarly the edges of Γ_2 (and hairs of Γ_1) are given by trees of the form



,

for some output vertices w_1, w_2 and input vertex v in G.

The properadic compositions in $\mathbf{Gra}_{c,d}$ are defined to make this map a morphism of properads. More precisely given $F(G_1) = (\Gamma_1, \Gamma_2)$ and given $F(G_2) = (\gamma_1, \gamma_2)$ we see that in the properadic composition $G_1 \; {}_{i_1,\cdots,i_k}\circ_{j_1,\cdots,j_k} G_2$ deleting the input i of G_1 and summing over all attachments to the input vertices of G_2 corresponds to deleting the vertex i of Γ_1 and summing over all attachments of the half-edges of i to the vertices of γ_1.

□

Using the maps

$$\mathbf{Lieb}_{c,d} \to \mathcal{D}_c(\mathbf{Lieb}_{c,d}) \to \mathcal{D}_c(\mathbf{Lieb}_{c,d})/I \to \mathbf{Gra}_{c,d}$$

we define the *complex of entangled graphs* by the deformation complex $\mathbf{fcGC}_{c,d} := \mathbf{Def}(\mathbf{Lieb}_{c,d} \to \mathbf{Gra}_{c,d})$ of the above composition. Its elements are given by pairs of graphs (Γ_1, Γ_2) with (skew)symmetrized vertices depending on the parity of c respectively d. The differential δ is given by

$$\delta(\Gamma_1, \Gamma_2) = \delta'(\Gamma_1, \Gamma_2) \pm \delta''(\Gamma_1, \Gamma_2),$$

where

$$\delta'(\Gamma_1,\Gamma_2) = \sum_{\substack{v\in V(\Gamma_1)\\ w\in V(\Gamma_2)}} \Big(\overset{\circ}{\underset{\textcircled{\scriptsize Γ}}{|}}, \Gamma_2^w\Big) \pm \sum_{\substack{v\in V(\Gamma_1)\\ w\in V(\Gamma_2)}} (\Gamma_1 \circ_v \circ\!\!-\!\!\circ, \Gamma_2^w),$$

i.e. the first sum is given by attaching a univalent vertex with no hairs to any vertex of Γ_1, while the term $\Gamma_1 \circ_v \circ\!\!-\!\!\circ$ is given by splitting the vertex v and summing over all possible attachments of its half-edges (including hairs) to the newly created vertices. For a vertex w of Γ_2 the graph Γ_2^w is obtained by attaching an hair labelled by the new edge in Γ_1 to the vertex w. Similarly $\delta''(\Gamma_1,\Gamma_2)$ is given by switching the roles of Γ_1 and Γ_2.

Remark 5.3.4. a) We do not count hairs for the valency of a vertex.

b) If a vertex $v \in V(\Gamma_1)$ is of valency at least one then the terms adding a univalent vertex with no hairs are canceled with the similar terms created by splitting v. Note however that univalent vertices with hairs can be created by the differential.

The cohomology of $\mathbf{fcGC}_{c,d}$ is surprisingly simple as shown by the following theorem:

Theorem 5.3.5. *The cohomology of* $\mathbf{fcGC}_{c,d}$ *is two-dimensional and generated by* $(\overset{|}{\circ}, \circ\!\!-\!\!\circ)$ *and* $(\circ\!\!-\!\!\circ, \overset{|}{\circ})$.

Proof. Let $\mathbf{fcGC}^0_{c,d}$ be the differential closure of the subspace

$$\operatorname{span}\left\langle (\overset{|}{\circ}, \circ\!\!-\!\!\circ), (\Gamma_1, \overset{E(\Gamma_1)}{\circ}) \text{ for } \Gamma_1 \in \mathbf{fcGC}_c \right\rangle.$$

Then the cohomology of $\mathbf{fcGC}^0_{c,d}$ is two-dimensional. Indeed we first observe that $\delta(\overset{|}{\circ}, \circ\!\!-\!\!\circ) = 0$ and as by connectivity of the elements in $\mathbf{fcGC}_{c,d}$ there has to be always an edge it has to be a cohomology class. Similarly $(\circ\!\!-\!\!\circ, \overset{|}{\circ})$ is a cohomology class. In addition if $\#E(\Gamma_1) \geqslant 2$ we have

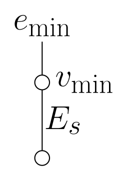
$$\delta(\Gamma_1, \overset{E(\Gamma_1)}{\circ}) = (\delta\Gamma_1, \overset{E(\delta\Gamma_1)}{\circ}) \pm \sum_{\substack{I\sqcup J = E(\Gamma_1)\\ I,J\neq\varnothing}} (\Gamma_1^e, \overset{I}{\circ}\!\!-\!\!\overset{J}{\circ}).$$

The last sum is never zero and thus $(\Gamma_1, \overset{E(\Gamma_1)}{\circ})$ is never a cycle.

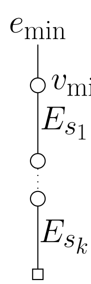
The theorem is proven if we show that the inclusion $\mathbf{fcGC}^0_{c,d} \hookrightarrow \mathbf{fcGC}_{c,d}$ is a quasi-isomorphism or equivalently if we show that the quotient complex $\mathbf{fcGC}^{\geqslant 1}_{c,d} := \mathbf{fcGC}_{c,d}/\mathbf{fcGC}^0_{c,d}$ is acyclic. We consider on $\mathbf{fcGC}^{\geqslant 1}_{c,d}$ a filtration on the number of vertices of Γ_2, i.e. for $p \geqslant 0$

$$F_{-p} = \operatorname{span}\langle (\Gamma_1,\Gamma_2) \text{with } \#V(\Gamma_2) \geqslant p\rangle.$$

Given a pair (Γ_1, Γ_2) in the associated graded complex $\mathbf{grfcGC}_{c,d}^{\geqslant 1}$ the induced differential acts by splitting vertices in Γ_1 and attaching hairs to Γ_2. In particular the vertices of edges of Γ_2 stay invariant and we can consider the larger complex $\widetilde{\mathbf{grfcGC}}_{c,d}^{\geqslant 1}$ spanned by pairs (Γ_1, Γ_2) with the vertices and edges of Γ_2 being totally ordered. Denote by $w_{\min}$ respectively $e_{\min}$ the minimal vertex respectively minimal edge of Γ_2. As the differential increases the number of hairs in Γ_2 we can consider the filtration on $\widetilde{\mathbf{grfcGC}_{c,d}}^{\geqslant 1}$ by the total number of hairs attached to the to the vertices in the set $V(\Gamma_2) \backslash \{w_{\min}\}$. Let $\widehat{\mathbf{grfcGC}_{c,d}}^{\geqslant 1}$ be the associated graded complex. For a pair (Γ_1, Γ_2) the induced differential acts as above on Γ_1 but the new hair is only attached to the vertex $w_{\min}$. Call a vertex $v_{\min} \in V(\Gamma_1)$ *very special* if it is of the form

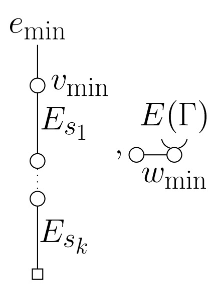


and the edge E_s is uniquely attached as a hair to $w_{\min}$. Note that $v_{\min}$ being very special is only defined in a pair (Γ_1, Γ_2). Next call a maximal chain of bivalent and bald vertices of the form, for $k \geqslant 1$,



the minimal antenna if all edges $E_{s_1}, E_{s_2}, \cdots, E_{s_k}$ are uniquely attached as hairs to the vertex $w_{\min}$. The ending vertex of the minimal antenna is either of valency different from 2, not bald or one of its edges is attached as a hair to a vertex different from $w_{\min}$. Call the vertices of the above *special* and consider a filtration on $\widehat{\mathbf{grfcGC}_{c,d}}^{\geqslant 1}$ by the number of non-special vertices in pairs (Γ_1, Γ_2). Let $\overline{\mathbf{grfcGC}_{c,d}}^{\geqslant 1}$ be the associated graded complex. We finally show that $\overline{\mathbf{grfcGC}_{c,d}}^{\geqslant 1}$ is acyclic.

