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Deformation theory of representations of prop(erad)s II

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Abstract. This paper is the follow-up of [39].

Introduction

In Section 1, we give another, more geometric, interpretation of the L_∞-algebra structure on the homotopy convolution properad introduced in [39], Section 4.5. In Section 2, we define the deformation theory of morphisms of properads following Quillen's method. It is identified with a homotopy convolution properad, so it carries a natural L_∞-algebra structure in general and a strict Lie algebra structure only in the Koszul case. Using explicit resolutions of properads, one can make the associated chain complexes explicit. Section 3 is devoted to examples. For instance, we show that for any minimal resolution of the properad encoding associative bialgebras, the deformation complex is isomorphic to Gerstenhaber-Schack bicomplex. As a corollary, this proves the existence of an L_∞-algebra structure on the Gerstenhaber-Schack bicomplex associated to the deformations of associative bialgebras. In the appendix, we endow the category of dg properads with a model category structure.

Contents

1. L_∞-algebras, dg manifolds, dg affine schemes and morphisms of prop(erad)s

1.1. L_∞-algebras, dg manifolds and dg affine schemes. A structure of an L_∞-algebra on a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ is, by definition, a degree -1 coderivation, $Q : \odot^{\geqq 1} s\mathfrak{g} \to \odot^{\geqq 1} s\mathfrak{g}$, of the free cocommutative coalgebra without counit,

$$\odot^{\geqq 1} s\mathfrak{g} := \bigoplus_{n \geqq 1} \odot^n(s\mathfrak{g}) \subset \odot^\bullet s\mathfrak{g} := \bigoplus_{n \geqq 0} \odot^n(s\mathfrak{g}),$$

which satisfies the condition $Q^2 = 0$. It is often very helpful to use geometric intuition and language when working with L_∞-algebras. Let us view the vector space $s\mathfrak{g}$ as a formal graded manifold (so that a choice of a basis in $\mathfrak{g}$ provides us with natural smooth coordinates on $s\mathfrak{g}$). If $\mathfrak{g}$ is finite-dimensional, then the structure ring, $\mathcal{O}_{s\mathfrak{g}}$, of formal smooth functions on the formal manifold $s\mathfrak{g}$ is equal to the completed graded commutative algebra $\widehat{\odot^\bullet}(s\mathfrak{g})^* := \prod_{n \geqq 0} \odot^n(s\mathfrak{g})^*$ which is precisely the dual of the coalgebra $\odot^\bullet s\mathfrak{g}$. This dualization sends the augmentation $\odot^{\geqq 1} s\mathfrak{g}$ of the latter into the ideal $I := \prod_{n \geqq 1} \odot^n(s\mathfrak{g})^*$ of the distinguished point $0 \in s\mathfrak{g}$, while the dual of the coderivation Q is sent into a degree -1 derivation of $\mathcal{O}_{s\mathfrak{g}}$, i.e. into a formal vector field (denoted by the same letter Q) on the manifold $s\mathfrak{g}$ which vanishes at the distinguished point (as $QI \subset I$) and satisfies the condition $[Q, Q] = 2Q^2 = 0$. Such vector fields are often called *homological*.

In this geometric picture of L_∞-algebra structures on $\mathfrak{g}$, the subclass of dg Lie algebra structures gets represented by at most quadratic homological vector fields Q, that is $Q\big((s\mathfrak{g})^*\big) \subset (s\mathfrak{g})^* \oplus \odot^2(s\mathfrak{g})^*$. Such a vector field has a well-defined value at an arbitrary point $s\gamma \in s\mathfrak{g}$, not only at the distinguished point $0 \in s\mathfrak{g}$, i.e. it defines a smooth homological vector field on $s\mathfrak{g}$ viewed as an ordinary (non-formal) graded manifold. Given a particular dg Lie algebra $(\mathfrak{g}, d, [\,,\,])$, the associated homological vector field Q on $s\mathfrak{g}$ has the value at a point $s\gamma \in s\mathfrak{g}$ given explicitly by

$$Q(\gamma) := d\gamma + \frac{1}{2}[\gamma, \gamma], \tag{1}$$

where we used a canonical identification of the tangent space $\mathcal{T}_\gamma$ at $s\gamma \in s\mathfrak{g}$ with $\mathfrak{g}$. One checks

$$\begin{aligned} Q^2(\gamma) &= Q\left(d\gamma + \frac{1}{2}[\gamma,\gamma]\right) \\ &= -d\big(Q(\gamma)\big) + [Q(\gamma),\gamma] \\ &= -d\left(d\gamma + \frac{1}{2}[\gamma,\gamma]\right) + \left[d\gamma + \frac{1}{2}[\gamma,\gamma],\gamma\right] \\ &= 0. \end{aligned}$$

Notice that the zero locus of Q is the set of Maurer-Cartan elements in $\mathfrak{g}$.

A serious deficiency of the above geometric interpretation of L_∞-algebras is the necessity to work with the dual objects $(\mathcal{O}_{s\mathfrak{g}}, Q)$ which make sense only for finite dimensional $\mathfrak{g}$. So we follow a suggestion of Kontsevich [23] and understand from now on a dg (*smooth formal*) *manifold* as a pair $(\odot^{\geqq 1}X, Q)$ consisting of a cofree cocommutative algebra on a $\mathbb{Z}$-graded vector space X together with a degree -1 codifferential Q. Note that the dual of $\odot^{\geqq 1}X$ is a well defined graded commutative algebra (without assumption on finite-dimensionality of X) and that the dual of Q is a well-defined derivation of the latter. We identify from now on Q with its dual and call it a *homological* vector field on the dg manifold[1] X. This abuse of terminology is very helpful as it permits us to employ geometric intuition and use simple formulae of type (1) to define (in a mathematically rigorous way!) codifferentials Q on $\odot^{\geqq 1}X$. Such codifferentials, $Q : \odot^{\geqq 1}X \to \odot^{\geqq 1}X$, are completely determined by the associated compositions

$$Q_{\mathrm{proj}} : \odot^{\geqq 1}X \xrightarrow{Q} \odot^{\geqq 1}X \xrightarrow{\mathrm{proj}} X.$$

The restriction of Q_{proj} to $\odot^n X \subset \odot^{\geqq 1}X$ is denoted by $Q^{(n)}$, $n \geqq 1$.

Since we work with dual notions (coalgebras, coderivations), we will need the notion of *coideal*, which is the categorical dual to the notion of ideal. Hence, a coideal I of a coalgebra C is defined to be a quotient of C such that the kernel of the associated projection $C \twoheadrightarrow I$ is a subcoalgebra of C. For a complete study of this notion, we refer the reader to Appendix B "Categorical Algebra" of [51]. This notion should not be confused with the notion of coideal used in Hopf algebra theory. Since a Hopf is an algebra and a coalgebra at the same time, a coideal in that sense is a submodule such that the induced quotient carries again a bialgebra structure.

If I is a coideal of the coalgebra $\odot^{\geqq 1}X$, we denote the associated sub-coalgebra of $\odot^{\geqq 1}X$ by $(\mathcal{O}_I := I\backslash\odot^{\geqq 1}X, Q)$. The latter is defined by the push-out diagram in the category of coalgebras

$$\begin{array}{ccc} \mathcal{O}_I & \longrightarrow & \odot^{\geqq 1}X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & I. \end{array}$$

[1] A warning about shift of grading: according to our definitions, a homological vector field on a graded vector space X is the same as an L_∞-structure on $s^{-1}X$.

If the coideal I is preserved by Q (i.e. admits a codifferential such that the right vertical arrow is a morphism of dg coalgebras), then the data $(\mathcal{O}_I, Q)$ is naturally a differential graded coalgebra which we often call a dg *affine scheme* (cf. [3]). The coideal may not, in general, be homogeneous so the "weight" gradation $\bigoplus_n \odot^n X$ may not survive in $\mathcal{O}_I$. A generic dg affine scheme by no means corresponds to an L_∞-algebra but, as we shall see below, some interesting examples (with non-trivial and non-homogeneous coideals) do.

A *morphism* of dg affine schemes is, by definition, a morphism of the associated dg coalgebras, $(I_1 \backslash \odot^{\geqq 1} X_1, Q_1) \to (I_2 \backslash \odot^{\geqq 1} X_2, Q_2)$.

1.2. Another geometric model for an L_∞-structure. One can interpret an L_∞-structure on a graded vector space $\mathfrak{g}$ as a linear total degree 1 polyvector field on the dual vector space $\mathfrak{g}^*$ viewed as a graded affine manifold. Note that there is no need to employ the degree shifting functors s and s^{-1} in this approach.

Indeed, let $(\wedge^\bullet \mathcal{T}_{\mathfrak{g}^*}, [\,,\,]_S)$ be the Schouten Lie algebra of polynomial polyvector fields on the affine manifold $\mathfrak{g}^*$. A generic total degree 1 polynomial polyvector field, $v = \{v_n \in \wedge^n \mathcal{T}_{\mathfrak{g}^*}\}_{n \geqq 0}$, can be identified with a collection of its Taylor components with respect to affine coordinates on $\mathfrak{g}^*$, i.e. with a collection of linear maps,

$$v_{m,n} : \odot^m \mathfrak{g}^* \to \wedge^n \mathfrak{g}^*, \quad m \geqq 0, n \geqq 0,$$

of degree $n - 2$. If v is a linear polyvector field and lies in the Lie subalgebra $\wedge^{\geqq 1} \mathcal{T}_{\mathfrak{g}^*}$, then only the Taylor components $\{v_{1,n}\}_{n \geqq 1}$ can be non-zero. Their duals, $v_n := (v_{1,n})^*$, is a collection of linear maps, $v_n : \wedge^n \mathfrak{g} \to \mathfrak{g}$, $n \geqq 1$, of degree $2 - n$. It is easy to check the following

Proposition 1. *The data $\{v_n\}_{n \geqq 1}$ defines a structure of L_∞-algebra on $\mathfrak{g}$ if and only if the linear polyvector field v on $\mathfrak{g}^*$ satisfies the equation $[v, v]_S = 0$.*

Corollary 2. *There is a one-to-one correspondence between structures of L_∞-algebra on a finite-dimensional vector space $\mathfrak{g}$ and linear degree one polyvector fields, $v \in \wedge^{\geqq 1} \mathcal{T}_{\mathfrak{g}^*}$, satisfying the equation $[v, v]_S = 0$.*

Kontsevich's formality morphism [23], $\mathcal{F}$, associates to an arbitrary Maurer-Cartan element in the Schouten Lie algebra $v \in \wedge^\bullet \mathcal{T}_{\mathfrak{g}^*}$ a Maurer-Cartan element, $\mathcal{F}(v)$, in the Hochschild dg Lie algebra, $\bigoplus_{n \geqq 0} \mathrm{Hom}_{\mathrm{poly}}(\mathcal{O}_{\mathfrak{g}^*}^{\otimes n}, \mathcal{O}_{\mathfrak{g}^*})[[\hbar]]$, of polydifferential operators on the graded commutative algebra, $\mathcal{O}_{\mathfrak{g}^*} := \odot^\bullet \mathfrak{g}$, of smooth functions on the affine manifold $\mathfrak{g}^*$. If v is a linear polyvector field on $\mathfrak{g}^*$ satisfying the equation $[v, v]_S = 0$, then one can set to zero all contributions to the formality morphism $\mathcal{F}$ coming from graphs with closed directed paths (wheels) [43] and the resulting element $\mathcal{F}_{\text{no-wheels}}(v) \in \bigoplus_{n \geqq 0} \mathrm{Hom}_{\mathrm{poly}}(\mathcal{O}_{\mathfrak{g}^*}^{\otimes n}, \mathcal{O}_{\mathfrak{g}^*})[[\hbar]]$ is still Maurer-Cartan. It is easy to check that $\mathcal{F}_{\text{no-wheels}}(v)$ has no summand with weight $n = 0$ and hence defines an A_∞-structure on $\odot^\bullet \mathfrak{g}$ which also makes sense for $\hbar = 1$. Moreover, as $\mathcal{F}_{\text{no-wheels}}(v)$ involves no wheels (and hence no associated traces of linear maps), this A_∞-structure makes sense for arbitrary (not necessarily finite-dimensional) L_∞-algebra.

Definition. Let $\{v_n : \wedge^n \mathfrak{g} \to \mathfrak{g}\}_{n \geqq 1}$ be an L_∞-structure on a graded vector space $\mathfrak{g}$. The A_∞-structure, $\mathcal{F}_{\text{no-wheels}}(v)$, on $\odot^\bullet \mathfrak{g}$ obtained via Kontsevich's "no-wheels" quantization of the associated linear polyvector field v is called the *universal enveloping algebra* of the L_∞-algebra.

In a recent interesting paper [4] Baranovsky also defined a universal enveloping for an L_∞-algebra $\mathfrak{g}$ as a certain A_∞-structure on the space $\odot^\bullet \mathfrak{g}$. In his approach the A_∞-structure is constructed with the help of the homological perturbation and the natural homotopy transfer of the canonical dg associative algebra structure on the cobar construction on the dg coalgebra $\odot^\bullet s\mathfrak{g}$.

1.3. Maurer-Cartan elements in a filtered L_∞-algebra. An L_∞-algebra

$$(\mathfrak{g}, Q = \{Q^{(n)}\}_{n \geqq 1})$$

is called *filtered* if $\mathfrak{g}$ admits a non-negative decreasing Hausdorff filtration,

$$\mathfrak{g}_0 = \mathfrak{g} \supseteqq \mathfrak{g}_1 \supseteqq \cdots \supseteqq \mathfrak{g}_i \supseteqq \cdots,$$

such that the linear map $Q^{(n)} : \odot^n(s\mathfrak{g}) \to s\mathfrak{g}$ takes values in $s\mathfrak{g}_n$ for all $n \geqq n_0$ and some $n_0 \in \mathbb{N}$. In this case Q extends naturally to a coderivation of the cocommutative coalgebra, $\odot^{\geqq 1} s\hat{\mathfrak{g}}$, with $\hat{\mathfrak{g}}$ being the completion of $\mathfrak{g}$ with respect to the topology induced by the filtration, and the equation

$$Q\left(\sum_{n \geqq 1} \frac{1}{n!} \gamma^{\odot n}\right) = 0$$

for a degree zero element $\gamma \in s\hat{\mathfrak{g}}$ (i.e. for a degree -1 element in $\hat{\mathfrak{g}}$) makes sense. Its solutions are called (*generalized*) *Maurer-Cartan elements* (or, shortly, MC *elements*) in $(\mathfrak{g}, Q)$. Geometrically, an MC element is a degree -1 element in $\hat{\mathfrak{g}}$ at which the homological vector field Q vanishes. From now on we do not distinguish between $\mathfrak{g}$ and its completion $\hat{\mathfrak{g}}$.

To every MC element γ in a filtered L_∞-algebra $(\mathfrak{g}, Q)$ there corresponds, by [36], Theorem 2.6.1, a twisted L_∞-algebra, $(\mathfrak{g}, Q_\gamma)$, with

$$Q_\gamma(\alpha) := Q\left(\sum_{n \geqq 0} \frac{1}{n!} \gamma^{\odot n} \odot \alpha\right)$$

for an arbitrary $\alpha \in \odot^{\geqq 1} s\mathfrak{g}$. The geometric meaning of this twisted L_∞-structure is simple [36]: if a homological vector field Q vanishes at a degree 0 point $\gamma \in s\mathfrak{g}$, then applying to Q a formal diffeomorphism, ϕ_γ, which is a translation sending γ into the origin 0 (and which is nothing but the unit shift, $e^{\mathrm{ad}\,\gamma}$, along the formal integral lines of the constant vector field $-\gamma$) will give us a new formal vector field, $Q_\gamma := d\phi_\gamma(Q)$, which is *homological* and *vanishes* at the distinguished point; thus Q_γ defines an L_∞-structure on the underlying space $\mathfrak{g}$. In fact, we can apply this "translation diffeomorphism" trick to arbitrary (i.e. not necessarily MC) elements γ of degree 0 in $s\mathfrak{g}$ and get *homological* vector fields, $Q_\gamma := d\phi_\gamma(Q)$, which do not vanish at 0 and hence define generalized L_∞-structures on $\mathfrak{g}$ with "zero term" $Q_\gamma^{(0)} \neq 0$.

1.4. Extended morphisms of dg props as a dg affine scheme. Let $(\mathscr{P}, \partial_{\mathscr{P}})$ and $(\mathscr{E}, \partial_{\mathscr{E}})$ be dg prop(erad)s with differentials $\partial_{\mathscr{P}}$ and $\partial_{\mathscr{E}}$ of degree -1. Let $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$ denote the graded vector space of all possible morphisms $\mathscr{P} \to \mathscr{E}$ in category of $\mathbb{Z}$-graded $\mathbb{S}$-bimodules, and let $\mathrm{Mor}(\mathscr{P}, \mathscr{E})$ denote the set of all possible morphisms $\mathscr{P} \to \mathscr{E}$ in category of prop(erad)s (note that we do *not* assume that elements of $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ or $\mathrm{Mor}(\mathscr{P}, \mathscr{E})$ respect differentials). It is clear that

$$\mathrm{Mor}(\mathscr{P}, \mathscr{E}) = \big\{\gamma \in \mathrm{Hom}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) \,|\, \gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) = \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big) \text{ and } |\gamma| = 0\big\}.$$

We need a $\mathbb{Z}$-graded extension of this set,

$$\text{(2)} \quad \mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) := \big\{\gamma \in \mathrm{Hom}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) \,|\, \gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) = \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big)\big\},$$

which we define by the same algebraic equations but dropping the assumption on the degree and homogeneity of γ.

Lemma 3. *The vector space* $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ *is naturally a* dg *manifold.*

Proof. We define a degree -1 coderivation of the free cocommutative coalgebra, $\odot^{\geqq 1} \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ by setting (in the dual picture, cf. §1.1)

$$\text{(3)} \qquad Q(\gamma) := \partial_{\mathscr{E}} \circ \gamma - (-1)^{\gamma} \gamma \circ \partial_{\mathscr{P}}$$

for an arbitrary $\gamma \in \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$. As

$$\begin{aligned} Q^2(\gamma) &= Q\big(\partial_{\mathscr{E}} \circ \gamma - (-1)^{\gamma} \gamma \circ \partial_{\mathscr{P}}\big) \\ &= -\partial_{\mathscr{E}} \circ Q(\gamma) - (-1)^{\gamma} Q(\gamma) \circ \partial_{\mathscr{P}} \\ &= -(-1)^{\gamma} \partial_{\mathscr{E}} \circ \gamma \circ \partial_{\mathscr{P}} + (-1)^{\gamma} \partial_{\mathscr{E}} \circ \gamma \circ \partial_{\mathscr{P}} \\ &= 0, \end{aligned}$$

Q is a linear homological field on $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$. (By the way, the zero locus of Q is a linear subspace of $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ describing morphisms of *complexes*.) $\square$

Proposition 4. *The set* $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$ *is naturally a* dg *affine scheme.*

Proof. Let I be the coideal in $\odot^{\geqq 1} \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ cogenerated by the algebraic relations

$$\text{(4)} \qquad \gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) - \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big),$$

on the "variable" $\gamma \in \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$. The sub-coalgebra

$$\mathscr{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})} := I \backslash \odot^{\geqq 1} \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$$

of $\odot^{\geqq 1} \mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ makes the set $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$ into a $\mathbb{Z}$-graded affine scheme. Next we show that the homological vector field Q defined in Lemma 3 is tangent to $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$. Indeed, identifying Q and I with their duals (as in Subsection 1.1 and the proof of Lemma 3), we have

$$
\begin{aligned}
&Q\big(\gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) - \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big)\big) \\
&\quad = Q(\gamma) \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) - \mu_{\mathscr{E}}\big(Q(\gamma)(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big) \\
&\quad\quad - (-1)^{|\gamma|}\mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} Q(\gamma)(\mathscr{P})\big).
\end{aligned}
$$

Consistency of $\partial_{\mathscr{P}}$ and $\partial_{\mathscr{E}}$ with $\mu_{\mathscr{P}}$ and, respectively, $\mu_{\mathscr{E}}$ implies

$$
\begin{aligned}
Q(\gamma) \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) &= \partial_{\mathscr{E}} \circ \gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) - (-1)^{\gamma}\gamma \circ \partial_{\mathscr{P}} \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) \\
&= \partial_{\mathscr{E}} \circ \gamma \circ \mu_{\mathscr{P}}(\mathscr{P} \boxtimes_{(1,1)} \mathscr{P}) - (-1)^{\gamma}\gamma \circ \mu_{\mathscr{P}}\big(\partial_{\mathscr{P}}(\mathscr{P}) \boxtimes_{(1,1)} \mathscr{P}\big) \\
&\quad - (-1)^{\gamma}\gamma \circ \mu_{\mathscr{P}}\big(\mathscr{P} \boxtimes_{(1,1)} \partial_{\mathscr{P}}(\mathscr{P})\big) \\
&=_{\mathrm{mod}\, I} \partial_{\mathscr{E}} \circ \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big) - (-1)^{\gamma}\mu_{\mathscr{E}}\big(\gamma \circ \partial_{\mathscr{P}}(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big) \\
&\quad - \mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} \gamma \circ \partial_{\mathscr{P}}(\mathscr{P})\big) \\
&=_{\mathrm{mod}\, I} \mu_{\mathscr{E}}\big(Q(\gamma)(\mathscr{P}) \boxtimes_{(1,1)} \gamma(\mathscr{P})\big) + (-1)^{|\gamma|}\mu_{\mathscr{E}}\big(\gamma(\mathscr{P}) \boxtimes_{(1,1)} Q(\gamma)(\mathscr{P})\big).
\end{aligned}
$$

Thus $Q(I) \subset I$, and hence Q gives rise to a degree -1 codifferential on the coalgebra $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})}$ proving the claim. $\square$

In the following theorem, we study the properties of the convolution L_∞-algebra defined in [39], Theorem 28.

Theorem 5. *Let $\big(\mathscr{P} = \mathscr{F}(s^{-1}\mathscr{C}), \partial_{\mathscr{P}}\big)$ be a quasi-free prop(erad) generated by an $\mathbb{S}$-bimodule $s^{-1}\mathscr{C}$ (so that $\mathscr{C}$ is a homotopy coprop(erad)), and let $(\mathscr{E}, \partial_{\mathscr{E}})$ be an arbitrary* dg *prop(erad). Then*:

(i) *The graded vector space, $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{E})$, is canonically an L_∞-algebra.*

(ii) *The canonical L_∞-structure in* (i) *is filtered and its* MC *elements are morphisms, $(\mathscr{P}, \partial_{\mathscr{P}}) \to (\mathscr{E}, \partial_{\mathscr{E}})$, of* dg *prop(erad)s.*

(iii) *If $\partial_{\mathscr{P}}(s^{-1}\mathscr{C}) \subset \mathscr{F}(s^{-1}\mathscr{C})^{(\leqq 2)}$, where $\mathscr{F}(s^{-1}\mathscr{C})^{(\leqq 2)}$ is the subspace of $\mathscr{F}(s^{-1}\mathscr{C})$ spanned by decorated graphs with at most two vertices, then $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{E})$ is canonically a* dg *Lie algebra.*

Proof. (i) If $\mathscr{P}$ is free as a prop(erad), then extended morphisms from $\mathscr{P}$ to $\mathscr{Q}$ are uniquely determined by their values on the generators $s^{-1}\mathscr{C}$ so that $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})} = \odot^{\geqq 1}\,\mathrm{Hom}^{\mathbb{S}}_{\bullet}(s^{-1}\mathscr{C}, \mathscr{E})$ and the claim follows from the definition of L_∞-structure in §1.1.