Let $C = \bigoplus_{k \geqslant 1} C_k$ be the subcomplex where C_k is generated by the following pair

$(\Gamma :=$  $).$

Then C is acyclic as the differential $\delta : C_k \to C_{k+1}$ is injective for k odd and zero for k even. In addition the complex splits as $\overline{\mathbf{grfcGC}_{c,d}}^{\geqslant 1} = C \oplus C'$. For $k \geqslant 0$ denote by C'_k the subspace of C' spanned by pairs (Γ_1, Γ_2) with the minimal antenna being of length k. The differential $\delta : C'_k \to C'_{k+1}$ is an injection for k even and zero for k odd (which was the opposite for the complex C). Hence the complex C' is acyclic and the result is proven. □

As a small application of the above result, we will briefly discuss, what happens if we restrict ourselves to graphs with univalent vertices. More precisely, let $\mathbf{GC}^1_{c,d}$ be the subcomplex spanned by pairs (Γ_1, Γ_2) with at least one univalent vertex. Then, contrary to the standard graph complex, the cohomology of $\mathbf{GC}^1_{c,d}$ is highly non-trivial. To see this, we first need to investigate the quotient complex $\mathbf{GC}^{\geqslant 2}_{c,d} := \mathbf{fcGC}_{c,d}/\mathbf{GC}^1_{c,d}$. It is spanned by pairs (Γ_1, Γ_2) with all vertices bivalent and an induced differential which does not create univalent vertices. The cohomology of $\mathbf{GC}^{\geqslant 2}_{c,d}$ is unknown at the moment, but we have a partial result:

Proposition 5.3.6. *There is an inclusion of complexes*

$$\begin{array}{rccc} \mathbf{Sym}: & \mathbf{GC}_c \otimes \mathbf{GC}_d & \longrightarrow & \mathbf{GC}^{\geqslant 2}_{c,d} \\ & \Gamma_1 \otimes \Gamma_2 & \longmapsto & \sum_{\substack{f_1 : E(\Gamma_2) \to V(\Gamma_1) \\ f_2 : E(\Gamma_1) \to V(\Gamma_2)}} (\Gamma_1^{f_1}, \Gamma_2^{f_2}) \end{array}$$

i.e. we sum over all possible hair attachments. In addition the map induces an inclusion on the level of cohomology

$$H^\bullet(\mathbf{GC}_c) \otimes H^\bullet(\mathbf{GC}_d) \to H^\bullet(\mathbf{GC}^{\geqslant 2}_{c,d}).$$

Proof. We need to show that the map $\mathbf{Sym}$ respects the differentials. Indeed, while splitting a vertex in $\mathbf{fcGC}_{c,d}$, the only situation where we do not sum over all hair attachments, is when we create univalent vertices (as we do not create univalent bald vertices). As these terms disappear in $\mathbf{GC}^{\geqslant 2}_{c,d}$ we conclude that $\mathbf{Sym}$ is a morphism of complexes.

We observe that if $\Gamma \in \mathbf{GC}^{\geqslant 2}_{c,d}$ satisfies $\delta\Gamma \in \mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d)$ then either $\delta\Gamma = 0$ or $\Gamma \in \mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d)$. This follows from the fact that if there are hair attachments missing in Γ these cannot be created by the differential. In particular the morphism

$$H^\bullet(\mathbf{GC}^{\geqslant 2}_{c,d}/\mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d)) \longrightarrow H^{\bullet+1}(\mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d))$$

induced by the short exact sequence of complexes

$$0 \longrightarrow \mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d) \longrightarrow \mathbf{GC}^{\geqslant 2}_{c,d} \longrightarrow \mathbf{GC}^{\geqslant 2}_{c,d}/\mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d) \longrightarrow 0$$

is the zero morphism and thus by exactness we see that the morphism

$$H^\bullet(\mathbf{Sym}(\mathbf{GC}_c \otimes \mathbf{GC}_d)) \longrightarrow H^\bullet(\mathbf{GC}^{\geqslant 2}_{c,d})$$

is injective.

□

The cohomology of $\mathbf{GC}^1_{c,d}$ can now be finally computed:

Proposition 5.3.7. *The cohomology of* $\mathbf{GC}^1_{c,d}$ *is equal to the cohomology of* $\mathbf{GC}^{\geqslant 2}_{c,d}$ *shifted by* 1*, up to the additional classes* $(\stackrel{|}{\circ}, \circ\!-\!\circ)$ *and* $(\circ\!-\!\circ, \stackrel{|}{\circ})$*.*
In particular there is an inclusion in cohomology

$$H^\bullet(\mathbf{GC}_c \otimes \mathbf{GC}_d) \longrightarrow H^{\bullet+1}(\mathbf{GC}^1_{c,d})$$

Proof. Define $\overline{\mathbf{fcGC}_{c,d}} = \mathbf{fcGC}_{c,d} \backslash \{(\stackrel{|}{\circ}, \circ\!-\!\circ), (\circ\!-\!\circ, \stackrel{|}{\circ})\}$ and $\overline{\mathbf{GC}^1_{c,d}} = \mathbf{GC}^1_{c,d} \backslash \{(\stackrel{|}{\circ}, \circ\!-\!\circ), (\circ\!-\!\circ, \stackrel{|}{\circ})\}$. The result is now proven using the short exact sequence

$$0 \longrightarrow \overline{\mathbf{GC}^1_{c,d}} \longrightarrow \overline{\mathbf{fcGC}_{c,d}} \longrightarrow \overline{\mathbf{fcGC}_{c,d}} / \overline{\mathbf{GC}^1_{c,d}} \cong \mathbf{GC}^{\geqslant 2}_{c,d} \longrightarrow 0$$

and the fact that $\overline{\mathbf{fcGC}_{c,d}}$ is acyclic by Theorem 5.3.5.

□

5.4 Proof of Theorem 5.4.1

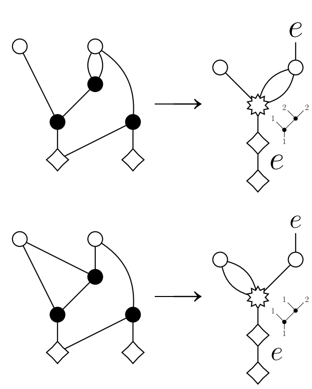
Recall that we are interested in the deformation complex of the morphism defined in Lemma 5.2.2. The main contribution is to compute its cohomology.

Theorem 5.4.1.

$$H^\bullet(\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))) \cong H^\bullet(\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d})).$$

Before we prove this, we will explain how to interpret elements of $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$ as (hairy) graphs with three types of vertices. Given $\Gamma \in \mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$ we first contract each irreducible connected component into a single vertex, called *star* vertex, which is decorated by the corresponding element in $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d})$ together with a partition of the input and output vertices. Next we delete each star vertex decorated by $\curlywedge$, add an edge between the two input vertices and add an hair decorated by said edge to the output vertex. Similarly for the star vertices decorated by $\curlyvee$.

Example 5.4.2.

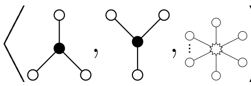


Note that there are no edges between two star vertices. Any edge attached to a star vertex will be called *star* edge.

The differential δ of $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$ can be written $\delta = \delta_{\text{Lieb}} + \delta_{\text{Graph}}$ where the first part increases the number of internal edges (and creates star vertices) while the second part acts by splitting the input and output vertices as in $\mathbf{fcGC}_{c,d}$.

Proof. Theorem 5.4.1
We can see $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d})$ in $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$ as the subspace spanned by the elements with all input and output vertices being univalent, i.e.

$$\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d}) \cong \text{span}\left\langle \quad , \quad , \quad \right\rangle.$$



As the graph part of the differential cannot create univalent bald vertices we have $\delta_{\text{Graph}} \equiv 0$ on $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d})$ and there is an inclusion of complexes $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d}) \hookrightarrow \mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$. In addition the differential cannot reduce the number of vertices of valency at least 2 and thus the complex $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d}))$ splits

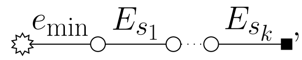
$$\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{i} \mathcal{D}_c(\mathbf{Lieb}_{c,d})) = \mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d}) \oplus C,$$

where C the complex spanned by elements with at least one non-star vertex of valency greater than 1. It remains to show that C is acyclic. Consider a filtration on the number of internal edges. Then the induced differential on the associated graded complex $\mathbf{gr}C$ cannot create any new star vertices and thus the complex splits

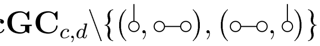
$$\mathbf{gr}C = \bigoplus_{N \geqslant 0} C_N,$$

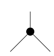
where C_N is spanned by graphs having N star vertices. Let $N \geqslant 1$. We show that C_N is acyclic. Since the differential cannot create star vertices, the set of star edges is invariant and thus we can consider the complex $\widetilde{C}_N$ where the star edges are totally ordered. As there is at least one non-star vertex of valency greater than 1 and by connectivity of the graphs, there is at least one star edge connected to a vertex of valency at least 2. Denote by $e_{\min}$ the minimal such edge. We can split the complex $\widetilde{C}_N = \widetilde{C}_{N,in} \oplus \widetilde{C}_{N,out}$ where $\widetilde{C}_{N,in}$ respectively $\widetilde{C}_{N,out}$ is spanned by graphs where $e_{\min}$ is connected to an input respectively output vertex. We will show that $\widetilde{C}_{N,in}$ is acyclic, the proof for $\widetilde{C}_{N,out}$ is obtained by changing the roles of input and output vertices.

Consider a filtration on $\widetilde{C}_{N,in}$ by the number of output vertices. By construction the differential on $\mathbf{gr}\widetilde{C}_{N,in}$ acts only by splitting input vertices and attaches hairs to output vertices. In particular the set of output vertices is invariant and thus we can consider the complex $\widehat{\mathbf{gr}C}_{N,in}$ where we assume a total ordering on the set of output vertices. Denote by $w_{\min}$ the minimal output vertex and define a minimal antenna as a maximal chain of bivalent bald input vertices as below ($k \geqslant 1$)



where we assume that all edges $E_{s_1}, E_{s_2}, \cdots, E_{s_k}$ are attached uniquely as hairs to the vertex $w_{\min}$. The square vertex is either a star vertex (and in that case we do not have a condition on E_{s_k}), an input vertex of valency different from 2 or a vertex of valency 2 which is either not bald or one of its edges is not uniquely attached as hair to $w_{\min}$. Call the vertices in the above antenna special and consider a filtration on the number of non-special vertices. Then the associated graded complex $\overline{\mathbf{gr}C}_{N,in} = \bigoplus_{k\geqslant 1} C'_k$ splits by the length of the minimal antenna. The differential $\delta_{\text{Graph}} : C'_k \to C'_{k+1}$ is an injection if k is odd and zero for k even. Hence we conclude that $\overline{\mathbf{gr}C}_{N,in}$ is acyclic.

It remains to show that C_0 is acyclic. As the elements of C_0 do not have star vertices, they are given by two disjoint hairy graphs and thus the complex is isomorphic to $\mathbf{fcGC}_{c,d}\setminus$ ,

as these elements correspond to  and  which are elements in $\mathbf{Def}(\mathbf{Lieb}_{c,d} \xrightarrow{id} \mathbf{Lieb}_{c,d})$. By Theorem 5.3.5 we conclude that C_0 is acyclic as we removed all cohomology classes. This finishes the proof of the theorem.

□

Remark 5.4.3. The above result implies that any homotopy non-trivial deformation γ of i comes from the infinitesimal deformations of the identity map $id : \mathbf{Lieb}_{c,d} \to \mathbf{Lieb}_{c,d}$. It has been proven in [MW18a] that every infinitesimal deformation of id exponentiates to a genuine deformation. As $\mathcal{D}$ is a functor, the latter result implies that every such γ exponentiates to a genuine deformation of i.

Chapter 6

The oriented graph complex revisited

The following text is taken from a joint work with Sergei Merkulov and Thomas Willwacher [MWW24]:

6.1 Introduction

The aim of this Chapter is to prove the following theorem:
Denote by $\mathbf{GC}^0_d \subset \mathbf{fGC}_d$ and $\mathbf{OGC}^0_{d+1} \subset \mathbf{fOGC}_{d+1}$ subcomplexes spanned by graphs Γ of loop order $g(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1 \geqslant 1$.

Theorem 6.1.1. *For any $d \in \mathbb{Z}$ there exists a dg Lie algebra $\widehat{\mathbf{OGC}}_{d,d+1}$ of graphs which fits into the following diagram of quasi-isomorphisms of dg Lie algebras*

$$\mathbf{GC}^0_d \xleftarrow{\pi_1} \widehat{\mathbf{OGC}}_{d,d+1} \xrightarrow{\pi_2} \mathbf{OGC}^0_{d+1}.$$

There is a grading by loop order on $\widehat{\mathbf{OGC}}_{d,d+1}$ and the arrows π_1, π_2 respect the loop order gradings.

The intertwining dg Lie algebra $\widehat{\mathbf{OGC}}_{d,d+1}$ is explicitly described in Section 6.1: it is generated by graphs with two types of vertices, and is equipped with a relatively non-trivial but explicitly described differential. This dg Lie algebra controls the deformation theory of a certain morphism of 2-coloured operads as explained in Section 6.1 of this paper. The morphisms π_1 and π_2 are very simple and explicit. The theorem is proven in Section 6.3.
By extending the zigzag with the inclusions $\mathbf{GC}^2_d \subset \mathbf{GC}^0_d$ and $\mathbf{OGC}^2_{d+1} \subset \mathbf{fOGC}_{d+1}$, which are known to be quasi-isomorphisms, we also obtain a zigzag of quasi-isomorphisms of dg Lie algebras

$$\mathbf{GC}^2_d \xrightarrow{\sim} \bullet \xleftarrow{\sim} \mathbf{OGC}^2_{d+1}$$

connecting the bivalent versions. It is furthermore known that the inclusions $\mathbf{GC}_d \subset \mathbf{GC}^2_d$ and $\mathbf{OGC}^3_{d+1} \subset \mathbf{OGC}^2_{d+1}$ are quasi-isomorphisms in loop orders $\geqslant 2$. Hence extending our zigzag and truncating to loop orders $\geqslant 2$ we also obtain a zigzag of quasi-isomorphisms of dg Lie algebras

$$\mathbf{GC}_d \xrightarrow{\sim} \bullet \xleftarrow{\sim} \mathbf{OGC}^3_{d+1}.$$

Here $\mathbf{OGC}^3_{d+1} \subset \mathbf{OGC}^2_{d+1}$ denotes the subcomplex spanned by graphs with all vertices of valency at least 3.