(ii) The canonical L_∞-structure on $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{E})$ is given by the restriction of the homological vector field (3) on $s\,\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{P}, \mathscr{E})$ to the subspace $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{E})$. This field is a formal power series in coordinates on $\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{E})$ and its part, $Q^{(n)}$, corresponding to monomials of (polynomial) degree n is given precisely by

$$
(5) \qquad Q^{(1)}(\gamma) := \partial_{\mathscr{E}} \circ \gamma - (-1)^{\gamma}\gamma \circ \partial^{(1)}_{\mathscr{P}} \quad \text{and} \quad Q^{(n)}(\gamma) := -(-1)^{\gamma}\gamma \circ \partial^{(n)}_{\mathscr{P}} \quad \text{for } n > 1,
$$

where $\partial_{\mathscr{P}}^{(n)}$ is the composition[2)]

$$\partial_{\mathscr{P}}^{(n)} : s^{-1}\mathscr{C} \xrightarrow{\partial_{\mathscr{P}}} \mathscr{F}(s^{-1}\mathscr{C}) \xrightarrow{\text{proj}} \mathscr{F}(s^{-1}\mathscr{C})^{(n)}.$$

Note that the first summand on the r.h.s. of (3) contributes only to $Q^{(1)}$.

Define an exhaustive increasing filtration on the $\mathbb{S}$-bimodule $\mathscr{C}$ by

$$\mathscr{C}_0 = 0, \quad \mathscr{C}_i := s\bigcap_{n \geqq i} \operatorname{Ker} \partial_{\mathscr{P}}^{(n)} \quad \text{for } i \geqq 1,$$

and the associated decreasing filtration on $\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$ by

$$\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})_i := \{\gamma \in \operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E}) \mid \gamma(v) = 0\ \forall v \in \mathscr{C}_i\}, \quad i \geqq 0.$$

Then, for all $n \geqq 2$ and any $f_1, \dots, f_n \in \operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$, equality (5) implies that the value of the map $Q^{(n)}(f_1, \dots, f_n) \in s^{-1}\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$ on arbitrary elements of $\mathscr{C}_n \subset \operatorname{Ker} \partial_{\mathscr{P}}^{(n)}$ is equal to zero, i.e.

$$Q^{(n)}(f_1, \dots, f_n) \in \operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})_n.$$

Which in turn implies the claim that the canonical L_∞-structure on $\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$ is filtered with respect to the constructed filtration. The claim about MC elements follows immediately from the definition (3) of the homological vector field.

(iii) As $\partial_{\mathscr{P}}^{(n)} = 0$ for $n > 2$ we conclude using formula (5) that $Q^{(n)} = 0$ for all $n > 2$. □

A special case of the above theorem when both $\mathscr{P}$ and $\mathscr{E}$ are operads was proven earlier by van der Laan [53] using different ideas.

The main point of our proof of Theorem 5 is an observation that, for a free prop(erad) $\mathscr{P} = \mathscr{F}(s^{-1}\mathscr{C})$, the set $\operatorname{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$ of extended morphisms from $\mathscr{P}$ to an arbitrary prop(erad) $\mathscr{E}$, i.e. the set of solutions of equation (2), can be canonically identified with the graded *vector* space $s\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$. This simple fact makes the dg affine scheme $\operatorname{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$ into a dg *smooth* manifold and hence provides us with a canonical L_∞-structure on $\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E})$. A similar phenomenon occurs for the set of extended morphisms $\operatorname{CoMor}_{\mathbb{Z}}(\mathscr{E}^{c}, \mathscr{P}^{c})$ from an arbitrary coprop(erad) $\mathscr{E}^{c}$ to a cofree coprop(erad) $\mathscr{P}^{c} := \mathscr{F}^{c}(s\mathscr{E})$, and hence the arguments very similar to the ones used in the proof of Theorem 5 (and which we leave to the reader as an exercise) establish the following

Theorem 6. *Let* $(\mathscr{P}^{c} = \mathscr{F}^{c}(s\mathscr{C}), d_{\mathscr{P}})$ *be a quasi-free coprop(erad), that is* $\mathscr{C}$ *is a homotopy prop(erad), and let* $(\mathscr{E}^{c}, d_{\mathscr{E}})$ *be an arbitrary* dg *coprop(erad). Then*:

(i) *The graded vector space*, $\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{E}^{c}, \mathscr{C})$, *is canonically an* L_∞*-algebra.*

[2)] Note that for any differential $\partial_{\mathscr{P}}$ in a free properad $\mathscr{F}(s^{-1}\mathscr{C})$ the induced map $\partial_{\mathscr{P}}^{(1)} : s^{-1}\mathscr{C} \to s^{-1}\mathscr{C}$ is also a differential.

(ii) *The canonical L_∞-structure in* (i) *is filtered and its* MC *elements are morphisms*, $(\mathscr{E}^c, d_{\mathscr{E}}) \to (\mathscr{P}^c, d_{\mathscr{P}})$, *of* dg *coprop(erad)s.*

(iii) *If $d_{\mathscr{P}}$ is quadratic, that is $\mathscr{C}$ is a usual prop(erad), then* $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{E}^c, \mathscr{C})$ *is canonically a* dg *Lie algebra.*

For finite-dimensional $\mathscr{Q}$ and $\mathscr{C}$, Theorems 5 and 6 are, of course, equivalent to each other.

A morphism of L_∞-algebras, $(\mathfrak{g}_1, \mathscr{Q}_1) \to (\mathfrak{g}_2, \mathscr{Q}_2)$, is, by definition [22], a morphism, $\lambda : \big(\odot^\bullet(s\mathfrak{g}_1), \mathscr{Q}_1\big) \to \big(\odot^\bullet(s\mathfrak{g}_2), \mathscr{Q}_2\big)$, of the associated dg coalgebras. It is called a *quasi-isomorphism* if the composition

$$s\mathfrak{g}_1 \xrightarrow{i} \odot^\bullet(s\mathfrak{g}_1) \xrightarrow{\lambda} \odot^\bullet(s\mathfrak{g}_2) \xrightarrow{p} s\mathfrak{g}_2$$

induces an isomorphism $H(s\mathfrak{g}_1, \mathscr{Q}_1^{(1)}) \to H(s\mathfrak{g}_2, \mathscr{Q}_2^{(1)})$ of the associated homology groups with respect to the *linear* (in cogenerators) parts of the codifferentials. Here i is a natural inclusion and p a natural projection.

By analogy, a map $\phi : \big(\mathscr{F}(s^{-1}\mathscr{C}_1), \partial_1\big) \to \big(\mathscr{F}(s^{-1}\mathscr{C}_2), \partial_2\big)$ of quasi-free properads is called a *tangent* quasi-isomorphism if the composition

$$s^{-1}\mathscr{C}_1 \xrightarrow{i} \mathscr{F}(s^{-1}\mathscr{C}_1) \xrightarrow{\phi} \mathscr{F}(s^{-1}\mathscr{C}_2) \xrightarrow{p} s^{-1}\mathscr{C}_2$$

induces an isomorphism of cohomology groups, $H(s^{-1}\mathscr{C}_2, \partial_2^{(1)}) \to H(s^{-1}\mathscr{C}_2, \partial_2^{(1)})$.

If we assume that properads $\mathscr{F}(s^{-1}\mathscr{C}_1)$ and $\mathscr{F}(s^{-1}\mathscr{C}_2)$ are completed by the number of vertices (see §5.4) and that their differentials are bounded,

$$\partial_i(s^{-1}\mathscr{C}_i) \subset \mathscr{F}(s^{-1}\mathscr{C}_i)^{(\leqq n_i)} \quad \text{for some } n_i \in \mathbb{N},\ i = 1, 2,$$

then it is not hard to show (using filtrations by the number of vertices as in §5.4 and the classical Comparison Theorem of spectral sequences) that any continuous tangent quasi-isomorphism $\phi : \big(\mathscr{F}(s^{-1}\mathscr{C}_1), \partial_1\big) \to \big(\mathscr{F}(s^{-1}\mathscr{C}_2), \partial_2\big)$ is actually a quasi-isomorphism in the ordinary sense.

Theorem 7. (i) *Let* $\big(\mathscr{P}_1 := \mathscr{F}(s^{-1}\mathscr{C}_1), \partial_1\big)$ *and* $\big(\mathscr{P}_2 := \mathscr{F}(s^{-1}\mathscr{C}_2), \partial_2\big)$ *be quasi-free prop(erad)s*, $(\mathscr{E}, \partial_{\mathscr{E}})$ *a* dg *prop(erad), and* $\big(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_1, \mathscr{E}), Q_1\big)$ *and* $\big(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_2, \mathscr{E}), Q_2\big)$ *the associated L_∞-algebras. Then any morphism*

$$\phi : (\mathscr{P}_1, \partial_1) \to (\mathscr{P}_2, \partial_2)$$

of dg *prop(erad)s induces canonically an associated morphism*

$$\phi_{\mathrm{ind}} : \big(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_2, \mathscr{E}), Q_2\big) \to \big(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_1, \mathscr{E}), Q_1\big)$$

of L_∞-algebras. Moreover, if ϕ is a tangent quasi-isomorphism of dg *prop(erad)s, then ϕ_{ind} is a quasi-isomorphism of L_∞-algebras.*

(ii) *Let* $(\mathscr{P} := \mathscr{F}(s^{-1}\mathscr{C}), \partial)$ *be a quasi-free prop(erad)*, $(\mathscr{E}_1, \partial_{\mathscr{E}_1})$ *and* $(\mathscr{E}_2, \partial_{\mathscr{E}_2})$ *arbitrary* dg *prop(erad)s, and* $(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E}_1), Q_1)$ *and* $(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E}_2), Q_2)$ *the associated* L_∞*-algebras. Then any morphism*

$$\psi : (\mathscr{E}_1, \partial_{\mathscr{E}_1}) \to (\mathscr{E}_2, \partial_{\mathscr{E}_2})$$

of dg *prop(erad)s induces canonically an associated morphism*

$$\psi_{\mathrm{ind}} : (\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E}_1), Q_1) \to (\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{E}_2), Q_2)$$

of L_∞*-algebras. Moreover, if* ψ *is a quasi-isomorphism of* dg *prop(erad)s, then* ψ_{ind} *is a quasi-isomorphism of* L_∞*-algebras.*

Proof. (i) The map ϕ induces a degree 0 linear map,

$$\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_2, \mathscr{E}) \to \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_1, \mathscr{E}),$$
$$\gamma \to \gamma \circ \phi.$$

Using definition (3) of the codifferentials Q_1 and Q_2, and the fact that ϕ respects differentials ∂_1 and ∂_2, we obtain, for any $\gamma \in \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_2, \mathscr{E})$,

$$\begin{aligned} Q_1(\gamma \circ \phi) &= \partial_{\mathscr{E}} \circ \gamma \circ \phi - (-1)^\gamma \gamma \circ \phi \circ \partial_1 \\ &= \partial_{\mathscr{E}} \circ \gamma \circ \phi - (-1)^\gamma \gamma \circ \partial_2 \circ \phi \\ &= Q_2(\gamma) \circ \phi, \end{aligned}$$

and hence conclude that ϕ induces a morphism of dg coalgebras

$$\phi_{\mathrm{ind}} : (\odot^{\geqq 1} \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_2, \mathscr{E}), Q_2) \to (\odot^{\geqq 1} \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_1, \mathscr{E}), Q_1).$$

As

$$\phi \circ \mu_{\mathscr{P}_1}(\mathscr{P}_1 \boxtimes_{(1,1)} \mathscr{P}_1) = \mu_{\mathscr{P}_2}(\phi(\mathscr{P}_1) \boxtimes_{(1,1)} \phi(\mathscr{P}_1)) \subset \mu_{\mathscr{P}_2}(\mathscr{P}_2 \boxtimes_{(1,1)} \mathscr{P}_2)$$

we have

$$\begin{aligned} &\gamma \circ \phi \circ \mu_{\mathscr{P}_1}(\mathscr{P}_1 \boxtimes_{(1,1)} \mathscr{P}_1) - \mu_{\mathscr{E}}(\gamma \circ \phi(\mathscr{P}_1) \boxtimes_{(1,1)} \gamma \circ \phi(\mathscr{P}_1)) \\ &\quad \subset \gamma \circ \mu_{\mathscr{P}_2}(\mathscr{P}_2 \boxtimes_{(1,1)} \mathscr{P}_2) - \mu_{\mathscr{E}}(\gamma(\mathscr{P}_2) \boxtimes_{(1,1)} \gamma(\mathscr{P}_1)). \end{aligned}$$

Thus the map ϕ_{ind} sends cogenerators (4) of the coideal I_2 in $\odot^{\geqq 1} \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_2, \mathscr{E})$ into cogenerators of the coideal I_1 in $\odot^{\geqq 1} \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_1, \mathscr{E})$, and hence gives rise to a morphism of dg coalgebras

$$\phi_{\mathrm{ind}} : (\mathscr{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}_2, \mathscr{E})}, Q_2) \to (\mathscr{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}_1, \mathscr{E})}, Q_1),$$

i.e. to a morphism of dg affine schemes, $\phi_{\mathrm{ind}} : (\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}_2, \mathscr{E}), Q_2) \to (\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}_1, \mathscr{E}), Q_1)$.

If the dg prop(erad)s $\mathscr{P}_1$ and $\mathscr{P}_2$ are quasi-free, then the above morphism of dg affine schemes is the same as a morphism of smooth dg manifolds, i.e. a morphism

$$\phi_{\mathrm{ind}} : (\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_2, \mathscr{E}), Q_2) \to (\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}_1, \mathscr{E}), Q_1)$$

of L_∞-algebras. The last statement of Theorem 7 follows immediately from the formulae (5) for $n = 1$ and the Künneth formula completing the proof of claim (i).

Claim (ii) is much easier than claim (i): it follows directly from the formulae (5) for $n = 1$. $\square$

1.5. Proofs via local coordinate computations. Calculations in local coordinates is a powerful and useful tool in differential geometry. In this section we show new proofs of Lemma 3, Proposition 4 and Theorem 5 by explicitly describing all the notions and constructions of §1.4 in local coordinates and justifying thereby the geometric language we used in that section. For simplicity, we show the proofs only for the case when $(\mathscr{P}, \partial_P)$ and $(\mathscr{E}, \partial_{\mathscr{E}})$ are dg associative algebras, that is, dg properads concentrated in biarity $(1,1)$ (a generalization to arbitrary dg (prop)erads is straightforward); moreover, to simplify Koszul signs in the formulae below we also assume that both $\mathscr{P}$ and $\mathscr{E}$ are free modules over a graded commutative ring, $R = \bigoplus_{i\in\mathbb{Z}} R^i$, with degree 0 generators $\{e_a\}_{a\in I}$ and, respectively, $\{e_\alpha\}_{\alpha\in J}$. Then multiplications and differentials in $\mathscr{P}$ and $\mathscr{E}$ have the following coordinate representations:

$$e_a \cdot e_b = \sum_{c\in I} \mu_{ab}^c e_c, \quad e_\alpha \cdot a_\beta = \sum_{\gamma\in J} \mu_{\alpha\beta}^\gamma e_\gamma,$$

$$\partial_{\mathscr{P}} e_a = \sum_{b\in I} D_a^b e_b, \quad \partial_{\mathscr{E}} e_\alpha = \sum_{\beta\in J} D_\alpha^\beta e_\beta,$$

for some coefficients $\mu_{ab}^c, \mu_{\alpha\beta}^\gamma \in R^0$ and $D_a^b, D_\alpha^\beta \in R^{-1}$. Equations $\partial_{\mathscr{P}}^2 = \partial_{\mathscr{E}}^2 = 0$ as well as equations for compatibility of differentials with products are given in coordinates as follows:

$$\sum_{b\in I} D_a^b D_b^c = 0, \quad \sum_{\beta\in J} D_\alpha^\beta D_\beta^\gamma = 0, \tag{6}$$

$$D_a^m \mu_{mb}^c + D_b^m \mu_{am}^c = \mu_{ab}^m D_m^c, \quad D_\alpha^\nu \mu_{\nu\beta}^\gamma + D_\beta^\nu \mu_{\alpha\nu}^\gamma = \mu_{\alpha\beta}^\nu D_\nu^\gamma. \tag{7}$$

A generic homogeneous map of graded vector space $\gamma : \mathscr{P} \to \mathscr{E}$ of degree $i \in \mathbb{Z}$ is uniquely determined by its values on the generators

$$\gamma(e_a) = \sum_{\alpha\in J} \gamma_{a(i)}^\alpha e_\alpha$$

for some coefficients $\gamma_{a(i)}^\alpha \in R^i$. We shall understand these coefficients as coordinates on the flat manifold $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$.

Consider now a completed free graded commutative algebra, $R[[\gamma_{a(i)}^\alpha]]$, generated by formal variables $\gamma_{a(i)}^\alpha$ to which we assign degree i. This algebra is precisely the algebra of smooth functions $\mathscr{O}_{\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P},\mathscr{E})}$ on the manifold $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$. Let us consider a degree -1 vector field (that is, a derivation of $\mathscr{O}_{\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P},\mathscr{E})}$)

$$Q = \left(\sum_{\alpha,\beta,a,i} D_\beta^\alpha \gamma_{a(i)}^\beta - \sum_{a,b,\alpha,i} (-1)^i D_a^b \gamma_{b(i)}^\alpha \right) \frac{\partial}{\partial \gamma_{a(i)}^\alpha},$$

on $\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E})$. In view of (6), we have

$$\begin{aligned}[Q,Q] &= 2\Bigg(\sum_{\alpha,\beta,\gamma,a,i} -D^\alpha_\beta D^\beta_\gamma \gamma^\gamma_{a(i)} - \sum_{\alpha,\beta,a,b,i}(-1)^i D^b_a D^\alpha_\beta \gamma^\beta_{b(i)}\Bigg)\frac{\partial}{\partial\gamma^\alpha_{a(i)}}\\ &\quad + 2\Bigg(-\sum_{\alpha,a,b,c,i}(-1)^i D^\alpha_\beta D^b_a \gamma^\beta_{b(i)} - \sum_{\alpha,\beta,\gamma,a,i} D^b_a D^c_b \gamma^\alpha_{c(i)}\Bigg)\frac{\partial}{\partial\gamma^\alpha_{a(i)}}\\ &= 0,\end{aligned}$$

proving thereby Lemma 3 which claims that $\big(\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E}),Q\big)$ is a dg manifold.

The space of extended morphisms of associative R-algebras $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})$ is, by its definition, a (singular, in general) subspace of the manifold $\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E})$ given explicitly by the equations

$$\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E}) := \Bigg\{\gamma^\alpha_{a(i)} \in \mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E}) : \sum_{c\in I}\mu^c_{ab}\gamma^\alpha_{c(i)} - \sum_{\substack{\beta,\gamma\in J\\ j+k=i}}\mu^\alpha_{\beta\gamma}\gamma^\beta_{a(j)}\gamma^\gamma_{b(k)} = 0\Bigg\}.$$

Let I be an ideal in $\mathcal{O}_{\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E})} = \mathbb{K}[[\gamma^\alpha_{a(i)}]]$ generated by the functions

$$\Bigg\{\sum_{c\in I}\mu^c_{ab}\gamma^\alpha_{c(i)} - \sum_{\substack{\beta,\gamma\in J\\ j+k=i}}\mu^\alpha_{\beta\gamma}\gamma^\beta_{a(j)}\gamma^\gamma_{b(k)}\Bigg\}.$$

Then the structure sheaf $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})}$ of the scheme $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})$ is given, by definition, by the quotient algebra $\mathcal{O}_{\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E})}/I$ (which, in general, is not freely generated, i.e. is not smooth). We claim that the vector field Q on $\mathrm{Hom}^{\mathbb{S}}_\bullet(\mathscr{P},\mathscr{E})$ is tangent to the subspace $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})$. Indeed, in view of (7), we have

$$\begin{aligned}Q\Bigg(\sum_{c\in I}\mu^c_{ab}\gamma^\alpha_{c(i)} - \sum_{\substack{\beta,\gamma\in J\\ j+k=i}}\mu^\alpha_{\beta\gamma}\gamma^\beta_{a(j)}\gamma^\gamma_{b(k)}\Bigg) &= \sum_{\sigma\in J} D^\alpha_\sigma\Bigg(\sum_{c\in I}\mu^c_{ab}\gamma^\sigma_{c(i)} - \sum_{\substack{\beta\gamma\in J\\ j+k=i}}\mu^\sigma_{\beta\gamma}\gamma^\beta_{a(j)}\gamma^\gamma_{b(k)}\Bigg)\\ &\quad - (-1)^i\sum_{e\in E} D^m_a\Bigg(\sum_{c\in I}\mu^c_{mb}\gamma^\sigma_{c(i)} - \sum_{\substack{\beta\gamma\in J\\ j+k=i}}\mu^\sigma_{\beta\gamma}\gamma^\beta_{m(j)}\gamma^\gamma_{b(k)}\Bigg)\\ &\quad - (-1)^i\sum_{e\in E} D^m_b\Bigg(\sum_{c\in I}\mu^c_{am}\gamma^\sigma_{c(i)} - \sum_{\substack{\beta\gamma\in J\\ j+k=i}}\mu^\sigma_{\beta\gamma}\gamma^\beta_{a(j)}\gamma^\gamma_{m(k)}\Bigg).\end{aligned}$$

Thus $Q(I) \subset I$ so that Q makes $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P},\mathscr{E})}$ into a *differential* graded algebra proving thereby Proposition 4.

To prove Theorem 5 we have to assume from now on that $\mathscr{P}$ is a free algebra, $\otimes^\bullet V$, generated by some free R-module V. Let $\{e_A\}_{A\in K}$ stand for a set of generators of V so that the basis $\{e_a\}$ we used above can be identified with the set

$$\{e_a\}_{a\in I} = \{e_A, e_{A_1A_2} := e_{A_1}\otimes e_{A_2}, e_{A_1A_2A_3} = e_{A_1}\otimes e_{A_2}\otimes e_{A_3}, \ldots\}_{A_\bullet\in K}.$$

The differential $\partial_{\mathscr{P}}$ is now completely determined by its values on the generators $\{e_A\}$,

$$\partial_{\mathscr{P}} e_A = \sum_{k \geqq 1} \sum_{A_1, \dots, A_k \in K} D_A^{A_1 \dots A_k} e_{A_1 \dots A_k},$$

for some coefficients $D_A^{A_1 \dots A_k} \in R^{-1}$. On the other hand, the R-algebra of smooth formal functions on the manifold $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$ gets the following explicit representation:

$$\mathcal{O}_{\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})} = R[[\gamma_{A(i)}^\alpha, \gamma_{A_1 A_2(i)}^\alpha, \gamma_{A_1 A_2 A_3(i)}^\alpha, \dots]].$$

The key point is that the system of equations defining the subspace

$$\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) \subset \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$$

can now be easily solved,

$$\gamma_{A_1 A_2(i)}^\alpha = \sum_{\substack{\beta_1, \beta_2 \in J \\ i_1 + i_2 = i}} \mu_{\beta_1 \beta_2}^\alpha \gamma_{A_1(i_1)}^{\beta_1} \gamma_{A_2(i_2)}^{\beta_2},$$

$$\gamma_{A_1 A_2 A_3(i)}^\alpha = \sum_{\substack{\beta_\bullet, \gamma \in J \\ i_1 + i_2 + i_3 = i}} \mu_{\beta_1 \gamma}^\alpha \mu_{\beta_2 \beta_3}^\gamma \gamma_{A_1(i_1)}^{\beta_1} \gamma_{A_2(i_2)}^{\beta_2} \gamma_{A_3(i_3)}^{\beta_3},$$

$$\dots$$

in terms of the independent variables $\gamma_{A(i)}^\alpha$. Thus $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})$ is itself a smooth formal manifold with the structure sheaf $\mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})} \simeq R[[\gamma_{A(i)}^\alpha]]$. The vector field Q on the manifold $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$ restricts to a smooth degree -1 homological vector field on the subspace $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) \subset \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}, \mathscr{E})$ which is given explicitly as follows:

$$Q|_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})} = \left(\sum_{\alpha, \beta, a, i} D_\beta^\alpha \gamma_{A(i)}^\beta - \sum_{A, A_\bullet, \alpha, i} (-1)^i D_A^{A_1 \dots A_k} \gamma_{A_1 \dots A_k(i)}^\alpha \right) \frac{\partial}{\partial \gamma_{A(i)}^\alpha}$$

where, for $k \geqq 2$,

$$\gamma_{A_1 A_2 \dots A_k(i)}^\alpha = \sum_{\substack{\beta_\bullet, \gamma_\bullet \in J \\ i_1 + \dots + i_k = i}} \mu_{\beta_1 \gamma_1}^\alpha \mu_{\beta_2 \gamma_2}^{\gamma_1} \dots \mu_{\beta_{k-1} \beta_k}^{\gamma_{k-2}} \gamma_{A_1(i_1)}^{\beta_1} \gamma_{A_2(i_2)}^{\beta_2} \dots \gamma_{A_k(i_k)}^{\beta_k}.$$

Thus $\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E}) = \mathrm{Hom}_{\mathbb{Z}}(V, \mathscr{E})$ is canonically a dg manifold, i.e. $s\,\mathrm{Hom}_{\mathbb{Z}}(V, \mathscr{E})$ is canonically an L_∞-algebra, and Theorem 5(i) is proved. Theorem 5(ii) follows from the above explicit expression for the homological vector field $Q|_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})}$ as its zero set is precisely the set of morphisms $\mathscr{P} \to \mathscr{E}$ which commute with the differentials. Finally, if the differential $\partial_{\mathscr{P}}$ is at most quadratic in generators, then $D_A^{A_1 \dots A_k} = 0$ for $k \geqq 3$ and hence $Q|_{\mathrm{Mor}_{\mathbb{Z}}(\mathscr{P}, \mathscr{E})}$ is evidently at most quadratic homological vector field so that Theorem 5(iii) is also done.

In a similar purely geometric way one can prove a new Theorem 7. We leave the details as an exercise to the interested reader.

1.6. Enlarged category of dg prop(erad)s. For any dg prop(erad)s $(\mathscr{P}_1, \partial_1)$, $(\mathscr{P}_2, \partial_2)$ and $(\mathscr{P}_3, \partial_3)$, the natural composition map

$$\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_2, \mathscr{P}_3) \otimes \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_1, \mathscr{P}_2) \to \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{P}_1, \mathscr{P}_3),$$

$$\gamma_2 \otimes \gamma_1 \to \gamma_2 \circ \gamma_1$$

respects the relations (4) and hence induces a map of coalgebras (cf. the proof of Theorem 7(i))

$$\circ : \mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_2,\mathcal{P}_3)} \otimes \mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_1,\mathcal{P}_2)} \to \mathcal{O}_{\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_1,\mathcal{P}_3)}.$$

Proposition 8. *The map* $\circ$ *respects the codifferentials* (3), *i.e. induces a morphism of* dg *affine schemes*,

$$\big(\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_2,\mathcal{P}_3), Q_{23}\big) \times \big(\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_1,\mathcal{P}_2), Q_{12}\big) \to \big(\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_1,\mathcal{P}_3), Q_{13}\big).$$

Proof. We have, for any $\gamma_1 \in \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathcal{P}_1,\mathcal{P}_2)$ and $\gamma_2 \in \mathrm{Hom}_\bullet^{\mathbb{S}}(\mathcal{P}_2,\mathcal{P}_3)$,

$$\begin{aligned} Q_{13}(\gamma_2\circ\gamma_1) &\overset{\text{by (3)}}{=} \partial_3\circ\gamma_2\circ\gamma_1 - (-1)^{\gamma_1+\gamma_2}\gamma_2\circ\gamma_1\circ\partial_1 \\ &= \partial_3\circ\gamma_2\circ\gamma_1 - (-1)^{\gamma_2}\gamma_2\circ\partial_2\circ\gamma_1 \\ &\quad + (-1)^{\gamma_2}\circ\partial_2\circ\gamma_1 - (-1)^{\gamma_1+\gamma_2}\gamma_2\circ\gamma_1\circ\partial_1 \\ &= Q_{23}(\gamma_2)\circ\gamma_1 + (-1)^{\gamma_2}\gamma_2\circ Q_{12}(\gamma_1). \quad \square \end{aligned}$$

As the composition $\circ$ is obviously associative, we end up with the following canonical enlargement of the category of dg prop(erad)s.

Corollary 9. *The data*

$$\mathsf{Objects} := \text{dg } \textit{prop(erad)s},$$
$$\mathsf{Hom}(\mathcal{P}_1,\mathcal{P}_2) := \textit{the } \text{dg } \textit{affine scheme } \big(\mathrm{Mor}_{\mathbb{Z}}(\mathcal{P}_1,\mathcal{P}_2), Q_{12}\big)$$

is a category. Moreover, the composition

$$\circ : \mathsf{Hom}(\mathcal{P}_2,\mathcal{P}_3) \times \mathsf{Hom}(\mathcal{P}_1,\mathcal{P}_2) \to \mathsf{Hom}(\mathcal{P}_1,\mathcal{P}_3)$$

is a morphism of dg *affine schemes.*

Note that if $\mathcal{P}_1$ is quasi-free then, by Theorem 5, $\mathsf{Hom}(\mathcal{P}_1,\mathcal{P}_2)$ is precisely the filtered L_∞-algebra whose Maurer-Cartan elements are ordinary morphisms of dg prop(erad)s from $\mathcal{P}_1$ to $\mathcal{P}_2$. Note also that if $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is an ordinary morphism of quasi-free dg prop(erad)s, then the composition map

$$\circ : \mathsf{Hom}(\mathcal{P}_2,\mathcal{P}_3) \times \phi \to \mathsf{Hom}(\mathcal{P}_1,\mathcal{P}_3)$$

is precisely the L_∞-morphism of Theorem 7(i).

1.7. Families of natural L_∞-structures on $\bigoplus\mathcal{P}$. It was shown in [39], Section 4.5, that for any homotopy properad $\mathcal{P}$ the associated direct sum $\bigoplus\mathcal{P} := \bigoplus_{m,n}\mathcal{P}(m,n)$ has a natural structure of L_∞-algebra which encodes all possible compositions in $\mathcal{P}$. In this section we show a new proof of this result which is independent of [39], Section 4.5, and the earlier works [20], [53] which treated the special case of operads. The present approach is based on certain universal properties of the properad of Frobenius algebras (and its non-commutative versions) and Theorem 5; it provides a conceptual explanation of the phenomenon in terms of convolution properads.

Theorem 10. *Let $\mathscr{P} = \{P(m,n)\}$ be a homotopy prop(erad). Then*:

(i) $\bigoplus_{m,n} P(m,n)$ *is canonically an L_∞-algebra.*

(ii) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_m}$ *is canonically an L_∞-algebra.*

(iii) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_n}$ *is canonically an L_∞-algebra.*

(iv) $\bigoplus_{m,n} P(m,n)^{\mathbb{S}_m\times\mathbb{S}_n}$ *is canonically an L_∞-algebra.*

(v) *There is a natural commutative diagram of L_∞-morphisms*

$$\begin{tikzcd} & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m} \arrow[dr] & \\ \bigoplus_{m,n} P(m,n) \arrow[ur] \arrow[dr] & & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m\times\mathbb{S}_n} \\ & \bigoplus_{m,n} P(m,n)^{\mathbb{S}_n}. \arrow[ur] & \end{tikzcd}$$

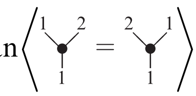
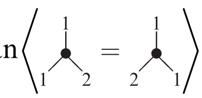
Finally, if $\mathscr{P}$ is a dg *properad, then all the above data are* dg *Lie algebras and morphisms of* dg *Lie algebras.*

Proof. Recall that the prop(erad) of Frobenius algebras can be defined as a quotient

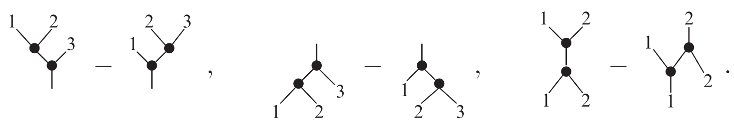
$$\mathscr{F}rob := \mathscr{F}\langle V\rangle/(R)$$

of the free prop(erad) $\mathscr{F}(V)$, generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}$,

$V(m,n) :=$

- $\mathrm{Id}_2 \otimes \mathrm{Id}_1 \equiv \mathrm{span}$  if $m = 2$, $n = 1$,
- $\mathrm{Id}_1 \otimes \mathrm{Id}_2 \equiv \mathrm{span}$  if $m = 1$, $n = 2$,
- 0 otherwise

modulo the ideal generated by relations



Here Id_n stands for the trivial one dimensional representation of the group $\mathbb{S}_n$. It is clear that $\mathscr{F}rob(m,n) = \mathrm{Id}_m \otimes \mathrm{Id}_n$ and the compositions in $\mathscr{F}rob$ are determined by the canonical isomorphism $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ (thus $\mathscr{F}rob$ is a prop(erad)ic analogue of $\mathscr{C}om$ in the theory of operads). The dual space, $\mathscr{F}rob^*$, is naturally a coprop(erad)[3]. Homotopy prop(erad) structure on $\mathscr{P}$ is the same as a differential, $d_{\mathscr{P}}$, in the free coprop(erad) $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$. Theorem 6(i) applied to the coprop(erad)s $\mathscr{F}rob^*$ and $\big(\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}}), d_{\mathscr{P}}\big)$, asserts that the vector space

$$\mathrm{Hom}_{\bullet}^{\mathbb{S}}(\mathscr{F}rob^*, \mathscr{P}) = \bigoplus_{m,n} (\mathrm{Id}_m \otimes \mathrm{Id}_n) \otimes_{\mathbb{S}_m \times \mathbb{S}_n} \mathscr{P}(m,n) = \bigoplus_{m,n} \mathscr{P}(m,n)^{\mathbb{S}_m \times \mathbb{S}_n}$$

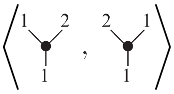
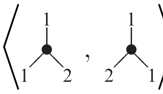
is canonically an L_∞-algebra. Hence the claim (iv).

Let us next define a non-commutative analogue of $\mathscr{F}rob$ as a quotient,

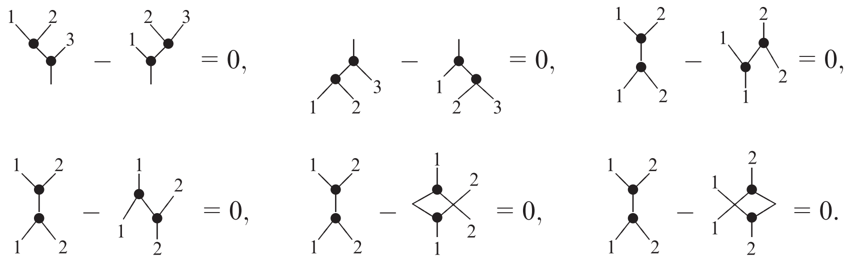
$$\mathscr{F}rob_+^+ := \mathscr{F}(V)/(R)$$

of the free prop(erad) $\mathscr{F}(V)$, generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}$,

$V(m,n) :=$

- $\mathbb{K}[\mathbb{S}_2] \otimes \mathrm{Id}_1 \equiv \mathrm{span}\big\langle$  $\big\rangle$ if $m = 2$, $n = 1$,
- $\mathrm{Id}_1 \otimes \mathbb{K}[\mathbb{S}_2] \equiv \mathrm{span}\big\langle$  $\big\rangle$ if $m = 1$, $n = 2$,
- 0 otherwise

modulo the ideal generated by relations



It is clear that $\mathscr{F}rob_+^+(m,n) = \mathbb{K}[\mathbb{S}_m] \otimes \mathbb{K}[\mathbb{S}_n]$. Analogously one defines two other versions of $\mathscr{F}rob$,

$$\mathscr{F}rob_+ = \{\mathscr{F}rob_+(m,n) = \mathrm{Id}_m \otimes \mathbb{K}[\mathbb{S}_n]\} \quad \text{and} \quad \mathscr{F}rob^+ = \{\mathscr{F}rob_+(m,n) = \mathbb{K}[\mathbb{S}_m] \otimes \mathrm{Id}_n\},$$

with comultiplication (resp. multiplication) commutative but multiplication (resp. comultiplication) noncommutative. Then applying again Theorem 6(i) or [39], Theorem 27, to $\mathscr{Q}$ being $(\mathscr{F}rob_+^+)^*$, $(\mathscr{F}rob_+)^*$ or $(\mathscr{F}rob^+)^*$ and $\mathscr{D}$ being $\mathscr{F}^{\mathrm{c}}(s\overline{\mathscr{P}})$ we conclude that the vector spaces

[3] In fact, $\mathscr{F}rob^*$ is a completed coproperad with respect to the topology induced by the number of vertices. The formulae for the composite coproduct in infinite. But since we 'dualize' it by considering the convolution homotopy properad $\mathrm{Hom}^{\mathbb{S}}(\mathscr{F}rob^*, -)$ it does not matter.

$$\mathrm{Hom}_\bullet^{\mathbb{S}}\big((\mathscr{F}rob_+^+)^*, \mathscr{P}\big) = \bigoplus_{m,n} \mathscr{P}(m,n),$$

$$\mathrm{Hom}_\bullet^{\mathbb{S}}\big((\mathscr{F}rob_+)^*, \mathscr{P}\big) = \bigoplus_{m,n} \mathscr{P}(m,n)^{\mathbb{S}_m},$$

$$\mathrm{Hom}_\bullet^{\mathbb{S}}\big((\mathscr{F}rob^+)^*, \mathscr{P}\big) = \bigoplus_{m,n} \mathscr{P}(m,n)^{\mathbb{S}_n}$$

admit canonically L_∞-structures proving thereby claim (i)–(iii).

Finally, the natural commutative diagram of morphisms of properads,

$$\begin{tikzcd}
 & \mathscr{F}rob_+ \arrow[dr] & \\
\mathscr{F}rob \arrow[ur] \arrow[dr] & & \mathscr{F}rob_+^+ \\
 & \mathscr{F}rob^+ \arrow[ur] &
\end{tikzcd}$$

proves claim (v). $\square$

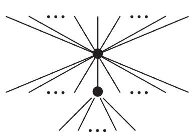
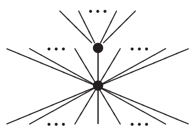
The prop(erad) $\mathscr{F}rob_+^+$ was designed so that it is generated as an $\mathbb{S}$-module by arbitrary (m,n)-corollas, and the comultiplication $\Delta_{(1,1)}$ in its dual, $(\mathscr{F}rob_+^+)^*$ splits such a corolla into *all* possible two vertex (m,n)-graphs. Hence the L_∞-structure claimed in Theorem 10 is exactly the same as in [39], Theorem 25.

The L_∞-structures on the direct sum $\bigoplus P$ and its subspaces of invariants constructed in the proof of Theorem 10 are the most natural ones to consider as they involve *all* possible compositions in $\mathscr{P}$. However, they are by no means unique in the case of prop(erad)s (as opposite to the case of operads). For example, the part of prop(erad) compositions which correspond to so called $\frac{1}{2}$-propic graphs or dioperadic graphs, that is graphs of genus 0 (see pictures (8) below), also combine into an L_∞-structure on $\bigoplus P$ (and its subspaces of invariants) as the following argument shows.

For a prop(erad) $\mathscr{P} = \{\mathscr{P}(m,n)\}$ we denote by $\mathscr{P}^\dagger = \{\mathscr{P}^\dagger(m,n)\}$ the associated "flow reversed" prop(erad) with $\mathscr{P}^\dagger(m,n) := \mathscr{P}(n,m)$. Let $\mathscr{A}ss$ be the operad of associative algebras and define the properad, $\mathscr{A}ss^\dagger \bullet \mathscr{A}ss = \{\mathscr{A}ss^\dagger \bullet \mathscr{A}ss(m,n)\}$ by setting

$$Ass^\dagger \bullet \mathscr{A}ss(m,n) := \mathscr{A}ss^\dagger(m) \otimes \mathscr{A}ss(n) \simeq \mathbb{K}[\mathbb{S}_m] \otimes \mathbb{K}[\mathbb{S}_n]$$

and defining the compositions $\mu_{(1,1)}$ to be non-zero only on decorated graphs of the form

(8)  and 

on which it is equal to the operadic compositions in $\mathscr{A}ss$. The properad $\mathscr{A}ss^\dagger \bullet \mathscr{A}ss$ corresponds to the $\frac{1}{2}$-prop $\mathscr{U}^{\text{properad}}_{\frac{1}{2}\text{-prop}}(\mathscr{F}rob^+_+)$.

Let $\mathscr{C}om$ be the properad of commutative algebras, and define the properads $\mathscr{C}om^\dagger \bullet \mathscr{A}ss$, $\mathscr{A}ss^\dagger \bullet \mathscr{C}om$, and $\mathscr{C}om^\dagger \bullet \mathscr{C}om$ by analogy to $\mathscr{A}ss^\dagger \bullet \mathscr{A}ss$. Similarly, they correspond to the $\frac{1}{2}$-props $\mathscr{U}^{\text{properad}}_{\frac{1}{2}\text{-prop}}(\mathscr{F}rob^+)$, $\mathscr{U}^{\text{properad}}_{\frac{1}{2}\text{-prop}}(\mathscr{F}rob_+)$ and $\mathscr{U}^{\text{properad}}_{\frac{1}{2}\text{-prop}}(\mathscr{F}rob)$. Applying Theorem 6(i) to $\mathscr{Q}$ being coproperads $(\mathscr{A}ss^\dagger \bullet \mathscr{A}ss)^*$, $(\mathscr{A}ss^\dagger \bullet \mathscr{C}om)^*$, $(\mathscr{C}om^\dagger \bullet \mathscr{A}ss)^*$, or $(\mathscr{C}om^\dagger \bullet \mathscr{C}om)^*$, we conclude that the vector spaces,

$$\bigoplus_{m,n} P(m,n), \quad \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m}, \quad \bigoplus_{m,n} P(m,n)^{\mathbb{S}_n}, \quad \bigoplus_{m,n} P(m,n)^{\mathbb{S}_m \times \mathbb{S}_n},$$

admit canonically L_∞-structures encoding $\frac{1}{2}$-prop compositions of the form (8). The natural morphism of operads,

$$\mathscr{A}ss \to \mathscr{C}om,$$

implies that these L_∞-structures are related to each other via the same commutative diagram of L_∞ morphisms as in Theorem 10(v). In the case of operads the constructed L_∞-structures are exactly the same as in Theorem 10 but for prop(erad)s they are different.

2. Deformation theory of morphisms of prop(erad)s

In this section, we define the deformation theory of morphisms of prop(erad)s. We follow the conceptual method proposed by Quillen in [41], [42].

2.1. Basic definition. Let $(\mathscr{P}, d_{\mathscr{P}}) \xrightarrow{\varphi} (\mathscr{Q}, d_{\mathscr{Q}})$ be a morphism of dg prop(erad)s. We would like to define a chain complex with which we could study the deformation theory of this map. Following Quillen [42], the conceptual method is to take the total right derived functor of the functor Der of derivations from the category of prop(erad)s above $\mathscr{Q}$ (see also [29], [53]). That is, we consider a cofibrant replacement $(\mathscr{R}, \partial)$ of $\mathscr{P}$ is the category of dg prop(erad)s

$$\begin{tikzcd} \mathscr{R} \arrow[r, "\varepsilon"] \arrow[dr, "\gamma"'] & \mathscr{P} \arrow[d, "\varphi"] \\ & \mathscr{Q}. \end{tikzcd}$$

Recall that $\mathscr{Q}$ has an infinitesimal $\mathscr{P}$-bimodule (respectively infinitesimal $\mathscr{R}$-bimodule) structure given by φ (respectively γ).

Lemma 11. *Let $(\mathscr{R}, \partial)$ be a resolution of $\mathscr{P}$ and let f be a homogeneous derivation of degree n in $\mathrm{Der}_n(\mathscr{R}, \mathscr{Q})$, the derivative $D(f) = d_{\mathscr{Q}} \circ f - (-1)^{|f|} f \circ \partial$ is a derivation of degree $n-1$ of $\mathrm{Der}_{n-1}(\mathscr{R}, \mathscr{Q})$.*

Proof. The degree of $D(f)$ is $n-1$. It remains to show that it is a derivation. For every pair r_1 and r_2 of homogeneous elements of $\mathscr{R}$, $D(f)\big(\mu_{\mathscr{R}}(r_1 \boxtimes_{(1,1)} r_2)\big)$ is equal to

$$\begin{aligned}
D(f)\big(\mu_{\mathscr{R}}(r_1 \boxtimes_{(1,1)} r_2)\big) &= \big(d_{\mathscr{Q}} \circ f - (-1)^n f \circ \partial\big)\big(\mu_{\mathscr{R}}(r_1 \boxtimes_{(1,1)} r_2)\big)\\
&= d_{\mathscr{Q}}\big(\mu_{\mathscr{Q}}\big(f(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{n|r_1|}\gamma(r_1) \boxtimes_{(1,1)} f(r_2)\big)\big)\\
&\quad - (-1)^n f\big(\mu_{\mathscr{R}}\big(\partial(r_1) \boxtimes_{(1,1)} r_2 + (-1)^{|r_1|} r_1 \boxtimes_{(1,1)} \partial(r_2)\big)\big)\\
&= \mu_{\mathscr{Q}}\big((d_{\mathscr{Q}} \circ f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{n+|r_1|} f(r_1) \boxtimes_{(1,1)} (d_{\mathscr{Q}} \circ \gamma)(r_2)\\
&\quad + (-1)^{n|r_1|}(d_{\mathscr{Q}} \circ \gamma)(r_1) \boxtimes_{(1,1)} f(r_2)\\
&\quad + (-1)^{|r_1|(n-1)}\gamma(r_1) \boxtimes_{(1,1)} (d_{\mathscr{Q}} \circ f)(r_2)\big)\\
&\quad - (-1)^n \mu_{\mathscr{Q}}\big((f \circ \partial)(r_1) \boxtimes_{(1,1)} \gamma(r_2)\\
&\quad + (-1)^{n(|r_1|-1)}(\gamma \circ \partial)(r_1) \boxtimes_{(1,1)} f(r_2)\\
&\quad + (-1)^{|r_1|} f(r_1) \boxtimes_{(1,1)} (\gamma \circ \partial)(r_2)\\
&\quad + (-1)^{(n-1)|r_1|}\gamma(r_1) \boxtimes_{(1,1)} (f \circ \partial)(r_2)\big).
\end{aligned}$$

Since γ is morphism of dg prop(erad)s, it commutes with the differentials, that is $\gamma \circ \partial = d_{\mathscr{Q}} \circ \gamma$. This gives

$$\begin{aligned}
D(f)\big(\mu_{\mathscr{R}}(r_1 \boxtimes_{(1,1)} r_2)\big) &= \mu_{\mathscr{Q}}\big((d_{\mathscr{Q}} \circ f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) - (-1)^n (f \circ \partial)(r_1) \boxtimes_{(1,1)} \gamma(r_2)\\
&\quad + (-1)^{|r_1|(n-1)}\big(\gamma(r_1) \boxtimes_{(1,1)} (d_{\mathscr{Q}} \circ f)(r_2)\\
&\quad - (-1)^n \gamma(r_1) \boxtimes_{(1,1)} (f \circ \partial)(r_2)\big)\big)\\
&= \mu_{\mathscr{Q}}\big(D(f)(r_1) \boxtimes_{(1,1)} \gamma(r_2) + (-1)^{|r_1|(n-1)}\gamma(r_1) \boxtimes_{(1,1)} D(f)(r_2)\big). \qquad \square
\end{aligned}$$

In other words, the space of derivations $\mathrm{Der}(\mathscr{R}, \mathscr{Q})$ is a sub-dg-module of the space of morphisms $\mathrm{Hom}^{\mathbb{S}}(\mathscr{R}, \mathscr{Q})$. We define the deformation complex of the morphism φ by $C_\bullet(\varphi) := \big(\mathrm{Der}_\bullet(\mathscr{R}, \mathscr{Q}), D\big)$. By Theorem 41 and Theorem 42, there always exists a quasi-free cofibrant resolution. For instance, we can consider the bar-cobar resolution by [39], Theorem 19. This will produce an explicit but huge complex which is difficult to compute. Instead of that, we will work with the chain complex obtained from a minimal model of $\mathscr{P}$ when it exists. Its size is much smaller but its differential can be not so easy to make explicit. In this sequel, our main example be the deformation theory of representations of $\mathscr{P}$ of the form $\mathscr{Q} = \mathrm{End}_X$, that is $\mathscr{P}$-gebras.