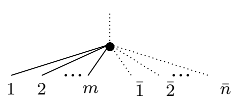
6.2 A 2-coloured operad of oriented graphs

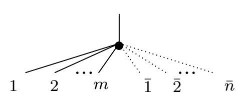
6.2.1 A 2-coloured operad of homotopy Lie algebras

Let $\mathbf{HoLie}^+_{d,d+1}$ be the 2-coloured operad controlling $\mathbf{HoLie}^+_1$-algebra structures in the dg vector spaces which are direct sums

$$V[d] \oplus W[d-1]$$

for arbitrary dg spaces V and W; put another way, a representation of $\mathbf{HoLie}^+_{d,d+1}$ in the 2-coloured endomorphism operad $\mathbf{End}_{V,W}$ is the same as a $\mathbf{HoLie}^+_1$-algebra structure in the above direct sum. Thus $\mathbf{HoLie}^+_{d,d+1}$ is a dg free 2-coloured operad generated by two sets of corollas, one set having the output in, say, white colour (shown in pictures as the dotted legs), and the other set having an output in black colour (shown as solid legs)

$\mathfrak{C}^\circ_{m+n} :=$  $\in sgn_m^{\otimes|d+1|} \otimes sgn_n^{|d|}[(d+1)m + dn - 2 - d], \quad m+n \geqslant 1,$

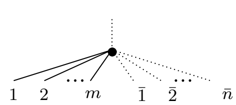
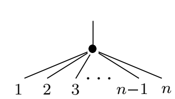
$\mathfrak{C}^\bullet_{m+n} =$  $\in sgn_m^{\otimes|d+1|} \otimes sgn_n^{|d|}[(d+1)m + dn - 1 - d], \quad m+n \geqslant 1.$

We understand these corollas as generators of the 1-dimensional $\mathbb{S}_m \times \mathbb{S}_n$ modules shown above. The differential in $\mathbf{HoLie}^+_{d,d+1}$ is given by splitting each corolla into two sums, one is via the substitution of the dotted edge inside the unique vertex, and the other is via the substituting the solid edge (cf. (2.2)).

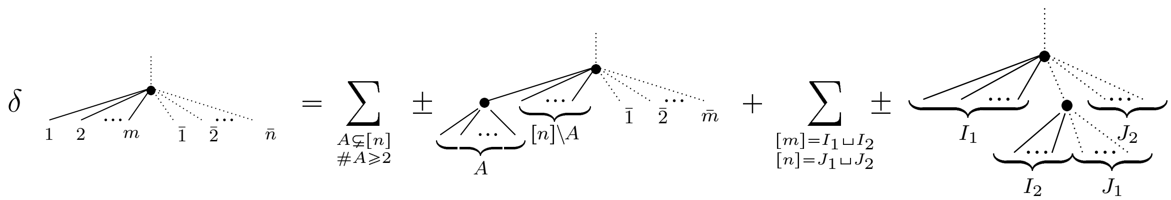
Let I be the ideal in $\mathbf{HoLie}^+_{d,d+1}$ generated by corollas $\mathfrak{C}^\circ_{1+0}$, $\mathfrak{C}^\bullet_{1+0}$, and all corollas $\mathfrak{C}^\bullet_{m+n}$ with $m \geqslant 1$. It is closed under the differential so that the quotient 2-coloured operad

$$\mathbf{HoLie}_{d,d+1} := \mathbf{HoLie}^+_{d,d+1}/I$$

is a dg *free* 2-colored operad with the following generators,

$\Bigg\{$  $m+n \geqslant 2,$  ,  $n \geqslant 2 \Bigg\}$

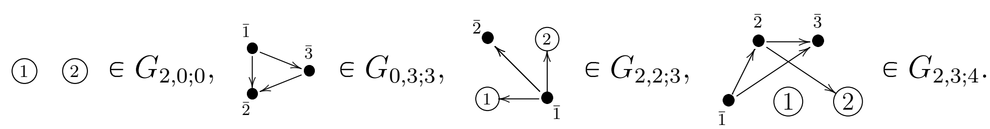
and with the differential given by (2.2) on generators with black output, and by the following formula



on the generators with white output. We use the operad $\mathbf{HoLie}_{d,d+1}$ together with a certain 2-coloured operad of graphs to build a new graph complex $\widehat{\mathbf{OGC}}_{d,d+1}$ below, the main gadget which solves the long standing problem as stated in Theorem 6.1.1.

6.2.2 A 2-coloured dg operad of graphs $\mathbf{Gra}_{d,d+1}$

Let $G_{m,n;p}$ be the set of connected graphs Γ with m labelled (by integers from $\{1,\ldots,m\}$) white vertices, n labelled (by integers from $\{\bar{1},\ldots,\bar{n}\}$) black vertices, and p labelled directed solid edges, e.g.



We also assume that the generators Γ

(a) have no closed paths of directed edges, and

(b) have no *outgoing* edges attached to white vertices.

Let us call a *sink* any vertex which has no outgoing edges attached; then every white vertex of a graph from $G_{m\geqslant 1,n;p,q}$ is a sink (we put no such restriction on black vertices).

If we denote by $E(\Gamma)$ the set of edges of $\Gamma\in G_{m,n;p}$, then the *labelling* of edges means that some isomorphism

$$E(\Gamma)\to [p],$$

is fixed; we do not show these labellings in our pictures as soon we shall get rid of this extra datum.

The group $\mathbb{S}_p$ acts on the $G_{m,n;p}$ by permuting labels of edges. Hence it makes sense to consider, for any integer $d\in\mathbb{Z}$ and any fixed pair of natural numbers $m,n\in\mathbb{N}$, a dg vector space (in fact, a dg $\mathbb{S}_m\times\mathbb{S}_n$-module),

$$\mathbf{Gra}^{\circ}_{d,d+1}(m,n) := \prod_{p\geqslant 0} \mathrm{span}\langle G_{m,n;p}\rangle \otimes_{\mathbb{S}_p} \mathrm{sgn}_p^{|d|}[pd]$$

equipped with the zero differential.

One can understand a generator of the $\mathbb{S}_m\times\mathbb{S}_n$-module as a graph Γ with

(i) m labeled white vertices,

(ii) n labelled black vertices,

(iii) some number of unlabeled edges whose directions are *fixed*,

(iv) a choice (up to sign) of a unital basis vector (called the *orientation*) of the following 1-dimensional Euclidean vector space

$$or(\Gamma) \in \begin{cases} \det(E(\Gamma)) & \text{if } d \text{ is even} \\ \mathbb{K} & \text{if } d \text{ is odd} \end{cases}$$

that is an ordering of edges for d even.

A generator $\Gamma \in \mathbf{Gra}^{\circ}_{d,d+1}(m,n)$ is assigned the cohomological degree

$$|\Gamma| = -d\#E(\Gamma),$$

i.e. every edge is assigned the degree $-d$.