2.2. Deformation theory of representations of prop(erad)s. Let $(\mathscr{P}, d_{\mathscr{P}})$ be a dg prop(erad), let (X, d_X) be an arbitrary dg $\mathscr{P}$-gebra and let $\big(\mathscr{P}_\infty := \Omega(\mathscr{C}), \partial\big)$ be a cofibrant quasi-free resolution of $\mathscr{P}$ and

$$\begin{tikzcd}
\Omega(\mathscr{C}) \arrow[r, "\varepsilon"] \arrow[dr, "\gamma"'] & \mathscr{P} \arrow[d, "\varphi"] \\
& \mathrm{End}_X.
\end{tikzcd}$$

Definition (deformation complex). We define the *deformation complex of the $\mathscr{P}$-gebra structure of X* by $C_\bullet(\mathscr{P}, X) := \big(\mathrm{Der}_\bullet\big(\Omega(\mathscr{C}), \mathrm{End}_X\big), D\big)$.

Theorem 12. *The deformation complex* $\left(\mathrm{Der}_\bullet(\Omega(\mathscr{C}),\mathscr{Q}),D\right)$ *is isomorphic to* $\mathrm{Hom}_\bullet^{\mathbb{S}}(\overline{\mathscr{C}},\mathscr{Q})$ *with* $D=Q^\gamma$ *for* $\gamma=\varphi\circ\varepsilon_{|\overline{\mathscr{C}}}$.

Proof. [39], Lemma 14 proves the identification between the two spaces. Since γ is a morphism of dg prop(erad)s from a quasi-free prop(erad), it is a solution of the Maurer-Cartan equation $Q(\gamma)=0$ in the convolution L_∞-algebra $\mathrm{Hom}_\bullet^{\mathbb{S}}(\overline{\mathscr{C}},\mathscr{Q})$ by Theorem 5. Let f be an element of $\mathrm{Hom}_n^{\mathbb{S}}(\overline{\mathscr{C}},\mathscr{Q})$. Following [39], Lemma 14, we denote by ∂_f the unique derivation of $\mathrm{Der}_n\left(\Omega(\mathscr{C}),\mathscr{Q}\right)$ induced by f. We have to show that $D(\partial_f)_{s^{-1}\overline{\mathscr{C}}}=Q^\gamma(f)$. For an element $s^{-1}c\in s^{-1}\mathscr{C}$, we use the Sweedler type notation for $\partial(s^{-1}c)=\sum\limits_{\mathscr{G}}\mathscr{G}(s^{-1}c_1,\dots,s^{-1}c_n)$. By [39], Lemma 14, we have

$$\begin{aligned}
&\partial_f\left(\mathscr{G}(s^{-1}c_1,\dots,s^{-1}c_n)\right)\\
&\quad=\sum_{i=1}^{n}(-1)^{n(|c_1|+\cdots+|c_{i-1}|+i-1)}\\
&\quad\quad\times\mu_{\mathscr{Q}}\left(\mathscr{G}\left(\gamma(s^{-1}c_1),\dots,\gamma(s^{-1}c_{i-1}),f(s^{-1}c_i),\gamma(s^{-1}c_{i+1}),\dots,\gamma(s^{-1}c_n)\right)\right).
\end{aligned}$$

Therefore, $D(\partial_f)_{s^{-1}\overline{\mathscr{C}}}$ is equal to

$$\begin{aligned}
D(\partial_f)(s^{-1}c)&=\left(d_{\mathscr{Q}}\circ\partial_f-(-1)^n\partial_f\circ\partial\right)(s^{-1}c)=d_{\mathscr{Q}}\left(f(s^{-1}c)\right)\\
&\quad-(-1)^n\sum_{\mathscr{G}}\sum_{i=1}^{n}(-1)^{n(|c_1|+\cdots+|c_{i-1}|+i-1)}\\
&\quad\times\mu_{\mathscr{Q}}\left(\mathscr{G}\left(\gamma(s^{-1}c_1),\dots,\gamma(s^{-1}c_{i-1}),f(s^{-1}c_i),\gamma(s^{-1}c_{i+1}),\dots,\gamma(s^{-1}c_n)\right)\right)\\
&=Q^\gamma(f).\quad\square
\end{aligned}$$

In other words, the deformation complex is equal to the convolution L_∞-algebra $\mathrm{Hom}_\bullet^{\mathbb{S}}(\overline{\mathscr{C}},\mathscr{Q})$ twisted by the Maurer-Cartan element γ.

Remark. It is natural to consider the augmentation of this chain complex by $\mathrm{Hom}^{\mathbb{S}}(I,\mathscr{Q})$, that is $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C},\mathscr{Q})$.

In summary, by Theorem 5 the vector space $\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C},\mathscr{Q})$ has a canonical filtered L_∞-structure, Q, whose Maurer-Cartan elements are morphisms of dg prop(erad)s, $\mathscr{P}_\infty\to\mathscr{Q}$, that is representations of $\mathscr{P}_\infty$ in $\mathscr{Q}$. Then let γ be one of these morphisms, and let Q^γ be the associated twisting of the canonical L_∞-algebra by γ (see §1.3). This defines the deformation complex of γ.

Definition (deformation complex). The deformation complex of a morphism of prop(erad)s $\gamma:\mathscr{P}_\infty\to\mathscr{Q}$ is the twisted L_∞-algebra $\left(\mathrm{Hom}_\bullet^{\mathbb{S}}(\mathscr{C},\mathscr{Q}),Q^\gamma\right)$.

This definition extends to the case of prop(erad)s the deformation complex of algebras over operads introduced in [25], [53].

With the results on the model category structure on prop(erad)s (see Appendix), we can now prove the independence of this construction in the homotopy category of homotopy prop(erad)s and homotopy Lie algebras.

Theorem 13. *Let $\Omega(\mathscr{C}_1)$ and $\Omega(\mathscr{C}_2)$ be two quasi-free cofibrant resolutions of a* dg *prop(erad) $\mathscr{P}$. For any* dg *prop(erad) $\mathscr{Q}$, the homotopy convolution prop(erad)s* $\mathrm{Hom}(\mathscr{C}_1, \mathscr{Q})$ *and* $\mathrm{Hom}(\mathscr{C}_2, \mathscr{Q})$ *are linked by two quasi-isomorphisms of homotopy prop(erad)s*

$$\mathrm{Hom}(\mathscr{C}_1, \mathscr{Q}) \leftrightarrows \mathrm{Hom}(\mathscr{C}_2, \mathscr{Q}),$$

and the natural maps

$$\left(\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}_1, \mathscr{Q}), Q^{\gamma_1}\right) \leftrightarrows \left(\mathrm{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}_2, \mathscr{Q}), Q^{\gamma_2}\right)$$

are a quasi-isomorphism of homotopy Lie algebras.

Proof. We apply the left lifting property in the model category of dg prop(erad)s to the following diagram :

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \Omega(\mathscr{C}_1) \arrow[d, two heads, "\sim"] \arrow[dl, dotted, shift left] \\ \Omega(\mathscr{C}_2) \arrow[r, two heads, "\sim"] \arrow[ur, dotted, shift left] & \mathscr{P}, \end{tikzcd}$$

to get the two dotted quasi-isomorphisms of prop(erad)s. By Proposition 43, they induce quasi-isomorphisms of the level of the homotopy coprop(erad)s $\mathscr{C}_1 \leftrightarrows \mathscr{C}_2$. We conclude by [39], Theorem 33 and Corollary 35. $\square$

The homology groups

$$H^{\gamma}_{\bullet}(\mathscr{Q}) := H_{\bullet}\left(\mathrm{Hom}^{\mathbb{S}}(\mathscr{C}, \mathscr{Q}), Q^{\gamma}_1\right)$$

are independent of the choice of a cofibrant quasi-free resolution of $\mathscr{P}$ and are called *homology groups of the $\mathscr{P}_\infty$-representation $\mathscr{Q}$*. In the case where $\mathscr{Q} = \mathrm{End}_X$, they are called the homology groups of the $\mathscr{P}_\infty$-gebra (X, γ).

Proposition 14. *The Maurer-Cartan elements Γ of Q^{γ} are in one-to-one correspondence with those $\mathscr{P}_\infty$-structures on X,*

$$\rho : (\mathscr{P}_\infty, \partial) \to (\mathrm{End}_X, d),$$

whose restrictions to the generating space $s^{-1}\mathscr{C}$ of $\mathscr{P}_\infty$ are equal precisely to the sum $\gamma + \Gamma$.

This proposition justifies the name 'deformation complex' because the L_∞-algebra *Q^{γ} controls the deformations of γ in the class of homotopy $\mathscr{P}$-structures.* When applied to $\mathscr{Q} = \mathrm{End}_X$ and $\gamma : \mathscr{P}_\infty \to \mathrm{End}_X$, this defines the deformation complex of the $\mathscr{P}_\infty$-gebra structure γ on X. (Some author calls this the "cohomology of X with coefficients into itself" but we are reluctant to make this choice and prefer to view it as a deformation complex.) This definition applies to any homotopy algebra over an operad (associative algebras, Lie algebras, commutative algebras, preLie algebras, Poisson or Gerstenhaber algebras, etc. ...) as well as to any homotopy (bial)gebra over a properad (Lie bialgebras, associative bialgebras, etc. ...) in order to give, for the first time, a cohomology theory for homotopy $\mathscr{P}$-(bial)gebras.

2.3. Koszul case and cohomology operations. In [39], Theorem 39, we have seen that a properad $\mathscr{P}$ is Koszul if and only if it admits a quadratic model $\Omega(\mathscr{P}^{\text{!`}}) \xrightarrow{\sim} \mathscr{P}$, where $\mathscr{P}^{\text{!`}}$ the Koszul dual (strict) coproperad. In this case, by Theorem 12, the deformation complex of a $\mathscr{P}_\infty$-gebra $\mathrm{Hom}^{\mathbb{S}}(\mathscr{P}^{\text{!`}}, \mathrm{End}_X)$ is dg Lie algebra where the boundary map is equal to the twisted differential $D(f) = d(f) + [\gamma, f]$.

The first definition of this kind of preLie operation appeared in the seminal paper of M. Gerstenhaber [12] in the case of the cohomology of associative algebras. In the case treated by M. Gerstenhaber, the cooperad $\mathscr{C}$ is the Koszul dual cooperad $\mathscr{A}s^{\text{!`}}$ of the operad $\mathscr{A}s$ coding associative algebras and the operad $\mathscr{P}$ is the endomorphism operad End_A. The induced Lie bracket is the *intrinsic Lie bracket* of Stasheff [45]. It is equal to the Lie bracket of Gerstenhaber [12] on Hochschild cochain complex of associative algebras, the Lie bracket of Nijenhuis-Richardson [40] on Chevalley-Eilenberg cochain complex of Lie algebras and the Lie bracket of Stasheff on Harrison cochain complex of commutative algebras. It is proven by Balavoine in [2] that the deformation complex of algebras over any Koszul operad admits a Lie structure. This statement was made more precise by Markl, Shnider and Stasheff in [33], Section 3.9, Part II, where they prove that this Lie bracket comes from a preLie product. This result on the level of operads was proved using the space of coderivations of the cofree $\mathscr{P}^{\text{!`}}$-coalgebra, which is shown to be a preLie algebra. Such a method is impossible to generalize to prop(erad)s simply because there exists no notion of (co)free gebra.

As explained here, one has to work with convolution prop(erad) to prove a similar result. Actually, this method gives a stronger statement.

Theorem 15. *Let $\mathscr{P}$ be a Koszul properad and let $\varphi : \mathscr{P} \to \mathscr{Q}$ be a morphism of properads, the deformation complex of φ is an* LR-*algebra.*

In the non-symmetric case, when $\mathscr{P}$ and $\mathscr{Q}$ are non-symmetric properads, the deformation complex is a non-symmetric properad.

Proof. It is a direct consequence of the definition of the deformation complex and Theorem 13 of [39]. In the non-symmetric case, the deformation complex is directly a non-symmetric convolution properad, since it is not restricted to invariant elements. □

This result provides higher braces or LR-operations (see [39], Section 2.4). Recall that non-symmetric braces play a fundamental role in the proof of Deligne's conjecture for associative algebras (see [47], [54], [25], [35], [5]) and in the extension of it to other kind of algebras (see [51], Section 5.5). From this rich structure, we derive a Lie-admissible bracket and then a Lie bracket which can be used to study the deformations of X. We expect the LR-operations to be used in the future for a better understanding of deformation theory (in the context of a Deligne conjecture for associative bialgebras and Gerstenhaber-Schack bicomplex, for instance).

Notice that this Lie bracket was found by hand in one example of gebras over a properad before this general theory. The properad of Lie bialgebras is Koszul. Therefore, on the deformation (bi)complex of Lie bialgebras, there is a Lie bracket. The construction of this Lie bracket was given by Kosmann-Schwarzbach in [24]. (See also Ciccoli-Guerra [8] for the interpretation of this bicomplex in terms of deformations.)

2.4. Definition à la Quillen. In the previous sections, we defined the deformation complex of representations of a prop(erad) $\mathscr{P}$ that admits a quasi-free model and proved the independence of this definition in the categories of homotopy prop(erad)s (and homotopy Lie algebras). In this section, we generalize the definition of the deformation theory of a morphism of commutative rings due to Quillen [42] to the case of prop(erad)s. (See L. Illusie [19] for a generalization in the context of topoi and schemes.) Hence, it defines a (relative) deformation complex for representations of any prop(erad). It also gives rise to the *cotangent complex* associated to any morphism of prop(erad)s.

Since a commutative algebra is an associative algebra, an associative algebra an operad and an operad a prop(erad), this generalization of Quillen theory can be seen as a way to extend results of (commutative) algebraic geometry to non-commutative non-linear geometry. It is non-linear because the monoidal product $\boxtimes$ defining prop(erad)s is neither linear on the left nor on the right, contrary to the tensor product $\otimes$ of vector spaces.

Let $\mathscr{I}$ be a 'ground' prop(erad) (to recover the previous section, consider $\mathscr{I} = I$, the unit of the monoidal category of $\mathbb{S}$-bimodules). We look at prop(erad)s $\mathscr{P}$ under $\mathscr{I}$, $\mathscr{I} \to \mathscr{P}$. And for such a prop(erad) $\mathscr{P}$, we consider the category of prop(erad)s over $\mathscr{P}$, that is

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "f"] \\ \mathscr{I} \arrow[ur, "u"] \arrow[r] & \mathscr{P}. \end{tikzcd}$$

We denote this category by $\mathrm{Prop(erad)}/\mathscr{P}$. Let M be an infinitesimal bimodule over $\mathscr{P}$ (see [39], Section 3.1). The infinitesimal $\mathscr{P}$-bimodule M is also an infinitesimal bimodule over any prop(erad) $\mathscr{X}$ over $\mathscr{P}$, by pulling back along $\mathscr{X} \to \mathscr{P}$. Hence, we can consider the space of $\mathscr{I}$-derivations from $\mathscr{X}$ to M, that is derivations from $\mathscr{X}$ to M which vanish on $\mathscr{I}$. We denoted this space by $\mathrm{Der}_{\mathscr{I}}(\mathscr{X}, M)$.

We aim now to represent this bifunctor on the left and on the right. To represent it on the left, we introduce the *square-zero* (*or infinitesimal*) *extension of* $\mathscr{P}$ *by* M : $\mathscr{P} \ltimes M := \mathscr{P} \oplus M$ with the following structure of prop(erad) over $\mathscr{P}$. The monoidal product $(\mathscr{P} \oplus M) \boxtimes (\mathscr{P} \oplus M)$ is equal to

$$\mathscr{P} \boxtimes \mathscr{P} \oplus \mathscr{P} \boxtimes (\mathscr{P} \oplus \underbrace{M}_{1}) \oplus (\mathscr{P} \oplus \underbrace{M}_{1}) \boxtimes \mathscr{P} \oplus \mathscr{M},$$

where $\mathscr{M}$ is the sub-$\mathbb{S}$-bimodule of $(\mathscr{P} \oplus M) \boxtimes (\mathscr{P} \oplus M)$ composed by at least two elements from M. On the first component $\mathscr{P} \boxtimes \mathscr{P}$, the product of $\mathscr{P} \ltimes M$ is defined by the product of $\mathscr{P}$. On the second component, it is defined by the left action of $\mathscr{P}$ on M. On the third one, it is defined by the right action of $\mathscr{P}$ on M. Finally, the product on $\mathscr{M}$ is null.

Lemma 16. *For any prop(erad) $\mathscr{P}$ and any infinitesimal $\mathscr{P}$-bimodule M, the infinitesimal extension $\mathscr{P} \ltimes M$ is a prop(erad).*

Proof. The definition of infinitesimal $\mathscr{P}$-bimodule directly implies the associativity of the prop(erad)ic composition of $\mathscr{P} \ltimes M$. $\square$

The purpose of this definition is in the following result, which states that infinitesimal $\mathscr{P}$-bimodules are abelian group objects in the category of prop(erad)s over $\mathscr{P}$.

Proposition 17. *There is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Prop(erad)}/\mathscr{P}}(\mathscr{X}, \mathscr{P} \ltimes M) \cong \mathrm{Der}_{\mathscr{I}}(\mathscr{X}, M),$$

where $\mathrm{Der}_{\mathscr{I}}(\mathscr{X}, M)$ *is the space of* $\mathscr{I}$*-derivations from* $\mathscr{X}$ *to* M.

Proof. Let us denote by $f : \mathscr{X} \to \mathscr{P}$ any morphism $\mathscr{X} \to \mathscr{P} \ltimes M = \mathscr{P} \oplus M$. The category of prop(erad)s over $\mathscr{P}$ is the sum of f with its component on M, which we denote by D. Finally, $f \oplus D : \mathscr{X} \to \mathscr{P} \ltimes M$ is a morphism of prop(erad)s if and only if D is a derivation $\mathscr{X} \to M$. $\square$

To represent the space of derivations on the right, we introduce the *module of Kähler differentials of a prop*(*erad*). It is a quotient of the free infinitesimal $\mathscr{X}$-bimodule over $\mathscr{I}$ on $\mathscr{X}$ by suitable relations. We recall from [52], Section 2.5, that the relative composition product is defined by the following coequalizer

$$M \boxtimes \mathscr{P} \boxtimes N \underset{\rho \boxtimes N}{\overset{M \boxtimes \lambda}{\rightrightarrows}} M \boxtimes N \twoheadrightarrow M \boxtimes_{\mathscr{P}} N,$$

where λ is the left action of $\mathscr{P}$ on N, $\mathscr{P} \boxtimes N \to N$ and ρ the right action of $\mathscr{P}$ on M, $M \boxtimes \mathscr{P} \to M$.

Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of prop(erad)s. There is a natural functor from the category of infinitesimal $\mathscr{Q}$-bimodules to the category of infinitesimal $\mathscr{P}$-bimodules by pulling back along f. We denote it by $f^* : \mathrm{Inf.}\,\mathscr{Q}\text{-biMod} \to \mathrm{Inf.}\,\mathscr{P}\text{-biMod}$.

Proposition 18. *The functor* $f^* : \mathrm{Inf.}\,\mathscr{Q}\text{-biMod} \to \mathrm{Inf.}\,\mathscr{P}\text{-biMod}$ *admits a left adjoint*

$$f_! : \mathrm{Inf.}\,\mathscr{P}\text{-biMod} \leftrightharpoons \mathrm{Inf.}\,\mathscr{Q}\text{-biMod} : f^*,$$

which is explicitly given by $f_!(M) = \mathscr{Q} \boxtimes_{\mathscr{P}} \underbrace{M}_{1} \boxtimes_{\mathscr{P}} \mathscr{Q}$, *for any infinitesimal* $\mathscr{P}$*-bimodule* M. *The* $\mathbb{S}$*-bimodule* $\mathscr{Q} \boxtimes_{\mathscr{P}} \underbrace{M}_{1} \boxtimes_{\mathscr{P}} \mathscr{Q}$ *is the coequalizer*

$$(\mathscr{Q} \boxtimes \mathscr{P}) \boxtimes \underbrace{M}_{1} \boxtimes (\mathscr{P} \boxtimes \mathscr{Q}) \underset{\mathscr{Q} \boxtimes (\lambda \circ (\mathscr{P} \boxtimes \rho)) \boxtimes \mathscr{Q}}{\overset{\rho_{\mathscr{Q}} \boxtimes M \boxtimes \lambda_{\mathscr{Q}}}{\rightrightarrows}} \mathscr{Q} \boxtimes \underbrace{M}_{1} \boxtimes \mathscr{Q} \twoheadrightarrow \mathscr{Q} \boxtimes_{\mathscr{P}} \underbrace{M}_{1} \boxtimes_{\mathscr{P}} \mathscr{Q},$$

where the notation $\mathscr{Q} \boxtimes \underbrace{M}_{1} \boxtimes \mathscr{Q}$ *stands for* 3*-levels graphs with only one element of* M *labelling a vertex on the second level and such that every element of* Q *on the first and third level have a common internal edge with this element of* M. (*The action of* $\mathscr{P}$ *on* $\mathscr{Q}$ *is given by the morphism* f.)

Proof. We have the natural bijection

$$\mathrm{Hom}_{\mathrm{Inf.}\,\mathscr{Q}\text{-biMod}}(\mathscr{Q} \boxtimes_{\mathscr{P}} \underbrace{M}_{1} \boxtimes_{\mathscr{P}} \mathscr{Q}, N) \cong \mathrm{Hom}_{\mathrm{Inf.}\,\mathscr{P}\text{-biMod}}\big(M, f^*(N)\big).$$

Let $\Phi : \mathcal{Q} \boxtimes_{\mathcal{P}} \underbrace{M}_{1} \boxtimes_{\mathcal{P}} \mathcal{Q} \to N$ be a morphism of infinitesimal $\mathcal{Q}$-bimodules. It is characterized by the image of the projection of the element of $I \boxtimes \underbrace{M}_{1} \boxtimes I$ in $\mathcal{Q} \boxtimes_{\mathcal{P}} \underbrace{M}_{1} \boxtimes_{\mathcal{P}} \mathcal{Q}$. Let us call $\varphi : M \to N$ this map. It is then easy to see that φ is a morphism of infinitesimal $\mathcal{P}$-bimodules. $\square$

Example. If we apply the preceding proposition to the unit $u : I \to \mathcal{P}$ of a prop(erad) $\mathcal{P}$, the functor $u^* : \mathrm{Inf.}\,\mathcal{P}\text{-biMod} \to \mathbb{S}\text{-biMod}$ is the classical forgetful functor. Hence $u_!(M) = \mathcal{P} \boxtimes \underbrace{M}_{1} \boxtimes \mathcal{P}$ is the free infinitesimal $\mathcal{P}$-bimodule associated to any $\mathbb{S}$-bimodule M.

Definition (module of Kähler differentials). Let us denote $u : \mathcal{I} \to \mathcal{X}$. The module of Kähler differentials of a prop(erad) $\mathcal{X}$ over $\mathcal{I}$ is the quotient of the free infinitesimal $\mathcal{X}$-bimodule over the infinitesimal $\mathcal{I}$-bimodule $u^*(\mathcal{X})$, that is

$$u_!\big(u^*(\mathcal{X})\big) = \mathcal{X} \boxtimes_{\mathcal{I}} \underbrace{u^*(\mathcal{X})}_{1} \boxtimes_{\mathcal{I}} \mathcal{X},$$

by the relations

$$\pi\big(I \boxtimes \mu_{\mathcal{X}}(x_1 \boxtimes_{(1,1)} x_2) \boxtimes I - I \boxtimes x_1 \boxtimes_{(1,1)} x_2 - (-1)^{|x_1|} x_1 \boxtimes_{(1,1)} x_2 \boxtimes I\big),$$

where π is the canonical projection of $\mathcal{X} \boxtimes \underbrace{\mathcal{X}}_{1} \boxtimes \mathcal{X}$ on the coequalizer

$$\mathcal{X} \boxtimes_{\mathcal{I}} \underbrace{u^*(\mathcal{X})}_{1} \boxtimes_{\mathcal{I}} \mathcal{X}.$$

We denote it by $\Omega_{\mathcal{X}/\mathcal{I}}$.

We define the universal derivation $D : \mathcal{X} \to \Omega_{\mathcal{X}/\mathcal{I}}$ by $D(x)$ equal to the class of $I \boxtimes x \boxtimes I$ in $\Omega_{\mathcal{X}/\mathcal{I}}$. Like in the case of commutative algebras (see J.-L. Loday [27], Section 1.3) or associative algebras (see [9], [21], [27], the module of Kähler differentials represents the derivations.

Proposition 19. *There is a natural bijection*

$$\mathrm{Der}_{\mathcal{I}}(\mathcal{X}, M) \cong \mathrm{Hom}_{\mathrm{Inf.}\,\mathcal{X}\text{-biMod}}(\Omega_{\mathcal{X}/\mathcal{I}}, M).$$

Proof. Let d be a derivation in $\mathrm{Der}_{\mathcal{I}}(\mathcal{X}, M)$. There is a unique morphism of infinitesimal $\mathcal{X}$-bimodules $\theta : \Omega_{\mathcal{X}/\mathcal{I}} \to M$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "D"] \arrow[dr, "d"'] & \Omega_{\mathcal{X}/\mathcal{I}} \arrow[d, dashed, "\theta"] \\ & M. \end{tikzcd}$$

The image of the class of $I \boxtimes x \boxtimes I$ in $\Omega_{\mathcal{X}/\mathcal{I}}$ under θ is defined by $d(x)$. It extends freely to the infinitesimal $\mathcal{X}$-bimodule $\mathcal{X} \boxtimes_{\mathcal{I}} \underbrace{u^*(\mathcal{X})}_{1} \boxtimes_{\mathcal{I}} \mathcal{X}$ and then passes to the quotient thanks to the Leibniz relation verified by d. $\square$

The module of Kähler differentials of an associative algebra is the non-commutative analog of classical differential forms (see A. Connes [9]). Since operads and prop(erad)s can also be used to encode geometry (see [37], [38]), the module of Kähler differentials for prop(erad)s seems a promising tool to study non-linear properties in non-commutative geometry.

Theorem 20. *For any infinitesimal $\mathscr{P}$-bimodule M, the following adjunction holds*:

$$\mathrm{Hom}_{\mathrm{Prop(erad)}/\mathscr{P}}(\mathscr{X}, \mathscr{P} \ltimes M) \cong \mathrm{Der}_{\mathscr{I}}(\mathscr{X}, M) \cong \mathrm{Hom}_{\mathrm{Inf.}\,\mathscr{P}\text{-biMod}}(\mathscr{P} \boxtimes_{\mathscr{X}} \underbrace{\Omega_{\mathscr{X}/\mathscr{I}}}_{1} \boxtimes_{\mathscr{X}} \mathscr{P}, M).$$

Proof. It is a direct corollary of Proposition 17 and Proposition 19. The last natural bijection

$$\mathrm{Hom}_{\mathrm{Inf.}\,\mathscr{X}\text{-biMod}}\big(\Omega_{\mathscr{X}/\mathscr{I}}, f^*(M)\big) \cong \mathrm{Hom}_{\mathrm{Inf.}\,\mathscr{P}\text{-biMod}}\big(f_!(\Omega_{\mathscr{X}/\mathscr{I}}), N\big)$$

is provided by Proposition 18 applied to the morphism $f : \mathscr{X} \to \mathscr{P}$. $\square$

In other words, the following functors form a pair of adjoint functors

$$\mathscr{P} \boxtimes_{-} \underbrace{\Omega_{-/\mathscr{I}}}_{1} \boxtimes_{-} \mathscr{P} : \mathrm{Prop(erad)}/\mathscr{P} \rightleftharpoons \mathrm{Inf.}\,\mathscr{P}\text{-biMod} : \mathscr{P} \ltimes -.$$

The model category structure on prop(erad)s induces a model category structure on $\mathrm{Prop(erad)}/\mathscr{P}$.

Lemma 21. *The category of infinitesimal $\mathscr{P}$-bimodules is endowed with a cofibrantly generated model category structure.*

Proof. We use the same arguments as in Appendix A, that is the Transfer Theorem 32 along the free infinitesimal $\mathscr{P}$-bimodule functor $\eta_! : \mathbb{S}\text{-biMod} \to \mathrm{Inf.}\,\mathscr{P}\text{-biMod}$. The forgetful functor η^* creates limits and colimits which proves (1) and (2). A relative $\eta_!(J)$-cell complex has the form $A_0 \to A_0 \oplus \mathscr{P} \boxtimes \underbrace{\left(\bigoplus_{i \geqq 0} D^{k_i}_{m_i, n_i}\right)}_{1} \boxtimes \mathscr{P}$, which is a quasi-isomorphism of dg $\mathbb{S}$-bimodules since the right-hand term $\mathscr{P} \boxtimes \underbrace{\left(\bigoplus_{i \geqq 0} D^{k_i}_{m_i, n_i}\right)}_{1} \boxtimes \mathscr{P}$ is acyclic. $\square$

Proposition 22. *The pair of adjoint functors*

$$\mathscr{P} \boxtimes_{-} \underbrace{\Omega_{-/\mathscr{I}}}_{1} \boxtimes_{-} \mathscr{P} : \mathrm{Prop(erad)}/\mathscr{P} \rightleftharpoons \mathrm{Inf.}\,\mathscr{P}\text{-biMod} : \mathscr{P} \ltimes -$$

forms a Quillen adjunction.

Proof. By [17], Lemma 1.3.4, it is enough to prove that the right adjoint $\mathscr{P} \ltimes -$ preserves fibrations and acyclic fibrations. Let $f : M \twoheadrightarrow M'$ be a fibration (resp. acyclic fibration) between two infinitesimal $\mathscr{P}$-bimodules, that is f is degreewise surjective (resp. and a quasi-isomorphism). Since $\mathscr{P} \ltimes (f)$ is the morphism of properads on $\mathrm{Id}_{\mathscr{P}} \oplus f : \mathscr{P} \oplus M \to \mathscr{P} \oplus M'$, it is degreewise surjective (resp. and a quasi-isomorphism), which concludes the proof. $\square$

Therefore, we can derive them in the associated homotopy categories.

This proves that the homology of $\mathrm{Der}_{\mathscr{I}}(\mathscr{R}, M)$ is independent of the choice of the cofibrant resolution of $\mathscr{P}$ because it is well defined in the homotopy category of prop(erad)s over $\mathscr{P}$ and in the homotopy category of infinitesimal $\mathscr{P}$-bimodules.

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Ho}(\mathrm{Prop(erad)}/\mathscr{P})}(\mathscr{X}, \mathscr{P} \ltimes M) &\cong \mathrm{Der}_{\mathscr{I}}(\mathscr{X}, M)\\ &\cong \mathrm{Hom}_{\mathsf{Ho}(\mathrm{Inf.}\,\mathscr{P}\text{-biMod})}(\mathscr{P} \boxtimes_{\mathscr{X}} \underbrace{\Omega_{\mathscr{X}/\mathscr{I}}}_{1} \boxtimes_{\mathscr{X}} \mathscr{P}, M).\end{aligned}$$

Definition (cotangent complex). The *cotangent complex* of $\mathscr{P}$ is the total left derived functor of the right adjoint, that is

$$\mathbb{L}_{\mathscr{P}/\mathscr{I}} := \mathscr{P} \boxtimes_{\mathscr{R}} \underbrace{\Omega_{\mathscr{R}/\mathscr{I}}}_{1} \boxtimes_{\mathscr{R}} \mathscr{P},$$

for $\mathscr{R}$ a cofibrant resolution of $\mathscr{P}$.

Since on the homology of the cotangent complex, in the classical case of commutative rings, one can read the properties of the morphism $\mathscr{I} \to \mathscr{P}$ (smooth, locally complete intersection, etc....), we expect to be able to read such properties on the generalized version defined here. In the same way, transitivity and flat base change theorems should be proved for this cotangent complex but it is not our aim here and will be studied in a future work.

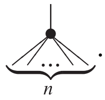
Remark. This section is written in the category of dg-prop(erad)s since we work in this paper over a field of characteristic 0. Therefore, to explicit the cotangent complex and the (co)homology of prop(erad)s, we have to use cofibrant resolutions in the category of dg prop(erad)s, for instance quasi-free resolutions (Koszul or homotopy Koszul). One can extend this section and the Appendix when the characteristic of the ground ring is not 0. In this case, one has to use simplicial resolutions like in M. André [1] and D. Quillen [42].

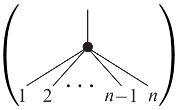
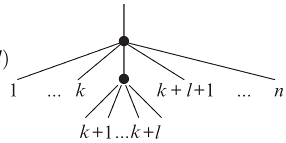
3. Examples of deformation theories

In this section, we show that the conceptual deformation theory defined here coincides to well known theories in the case of associative algebras, Lie algebras, commutative algebras, Poisson algebras. As a corollary, we get classical Lie brackets on these cohomology theory as well as classical Lie brackets in differential geometry. More surprisingly, we make deformation theory explicit in the case of associative bialgebras and show that it corresponds to Gerstenhaber-Schack type bicomplex.

3.1. Associative algebras. If P is the properad Ass of associative algebras, it is generated by a non-symmetric operad still denoted by Ass. This operad is Koszul that is, its minimal resolution exists and is generated by the (strict) cooperad Ass^{i} with

$$\mathsf{Ass}^{\text{i}}(m,n) = \begin{cases} s^{n-2}\mathbb{K}[\mathbb{S}_1] \otimes \mathbb{K}[\mathbb{S}_n] & \text{for } m = 1,\ n \geqq 2,\\ 0 & \text{otherwise.}\end{cases}$$

We represent the generating element of $\mathsf{Ass}(1,n)$ by a corolla . The partial coproduct of this cooperad is given by the formula

$\Delta_{(1,1)}\Bigg($  $\Bigg) = \sum_{k=0}^{n-1}\sum_{l=1}^{n-k}(-1)^{(l-1)(n-k-l)}$  .

Proposition 23. *Let $\mathsf{Ass} \xrightarrow{\varphi} \mathscr{Q}$ be a map of (non-symmetric) operads. The deformation complex of this map is isomorphism to $\mathscr{Q}$ up to the following shift of degree*

$$C^{\varphi}_{\bullet}(\mathsf{Ass},\mathscr{Q})(n) = s^{-1}\operatorname{Hom}^{\mathbb{S}}_{\bullet}(\mathsf{Ass}^{\mathsf{i}},\mathscr{Q})(n) = s^{-n}\mathscr{Q}_{\bullet}(n).$$

The boundary map is given by

$$D(q) = d(q) + \mu_{\mathscr{Q}}\big(\varphi(v); I, q\big) + \sum_{i=1}^{n}(-1)^{i}\mu_{\mathscr{Q}}\big(q; I,\dots,I,\underbrace{\varphi(v)}_{i},I,\dots,I\big)$$
$$+ (-1)^{n+1}\mu_{\mathscr{Q}}\big(\varphi(v); q, I\big),$$

for $q \in \mathscr{Q}(n)$ if we denote by v the generating binary operation of $\mathsf{Ass}(2)$.

Proof. There is a one-to-one correspondence between $\mathbb{S}_n$-equivariant maps from $\mathsf{Ass}^{\mathsf{i}}(n)$ to $\mathscr{Q}(n)$ and elements of $\mathscr{Q}(n)$. Let us denote by f_q the unique map determined by $q \in \mathscr{Q}(n)$. Since $\mathsf{Ass}^{\mathsf{i}}$ is a cooperad and $\mathscr{Q}$ is an operad, the convolution operad $\operatorname{Hom}(\mathsf{Ass}^{\mathsf{i}},\mathscr{Q})$ is preLie algebra with product denoted $\star$ (see [39], Section 2). By Theorem 12 and Section 2.3, we have $D(f_q) = d(f_q) + \gamma \star f_q - (-1)^{|f_q|} f_q \star \gamma$. Since f_q vanishes on $\mathsf{Ass}^{\mathsf{i}}(m)$ for $m \neq n$ and since γ vanishes on $\mathsf{Ass}^{\mathsf{i}}(m)$ for $m \neq 2$, the only non-vanishing component of $\gamma \star f_q = \mu_{\mathscr{Q}} \circ (\gamma \boxtimes_{(1,1)} f_q) \circ \Delta_{(1,1)}$ is

$$-\mu_{\mathscr{Q}}\big(\varphi(v); I, q\big) + (-1)^{n}\mu_{\mathscr{Q}}\big(\varphi(v); q, I\big) \quad \text{on } \operatorname{Hom}^{\mathbb{S}}_{\bullet}(\mathsf{Ass}^{\mathsf{i}},\mathscr{Q}).$$

And the only non-vanishing component of $f_q \star \gamma = \mu_{\mathscr{Q}} \circ (f_q \boxtimes_{(1,1)} \gamma) \circ \Delta_{(1,1)}$ is

$$\sum_{i=1}^{n}(-1)^{i+1}\mu_{\mathscr{Q}}\big(q; I,\dots,I,\underbrace{\varphi(v)}_{i},I,\dots,I\big) \quad \text{on } \operatorname{Hom}^{\mathbb{S}}_{\bullet}(\mathsf{Ass}^{\mathsf{i}},\mathscr{Q}),$$

which concludes the proof. $\square$

This deformation complex appears in many places in the literature under different names. When $\mathscr{Q} = \operatorname{End}_X$ with X an associative algebra, it is the Hochschild (co)chain complex of X (*with coefficient in* X): $s^{-1}\operatorname{Hom}(V,\operatorname{End}_X) = \bigoplus_{n \geqq 2} s^{1-n}\operatorname{Hom}(X^{\otimes n}, X)$. The induced L_∞-algebra, Q on it is strict since the operad Ass is Koszul. It is precisely the Gerstenhaber Lie algebra [12] and Q^{γ} is the Hochschild dg Lie algebra controlling deformations of a particular associative algebra structure $\gamma : \mathsf{Ass} \to \operatorname{End}_X$ on a vector space X.

In the work of McClure-Smith on Deligne's conjecture [35], an operad $\mathscr{Q}$ with a morphism of operads $\mathsf{Ass} \to \mathscr{Q}$ is called a *multiplicative operad*. The simplicial complex that they define on such an operad is exactly the deformation complex of this map. For the operad $\mathscr{Q} = \mathit{Poisson}$, this complex is related to the homology of long knots (see [49]). More generally, Maxim Kontsevich proposed the conjecture that the deformation complex of $\mathsf{Ass} \to \mathrm{End}_X$ is a $d+1$-algebra when X is a d-algebra in [22]. This conjecture was proved by Tamarkin in [48], see also Hu, Kriz and Voronov [18]. In this context, this chain complex is often called the *Hochschild complex of* $\mathscr{Q}$.

Since this (co)chain complex comes from the general theory of (co)homology of Quillen, it would be better to call its (co)homology the *cohomology of* Ass *with coefficients in* $\mathscr{Q}$ or the chain complex, the *deformation complex of the map* φ.

Analogously one recovers other classical examples—Harrison complex/cohomology and Chevalley-Eilenberg complex/cohomology—from the operads of commutative algebras and, respectively, Lie algebras.

3.2. Poisson structures. A *Lie* 1-*bialgebra* is, by definition, a graded vector space V together with two linear maps

$$\begin{aligned} \delta : V &\to \wedge^2 V, & [\bullet] : \odot^2 V &\to V, \\ a &\to \sum a_1 \wedge a_2, & a \otimes b &\to (-1)^{|a|}[a \bullet b], \end{aligned}$$

of degrees 0 and -1 respectively which satisfy the identities

(i) $(\delta \otimes \mathrm{Id})\delta a + \tau(\delta \otimes \mathrm{Id})\delta a + \tau^2(\delta \otimes \mathrm{Id})\delta a = 0$, where τ is the cyclic permutation (123) represented naturally on $V \otimes V \otimes V$ (co-Jacobi identity);

(ii) $[[a \bullet b] \bullet c] = [a \bullet [b \bullet c]] + (-1)^{|b|\,|a|+|b|+|a|}[b \bullet [a \bullet c]]$ (Jacobi identity);

(iii)

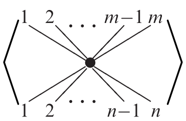
$$\delta[a \bullet b] = \sum a_1 \wedge [a_2 \bullet b] - (-1)^{|a_1|\,|a_2|} a_2 \wedge [a_1 \bullet b] + [a \bullet b_1] \wedge b_2 - (-1)^{|b_1|\,|b_2|}[a \bullet b_2] \wedge b_1$$

(Leibniz type identity).

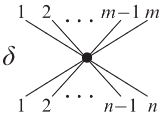
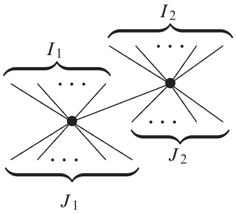
This notion of Lie 1-bialgebras is similar to the well-known notion of Lie bialgebras except that in the latter case both operations, Lie and co-Lie brackets, have degree 0.

Let $\mathscr{L}ieBi$ be the properad whose representations are Lie 1-bialgebras. It is Koszul contractible, that is its minimal resolution $(\mathscr{L}ieBi_\infty, \delta)$ exists and is generated by the $\mathbb{S}$-bimodule $V = \{V(m,n)\}_{m,n \geqq 1, m+n \geqq 3}$ with

$$V(m,n) := s^{m-2}\,\mathrm{sgn}_m \otimes \mathbf{1}_n = \mathrm{span}\langle \ \rangle,$$



where sgn_m stands for the sign representation of $\mathbb{S}_m$ and $\mathbf{1}_n$ for the trivial representation of $\mathbb{S}_n$. The differential is given on generators by [38]

δ  $= \sum_{\substack{I_1 \sqcup I_2 = (1,\dots,m) \\ J_1 \sqcup J_2 = (1,\dots,n) \\ |I_1| \geqq 0, |I_2| \geqq 1 \\ |J_1| \geqq 1, |J_2| \geqq 0}} (-1)^{\sigma(I_1 \sqcup I_2) + |I_1|\,|I_2|}$ 

where $\sigma(I_1 \sqcup I_2)$ is the sign of the shuffle $I_1 \sqcup I_2 = (1, \dots, m)$.

Hence, for an arbitrary dg vector space X,

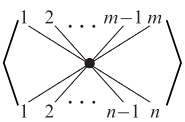
$$s^{-1} \operatorname{Hom}(V, \operatorname{End}_X) = \bigoplus_{m,n \geqq 1} s^{1-m} \wedge^m X \otimes \odot^n X \simeq \wedge^\bullet T_X,$$

where $\wedge^\bullet T_X$ is the vector space of formal germs of polyvector fields at $0 \in X$ when we view X as a formal graded manifold. It is not hard to show using the above explicit formula for the differential δ that the canonically induced, in accordance with Theorem 5(i), L_∞-structure on $s^{-1} \operatorname{Hom}(V, \operatorname{End}_X)$ is precisely the classical Schouten Lie algebra structure on polyvector fields. Thus our theory applied to Lie 1-bialgebras reproduces deformation theory of Poisson structures, and $\mathscr{L}ie\mathscr{B}i$-homology is precisely Poisson homology.

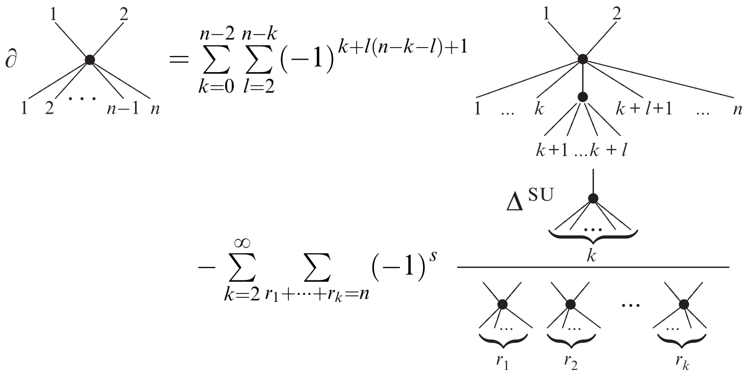
In a similar way one can check that our construction of L_∞-algebras applied to the minimal resolution of so called pre-Lie^2-algebras [37] gives rise to another classical geometric object—the Frölicher-Nijenhuis Lie brackets on the sheaf $T_X \otimes \Omega_X^\bullet$ of tangent vector bundle valued differential forms. Thus the associated deformation theory describes deformations of integrable Nijenhuis structures.

3.3. Associative bialgebras. In this section, we make explicit the deformation theory of representation of the properad $\mathscr{A}ss\mathscr{B}i$ of associative bialgebras. As this example has never been rigorously treated in the literature before, we show full details here.

As the properad $\mathscr{A}ss\mathscr{B}i$ is homotopy Koszul (see [39], Section 5.4) it admits a minimal resolution $\mathscr{A}ss\mathscr{B}i_\infty = \big(\mathscr{F}(\mathscr{C}), \partial\big)$ which is generated by a relatively small $\mathbb{S}$-bimodule $\mathscr{C} = \{\mathscr{C}(m,n)\}_{m,n \geqq 1, m+n \geqq 3}$,

$\mathscr{C}(m,n) := s^{m+n-3}\mathbb{K}[\mathbb{S}_m] \otimes \mathbb{K}[\mathbb{S}_n] = \operatorname{span}\Big\langle$  $\Big\rangle.$

The differential ∂ in $\mathscr{A}ss\mathscr{B}i_\infty$ is neither quadratic nor of genus 0. The derivation ∂ on $\mathscr{F}(\mathscr{C})$ is equivalent to a structure of homotopy coproperad on $s^{-1}\mathscr{C}$. The values of ∂ on $(1,n)$- and $(m,1)$-corollas are given, of course, by the well-known A_∞-formulae, while

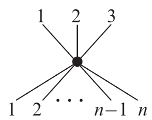


where

$$s = (k-1)(r_1-1) + (k-2)(r_2-1) + \cdots + 1(r_k-1),$$

Δ^{SU} is the Saneblidze-Umble diagonal, and the horizontal line means *fraction* composition from [31]. The meaning of this part of the differential is clear: it describes A_∞-morphisms between an A_∞-structure on X and the associated Saneblidze-Umble diagonal A_∞-structure on $X \otimes X$. Explicitly, this formula is obtained by first considering the quasi-free resolution of the 2-colored operad coding two associative algebras and a morphism between them. While the resolution of the associative operad is given by the associahedra, this resolution is given by the multiplihedra. This resolution gives the relaxed notion of A_∞-algebra and morphism of A_∞-algebras at the same time. Then, to get the formula above, we applied this resolution to the A_∞-algebra X and to $X \otimes X$ with the A_∞-algebra structure induced by the Saneblidze-Umble diagonal.