Next we define a dg 2-coloured (with colours called *white* and *black*) operad of graphs

$$\mathbf{Gra}_{d,d+1} = \left\{\mathbf{Gra}_{d,d+1}(p) := \mathbf{Gra}^{\circ}_{d,d+1}(p) \oplus \mathbf{Gra}^{\bullet}_{d,d+1}(p)\right\}_{p\geqslant 1},$$

where $\mathbf{Gra}^{\circ}_{d,d+1}(p)$ (resp., $\mathbf{Gra}^{\bullet}_{d,d+1}(p)$) stands for the dg $\mathbb{S}_p$-module spanned by elements with so called white (resp. black) output; more precisely,

$$\mathbf{Gra}^{\circ}_{d,d+1}(p) := \bigoplus_{p=m+n} \mathrm{Ind}^{\mathbb{S}_p}_{\mathbb{S}_m\times\mathbb{S}_n} \mathbf{Gra}^{\circ}_{d,d+1}(m,n), \quad \mathbf{Gra}^{\bullet}_{d,d+1}(p) := \mathbf{Gra}^{or}_{d+1}(p).$$

Thus a generator of $\mathbf{Gra}_{d,d+1}$ is a pair of graphs

$$\left(\mathbf{\Gamma}^{\circ} \in \mathbf{Gra}_{d,d+1}(m,n)^{\circ},\ \mathbf{\Gamma}^{\bullet} \in \mathbf{Gra}^{or}_{d+1}\right)$$

and the operadic composition, $\forall\ i \in [p]$,

$$\begin{array}{rccc} \circ_i : & \mathbf{Gra}_{d,d+1}(p) \times \mathbf{Gra}_{d,d+1}(q) & \longrightarrow & \mathbf{Gra}_{d,d+1}(p+q-1), \\ & (\mathbf{\Gamma}_1 = (\mathbf{\Gamma}^{\circ}_1, \mathbf{\Gamma}^{\bullet}_1), \mathbf{\Gamma}_2 = (\mathbf{\Gamma}^{\circ}_2, \mathbf{\Gamma}^{\bullet}_2)) & \longrightarrow & \mathbf{\Gamma}_1 \circ_i \mathbf{\Gamma}_2, \end{array}$$

is given by

$$\mathbf{\Gamma}_1 \circ_i \mathbf{\Gamma}_2 := \begin{cases} (\mathbf{\Gamma}^{\circ}_1 \circ_i \mathbf{\Gamma}^{\circ}_2, 0) & \text{if } i \text{ is a white vertex in } \Gamma^{\circ}_1, \\ (\mathbf{\Gamma}^{\circ}_1 \circ_i \mathbf{\Gamma}^{\bullet}_2, 0) & \text{if } i \text{ is a black vertex in } \mathbf{\Gamma}^{\circ}_1, \\ (0, \mathbf{\Gamma}^{\bullet}_1 \circ_i \mathbf{\Gamma}^{\bullet}_2) & \text{if } i \text{ is a black vertex in } \mathbf{\Gamma}^{\bullet}_1. \end{cases}$$

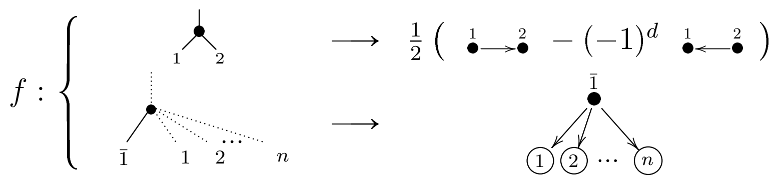
The symbol $A \circ_i B$ in the right hand side of the above equality stands for the substitution of a graph B into the i-labelled vertex v of a graph A and taking a sum over all possible re-attachments of dangling edges (attached before to v) to the vertices of B (cf. §2.8).

6.2.3 Proposition

There is a morphism of dg 2-coloured operads

$$f : \mathbf{HoLie}_{d,d+1} \longrightarrow \mathbf{Gra}_{d,d+1}$$

which vanishes on all generators of $\mathbf{HoLie}_{d,d+1}$ *except the following ones,*



This Proposition can be proven in two ways. The first proof goes via a direct checking that the above formulae respect the differentials in both sides of the morphism f (which is a straightforward but a bit tedious calculation).

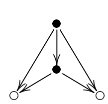
We choose another approach which is more suitable for our purposes in the next section, an approach which starts with a consideration of the dg Lie algebra

$$\widehat{\mathbf{OGC}}_{d,d+1} := \mathbf{Def}\left(\mathbf{HoLie}_{d,d+1} \stackrel{0}{\longrightarrow} \mathbf{Gra}_{d,d+1}\right) \tag{6.1}$$

controlling loop order $g \geqslant 1$ deformations of the *zero* morphism of the aforementioned operads; the differential in $\widehat{\mathbf{OGC}}_{d,d+1}$ is zero (for now). In view of the operadic composition formulae in $\mathbf{Gra}_{d,d+1}$ described above, we conclude that the dg Lie algebra $\widehat{\mathbf{OGC}}_{d,d+1}$ decomposes into a semidirect product,

$$\widehat{\mathbf{OGC}}_{d,d+1} = \mathbf{OGC}_{d,d+1} \rtimes \mathbf{OGC}^0_{d+1}, \tag{6.2}$$

of the dg Lie algebra $\mathbf{OGC}^0_{d+1}$ studied above and the dg Lie algebra $\mathbf{OGC}_{d,d+1}$ generated by graphs Γ° with two types of unlabelled vertices, black and white ones.

 $\in \mathbf{OGC}_{d,d+1}$.

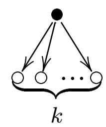
The cohomological degree is given by the formula

$$|\Gamma^\circ| = d\#V_\circ(\Gamma) + (d+1)\#V_\bullet(\Gamma^\circ) - d\#E(\Gamma^\circ) - d.$$

The graded Lie algebra $\widehat{\mathbf{OGC}}_{d,d+1}$ has an underlying dg pre-Lie algebra structure which is fully analogous to the one (3.2) in $\mathbf{GC}^0_d$. If we represent a generic element Γ of the graded Lie algebra $\widehat{\mathbf{OGC}}_{d,d+1}$ as a pair $\Gamma = (\Gamma^\circ, \Gamma^\bullet)$, with $\Gamma^\circ \in \mathbf{OGC}_{d,d+1}$ and $\Gamma^\bullet \in \mathbf{OGC}^0_{d+1}$, then the pre-Lie composition is given by

$$(\Gamma^\circ_1, \Gamma^\bullet_1) \circ (\Gamma^\circ_2, \Gamma^\bullet_2) = \left(\sum_{v \in V_\circ} \Gamma^\circ_1 \circ_v \Gamma^\circ_2 + \sum_{v \in V_\bullet} \Gamma^\circ_1 \circ_v \Gamma^\bullet_2,\ \sum_{v \in V(\Gamma^\bullet_1)} \Gamma^\bullet_1 \circ_v \Gamma^\bullet_2\right).$$

The calculations given in §5.1 and §5.2 of [Mer25] imply that the degree 1 element

$$\gamma = \left(\gamma^{\circ} := \bullet + \sum_{k=1}^{\infty} \frac{1}{k!} \right.$$  $$, \qquad \gamma^{\bullet} :=$$  $$\left. \right)$$

in $\widehat{\mathbf{OGC}}_{d,d+1}$ satisfies the equation,

$$\gamma \circ \gamma = 0,$$

i.e. that Γ is a Maurer-Cartan element of the deformation complex (6.1). Any such a MC element corresponds to some morphism from $\mathbf{HoLie}_{d,d+1}$ to $\mathbf{Gra}_{d,d+1}$, and the one shown just above corresponds precisely to the morphism f. The proposition is proven.
The twisted dg Lie algebra $(\widehat{\mathbf{OGC}}_{d,d+1}, [\ ,\ ], \delta := [\gamma,\ ])$ controls the deformation theory of the morphism f. From now one we abbreviate this structure to $\widehat{\mathbf{OGC}}_{d,d+1}$; it provides us with the main tool which we use to prove Theorem 6.1.1 in the next section.

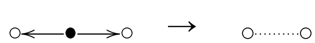
6.3 Proof of Theorem 6.1.1

We claim that there are morphisms of dg Lie algebras

$$\mathbf{GC}^0_d \xleftarrow{\pi_1} \widehat{\mathbf{OGC}}_{d,d+1} \xrightarrow{\pi_2} \mathbf{OGC}^0_{d+1}.$$

Here the right-hand arrow π_2 is the obvious projection onto the second factor in (6.2).
To construct the left-hand arrow π_1 we introduce the following notation. Let us call a black vertex of a graph $\Gamma \in \mathbf{OGC}_{d,d+1}$ (or in $\mathbf{OGC}^0_{d+1}$) *inessential* if it is of valence 2 with two outputs, and *essential* otherwise. Then we define π_1 as follows:

- $\pi_1(\mathbf{OGC}^0_{d+1}) = 0$.
- For $\Gamma \in \mathbf{OGC}_{d,d+1}$ with at least one essential black vertex we set $\pi_1(\Gamma) = 0$.
- Suppose $\Gamma \in \mathbf{OGC}_{d,d+1}$ has only inessential black vertices. Then necessarily each such vertex has two white neighbors. We may hence build a graph $\pi_1(\Gamma) \in \mathbf{GC}^0_d$ by retaining only the white vertices, and adding one edge between white vertices (u, v) for every inessential vertex connected to u and v. (Note that here possibly $u = v$.)



It is elementary to check that π_1 and π_2 are both morphisms of dg Lie algebras. The main Theorem is then an immediately consequence of the following two Propositions.

6.3.1 Proposition

The map π_2 is a quasi-isomorphism.

Proof. Since π_2 is surjective, the statement of the proposition is equivalent to the fact that $\ker \pi_2 = \mathbf{OGC}_{d,d+1}$ is acyclic, which we shall show. Let us consider a filtration of $\mathbf{OGC}_{d,d+1}$ by the total number of vertices. The induced differential in the associated graded complex $gr\mathbf{OGC}_{d,d+1}$ just makes white vertices (if any) into black ones, $\circ \to \bullet$. By Maschke's Theorem, to prove the acyclicity of $gr\mathbf{OGC}_{d,d+1}$ it is enough to prove the acyclicity of its version $gr\mathbf{OGC}^{marked}_{d,d+1}$ in which the generating graphs Γ have all their edges and vertices totally ordered, but the type of vertices is not fixed. Then there is an isomorphism of complexes

$$gr\mathbf{OGC}^{marked}_{d,d+1} = \prod_{\text{generators } \Gamma} \left(\bigotimes_{v \in V(\Gamma)} C_v \right),$$

where C_v is either a one-dimensional trivial complex (generated by a black vertex) in the case when v has at least one outgoing solid edge in Γ, or C_v is a two-dimensional acyclic complex (generated by one black and one white vertex) if v has no outgoing solid edges in Γ. As any generating graph Γ has least one vertex of the latter type since there cannot be directed cycles, the complex $gr\mathbf{OGC}^{marked}_{d,d+1}$ has at least one acyclic tensor factor. The claim is proven (cf. Lemma 6.2.1 in [Mer25]). □

6.3.2 Proposition

The map π_1 is a quasi-isomorphism.

Proof. Note that the map π_1 is also surjective. Hence it is sufficient to check that

$$\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1} := \ker \pi_1$$

is acyclic. Furthermore note that $\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1} \subset \widehat{\mathbf{OGC}}_{d,d+1}$ is the subcomplex spanned by graphs with at least one black essential vertex. Let us filter $\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1}$ by the total number of white vertices + twice the number of essential black vertices which are not univalent sources. Here we use the fact that the differential in $\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1}$ cannot create new univalent black sources, but can kill such vertices by attaching to them univalent white targets.
We claim that the associated graded with respect to this filtration is acyclic

$$H^{\bullet}(gr\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1}) = 0, \tag{6.3}$$

from which the proposition immediately follows. To see this, note that the differential on the complex $gr\widehat{\mathbf{OGC}}^{\bullet}_{d,d+1}$ has only the following two terms, $\delta = \delta_1 + \delta_2$ with

- δ_1 is the natural inclusion of $\mathbf{OGC}^0_{d+1}$ into (the kernel of π_1 inside) $\mathbf{OGC}_{d,d+1}$.

- δ_2 creates one new inessential vertex from an edge between two essential vertices:

$$\bullet\!\!\longrightarrow\!\!\circ \quad \to \quad \bullet\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\circ \ , \qquad\qquad \bullet\!\!\longrightarrow\!\!\bullet \quad \to \quad \bullet\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet \ .$$

The map $\delta_1 : \mathbf{OGC}^0_{d+1} \to \mathbf{OGC}_{d,d+1}$ is injective. Hence it is sufficient to check that

$$\mathrm{coker}(\delta_1) =: \mathbf{OGC}^{\bullet\circ}_{d,d+1}$$

is acyclic, i.e.,

$$H(\mathbf{OGC}^{\bullet\circ}_{d,d+1}, \delta_2) = 0.$$

Note that $\mathbf{OGC}^{\bullet\circ}_{d,d+1}$ can be identified with the subquotient of $\mathbf{OGC}_{d,d+1}$ spanned by graphs with at least one white vertex and at least one black essential vertex. Finally, the acyclicity of $(\mathbf{OGC}^{\bullet\circ}_{d,d+1}, \delta_2)$ can be easily shown using the argument of §6.2.3 in [Mer25]. Indeed, one can assume without loss of generality that the essential vertices of graphs Γ generating $\mathbf{OGC}^{\bullet\circ}_{d,d+1}$ are distinguished, say totally ordered. Every such graph Γ contains at least one black essential vertex and at least one white vertex which are connected by an edge, or are both neighbors of some (the same) inessential vertex, and we can assume without loss of generality that their labels are 1 and 2 respectively. Considering a filtration of $\mathbf{OGC}^{\bullet\circ}_{d,d+1}$ by the number of inessential vertices not between 1 and 2, we arrive at the associated graded complex which is the tensor product of a trivial complex and the complex C_{12} which controls the types of all possible "edges" between vertices 1 and 2. One has

$$C_{12} = \bigoplus_{k\geqslant 1} \odot^k C$$

where C is a 2-dimensional complex generated by the two possible connections,

$$C = \mathrm{span}\left\langle \overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\circ} \ , \quad \overset{1}{\bullet}\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\overset{2}{\circ} \right\rangle$$

with the differential sending the first element to the second. This complex is acyclic implying the acyclicity of $\mathbf{OGC}^{\bullet\circ}_{d,d+1}$. The Proposition is proven. □

The proof of Theorem 6.1.1 is hence completed.

Chapter 7

Lie brackets on reduced graph complexes

The aim of this chapter is to introduce and compare two dg Lie algebra structures on $\overline{\mathbf{GC}}_{d+1}$ and to show that the two induced Lie algebra structures on cohomology are isomorphic.

7.1 A reduced operad of graphs

Let $\overline{\mathbf{Gra}}_{d+1}$ be an operad spanned by connected graphs with labelled vertices and two types of edges: solid edges $\longrightarrow$ with fixed direction and of degree $-d$ and dotted edges $\dashrightarrow = (-1)^d \dashleftarrow$ of degree $1-d$. The operadic composition are given by graph insertions.

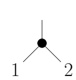
There is a differential δ' which acts on edges by

$$\longrightarrow \;\longmapsto\; \dashrightarrow .$$

Lemma 7.1.1. *There is a morphism of dg operads*

$$(\mathbf{Lie}_{d+1}, 0) \longrightarrow (\overline{\mathbf{Gra}}_{d+1}, \delta')$$

defined by



$$\longmapsto {}_1\bullet\!\!\longrightarrow\!\!\bullet_2 + (-1)^{d+1}\, {}_1\bullet\!\!\longleftarrow\!\!\bullet_2 .$$

Proof. This follows from the computation

$$\delta'\big({}_1\bullet\!\!\longrightarrow\!\!\bullet_2 + (-1)^{d+1}\, {}_1\bullet\!\!\longleftarrow\!\!\bullet_2\big) = {}_1\bullet\!\!\dashrightarrow\!\!\bullet_2 + (-1)^{d+1}\, {}_1\bullet\!\!\dashleftarrow\!\!\bullet_2 = 0.$$

□

The *full reduced graph complex* is now defined as the deformation complex $\overline{\mathbf{fGC}}_{d+1} = \mathbf{Def}(\mathbf{Lie}_{d+1} \to \overline{\mathbf{Gra}}_{d+1})$. It comes equipped with a differential $\delta + \delta'$ where δ is given by vertex splitting and δ' changes the types of edges. We denote by $[\bullet, \bullet]_{st}$ the Lie bracket, which is given by graph insertions.