The values of ∂ on corollas of the form



describe a *homotopy* between two natural A_∞-morphisms from X to $X \otimes X \otimes X$, values on corollas with 4 output legs—homotopies between homotopies etc. We conjecture that $(\mathcal{A}ss\mathcal{B}i_\infty, \partial)$ is a one coloured version of a certain $\mathbb{N}$-coloured properad describing A_∞-algebras, morphisms of A_∞-algebras, homotopies between morphisms of A_∞-algebras, homotopies of homotopies etc., and we hope to describe it in a future publication.

It was proven in [31], [34] that there exists a minimal model $(\mathcal{A}ss\mathcal{B}i_\infty, \partial)$ such that the differential preserves Kontsevich's path grading of $\mathcal{A}ss\mathcal{B}i_\infty$ and has the form $\partial = \partial_0 + \partial_{\mathrm{pert}}$, where ∂_0 describes the minimal resolution $\frac{1}{2}\mathcal{A}ss\mathcal{B}_\infty$ of the prop of $\frac{1}{2}$-bialgebras (these facts follow also immediately from [39], Corollary 42). The perturbation part ∂_{pert} is a linear combination of so called *fractions* and their compositions. We shall assume from now on that ∂ has all these properties. By checking genus of these fractions (or by referring to our proof of homotopy Koszulness of $\mathscr{A}ss\mathscr{B}i$ in [39], Section 5.4) one can easily obtain the following useful (for our purposes)

Fact 24. *The differential ∂_0 is precisely the quadratic part of ∂, i.e. it is equal to the composition,*

$$\partial_0 : \mathscr{C} \xrightarrow{\partial} \mathscr{F}(\mathscr{C}) \xrightarrow{\text{proj}} \mathscr{F}(\mathscr{C})^{(2)}.$$

Let $\mathscr{Q}$ be a dg properad. By Theorem 5, the vector space,

$$s^{-1}\operatorname{Hom}^{\mathbb{S}}(\mathscr{C}, \mathscr{Q}) = \bigoplus_{\substack{n,m \geqq 1 \\ m+n \geqq 3}} s^{2-m-n}\mathscr{Q}(m,n) =: \mathfrak{g}_{GS}(\mathscr{Q}),$$

has a canonical homotopy non-symmetric properad and L_∞-structure Q, whose Maurer-Cartan elements are morphisms of properads $\mathscr{F}(\mathscr{C}) \to \mathscr{Q}$.

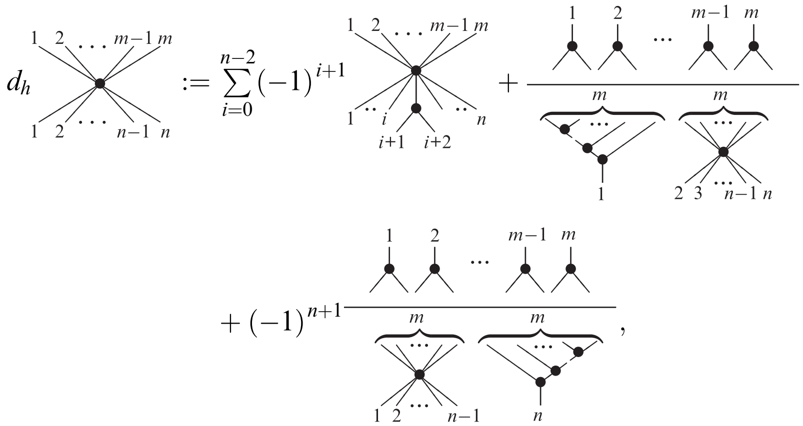
If $\gamma : \mathscr{A}ss\mathscr{B}i \to \mathscr{Q}$ is a representation of $\mathscr{A}ss\mathscr{B}i$, or more generally of

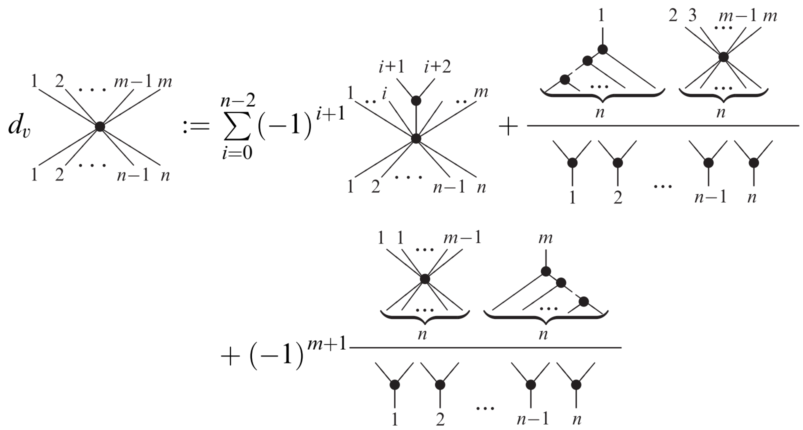
$$\mathscr{A}ss\mathscr{B}i_\infty : \mathscr{A}ss\mathscr{B}i_\infty \to \mathscr{Q},$$

then, by Definition 2.2, there exists an associated twisted L_∞-structure $Q^\gamma = \{Q_n^\gamma\}_{n\geqq 1}$ on $\mathfrak{g}_{GS}(\mathscr{Q})$ which controls deformations of γ in the class of representations of $\mathscr{A}ss\mathscr{B}i_\infty$. An explicit formula for the differential ∂ would induce an explicit L_∞-structure. Once again, our main example of this deformation theory is given by $\mathscr{Q} = \operatorname{End}_X$. In this case, the complex above is the deformation complex of associative bialgebra, or more generally of $\mathscr{A}ss\mathscr{B}i_\infty$-gebra, structure on X.

When X is an associative bialgebra, Gerstenhaber and Schack defined in [13] a bicomplex whose homology has nice properties with respect to the deformations of the associative bialgebra structure (see also [26]). Let us first extend this definition to any properad $\mathscr{Q}$ and not only End_X.

Definition. Let $\gamma : \mathscr{A}ss\mathscr{B}i \to \mathscr{Q}$ be a representation of $\mathscr{A}ss\mathscr{B}i$. We define the *Gerstenhaber-Schack bicomplex of* γ by $C^{m,n} := \mathscr{Q}(m,n)$ and the differentials by





where the general (m,n)-corollas have to be understood as elements of $\mathcal{Q}(m,n)$. The binary corollas are the image under γ of the generating product and coproduct of $\mathcal{A}ss\mathcal{B}i$. Finally, this pictures represent the composition of all these elements in $\mathcal{Q}$.

Let us compare these with the Gerstenhaber-Schack differential, d_{GS}, in the bicomplex $C_{\mathrm{GS}}^{m,n} := \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$ which is defined by [13]

$$d_{\mathrm{GS}} = d_h + d_v,$$

with $d_h : \mathrm{Hom}(X^{\otimes n}, X^{\otimes m}) \to \mathrm{Hom}(X^{\otimes n+1}, X^{\otimes m})$ given on an arbitrary

$$f \in \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$$

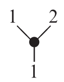
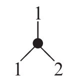
by

$$\begin{aligned}(d_h f)(a_0, a_1, \ldots, a_n) := \Delta^m(a_0) \square f(a_1, a_2, \ldots, a_n) - \sum_{i=0}^{n-1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \\ + (-1)^{n+1} f(a_1, a_2, \ldots, a_n) \square \Delta^m(a_n) \quad \forall a_i \in X.\end{aligned}$$

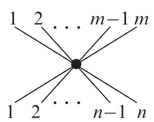
Here the multiplication in X is denoted by juxtaposition, the induced multiplication in the algebra $X^{\otimes m}$ by $\square$, the comultiplication in X by Δ, and

$$\Delta^n : (\Delta \otimes \mathrm{Id}^{\otimes m-2}) \circ (\Delta \otimes \mathrm{Id}^{\otimes m-3}) \circ \cdots \circ \Delta : X \to X^{\otimes m}.$$

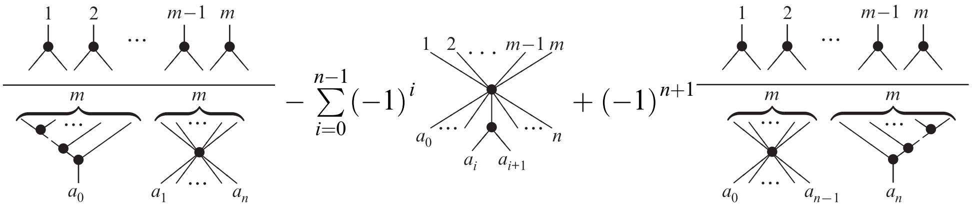
The expression for d_v is an obvious "dual" analogue of d_h. Now let us represent d_h in graphical terms by associating the graphs

 and 

to comultiplication and, respectively, multiplication while the corolla



to f. Then the r.h.s of the formula for d_h reads



which are precisely the first three summands in the previous definition. The other three terms correspond to d_v. Therefore, when $\mathscr{Q} = \mathrm{End}_X$ and $\gamma : \mathscr{A}ss\mathscr{B}i \to \mathrm{End}_X$ is an associative bialgebra structure on X, the preceding bicomplex is exactly Gerstenhaber-Schack bicomplex [13].

However, even without an explicit minimal model of $\mathscr{A}ss\mathscr{B}i$, we can show the following general result.

Theorem 25. *Let* $(\mathscr{A}ss\mathscr{B}i_\infty, \partial) \xrightarrow{\pi} \mathscr{A}ss\mathscr{B}i$, *be a minimal model of the properad of bialgebras and* $\gamma : \mathscr{A}ss\mathscr{B}i \to \mathscr{Q}$ *an arbitrary representation of* $\mathscr{A}ss\mathscr{B}i$. *Then the differential*

$$Q_1^\gamma = Q \circ e^{\gamma \odot}$$

associated to this minimal model in the twisted L_∞*-structure* Q^γ *on* $\mathfrak{g}_{\mathrm{GS}}$, *is isomorphic to the Gerstenhaber-Schack differential. Hence the deformation complex of representation of* $\mathscr{A}ss\mathscr{B}i$ *is isomorphic to the Gerstenhaber-Schack bicomplex.*

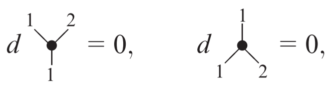
Proof. Let $\left(\mathscr{A}ss\mathscr{B}i_\infty = \mathscr{F}(\mathscr{C}), \partial\right)$ be a minimal model of the properad of bialgebras, and let I be the ideal in $\mathscr{F}(\mathscr{C})$ generated by graphs in $\mathscr{F}(\mathscr{C})^{(\geqq 2)}$ with at least two non-binary (i.e. neither  nor ) vertices, and let

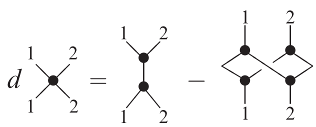
$$\mathsf{B} := \frac{\mathscr{A}ss\mathscr{B}i_\infty}{(I, \partial I)}$$

be the associated quotient dg properad. The induced differential in B we denote by ∂_{ind}. It is precisely this quotient part ∂_{ind} of the total differential ∂ which completely determines the L_∞-differential differential Q_1^γ. Thus our plan is the following: in the next lemma we present an explicit, up to an automorphism, form of the differential ∂_{ind} (despite the fact that ∂ is not explicit !) and thereafter compare the resulting Q_1^γ with the Gerstenhaber-Schack definition.

The major step in the proof is the following lemma (in its formulation we use fraction notations again).

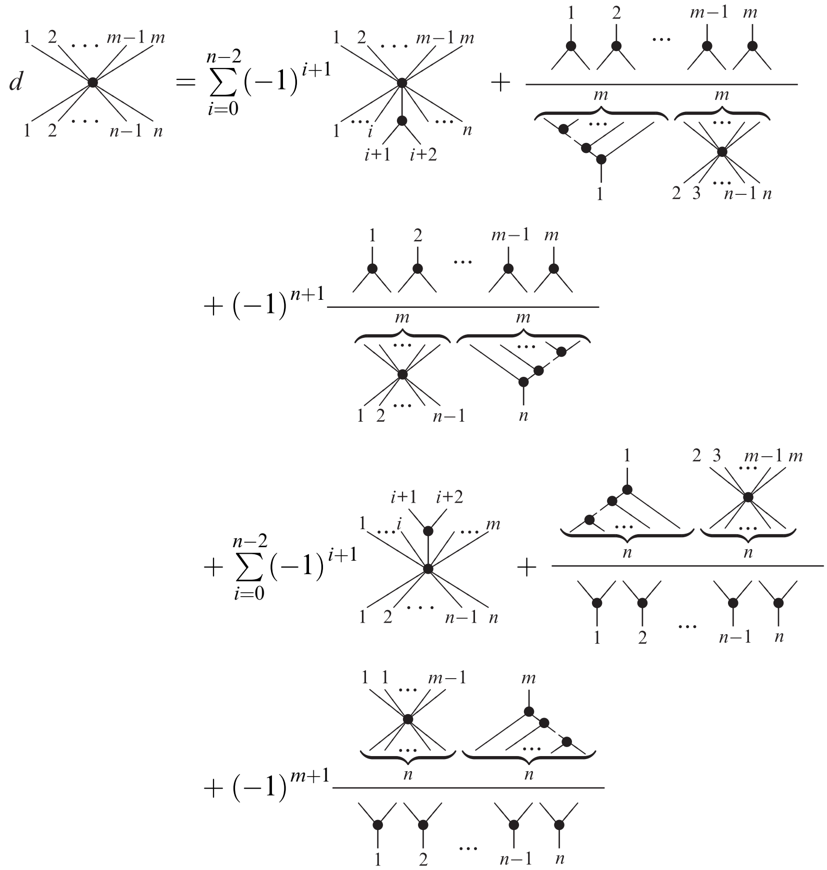
Lemma 26. (i) *The derivation d of* B *given on generators by*

(9) 

(10) 

and, for all other generators with $m + n \geqq 4$, *by*

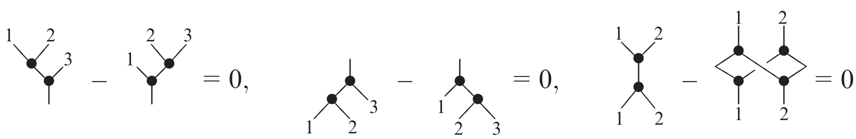
(11)



is a differential.

(ii) *The* dg *properads* $(\mathsf{B}, \partial_{\mathrm{ind}})$ *and* (B, d) *are isomorphic.*

Proof. (i) It is easy to see that among for 2-vertex connected binary graphs[4)] attached to any other graph in B the bialgebra relations



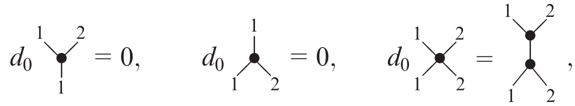
hold. Using this fact it is an easy and straightforward calculation to check that $d^2 = 0$. We omit the details. (In fact we shall show below that d is essentially a graph encoding of the Gerstenhaber-Schack differential d_{GS} so this calculation is essentially identical to the one which establishes $d_{\mathrm{GS}}^2 = 0$.)

(ii) We begin our proof of Lemma 26(ii) with the following

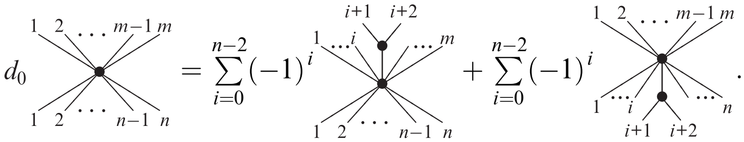
[4)] Equivalence classes of graphs in B we call simply graphs for shortness.

Claim 1. *The natural projection $p : (\mathsf{B}, d) \to \mathcal{A}ss\mathcal{B}i$ is a quasi-isomorphism.*

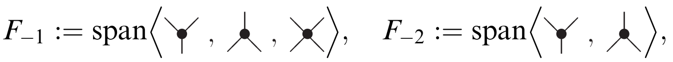
Indeed, the dg properad (B, d) has a natural increasing and bounded above filtration[5] $\{\mathsf{F}_{-p}\mathsf{B}\}_{p \geqq 0}$ with $\mathsf{F}_{-p}\mathsf{B}$ being the span of equivalence classes of graphs which admit a representative in $\mathcal{A}ss\mathcal{B}i_\infty^{(\geqq p)}$. As the differential d is connected and preserves the induced path gradation, the associated spectral sequence $(\mathsf{E}_r, \mathsf{d}_r)$ converges to $H^\bullet(\mathsf{B}, d)$. The 0th term $(\mathsf{E}_0, \mathsf{d}_0)$ has the differential given on generators by

(12) 

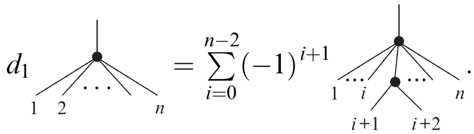
and, for all other generators with $m + n \geqq 4$,

(13) 

We want to compute homology $\mathsf{E}_1 = H^\bullet(\mathsf{E}_0, \mathsf{d}_0)$ of this complex and show that $\mathsf{E}_1 \simeq \mathcal{A}ss\mathcal{B}i$. For this purpose consider a 3-step filtration $0 \subset F_{-2} \subset F_{-1} \subset F_0 = \mathsf{E}_0$ of the complex $(\mathsf{E}_0, \mathsf{d}_0)$ with



and let $(\mathcal{E}_r, \partial_r)$ be the associated spectral sequence. The differential ∂_0 is zero on the generators of F_{-1} and is equal to d_0 on all the other generators. Thus, modulo shifts of gradings, actions of finite groups and tensor products by trivial (i.e. with zero differential) complexes, the complex $(\mathcal{E}_0, \partial_0)$ is isomorphic to the tensor product of two isomorphic operadic complexes (one with "time" flow reversed upside down relative to another) which were studied in [32], page 40, and which have the differential (in notations of that paper) given by



It is shown in [32] that the cohomology of this complex is concentrated in degree 0 and is isomorphic to the operad of associative algebras. In our context this result immediately implies that $(\mathcal{E}_1, \partial_1)$ is isomorphic to F_{-1} with the differential ∂_1 given on generators by (12). Its cohomology is obviously concentrated in degree 0 (and is equal, in fact, to the properad $\frac{1}{2}\mathsf{B}$ of infinitesimal bialgebras). Hence $H^\bullet(\mathsf{B}, d)$ is concentrated in degree 0 proving Claim 1.

[5] One might prove Claim 1 using another filtration of B by the number of vertices and Fact 24 provided one assumes (without any losses) that B is completed with respect to this filtration.

Claim 2. *The natural projection $\pi : (\mathsf{B}, \partial_{\mathrm{ind}}) \to \mathscr{A}ss\mathscr{B}i$ is a quasi-isomorphism.*

Indeed, the defined above filtration $\{\mathsf{F}_{-p}\mathsf{B}\}_{p \geqq 0}$ by the number of vertices is also compatible with the differential ∂_{ind}. Let (E_r, d_r) be the associated spectral sequence. Its first nontrivial term, (E_1, d_1) is, by Fact 24, isomorphic to the complex $(\mathsf{E}_1, \mathsf{d}_1)$ above. Hence we can apply the same reasoning as in the proof of Claim 1.

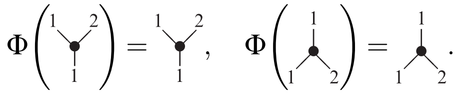
Claim 3. *There exits a morphism of* dg *properads Φ making the diagram*

$$\begin{tikzcd} & (\mathsf{B}, d) \arrow[d, "p"] \\ (\mathscr{A}ss\mathscr{B}i_\infty, \partial) \arrow[ur, "\Phi"] \arrow[r, "\pi"'] & (\mathscr{A}ss\mathscr{B}i, 0) \end{tikzcd}$$

commutative.

Since $\mathscr{A}ss\mathscr{B}i$ is a properad concentrated in degree 0, the map p is surjective. Since p is a quasi-isomorphism by Claim 2, it is an acyclic fibration in the model category of properads (see Appendix A). By Corollary 40, $\mathscr{A}ss\mathscr{B}i_\infty$ is a cofibrant properad. Finally, the morphism Φ is given by the left lifting property in the model category of properads. Hence, the existence of Φ is clear but we need to make it more precise. We construct it as follows and refine it in Claim 4.

As $\mathscr{A}ss\mathscr{B}i_\infty = \mathscr{F}(\mathscr{C})$ is a free properad, a morphism Φ is completely determined by its values on the generating (m, n)-corollas which span the vector space $\mathscr{C}$, and one can construct Φ by a simple induction[6] on the degree $r := m + n - 3 \geqq 0$ of such corollas. For $r = 0$ we set Φ to be identity, i.e.



Assume we constructed values of Φ on all corollas of degree $r \leqq N$. Let e be a generating corolla of non-zero weight $r = N + 1$. Note that δe is a linear combination of graphs whose vertices are decorated by corollas of weight $\leqq N$ (as differential δ has degree -1). Then, by induction, $\Phi(\delta e)$ is a well-defined element in B. As $\pi(e) = 0$, the element

$$\Phi(\delta e)$$

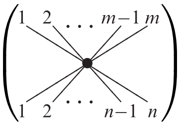
is a closed element in B which projects under p to zero. By Claim 1, the surjection p is a quasi-isomorphism. Hence this element is exact and there exists $\mathfrak{e} \in \mathsf{B}$ such that

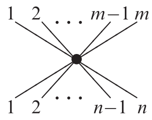
$$d\mathfrak{e} = \Phi(\delta e).$$

We set $\Phi(e) := \mathfrak{e}$ completing thereby inductive construction of Φ.

[6] This induction is a straightforward analogue of the Whitehead lifting trick in the theory of CW-complexes in algebraic topology.

Claim 4. *A morphism Φ can be chosen so that*

Φ (

) = 

+ *terms with* $\geqq 2$ *number of vertices.*

Indeed, the differential d in B has the form

$$d = \mathsf{d}_1 + d_{\text{rest}},$$

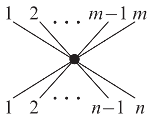
where d_1 is the quadratic *differential* in B defined by (13) and the part d_{part} corresponds to graphs lying in $F_{-3}\mathsf{B}$. We shall prove Claim 4 by induction on the degree $r = m + n - 3$ of the generating (m,n)-corollas in $\mathscr{A}ss\mathscr{B}i_\infty$ (cf. [31], proof of Theorem 43). For $r = 0$ the claim is true. Assume we have already constructed Φ such that the claim is true for values of Φ on corollas with non-zero degree $\leqq N$ and consider a generating corolla e of degree $N + 1$. The value $\mathfrak{e} := \Phi(e)$ is a solution of the equation

$$\mathsf{d}_1\mathfrak{e} + d_{\text{rest}}\mathfrak{e} = \Phi(\partial_0 e) + \Phi(\partial_{\text{pert}} e). \tag{14}$$

Let π_1 and π_2 denote projections in B to the subspaces spanned by equivalence classes of graphs with 1 and, respectively, 2 vertices. Then equation (14) implies,

$$\pi_2 \circ \mathsf{d}_1(\mathfrak{e}) = \mathsf{d}_1 \circ \pi_1(\mathfrak{e}) = \pi_2 \circ \Phi(\partial_0 e),$$

as both $d_{\text{rest}}\mathfrak{e}$ and $\Phi(\partial_{\text{pert}} e)$ are spanned by graphs lying in $F_{-3}\mathsf{B}$. Using now the explicit form for the differential ∂_0 (given, e.g., by [31], formula (14)) and the induction assumption we immediately conclude that

$\pi_1(\mathfrak{e}) =$ 

completing the proof of Claim 4.