The reduced graph complex $\overline{\mathbf{GC}}_{d+1}$ spanned by graphs with vertices being at least trivalent of Section 3.4.3 is a subcomplex of $\overline{\mathbf{fGC}}_{d+1}$ and more importantly a sub dg Lie algebra.

7.2 Another Lie bracket on $\overline{\mathbf{GC}}_{d+1}$

Recall the quasi-isomorphism (3.3)

$$f : \overline{\mathbf{GC}}_{d+1} \longrightarrow \mathbf{dGC}_{d+1}$$

sending every dotted edge to a linear combination

$$\bullet \cdots\!\!\to \bullet \longmapsto \frac{1}{2}(\bullet\!\!\longrightarrow\!\!\bullet\!\!\longleftarrow\!\!\bullet \; - \; \bullet\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet).$$

Theorem 7.2.1. *There is a Lie algebra structure* $[-,-]$ *on* $\overline{\mathbf{GC}}_{d+1}$ *such that the morphism* $f : \overline{\mathbf{GC}}_{d+1} \to \mathbf{dGC}_{d+1}$ *is a morphism of Lie algebras.*

Proof. As the map f is a monomorphism, it is enough to show that the image $f(\overline{\mathbf{GC}}_{d+1}) \subset \mathbf{dGC}_{d+1}$ is Lie subalgebra. In this case we define, for $\Gamma_1, \Gamma_2 \in \overline{\mathbf{GC}}_{d+1}$, the Lie bracket by $[\Gamma_1, \Gamma_2] := f^{-1}([f(\Gamma_1), f(\Gamma_2)])$.
Let $\Gamma_1, \Gamma_2 \in \overline{\mathbf{GC}}_{d+1}$. Summands in $[f(\Gamma_1), f(\Gamma_2)]$ have either all vertices at least trivalent and belong to the image or contain bivalent vertices. Any bivalent comes from some dotted edge and by the definition of the map (3.3) we have the same term except for a change of sign and the edges of the bivalent vertex being reversed. Hence the two bivalent vertices combine to a dotted edge and doing this for the remaining bivalent vertices yields the desired result.

□

The Lie bracket can be described intrinsically in the following way. There is a pre-Lie algebra structure $\Gamma_1 \circ \Gamma_2$ defined by

$$\Gamma_1 \circ \Gamma_2 = \sum_{v \in V(\Gamma_1)} \Gamma_1 \circ_v \Gamma_2 + \sum_{e \in E_{dot}(\Gamma_1)} \Gamma_1 \circ_e \Gamma_2.$$

First for a vertex $v \in V(\Gamma_1)$ we have

$$\Gamma_1 \circ_v \Gamma_2 = \sum_{k=0}^{\min(val(v), \#E_{dot}(\Gamma_2))} \frac{1}{2^k} \sum_{\substack{I \subset E_{dot}(\Gamma_2) \\ \#I = k}} \Gamma_1 \circ_{v,I} \Gamma_2,$$

where $\Gamma_1 \circ_{v,I} \Gamma_2$ is given by summing over all attachments of the half-edges of v to the vertices of Γ_2 and the middle vertices obtained by replacing each dotted edge in I by the linear combination $\bullet\!\!\longrightarrow\!\!\bullet\!\!\longleftarrow\!\!\bullet - \bullet\!\!\longleftarrow\!\!\bullet\!\!\longrightarrow\!\!\bullet$. We also require that each middle vertex is hit by at least one half-edge of v.

Second for a dotted edge $e \in E_{dot}(\Gamma_1)$ we have

$$\Gamma_1 \circ_e \Gamma_2 = \sum_{k=0}^{\min(2, \# E_{dot}(\Gamma_2))} \frac{1}{2^{k+1}} \sum_{\substack{I \subset E_{dot}(\Gamma_2) \\ \# I = k}} \Gamma_1 \circ_{e,I} \Gamma_2,$$

where $\Gamma_1 \circ_{e,I} \Gamma_2$ is obtained as above except that we replace the edge e by the linear combination $\bullet\!\longrightarrow\!\bullet\!\longleftarrow\!\bullet \; - \; \bullet\!\longleftarrow\!\bullet\!\longrightarrow\!\bullet$ and insert Γ_2 into the newly created bivalent vertex.
The Lie bracket is given by

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \circ \Gamma_2 - (-1)^{|\Gamma_1||\Gamma_2|} \Gamma_2 \circ \Gamma_1.$$

In particular we can write

$$[\Gamma_1, \Gamma_2] = [\Gamma_1, \Gamma_2]_{st} + [\Gamma_1, \Gamma_2]_{sp}$$

where $[\Gamma_1, \Gamma_2]_{st}$ is the standard graph insertions defined as in $\mathbf{dGC}_{d+1}$ while $[\Gamma_1, \Gamma_2]_{sp}$ consists of all terms which have additional vertices, i.e. where a dotted edge got destroyed.

Note that for graphs Γ_1, Γ_2 having no dotted edges, the two Lie brackets coincide. This observation can be used as follows. There is a quasi-isomorphism g from the usual graph complex $\mathbf{GC}_{d+1}$ to the reduced complex $\overline{\mathbf{GC}}_{d+1}$ by sending an (undirected) graph to the sum over all possible direction. As the Lie brackets in $\mathbf{GC}_{d+1}$ and $[-,-]_{st}$ are both given by graph insertions, the map g defines a quasi-isomorphism of dg Lie algebras

$$(\mathbf{GC}_{d+1}, [-,-]) \longrightarrow (\overline{\mathbf{GC}}_{d+1}, [-,-]_{st}).$$

In addition the summands to not contain any dotted edges and thus on the subcomplex $g(\mathbf{GC}_{d+1})$ the two Lie brackets on $\overline{\mathbf{GC}}_{d+1}$ are equal. In particular the map g defines a quasi-isomorphism of dg Lie algebras

$$(\mathbf{GC}_{d+1}, [-,-]) \longrightarrow (\overline{\mathbf{GC}}_{d+1}, [-,-]_{st}).$$

We can summarize the discussion above in the following theorem.

Theorem 7.2.2. *There is a zigzag of quasi-isomorphism of dg Lie algebras*

$$(\overline{\mathbf{GC}}_{d+1}, [-,-]_{st}) \longleftarrow (\mathbf{GC}_{d+1}, [-,-]) \longrightarrow (\mathbf{dGC}_{d+1}, [-,-]) \longleftarrow (\overline{\mathbf{GC}}_{d+1}, [-,-]).$$

In particular there is an isomorphism of Lie algebras

$$(H^\bullet(\overline{\mathbf{GC}}_{d+1}), [-,-]_{st}) \cong (H^\bullet(\overline{\mathbf{GC}}_{d+1}), [-,-]).$$

Remark 7.2.3. It is not hard to see that all the constructions above work if we consider oriented graphs and thus we have two dg Lie algebra structures on $\overline{\mathbf{OGC}}_{d+1}$. However the proof of Theorem 7.2.2 does not work here (as summing over all directions will create directed cycles). Hence whether the induced Lie algebra structures on $H^\bullet(\overline{\mathbf{OGC}}_{d+1})$ are isomorphic or not is left open.