Claim 5. *The morphism Φ induces a* dg *isomorphism* $(\mathsf{B}, \partial_{\text{ind}}) \to (\mathsf{B}, d)$.

Indeed, Φ sends the ideal I to zero. Since Φ respects differentials, it sends the ideal $(I, \partial I)$ to zero as well and hence induces, by Claims 3 and 4, a required isomorphism. This completes the proof of Lemma 26. $\square$

Now we continue with the proof of Theorem 25. The differential Q_1^γ in the graded vector space $\mathfrak{g}_{\text{GS}}(\mathscr{Q}) = \bigoplus_{m,n} s^{2-m-n}\mathscr{Q}$ is completely determined by the quotient differential ∂_{ind} of the full differential ∂ in $\mathscr{A}ss\mathscr{B}i_\infty$. By Lemma 26, this quotient differential is given, up to automorphisms, by formulae (9)–(11). The proof of the Theorem 25 is completed. $\square$

As a direct corollary, we have

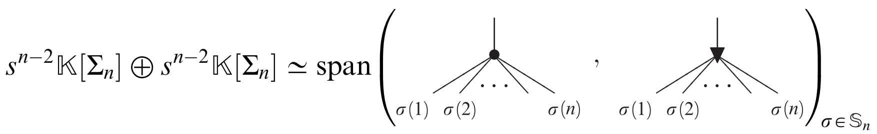
Corollary 27. *The Gerstenhaber-Schack bicomplex of an associative bialgebra X is a homotopy non-symmetric properad and a twisted L_∞-algebra whose Maurer-Cartan elements are deformations of the first structure.*

The homotopy non-symmetric properad structure induces, on this chain complex, (homotopy) LR-operations which play the same role as the non-symmetric braces for Hochschild cochain complex. They are expected to be used in the proof of a Deligne conjecture for associative bialgebras.

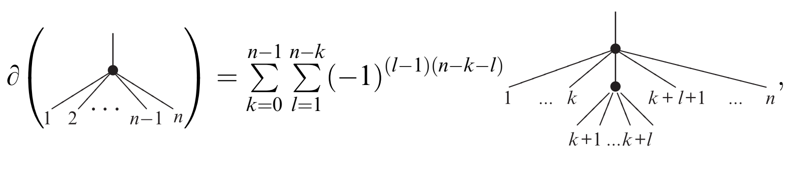
3.4. Twisted L_∞-algebras and dg prop(erad)s. For any quasi-free prop(erad) $(\mathscr{P}_\infty = \mathscr{F}(s^{-1}\mathscr{C}), \partial_{\mathscr{P}})$ and any prop(erad) $\mathscr{Q}$ there exists, in accordance with Theorem 5, a canonical L_∞-structure Q on the graded vector space $s^{-1}\operatorname{Hom}^{\mathbb{S}}_{\bullet}(\mathscr{C}, \mathscr{Q})$ whose Maurer-Cartan elements are in one-to-one correspondence with representation of P_∞ in $\mathscr{Q}$. If γ is any particular representation of P_∞, then the associated twisted L_∞-algebra Q^γ describes deformation theory of γ within the class of representation of $\mathscr{P}_\infty$ (see §7.2). Remarkably, *there always exists a quasi-free prop(erad) $(\mathscr{P}^{(2)}_\infty, \partial)$ whose representations in $\mathscr{Q}$ are in one-to-one correspondence with pairs (γ, Γ) where γ is a representation of $\mathscr{P}_\infty$ on $\mathscr{Q}$ and Γ is an* MC *element in Q^γ*. Thus the dg prop(erad) $\mathscr{P}^{(2)}_\infty$ gives a complete description of the deformation theory of a generic representation of $\mathscr{P}_\infty$. In fact, this constructions can be obviously iterated giving rise to quasi-free prop(erad)s $\mathscr{P}^{(3)}_\infty$, $\mathscr{P}^{(4)}_\infty$ etc.

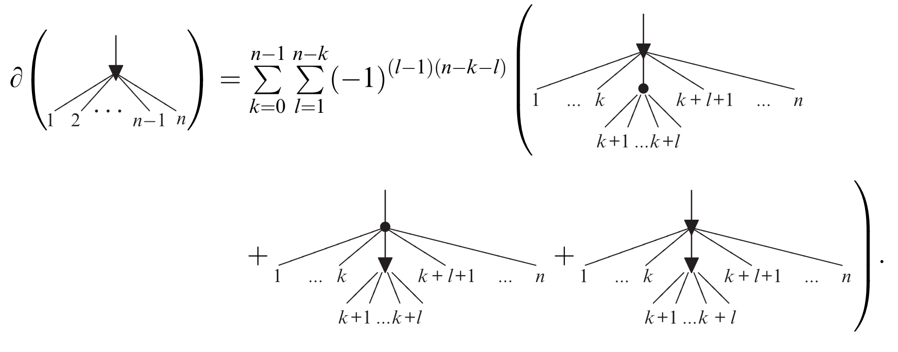
By definition, $\mathscr{P}^{(2)}_\infty$ is a free prop(erad) on the $\mathbb{S}$-bimodule $s^{-1}\mathscr{C} \oplus s^{-1}\mathscr{C}$ but the differential ∂ in $\mathscr{P}^{(2)}_\infty$ is *not* a direct sum $\partial_{\mathscr{P}} \oplus \partial_{\mathscr{P}}$ of differentials in $\mathscr{P}_\infty$. We illustrate the above claim in the case of $\mathscr{P} = \mathscr{A}ss$, the operad of associative algebras, before giving the general definition.

Let $\mathscr{A}ss^{(2)}_\infty$ be a quasi-free operad generated by an $\mathbb{S}$-module



and equipped with a differential given on generators by





Proposition 28. *There is a one-to-one correspondence between representations of the* dg *operad* $\mathscr{A}ss_\infty^{(2)}$ *in an operad* $\mathscr{Q}$ *and degree* -1 *elements* γ *and* Γ *in the deformation complex* (*Hochschild complex*) $\bigoplus_{n \geqq 2} s^{1-n}\mathscr{Q}(n)$ *such that*

$$[d+\gamma, d+\gamma]_{\mathrm{G}} = 0,$$

$$[d+\gamma, \Gamma]_{\mathrm{G}} + \frac{1}{2}[\Gamma, \Gamma]_{\mathrm{G}} = 0,$$

where $[\,,]_{\mathrm{G}}$ *stands for the Gerstenhaber brackets.*

Proof is obvious and hence omitted. The data (d, γ) describes a representation of $\mathscr{A}ss_\infty$ on $\mathscr{Q}$, and the data Γ describes a deformation of this representation.

In general, $\mathscr{P}_\infty^{(2)}$ is the prop(erad) given by $\mathscr{F}(s^{-1}\mathscr{C}^\bullet \oplus s^{-1}\mathscr{C}^\blacktriangledown)$. We denote by $\partial_{\mathscr{P}}(c) = \sum \mathscr{G}(c_1, \dots, c_n)$ the image under the differential $\partial_{\mathscr{P}}$ of an element c of $s^{-1}\mathscr{C}$, with $c_1, \dots, c_n \in \mathscr{C}$. The differential ∂ of $\mathscr{P}_\infty^{(2)}$ is defined by

$$\partial(c^\bullet) := \sum \mathscr{G}(c_1^\bullet, \dots, c_n^\bullet) \quad \text{for } c^\bullet \in s^{-1}\mathscr{C}^\bullet,$$

$$\partial(c^\blacktriangledown) := \sum \mathscr{G}(c^{i_1}, \dots, c_n^{i_n}) \quad \text{for } c^\blacktriangledown \in s^{-1}\mathscr{C}^\blacktriangledown,$$

where the $i_1, \dots, i_n$ are in $\{\bullet, \blacktriangledown\}$ with at least one equal to $\blacktriangledown$. It is easy to see that $\partial^2 = 0$.

Proof. The formula for $\partial(c^\bullet)$ gives the first relation. With the formula for $\partial(c^\blacktriangledown)$, it gives the second one. $\square$

Proposition 29. *There is a one-to-one correspondence between representations of the* dg *prop*(*erad*) $\mathscr{P}_\infty^{(2)}$ *in a prop*(*erad*) $\mathscr{Q}$ *and degree* -1 *elements* γ *and* Γ *in the deformation complex* $s^{-1}\operatorname{Hom}_\bullet^{\mathbb{S}}(\mathscr{C}, \mathscr{Q})$ *such that*

$$Q(\gamma) = 0,$$

$$Q^\gamma(\Gamma) = Q(\gamma + \Gamma) = 0,$$

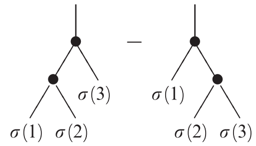
where Q *stands for the* L_∞*-algebra structure.*

In the following proposition, we interpret $\mathscr{A}ss_\infty^{(2)}$ as the Koszul resolution of a new operad, denoted by $\mathscr{A}ss^{(2)}$.

Proposition 30. *The* dg *operad* $(\mathscr{A}ss_\infty^{(2)}, \partial)$ *is a quadratic resolution of a quadratic operad* $\mathscr{A}ss^{(2)}$ *defined as the quotient of the free operad on the* $\mathbb{S}$*-module*

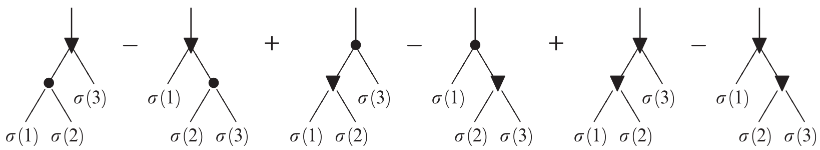
$$A(n) := \begin{cases} \mathbb{K}[\mathbb{S}_2] \oplus \mathbb{K}[\mathbb{S}_2] = \operatorname{span}\left(\begin{smallmatrix} & | & \\ & \bullet & \\ / & & \backslash \\ \sigma(1) & & \sigma(2) \end{smallmatrix} \;,\; \begin{smallmatrix} & | & \\ & \blacktriangledown & \\ / & & \backslash \\ \sigma(1) & & \sigma(2) \end{smallmatrix} \right)_{\sigma \in \Sigma_2} & \textit{for } n = 2, \\ 0 & \textit{otherwise,} \end{cases}$$

modulo the ideal generated by relations,



$$= 0 \quad \forall \sigma \in \mathbb{S}_3,$$

and



$$= 0 \quad \forall \sigma \in \mathbb{S}_3.$$

Proof. Let $F_{-p}(\mathscr{A}ss_\infty^{(2)})$ be the subspace of $\mathscr{A}ss_\infty^{(2)}$ spanned by trees with at least p internal edges between one vertex labelled by $\bullet$ and the other one labelled by $\blacktriangledown$. This defines an increasing filtration which is bounded on $\mathscr{A}ss_\infty^{(2)}(n)$ for each n. Therefore it converges to the homology of $\mathscr{A}ss_\infty^{(2)}(n)$ by the Classical Convergence Theorem 5.5.1 of [55]. The first term E^0_{pq} is equal to the subspace of $\mathscr{A}ss_\infty^{(2)}$ spanned by trees with exactly p internal edges between one vertex labelled by $\bullet$ and the other one labelled by $\blacktriangledown$. And the differential d^0 is equal to the sum of the differentials $\partial_{\mathscr{A}ss_\infty^\bullet} + \partial_{\mathscr{A}ss_\infty^\blacktriangledown}$, that is it splits $\bullet$ and $\blacktriangledown$ vertices into pure $\bullet$ and $\blacktriangledown$ trees. Hence (E^0, d^0) is the coproduct $\mathscr{A}ss_\infty^\bullet \vee \mathscr{A}ss_\infty^\blacktriangledown$ (see Section A.3) of two resolutions of $\mathscr{A}ss$, which is acyclic. Finally, we have $E^1_{pq} = 0$ for $p + q \neq 0$ and $\bigoplus_{p \geqq 0} E^1_{-pp} = \mathscr{A}ss^\bullet \vee \mathscr{A}ss^\blacktriangledown$. The spectral sequence collapses and the homology of $\mathscr{A}ss_\infty^{(2)}$ is concentrated in degree 0. Another presentation of this homology group is given by the quotient of the free operad on degree 0 elements, namely the two binary products $\bullet$ and $\blacktriangledown$, by the ideal generated by the image under ∂ of the degree 1 elements of $\mathscr{A}ss_\infty^{(2)}$. $\square$

In other words, the operad $\mathscr{A}ss^{(2)}$ is Koszul. A representation of $\mathscr{A}ss^{(2)}$ in a vector space X is equivalent to a pair of linear maps $\mu : X \otimes X \to X$ and $\nu : X \otimes X \to X$ such that both (X, μ) and $(X, \mu + \nu)$ are associative algebras. As a corollary, we get the following isomorphism of $\mathbb{S}$-modules $\mathscr{A}ss^{(2)} \cong \mathscr{A}ss \vee \mathscr{A}ss$.

Remark. The example of $\mathscr{A}ss^{(2)}$ is also interesting from the viewpoint of Koszul operad. It comes from a set theoretic operad. It is Koszul whereas the method of [50] cannot be applied because $\mathscr{A}ss^{(2)}$ is not basic set, that is the composition of operations is not injective. The product $\blacktriangledown$ has an "absorbing" effect.

In the same way, we define the operad $\mathscr{L}ie^{(2)}$ by

$$\mathscr{F}([\,,]_\bullet \oplus [\,,]_\blacktriangledown)/\big(\mathrm{Jac}_\bullet^\bullet \oplus (\mathrm{Jac}_\blacktriangledown^\bullet + \mathrm{Jac}_\bullet^\blacktriangledown + \mathrm{Jac}_\blacktriangledown^\blacktriangledown)\big),$$

where $[\,,]_\bullet$ and $[\,,]_\blacktriangledown$ stand for two skew-symmetric brackets and where Jac_a^b stands for the "Jacobi" relation $[[X, Y]_a, Z]_b + [[Y, Z]_a, X]_b + [[Z, X]_a, Y]_b = 0$. This operad enjoys the same properties with $\mathscr{L}ie$ as the non-symmetric operad $\mathscr{A}ss^{(2)}$ with $\mathscr{A}ss$ explained above.

More generally, to any binary quadratic operad $\mathscr{P}$ (eventually non-symmetric) with its minimal model $\mathscr{P}_\infty$, we can associate an operad $\mathscr{P}^{(2)}$ such that $\mathscr{P}_\infty^{(2)}$ is its minimal model.

We summarize this result with the explicit form of $\mathscr{P}^{(2)}$ in terms of Manin products in the following theorem. Let $\mathscr{P} = \mathscr{F}(V)/(R)$ and $\mathscr{Q} = \mathscr{F}(W)/(S)$ be two binary quadratic non-symmetric operads. There exists a morphism of

$$\psi : \mathscr{F}(V)(3) \otimes \mathscr{F}(W)(3) \to \mathscr{F}(V \otimes W)(3).$$

Manin's black square product of $\mathscr{P}$ and $\mathscr{Q}$ is equal to $\mathscr{P} \blacksquare \mathscr{Q} := \mathscr{F}(V \otimes W)/\big(\psi(R \otimes S)\big)$. In the symmetric case, the definition is similar but the morphism ψ is more involved and requires the use of signature representations (see [51] for more details).

Theorem 31. *For any binary quadratic non-symmetric operad $\mathscr{P}$ which admits a minimal model $\mathscr{P}_\infty$, the non-symmetric operad $\mathscr{P}_\infty^{(2)}$ is a minimal model* (*resolution*) *of $\mathscr{P} \blacksquare \mathscr{A}ss^{(2)}$, which is isomorphic, as a graded module, to $\mathscr{P} \vee \mathscr{P}$, where the coproduct has to be taken in the category of non-symmetric operads.*

For any binary quadratic operad $\mathscr{P}$ which admits a minimal model $\mathscr{P}_\infty$, the operad $\mathscr{P}_\infty^{(2)}$ is a minimal model (*resolution*) *of $\mathscr{P} \bullet \mathscr{L}ie^{(2)}$which is isomorphic, as an $\mathbb{S}$-module to $\mathscr{P} \vee \mathscr{P}$.*

Proof. By the same argument as in Proposition 30 above, the homology of $\mathscr{P}_\infty^{(2)}$ is concentrated in degree 0. If we denote the quadratic operad $\mathscr{P}$ by $F(V)/(\mathscr{R})$, this non-trivial homology group is equal to $F(V_\bullet \oplus V_\blacktriangledown)/(\mathscr{R}_\bullet^\bullet \oplus \mathscr{R}_\blacktriangledown^\bullet + \mathscr{R}_\bullet^\blacktriangledown + \mathscr{R}_\blacktriangledown^\blacktriangledown)$, which is equal to the black product of $\mathscr{P}$ with $\mathscr{L}ie^{(2)}$. $\square$

Appendix A. Model category structure for prop(erad)s

In this appendix, we prove that the categories of properads have a cofibrantly generated model category structure. We make precise coproducts, pushouts, cofibrations and cofibrant objects. We refer the reader to the book [17] of M. Hovey for a comprehensive treatment of model categories. (In order to be self-contained in this appendix, we will not avoid the prefix dg here.)

Let us denote by dg *Properads* the category of dg properads. It means either the category of reduced dg properads ($\mathscr{P}(m,n) = 0$ for $n = 0$ or $m = 0$) or the category of dg properads over a field of characteristic 0. By default, we work over unbounded chain complexes but the following proofs hold over bounded chain complexes as well. We transfer the cofibrantly generated model category structure of $\mathbb{S}$-bimodules to the category of properads via the free properad functor.

A.1. Model category structure on $\mathbb{S}$-bimodules. The category of dg $\mathbb{S}$-bimodules is endowed with a cofibrantly generated model category structure coming from the cofibrantly generated model category structure on dg $\mathbb{K}$-modules.

Recall from [17], Theorem 2.3.11, that the category of dg R-modules has a cofibrantly generated model category structure for any ring R. Quasi-isomorphisms form the class of weak equivalences and degreewise surjective maps form the class of fibrations. Let us make explicit the (generating) acyclic cofibrations. The model category of dg $\mathbb{K}$-modules is cofibrantly generated by the acyclic cofibrations $J^k : 0 \to D^k$, where D^k is the chain complex

$$\cdots \to 0 \to \underbrace{\mathbb{K}}_{k} \xrightarrow{\mathrm{Id}} \underbrace{\mathbb{K}}_{k-1} \to 0 \to \cdots$$

and by the cofibrations $I^k : S^{k-1} \to D^k$, where S^{k-1} is the following chain complex

$$\cdots \to 0 \to \underbrace{\mathbb{K}}_{k-1} \to 0 \to \cdots.$$

For any $m, n \in \mathbb{N}$, the category of left $\mathbb{S}_m$ and right $\mathbb{S}_n$-bimodules is the category of dg modules over the group ring $\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]$. By [17], Theorem 2.3.11, the preceding theorem, it has a cofibrantly generated model category structure, where the generating acyclic cofibrations are the maps $J_{m,n}^k : 0 \to D_{m,n}^k$, where $D_{m,n}^k$ is the acyclic dg $\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]$-module

$$\cdots \to 0 \to \underbrace{\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]}_{k} \xrightarrow{\mathrm{Id}} \underbrace{\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]}_{k-1} \to 0 \to \cdots$$

and where the generating cofibrations are the maps $I_{m,n}^k : S_{m,n}^{k-1} \to D_{m,n}^k$, with $S_{m,n}^{k-1}$ the following dg $\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]$-module

$$\cdots \to 0 \to \underbrace{\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]}_{k-1} \to 0 \to \cdots.$$

Since the category of dg $\mathbb{S}$-bimodules is the product over $(m,n) \in \mathbb{N}^2$ of the model categories of left $\mathbb{S}_m$ and right dg $\mathbb{S}_n$-bimodules, it is naturally endowed with a term-by-term cofibrantly generated model category structure. The set of generating acyclic cofibrations can be chosen to be $J = \{\tilde{J}_{m,n}^k \,|\, k \in \mathbb{Z}, m, n \in \mathbb{N}\}$, where $\tilde{J}_{m,n}^k$ is equal to $J_{m,n}^k : 0 \to D_{m,n}^k$ in arity (m,n) and 0 elsewhere. Similarly, the set of generating cofibrations can be chosen to be $I = \{\tilde{I}_{m,n}^k \,|\, k \in \mathbb{Z}, m, n \in \mathbb{N}\}$, where $\tilde{I}_{m,n}^k$ is equal to $I_{m,n}^k : S_{m,n}^{k-1} \to D_{m,n}^k$ in arity (m,n) and 0 elsewhere. Notice that the domains of elements of I or J are sequentially small with respect to any map in the category of dg $\mathbb{S}$-bimodules.

A.2. Transfer theorem. In the section, we recall the theorem of transfer, mainly due to Quillen [41], Section II.4 (see also S. E. Crans [10], Theorem 3.3, and M. Hovey [17], Proposition 2.1.19). We will use it to endow the category of dg prop(erad)s with a model category structure.

Definition (relative I-cell complexes). For every class I of maps of a category, a *relative* I-*cell complex* is a sequential colimit of pushouts of maps of I.

Let us make explicit this type of morphisms. A relative I-cell complex is a map $A_0 \xrightarrow{\varphi} A_\infty$ which comes from a sequential colimit

$$\begin{tikzcd}
A_0 \arrow[r, "\imath_0"] \arrow[d, "\varphi"] & A_1 \arrow[r, "\imath_1"] \arrow[dl] & \cdots \arrow[r] & A_n \arrow[r, "\imath_n"] \arrow[dlll] & A_{n+1} \arrow[r] \arrow[dllll] & \cdots \\
A_\infty := \mathrm{Colim}_{\mathbb{N}} A_n & & & & &
\end{tikzcd}$$

where each map $A_n \xrightarrow{i_n} A_{n+1}$ is defined by a pushout

$$\begin{tikzcd} \bigvee_\alpha S_\alpha \arrow[r] \arrow[d, "\bigvee_\alpha j_\alpha"] & A_n \arrow[d, "i_n"] \\ \bigvee_\alpha T_\alpha \arrow[r] & A_{n+1} \end{tikzcd}$$

with $j_\alpha \in I$. As usual, we denote the collection of relative I-cell complexes by I-cell.

Theorem 32 ([41], Section II.4, [10], Theorem 3.3, [17], Proposition 2.1.19). *Let $\mathscr{C}$ be a cofibrantly generated model category with I as the set of generating cofibrations and J as the set of generating acyclic cofibrations. Let $F : \mathscr{C} \rightleftharpoons \mathscr{D} : U$ be an adjunction, where F is the left adjoint and U the right adjoint. Suppose that*

(1) *$\mathscr{D}$ has finite limits and colimits,*

(2) *the functor U preserves filtered colimits,*

(3) *the image under U of any relative $F(J)$-cell complex is a weak equivalence in $\mathscr{C}$.*

A map f in $\mathscr{D}$ is defined to be a weak equivalence (resp. fibration) if the associated map $U(f)$ is a weak equivalence (resp. fibration) in $\mathscr{C}$. The class of cofibrations in $\mathscr{D}$ is the class of maps that verify the left lifting property (LLP) *with respect to acyclic fibrations.*

These three classes of maps provide the category $\mathscr{D}$ with a model category structure cofibrantly generated by $F(I)$ as the set of generating cofibrations and $F(J)$ as the set of generating acyclic cofibrations.