Bibliography

[AM22] Assar Andersson and Sergei Merkulov. From deformation theory of wheeled props to classification of Kontsevich formality maps. *Int. Math. Res. Not. IMRN*, (12):9275–9307, 2022.

[AT12] Anton Alekseev and Charles Torossian. The Kashiwara-Vergne conjecture and Drinfeld's associators. *Ann. of Math. (2)*, 175(2):415–463, 2012.

[AWZ20] Assar Andersson, Thomas Willwacher, and Marko Zivkovic. Oriented hairy graphs and moduli spaces of curves, 2020.

[Bar08] Vladimir Baranovsky. A universal enveloping for L_∞-algebras. *Math. Res. Lett.*, 15(6):1073–1089, 2008.

[BD82] A. A. Belavin and V. G. Drinfel'd. Solutions of the classical Yang-Baxter equation for simple Lie algebras. *Funktsional. Anal. i Prilozhen.*, 16(3):1–29, 96, 1982.

[Bek05] Xavier Bekaert. Universal enveloping algebras and some applications in physics. 2005.

[BFF$^+$78] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. I. Deformations of symplectic structures. *Ann. Physics*, 111(1):61–110, 1978.

[BM08] Dennis V. Borisov and Yuri I. Manin. *Generalized Operads and Their Inner Cohomomorphisms*, pages 247–308. Birkhäuser Basel, Basel, 2008.

[Bro12] Francis Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[CGP21] Melody Chan, Søren Galatius, and Sam Payne. Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$. *J. Amer. Math. Soc.*, 34(2):565–594, 2021.

[CS04] Moira Chas and Dennis Sullivan. *Closed String Operators in Topology Leading to Lie Bialgebras and Higher String Algebra*, pages 771–784. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[Dol21] V. A. Dolgushev. Stable formality quasi-isomorphisms for Hochschild cochains. *Mém. Soc. Math. Fr. (N.S.)*, (168):vi + 108, 2021.

[Dri87] V. G. Drinfel'd. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820. Amer. Math. Soc., Providence, RI, 1987.

[Dri90] V. G. Drinfel'd. On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.

[Dri92] V. G. Drinfeld. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

[EK96] Pavel Etingof and David Kazhdan. Quantization of Lie bialgebras. I. *Selecta Math. (N.S.)*, 2(1):1–41, 1996.

[FW20] Benoit Fresse and Thomas Willwacher. Mapping spaces for dg hopf cooperads and homotopy automorphisms of the rationalization of E_n-operads. *arXiv*, 2020.

[Gut11] Simone Gutt. Deformation quantisation of Poisson manifolds. In *Lectures on Poisson geometry*, volume 17 of *Geom. Topol. Monogr.*, pages 171–220. Geom. Topol. Publ., Coventry, 2011.

[Kat98] Vinay Kathotia. Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula, i. *arXiv*, 1998.

[Kon97] Maxim Kontsevich. Formality conjecture. In *Deformation theory and symplectic geometry (Ascona, 1996)*, volume 20 of *Math. Phys. Stud.*, pages 139–156. Kluwer Acad. Publ., Dordrecht, 1997.

[Kon03] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.

[KS97] Y. Kosmann-Schwarzbach. Lie bialgebras, Poisson Lie groups and dressing transformations. In *Integrability of nonlinear systems (Pondicherry, 1996)*, volume 495 of *Lecture Notes in Phys.*, pages 104–170. Springer, Berlin, 1997.

[KT23] Anton Khoroshkin and Pedro Tamaroff. Derived Poincaré–Birkhoff–Witt theorems. *Letters in Mathematical Physics*, 113(1):15, Jan 2023.

[KWŽ17] Anton Khoroshkin, Thomas Willwacher, and Marko Živković. Differentials on graph complexes. *Advances in Mathematics*, 307:1184–1214, 2017.

[LM95] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995.

[Mar04] Martin Markl. Homotopy algebras are homotopy algebras. *Forum Math.*, 16(1):129–160, 2004.

[Mer06] S. A. Merkulov. PROP profile of Poisson geometry. *Comm. Math. Phys.*, 262(1):117–135, 2006.

[Mer09] S. A. Merkulov. Graph complexes with loops and wheels. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 311–354. Birkhäuser Boston, Boston, MA, 2009.

[Mer10] Sergei A. Merkulov. Wheeled props in algebra, geometry and quantization. In *European Congress of Mathematics*, pages 83–114. Eur. Math. Soc., Zürich, 2010.

[Mer11] S. A. Merkulov. Permutahedra, HKR isomorphism and polydifferential Gerstenhaber-Schack complex. In *Higher structures in geometry and physics*, volume 287 of *Progr. Math.*, pages 293–314. Birkhäuser/Springer, New York, 2011.

[Mer23] Sergei A. Merkulov. Twisting of properads. *J. Pure Appl. Algebra*, 227(10):Paper No. 107388, 47, 2023.

[Mer25] Sergei A. Merkulov. On Interrelations Between Graph Complexes. *Int. Math. Res. Not. IMRN*, (2):rnae287, 2025.

[MF22] José Manuel Moreno Fernández. The Milnor-Moore theorem for L_∞ algebras in rational homotopy theory. *Math. Z.*, 300(3):2147–2165, 2022.

[MV09] Sergei Merkulov and Bruno Vallette. Deformation theory of representations of prop(erad)s. II. *J. Reine Angew. Math.*, 636:123–174, 2009.

[MW15] Sergei Merkulov and Thomas Willwacher. Props of ribbon graphs, involutive Lie bialgebras and moduli spaces of curves. *arXiv*, 2015.

[MW18a] Sergei Merkulov and Thomas Willwacher. Deformation theory of Lie bialgebra properads. In *Geometry and Physics: A Festschrift in honour of Nigel Hitchin.* Oxford University Press, 2018.

[MW18b] Sergei Merkulov and Thomas Willwacher. An explicit two step quantization of Poisson structures and Lie bialgebras. *Comm. Math. Phys.*, 364(2):505–578, 2018.

[MW20] Sergei Merkulov and Thomas Willwacher. Classification of universal formality maps for quantizations of Lie bialgebras. *Compos. Math.*, 156(10):2111–2148, 2020.

[MWW24] Sergei Merkulov, Thomas Willwacher, and Vincent Wolff. The oriented graph complex revisited. *arXiv*, 2024.

[MŽ22] Sergei Merkulov and Marko Živković. Quantizations of Lie bialgebras, duality involution and oriented graph complexes. *Letters in Mathematical Physics*, 112(1):13, Feb 2022.

[RW14] Carlo A Rossi and Thomas Willwacher. P. Etingof's conjecture about Drinfeld associators. *arXiv preprint arXiv:1404.2047*, 2014.

[Sho01] Boris Shoikhet. Vanishing of the Kontsevich integrals of the wheels. volume 56, pages 141–149. 2001. EuroConférence Moshé Flato 2000, Part II (Dijon).

[Sta63] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[Ž20] Marko Živković. Multi-directed graph complexes and quasi-isomorphisms between them I: oriented graphs. *High. Struct.*, 4(1):266–283, 2020.

[Wil15a] Thomas Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra. *Invent. Math.*, 200(3):671–760, 2015.

[Wil15b] Thomas Willwacher. The oriented graph complexes. *Comm. Math. Phys.*, 334(3):1649–1666, 2015.

[Wol23] Vincent Wolff. From the Lie Operad to the Grothendieck–Teichmüller Group. *International Mathematics Research Notices*, page rnad225, 09 2023.

[Wol25] Vincent Wolff. Polydifferential Lie bialgebras and graph complexes. *Letters in Mathematical Physics*, 115(2):28, Mar 2025.

[Yau16] Donald Yau. *Colored operads*, volume 170 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2016.