We also refer the reader to [6], Section 2.5, for the application of this theorem with stronger and sometimes more convenient hypotheses. Remark that Transfer Theorem 32 was used (and rephrased) by V. Hinich in [15] to provide a model category structure to the category of operads over unbounded chain complexes (see [15], Theorem 2.2.1, and the corrected version of [16], Theorem 6.6.1). M. Spitzweck also applied this theorem to prove a general result about model category structures on categories of algebras over a triple ([44], Theorem 1).

A.3. Limits and colimits of properads. In this section, we prove that the category of properads has all limits and finite colimits. We also make explicit the coproducts and pushouts of properads.

Proposition 33. *The category of properads has all limits.*

Proof. We recall from D. Borisov and Y. I. Manin [7] that the free properad functor induces a triple $\mathscr{F} : \mathbb{S}\text{-biMod} \to \mathbb{S}\text{-biMod}$ such that an algebra over it is a properad. Since the underlying category of $\mathbb{S}$-bimodules has limits, the category of properads has all limits ([14], Section 1.5). $\square$

To prove that the category of properads has finite colimits, we first make explicit coproducts and pushouts. This section is the generalization of [14], Section 1.5, from operads to properads. Once again, the situation is more subtle for properads than for operads since it requires the notion of adjacent vertices of a graph (see [39], Section 4.2).

Let $\mathscr{P}$ and $\mathscr{Q}$ be two properads. The coproduct of $\mathscr{P}$ and $\mathscr{Q}$ is given by a quotient of the free properad on their sum $\mathscr{F}(\mathscr{P} \oplus \mathscr{Q})$. On this space, we define an equivalence relation by the following generating relation: if a graph g, with vertices indexed by elements of $\mathscr{P}$ and $\mathscr{Q}$ has two adjacent vertices indexed elements of $\mathscr{P}$ (or $\mathscr{Q}$), it is equivalent to the same graph, where the two adjacent vertices are contracted and the new vertex is labelled by the composition in $\mathscr{P}$ (or $\mathscr{Q}$) of the two associated elements of $\mathscr{P}$ (or $\mathscr{Q}$). The quotient of $\mathscr{F}(\mathscr{P} \oplus \mathscr{Q})$ by this relation is the coproduct of $\mathscr{P}$ and $\mathscr{Q}$. We denote it by $\mathscr{P} \vee \mathscr{Q}$. This $\mathbb{S}$-bimodule has the following basis. It can be represented by the sum over connected graphs with vertices indexed by elements of $\mathscr{P}$ and $\mathscr{Q}$ such that no adjacent pair of vertices is labelled by the same kind of elements (see Figure 1).

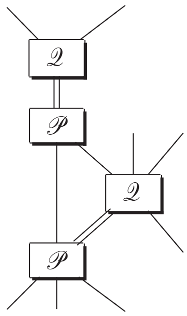


Figure 1. Element of the coproduct $\mathscr{P} \vee \mathscr{Q}$

Let $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ be three properads. Let $f : \mathscr{P} \to \mathscr{Q}$ and $g : \mathscr{P} \to \mathscr{R}$ be two morphisms of properads. Their pushout is isomorphic to the quotient of $\mathscr{Q} \vee \mathscr{R}$ by the ideal generated by $\{f(p) - g(p) \mid p \in \mathscr{P}\}$. (We refer to [51], Appendix B, for the notion of ideal of a prop(erad). The notion of ideal generated by a sub-$\mathbb{S}$-bimodule is also made explicit there.) The pushout $\mathscr{Q} \vee_{\mathscr{P}} \mathscr{R}$ is represented by labelled connected graphs as above but further quotient by the following relation: if a vertex is labelled by an element of the form $f(p)$ for $p \in \mathscr{P}$, it can be replaced by the same vertex labelled by the corresponding element $g(p)$ and vice-versa. When this operation generates two adjacent vertices indexed by elements of the same properad, they are to be composed.

Proposition 34. *The category of properads has finite colimits.*

Proof. This result can be proved with two methods.

First, recall that the free properad on an $\mathbb{S}$-bimodule V is given by the sum on connected graphs without level whose vertices are coherently labelled by elements of V (see [52], Section 2.7). We denote it by

$$\mathscr{F}(V) = \Bigg(\bigoplus_{g \in \mathscr{G}_c} \bigotimes_{v \in \mathscr{N}(g)} V\big(|\mathrm{Out}(v)|, |\mathrm{In}(v)|\big) \Bigg) / \approx,$$

in [52], Theorem 2.3, where $\mathcal{N}(g)$ is the set of vertices of a graph g. Since the tensor product of dg $\mathbb{S}$-bimodules preserves colimits, the functor

$$\mathbb{S}\text{-biMod} \to \mathbb{S}\text{-biMod},$$
$$V \mapsto \bigotimes_{v \in \mathcal{N}(g)} V\big(|\mathrm{Out}(v)|, |\mathrm{In}(v)|\big),$$

associated to any graph g, preserves filtered colimits (see [14], Lemma 1.14). Then the triple $\mathcal{F} : \mathbb{S}\text{-biMod} \to \mathbb{S}\text{-biMod}$ associated to the free properad functor preserves filtered colimits. The argument of [14], page 16, proves that the category of properads has filtered colimits. Since it has pushouts and filtered colimits, it has finite colimits by [28], Chapter IX.

We can also construct coequalizers in this category. Since it is an additive category, it is enough to construct cokernels. Let $f : \mathcal{P} \to \mathcal{Q}$ be a morphism of properads. Its cokernel is given by the quotient of $\mathcal{Q}$ with the ideal generated by the image of f. Since it has coproducts and coequalizers, this category has finite colimits by [28], Theorem 2.1, Chapter V. $\square$

A.4. Model category structure. In this section, we apply the Transfer Theorem 32 to provide a cofibrantly generated model category structure on the category of properads.

We consider the free properad adjunction $\mathcal{F} : \mathrm{dg}\,\mathbb{S}\text{-biMod} \rightleftharpoons \mathrm{dg}\,\mathrm{Properads} : U$. We proved in A.1 that the category on the left-hand side is a cofibrantly generated model category. We apply the Transfer Theorem 32 to this adjunction as follows. The generating acyclic cofibrations are $\mathcal{F}(J) = \{I \to \mathcal{F}(D^k_{m,n})\}$ and the generating cofibrations are $\mathcal{F}(I) = \{\mathcal{F}(S^{k-1}_{m,n}) \to \mathcal{F}(D^k_{m,n})\}$.

Lemma 35. *A morphism of* dg *properads is a relative $\mathcal{F}(J)$-cell complex if and only if it is a map $\mathcal{P} \to \mathcal{P} \vee \mathcal{F}(D)$, where $D = \bigoplus_{d \geqq 1} D_i$ is an acyclic* dg *$\mathbb{S}$-bimodule whose components are free $\mathbb{S}$-bimodules with each D_i equal to a direct sum of* dg *$\mathbb{S}$-bimodules $D^k_{m,n}$.*

A morphism of dg *properads is a relative $\mathcal{F}(I)$-cell complex if and only if it is a map $\mathcal{P} \to \mathcal{P} \vee \mathcal{F}(S)$, where S is a* dg *$\mathbb{S}$-bimodule, whose components are free $\mathbb{S}$-bimodules, endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{Colim}_i\, S_i = S$$

such that $d : S_i \to \mathcal{F}(S_{i-1})$ and such that $S_{i-1} \rightarrowtail S_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules with cokernels isomorphic to a free $\mathbb{S}$-bimodule.*

Proof. Pushouts of elements of $\mathcal{F}(J)$ are as follows:

$$\begin{array}{ccc} I & \longrightarrow & \mathcal{P} \\ \Big\downarrow{\scriptstyle \bigvee_\alpha \mathcal{F}(J^\alpha)} & & \Big\downarrow \\ \bigvee_\alpha \mathcal{F}(D^\alpha) & \longrightarrow & \mathcal{P} \vee \left(\bigvee_\alpha \mathcal{F}(D^\alpha)\right), \end{array}$$

with each D^α equal to a $D^k_{m,n}$. Since the coproduct of free properads is the free properad on the sum of their generating spaces $\mathscr{F}(V) \vee \mathscr{F}(V') \cong \mathscr{F}(V \oplus V')$, the composite of two such maps is equal to $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}\left(\bigoplus_\alpha D^\alpha \oplus \bigoplus_\beta D^\beta\right)$. Hence a sequential colimit of such pushouts has the form $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}(D)$, with $D = \bigoplus_{d \geqq 1} D_i$ an acyclic dg $\mathbb{S}$-bimodule whose components are free $\mathbb{S}$-bimodules.

A pushout of an element of $\mathscr{F}(I)$ is

$$\begin{tikzcd} \bigvee_\alpha \mathscr{F}(S^\alpha) \arrow[r, "f"] \arrow[d, "\bigvee_\alpha \mathscr{F}(I^\alpha)"', tail] & \mathscr{P} \arrow[d] \\ \bigvee_\alpha \mathscr{F}(D^\alpha) \arrow[r] & \mathscr{Q} \end{tikzcd}$$

with each S^α equal to an $S^k_{m,n}$ and D^α equal to a $D^k_{m,n}$. We denote by z the image under f of the generating element of $S^{k-1}_{m,n}$. Notice that z is a cycle in $\mathscr{P}$. If we denote by ξ and $d\xi$ the generating elements of $D^k_{m,n}$, the pushout $\mathscr{Q}$ is equal to $\mathscr{P} \vee \mathscr{F}\left(\bigoplus_\alpha \xi^\alpha.\mathbb{K}[\mathbb{S}^{\mathrm{op}}_m \times \mathbb{S}_n]\right)$ with $d\xi = z$. Therefore a relative $\mathscr{F}(I)$-cell complex is a map $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}(S)$, with S a dg $\mathbb{S}$-bimodule whose components are free $\mathbb{S}$-bimodules. Since a relative $\mathscr{F}(I)$-cell complex is a sequential colimit of such pushouts, the filtration of S is given by this sequential guing of cells. $\square$

Theorem 36. *The category of properads has a cofibrantly model category structure provided by the following three classes of morphisms. A map $\mathscr{P} \xrightarrow{f} \mathscr{Q}$ is a*

- *weak equivalence if and only if it is a quasi-isomorphism of* dg $\mathbb{S}$*-bimodules, that is a quasi-isomorphism in any arity,*

- *fibration if and only if it is a degreewise surjection in any arity,*

- *cofibration if and only if it has the left lifting property with respect to acyclic fibrations.*

The generating cofibrations are the maps $\mathscr{F}(I) = \{\mathscr{F}(S^{k-1}_{m,n}) \to \mathscr{F}(D^k_{m,n})\}$ and the generating acyclic cofibrations are the maps $\mathscr{F}(J) = \{I \to \mathscr{F}(D^k_{m,n})\}$.

Proof. The category of properads has finite limits and colimits (1) by the preceding section. To any dg $\mathbb{S}$-bimodule M, we can consider the trivial (abelian) prop(erad) structure on $I \oplus M$, that is the composite product is zero on M. So, it is easy to check that the forgetful functor preserves filtered colimits (2). Recall from A.3 that the coproduct $\mathscr{P} \vee \mathscr{F}(D)$ admits a basis composed by (connected) graphs with vertices indexed elements of $\mathscr{P}$ and D such that there is no pair of adjacent vertices indexed by two elements of $\mathscr{P}$. Therefore, $\mathscr{P} \vee \mathscr{F}(D)$ is equal to the directed sum $\mathscr{P} \oplus X$, where X has a basis given by graphs indexed by elements coming from $\mathscr{P}$ and at least one element from D. The map $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}(D)$ is the inclusion of $\mathscr{P}$ into the first summand so that it is enough to prove that X is an acyclic chain complex. For every graph g indexed by elements of $\mathscr{P}$ and at least one element of D, the resulting chain complex is isomorphic to a quotient by the action of some symmetric groups of tensor products of $\mathscr{P}$ and at least one D. Since D is an acyclic chain complex made of free $\mathbb{K}[\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n]$-modules, it is an acyclic projective chain complex over any ring of symmetric subgroup. Hence the chain complex associated to any graph g indexed by elements of $\mathscr{P}$ and at least one element of D is acyclic, which proves hypothesis (3) of Transfer Theorem 32. $\square$

A.5. Cofibrations and cofibrant objects. In this section, we make explicit the cofibrations and the cofibrant objects in the model category of dg properads. We refer to the Appendix of [11] for the case of operads.

Proposition 37. *A map $f : \mathscr{P} \rightarrowtail \mathscr{Q}$ is a cofibration in the model category of* dg *properads if and only if it is a retract of a map $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}(S)$, with isomorphisms on domains, where S is a* dg *$\mathbb{S}$-bimodule whose components are free $\mathbb{S}$-bimodules, endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{Colim}_i\, S_i = S$$

such that $d : S_i \to \mathscr{F}(S_{i-1})$ and such that $S_{i-1} \rightarrowtail S_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules with cokernels isomorphic to a free $\mathbb{S}$-bimodule.*

A map $f : \mathscr{P} \overset{\sim}{\rightarrowtail} \mathscr{Q}$ is an acyclic cofibration in the model category of dg *properads if and only if it is a retract of a map $\mathscr{P} \to \mathscr{P} \vee \mathscr{F}(D)$, with isomorphisms on domains, where $D = \bigoplus_{d \geqq 1} D_i$ is an acyclic* dg *$\mathbb{S}$-bimodule whose components are free $\mathbb{S}$-bimodules with each D_i equal to a direct sum of* dg *$\mathbb{S}$-bimodules $D^k_{m,n}$.*

Proof. The proposition follows from general results on the (acyclic) cofibrations of cofibrantly generated model categories. Explicitely, we apply [17], Proposition 2.1.18 to the cofibrantly generated model category of properads. This proposition gives explicitly that (acyclic) cofibrations of prop(erad)s are retracts of relative $\mathscr{F}(I)$-cell complexes (relative $\mathscr{F}(J)$-cell complexes). We conclude by Lemma 35. $\square$

Applied to $\mathscr{P} = I$, this proposition gives the following corollary.

Proposition 38. *A* dg *properad is cofibrant for this model category structure if and only if it is a retract of a quasi-free properad $\mathscr{F}(S)$, where the components of S are free $\mathbb{S}$-bimodules, endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{Colim}_i\, S_i = S$$

such that $d : S_i \to \mathscr{F}(S_{i-1})$ and such that $S_{i-1} \rightarrowtail S_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules with cokernels isomorphic to a free $\mathbb{S}$-bimodule.*

Remark. In the model category of dg properads on non-negatively graded dg $\mathbb{S}$-bimodules, a dg properad is cofibrant if and only if it is retract of a quasi-free properad $\mathscr{F}(S)$ whose components are free $\mathbb{S}$-bimodules. The extra assumption on the filtration is automatically given by the homological degree.

Recall that we are working over a field of characteristic 0.

Lemma 39. *Any quasi-free properad $\mathscr{F}(X)$ is a retract of a quasi-free properad $\mathscr{F}(S)$, where the components of S are free $\mathbb{S}$-bimodules. Moreover, if X is endowed with an exhaustive filtration*

$$X_0 = \{0\} \subset X_1 \subset X_2 \subset \cdots \subset \mathrm{Colim}_i\, X_i = X$$

such that $d : X_i \to \mathscr{F}(X_{i-1})$ and such that $X_{i-1} \rightarrowtail X_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules, then S can be chosen with the same property and such that the cokernels of the $S_{i-1} \rightarrowtail S_i$ are free $\mathbb{S}$-bimodules.*

Proof. Let $\overline{X}(m,n)$ denote the set of equivalence classes under the action of $\mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n$. For simplicity, we use the generic notation $\overline{X}$. We choose a set of representatives $\{\bar{x}_i\}_{i\in\mathscr{I}}$ of $\overline{X}$. Let S be the free $\mathbb{S}$-bimodule generated by the $\{\bar{x}_i\}_{i\in\mathscr{I}}$. The generator associated to $\bar{x}_i$ will be denoted by s_i. For any x in X, we consider the sub-group $\mathbb{S}_x := \{\sigma \in \mathbb{S}_m^{\mathrm{op}} \times \mathbb{S}_n \mid x.\sigma = \chi(\sigma)x, \chi(\sigma) \in \mathbb{K}\}$. In this case, χ is a character of $\mathbb{S}_x$. We define the following element of S:

$$N(\bar{x}_i) := \frac{1}{|\mathbb{S}_{\bar{x}_i}|} \sum \chi(\sigma^{-1}).s_i\sigma,$$

where the sum runs over $\sigma \in \mathbb{S}_{\bar{x}_i}$. The image under the boundary map ∂ of an $\bar{x}_i$ is a sum of graphs of the form $\sum \mathscr{G}(\bar{x}_{i_1}, \dots, \bar{x}_{i_k})$. We define the boundary map ∂' on $\mathscr{F}(S)$ by

$$\partial'(s_i) := \sum \frac{1}{|\mathbb{S}_{\bar{x}_i}|} \sum \chi(\sigma^{-1}).\mathscr{G}\big(N(\bar{x}_{i_1}), \dots, N(\bar{x}_{i_k})\big)\sigma,$$

where the second sum runs over $\sigma \in \mathbb{S}_{\bar{x}_i}$. Finally, we define the maps of dg prop(erad)s $\mathscr{F}(S) \to \mathscr{F}(X)$ by $s_i \mapsto \bar{x}_i$ and $\mathscr{F}(X) \to \mathscr{F}(S)$ by $\bar{x}_i \mapsto N(\bar{x}_i)$. They form a deformation retract, which preserves the filtration of X when it exists. $\square$

Corollary 40. *In the model category of* dg *properads, any quasi-free properad $\mathscr{F}(X)$, where X is endowed with an exhaustive filtration*

$$X_0 = \{0\} \subset X_1 \subset X_2 \subset \cdots \subset \mathrm{Colim}_i\, X_i = X$$

such that $d : X_i \to \mathscr{F}(X_{i-1})$ and such that the $X_{i-1} \rightarrowtail X_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules is cofibrant.*

Remark. In the non-negatively graded case, any quasi-free properad is cofibrant.

Proof. It is a direct corollary of Proposition 38 and Lemma 39. $\square$

Theorem 41. *Any* dg *properad $\mathscr{Q}$ admits a cofibrant replacement of the form $\mathscr{F}(S) \xrightarrow{\sim} Q$, where the components of S are free $\mathbb{S}$-bimodules, endowed with an exhaustive filtration*

$$S_0 = \{0\} \subset S_1 \subset S_2 \subset \cdots \subset \mathrm{Colim}_i\, S_i = S$$

such that $d : S_i \to \mathscr{F}(S_{i-1})$ and such that $S_{i-1} \rightarrowtail S_i$ are split monomorphisms of dg *$\mathbb{S}$-bimodules with cokernels isomorphic to a free $\mathbb{S}$-bimodule.*

Proof. Any dg properad $\mathscr{Q}$ admits a cofibrant replacement $I \rightarrowtail \mathscr{P} \xrightarrow{\sim} Q$. Since $\mathscr{P}$ is cofibrant, it is retract $\mathscr{P} \xrightarrow{\sim} \mathscr{F}(S) \xrightarrow{\sim} \mathscr{P}$ of such an $\mathscr{F}(S)$ by Proposition 38. $\square$

We can simplify such a cofibrant replacement as follows.

Theorem 42. *A quasi-free cofibrant replacement $\mathscr{F}(S) \xrightarrow{\sim} Q$ induces a quasi-free cofibrant replacement $\mathscr{F}(X) \xrightarrow{\sim} Q$, where the action of the symmetric groups on the components of X is the same as the action on their image in $\mathscr{Q}$. Moreover, X is endowed with an exhaustive filtration*

$$X_0 = \{0\} \subset X_1 \subset X_2 \subset \cdots \subset \mathrm{Colim}_i\, X_i = X$$

such that $d : X_i \to \mathscr{F}(X_{i-1})$ and such that $X_{i-1} \rightarrowtail X_i$ are split monomorphisms of dg $\mathbb{S}$-*bimodules.*

Proof. The dg $\mathbb{S}$-bimodule which generates the quasi-free cofibrant replacement $\mathscr{F}(S) \xrightarrow{\sim} Q$ is a free $\mathbb{S}$-bimodule. Let us denote by s_α the generators and q_α their image in $\mathscr{Q}$. We define X to be the $\mathbb{S}$-bimodule generated by the q_α and we consider the free properad $\mathscr{F}(X)$ on X. In $\mathscr{F}(S)$, the image of s_α under the differential map d is equal to $d(s_\alpha) = \sum \mathscr{G}(s_{\alpha_1}, \ldots, s_{\alpha_k})$. We define the differential map of $\mathscr{F}(X)$ by $d(q_\alpha) = \sum \mathscr{G}(q_{\alpha_1}, \ldots, q_{\alpha_k})$. The map $\mathscr{F}(S) \xrightarrow{\sim} Q$ factors through $\mathscr{F}(S) \xrightarrow{\sim} \mathscr{F}(X) \xrightarrow{\sim} Q$. Finally, $\mathscr{F}(X)$ is cofibrant by Corollary 40. $\square$

The difference between resolution $\mathscr{F}(S)$ and $\mathscr{F}(X)$ is that in $\mathscr{F}(S)$, the symmetry of the operations of $\mathscr{Q}$ is deformed up to homotopy whereas in $\mathscr{F}(X)$ only the relations are deformed up to homotopy. (The same phenomenon appears for resolutions of the operad *Com* of commutative algebras where the former corresponds to E_∞ operads and the later to C_∞.)

We can now choose to work with such cofibrant models. The extra filtration on the space of generators, which appears conceptually here, is similar to the one used by Sullivan [46] in rational homotopy theory and by Markl in [30] for operads.

Let $\mathscr{P}$ be a dg properad. Its space of *indecomposable elements* is the cokernel of the composite map with non-trivial elements, $\mu : \overline{\mathscr{P}} \boxtimes \overline{\mathscr{P}} \to \overline{\mathscr{P}}$. The space of indecomposable elements inherits a differential map from the one of $\mathscr{P}$ which makes it into a dg $\mathbb{S}$-bimodule. The associated functor $\mathrm{Indec} : \mathrm{dg\,properads} \to \mathrm{dg}\,\mathbb{S}\text{-bimodules}$ is left adjoint to the augmentation functor $M \mapsto M \oplus I$, where the properad structure on $M \oplus I$ is the trivial one.

The following last result will allow us to prove that the deformation complex defined in Section 2 does not depend on the quasi-free model chosen to make it explicit.

Proposition 43. *Any weak equivalence* (*quasi-isomorphism*) *between two quasi-free cofibrant* dg *properads $\mathscr{F}(X) \xrightarrow{\sim} \mathscr{F}(Y)$ induces a weak equivalence* (*quasi-isomorphism*) *between the spaces of indecomposable elements $X \xrightarrow{\sim} Y$.*

Proof. The two categories of dg properads and dg $\mathbb{S}$-bimodules have model categories structures. Since the augmentation functor preserves fibrations and acyclic fibrations, by [17], Lemma 1.3.4, the indecomposable functors, being its left adjoint, preserves cofibrations and acyclic cofibrations. And by Brown's Lemma ([17], Lemma 1.1.12), it preserves weak equivalences between cofibrant objects. □

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